Quasi-Isometries & Baumslag-Solitar Groups

How we can think about the large scale geometry of $BS(m, n)$?

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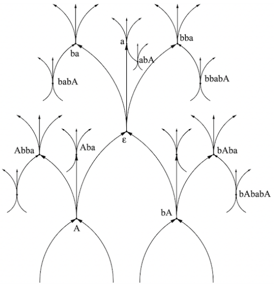
Geometric Group Theory · Bowdoin College



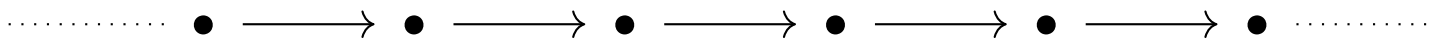
Overview

1. What is a Quasi Isometry?
    - Examples and Motivation
    - Definition
    - Some basic results
2. BS(m,n)
    - Main result
    - Introducing the argument
    - The level slide algorithm
3. Takeaways

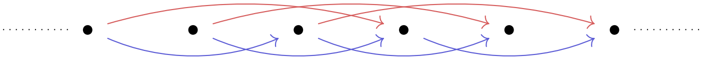
Motivations

How do we choose which generating set we work with for a group?

$\Gamma(\mathbb{Z}, \{1\})$



$\Gamma(\mathbb{Z}, \{2, 3\})$



Both are completely fine generating sets! On a **large scale** they are similar

Desiderata and Motivations

Can we find a good notion of **equivalence** between $\Gamma(\mathbb{Z}, \{1\})$ and $\Gamma(\mathbb{Z}, \{2, 3\})$?

What about more generally $\Gamma(G, S)$ and $\Gamma(G, S')$? This needs to be fine enough to capture the geometry of the group while ignoring small details.

Guiding Questions:

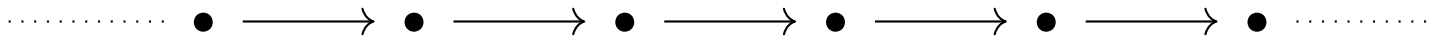
1. What properties of the Cayley Graph are **invariant** under this equivalence?
2. What do these properties then tell us about the group?

Bi-Lipschitz Equivalence

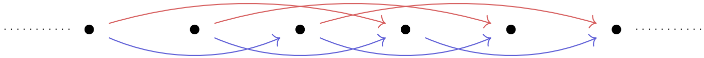
First Attempt: A bijection $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a **bi-lipschitz equivalence** if there is a constant $K \geqslant 1$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) \leqslant d_Y(f(x_1), f(x_2)) \leqslant K d_X(x_1, x_2)$$

$\Gamma(\mathbb{Z}, \{1\})$



$\Gamma(\mathbb{Z}, \{2, 3\})$



Are $\Gamma(\mathbb{Z}, \{1\})$ and $\Gamma(\mathbb{Z}, \{2, 3\})$ **bi-lipschitz equivalent**?

Bi-Lipschitz Equivalence Cont.

Claim: Let G be a group with finite generating sets S and S'. Then, (G, d_S) and $(G, d_{S'})$ are bi-lipschitz equivalent.

Proof Outline:

1. Let $M = \max_{s \in S \cup S^{-1}} d_{S'}(e, s)$
2. We claim that the identity map $\mathrm{Id} : (G, d_S) \to (G, d_{S'})$ is a bi-lipschitz equivalence.
3. We can prove the result for distances between the identity and an arbitrary element.
4. Consider an element $g \in G$ and consider a geodesic path $w_1 \dots w_k$ between v_e and v_g in $\Gamma(G, S')$ (i.e. g has length k in $(G, d_{S'})$).

Bi-Lipschitz Equivalence Cont.

Claim: Let G be a group with finite generating sets S and S'. Then, (G, d_S) and $(G, d_{S'})$ are bi-lipschitz equivalent.

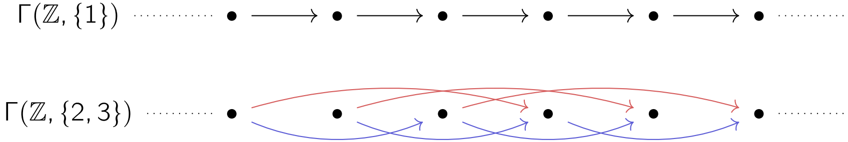
Proof Outline:

$$\begin{aligned}
d_{S'}(\mathrm{Id}(e), \mathrm{Id}(n)) &= d_{S'}(e, n) = d_{S'}(e, w_1 \dots w_k) \\
&\leqslant d_{S'}(e, w_1) + d_{S'}(w_1, w_1 w_2 \dots w_k) \\
&\leqslant d_{S'}(e, w_1) + d_{S'}(w_1^{-1} w_1, w_1^{-1} w_1 w_2 \dots w_k) \\
&\leqslant d_{S'}(e, w_1) + d_{S'}(e, w_2 \dots w_k) \\
&\vdots \\
&\leqslant d_{S'}(e, w_1) + \dots + d_{S'}(e, w_k) \\
&\leqslant Mk = M d_S(e, g)
\end{aligned}$$

Similarly for other direction.

