Convex Geometries and their Representations

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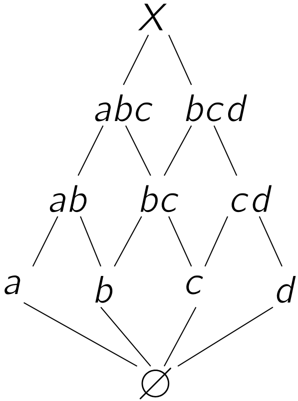
June 23, 2022

NYC Discrete Math REU

What is a convex geometry?

$$X = \{a, b, c, d\}, \quad \mathcal{F} = \{\emptyset, a, b, c, d, ab, bc, cd, abc, bcd, X\}$$



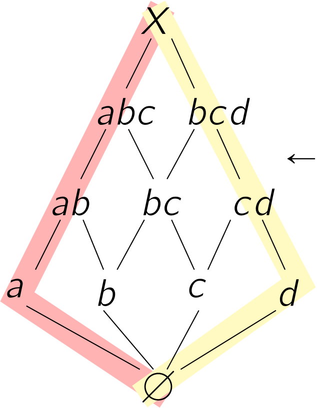
What is a convex geometry?

Properties: $(X, \mathcal{F})$

1. X and $\varnothing$ are in $\mathcal{F}$.
2. If $Y, X \in \mathcal{F}$, then $Y \cap Z \in \mathcal{F}$.
3. If $Y \in \mathcal{F}$, then there exists some element $a \in X$ where $Y \cup \{a\} \in \mathcal{F}$

What is a Convex Dimension?

What is the minimum number of chains needed to get all the elements in $\mathcal{F}$ through intersecting sets?



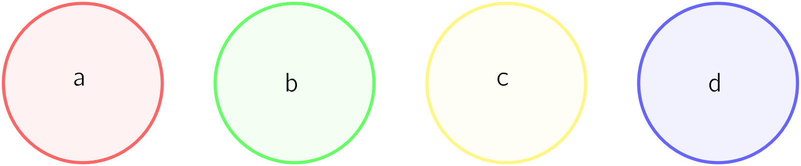
$\leftarrow$ convex dimension $= 2$!

Representation by Circles

1. Elements are represented by fixed circles in $\mathbb{R}^2$.

2. A subset is in $\mathcal{F}$ if the convex hull of circles does not contain another circle.

$X = \{a, b, c, d\}$ $\mathcal{F} = \{\emptyset, a, b, c, d, ab, bc, cd, abc, bcd, X\}$



Known and Past Results

Which dimensions can always be represented by circles in $\mathbb{R}^2$?

$CDim = 2$ ✓ *(G. Czédli 2014)*

$CDim \geqslant 4$ ✗ *(Polymath Jrs. 2020)*

$CDim = 3$?

Results in 3-Dimensions

5 element sets:

- 672 unique convex geometries
- 312 with convex dimension 3

Theorem (Polymath Jrs. 2021):
All convex geometries of dimension 3 on 5 elements can be represented by circles.

Main Result

6 element sets:

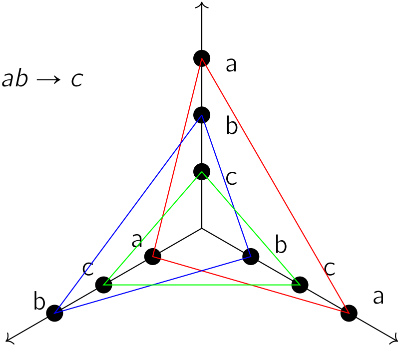
- 199,572 unique convex geometries
- $\approx$ 67,000 with convex dimension 3

Theorem (A&A&N 2022):
All convex geometries of dimension 3 on 6 element sets can be represented by circles.