Bi-Lipschitz Equivalence Cont.

Problem: This demonstrates that $(\mathbb{Z}, d_{\{1\}})$ and $(\mathbb{Z}, d_{\{2,3\}})$ are bi-lipschitz equivalent, but what about their cayley graphs? This map does not tell us where points on the edges of $\Gamma(\mathbb{Z}, \{1\})$ map to in $\Gamma(\mathbb{Z}, \{2, 3\})$. How can this preserve the metric on them?!?



Definition of Quasi-Isometry

Final Attempt: A map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a **quasi-isometric embedding** if there are constants $K \geqslant 1$ and $C \geqslant 0$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) - C \leqslant d_Y(f(x_1), f(x_2)) \leqslant K d_X(x_1, x_2) + C$$

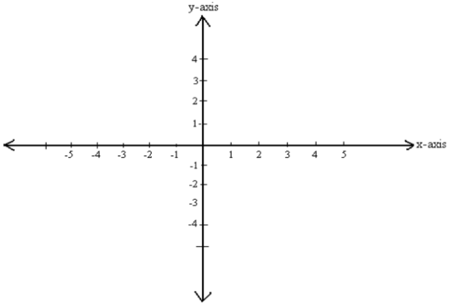
We say f is a **quasi-isometry** if there is a constant $D \geqslant 0$ such that for each $y \in Y$ there is some $x \in X$ where $d_Y(f(x), y) \leqslant D$.

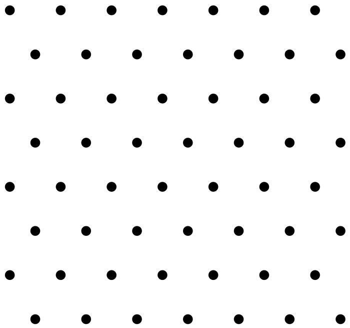
1. The first condition says the map is "coarse-bi-lipschitzïn that the map is bi-lipschitz except on balls of radius C. i.e. It ignores small-scale errors but ensure map is bi-lipschitz on the large scale.

2. The second condition says the map is almost surjective.

Examples of Quasi-Isometries

Examples of Quasi-Isometries: $\varphi : \mathbb{R}^2 \to \mathbb{Z}^2$ where φ sends any point to the nearest lattice point.





Basic Results about Quasi-Isometries

Theorem 1. *1. Quasi-Isometry is an equivalence relation.*

2. If G is finitely generated by both S and S', then $\Gamma(G, S)$ and $\Gamma(G, S')$ are Quasi-Isometric.

Proof Outline: We will only handle (2). Recall that (G, d_S) and $(G, d_{S'}$ are Quasi-Isometric. Moreover, we claim that $(G, d_{S'})$ and $\Gamma(G, S')$ are quasi-isometric as are (G, d_S) and $\Gamma(G, S)$. Since this is an equivalence relation, the result follows.

$$\begin{tikzcd}
(G, d_S) \arrow[r] & (G, d_{S'}) \arrow[d] \\
\Gamma(G, S) \arrow[u] & \Gamma(G, S')
\end{tikzcd}$$

2. Large Scale Geometry of BS(m,n)

Our goal is to study the **large scale geometry of** $\mathrm{BS}(m,n)$, which essentially means studying the Quasi-Isometry classes of $\mathrm{BS}(m,n)$. The first result in this vane comes in the following:

Theorem 2 (1998, Farb-Mosher)**.** $\mathrm{BS}(1,p)$ *is Quasi-Isometric to* $\mathrm{BS}(1,q)$ *if and only if* $p = n^r$ *and* $q = n^s$ *for some* $n, r, s \in \mathbb{Z}_{\geqslant 0}$*.*

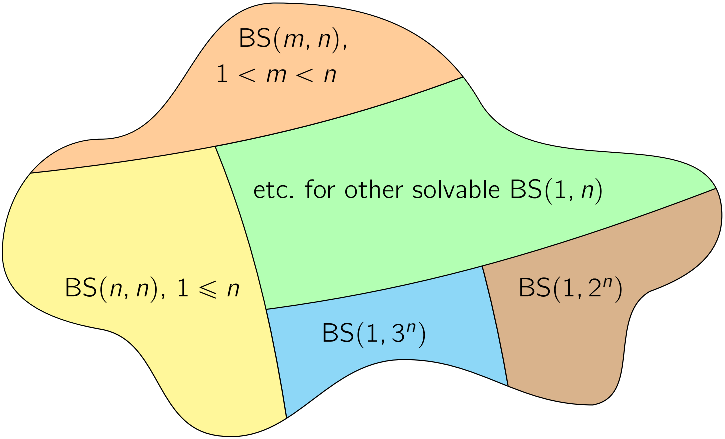
A Baumslag-Solitar Group is solvable if and only if it is of the form $\mathrm{BS}(1,p)$ for some $p \in \mathbb{Z}$. The result demonstrates that there are infinitely many distinct quasi-isometry classes of solvable Baumslag-Solitar groups.

Large Scale Geometry of BS(m,n) Cont.

The major results were given by Kevin Whyte in 2001.

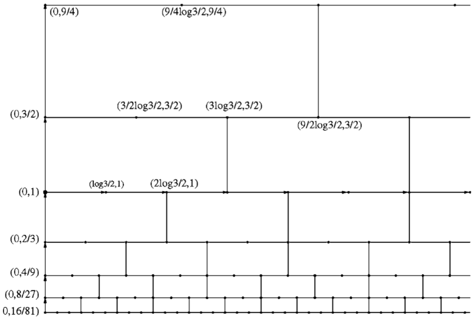
Theorem 3 (2001, Whyte)**.** $\mathrm{BS}(2,3)$ *is Quasi-Isometric to* $\mathrm{BS}(p,q)$ *for* $1 < p < q$*. This Quasi-Isometry class is distinct from those of the solvable Baumslag Solitar Groups.* $\mathrm{BS}(m,m)$ *is Quasi-Isometric to* $\mathrm{BS}(n,n)$ *for any* $n \geqslant n$*. Moreover, this Quasi-Isometry class is distinct from those discussed prior.*

Large Scale Geometry of BS(m, n)



Proof Techniques for Whyte's Theorem

Proposition 4. *The Cayley graph of $BS(p, q)$ is quasi-isometric to a corresponding two-complex X, obtained by replacing each sheet with a copy of the Euclidean plane ($p = q$) or the (upper half space model of the) hyperbolic ($p < q$).*



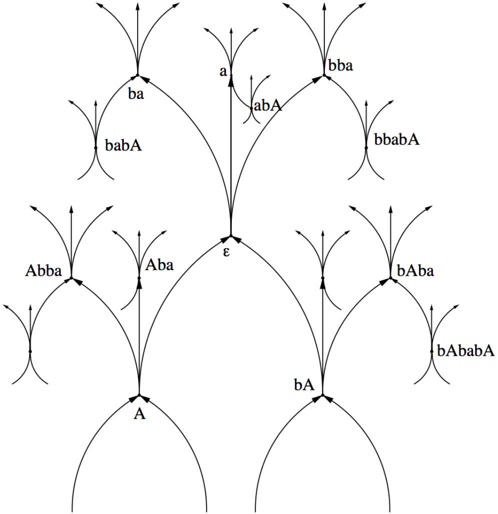
Proof Techniques for Whyte's Theorem

We will focus on the quasi-isometry class for $\mathrm{BS}(m, n)$ where $m \neq n$, and provide an outline of the major constructions utilized in this proof.

$$\begin{tikzcd} BS(m,n) \arrow[d] & & BS(m+1,n) \\ X_{m,n} \arrow[rr] & & X_{m+1,n} \arrow[u] \end{tikzcd}$$

Our immediate goal is to describe a quasi-isometric method of sliding certain in-branches of the spine tree T in a regular fashion in order to increase the in-valence of each vertex from p to $p+1$.

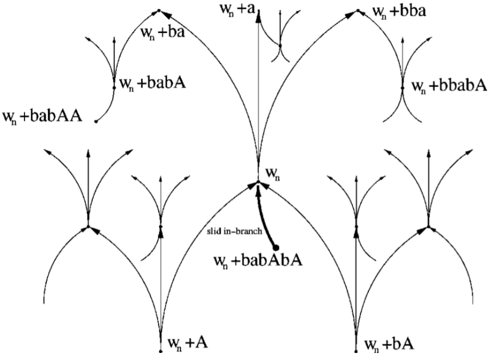
Proof Techniques for Whyte's Theorem



Proof Techniques for Whyte's Theorem

LEVEL SLIDE ALGORITHM 3.6. Assume $1 < p < q$. Given the directed tree T with vertices labelled by KBT reduced prefixes:

For each $n > 0$ do
  Find the *nth* vertex labelled w_n
  Delete the upward edge $[w_n + babAbA, w_n + babA]$ and add the upward edge $[w_n + babAbA, w_n]$
  If the in-valence of w_n equals p
    Delete the upward edge $[w_n + bbabAbA, w_n + bbabA]$ and add the upward edge $[w_n + bbabAbA, w_n]$
Increment n.



3. Takeaways and Larger Context

"Quasi isometry is an unreasonably effective tool in group theory."
- Nicholas Touikan

The key idea here is something called "quasi-isometric rigidity." We say a group G is **quasi-isometrically rigid** provided that every group H quasi-isometric to G must either have the property that (1) H contain a finite index subgroup isomorphic of G, or (2) G is isomorphic to a quotient of H by a finite group.

Translation: G is Quasi-isometrically rigid if

$$G \text{ talks like a duck and walks like a duck} \Rightarrow G \text{ is a duck}$$

When does this happen? **Really often!** For example, $\mathbb{Z}^n$, groups with polynomial growth, **solvable Baumslag-Solitar Groups**, etc.