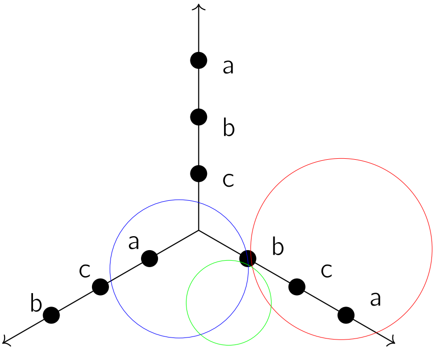
Useful Result

Theorem: (Richter and Rogers 2015) All 3-dimensional convex geometries can be represented by triangles.

Implication: $ab \to c$



Generalizing to Circles?



Rough Algorithm Idea

1. Create 3 chains for the geometry.
2. Place the chains in order on three rays.
3. Draw circles through the points for each element.
4. Fix the implications by moving points outwards.

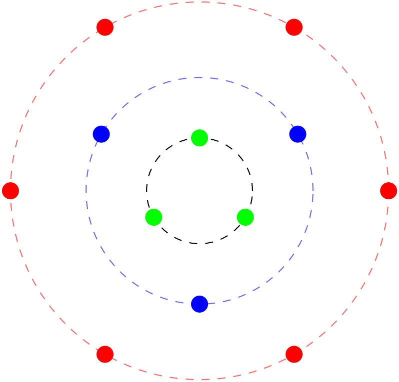
Ordering Implications

Rule: Every implication where an element is on the left should go before the implications where it is on the right.

$$\begin{array}{l} ae \to c \\ cd \to e \\ bd \to e \\ ad \to ce \\ bc \to de \\ ab \to cde \end{array} \quad \longrightarrow \quad \begin{array}{l} 1)\ cd \to e \\ 2)\ bd \to e \\ 3)\ bc \to de \\ 4)\ ad \to ce \\ 5)\ ae \to c \\ 6)\ ab \to cde \end{array}$$

Introducing Layers

Extreme Points: *elements that are maximal on at least one ray or chain.*



Fixing Implications by Layers

$$ae \to c$$
$$cd \to e$$
$$bd \to e$$
$$ad \to ce$$
$$bc \to de$$
$$ab \to cde$$

$$L_1 = \{a, b\}$$

$$L_2 = \{c, d\}$$

$$L_3 = \{e\}$$

Fixing Implications by Layers

Lemma 1: *There exists an element on the left side of every implication that is on a layer outside of the element on the right.*

Fixing Layer 3

1) $cd \to e$
2) $bd \to e$
3) $bc \to e$
4) $ad \to e$

Fixing Layer 2

5) $ae \to c$
6) $ad \to c$
7) $bc \to d$
8) $ab \to cde$

Fixing 2-Implications

Lemma 2: *Any 2-implication can be fixed by moving either element on the left of an implication.*

Option 1: Move b out on one ray.

Option 2: Move a out on two rays.

Fixing 3-implications

Lemma 3: *Any 3-implication can be fixed by moving any two elements on the left of an implication.*

Why only 6 Elements?

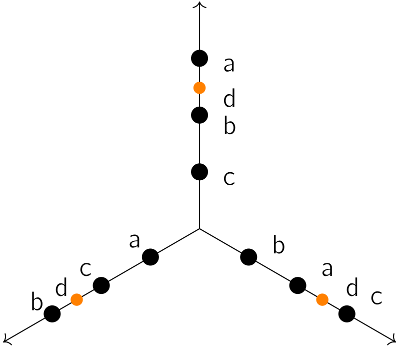
Issue Occurs When:

- Implications form a cycle.
- Cycle contains only 3-implications.
- Only 1 outer element on the left of each implication.

Theorem: *The situation above cannot happen on 6 element sets.*

Configuration of 3-implications

$$abc \to d$$



Useful Lemma

Lemma 4: A minimal cycle of length 2 with 3-implications is impossible.

Proof:

$$abc \to d$$
$$efd \to c$$

d must dominate c on two rays.

c must dominate d on two rays.

$\to$ Contradiction! There are only 3 rays