Density of zeroes of generalized Zeta Functions

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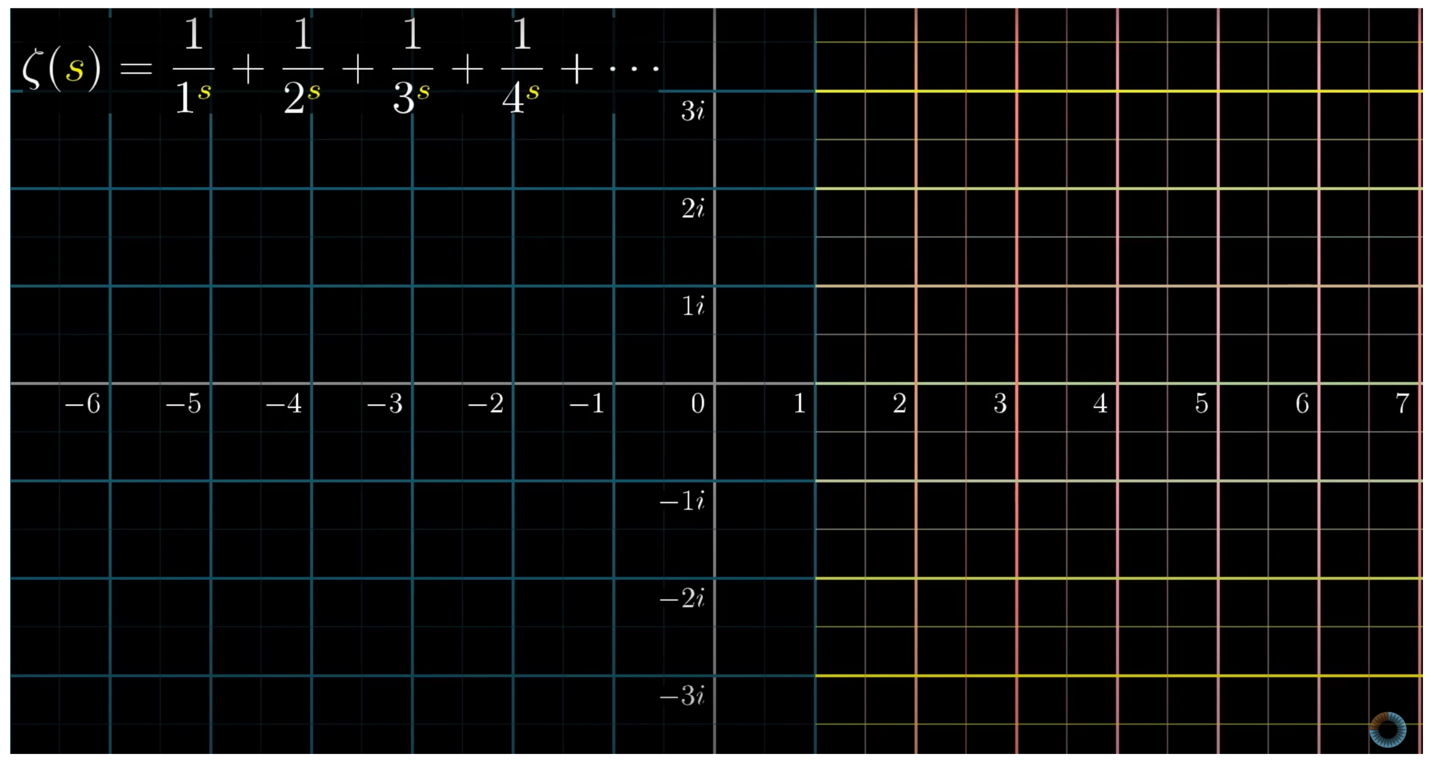
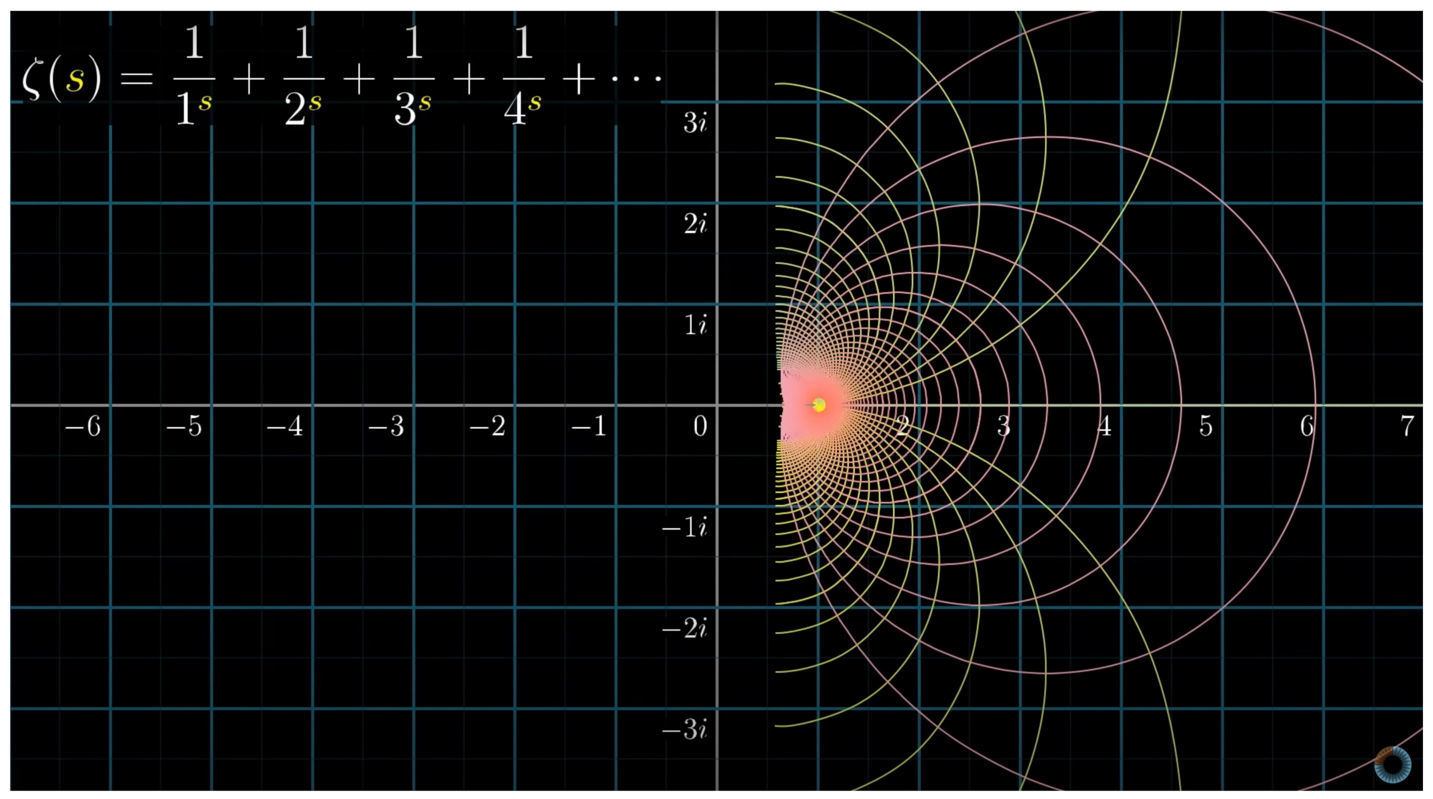
The Riemann-Zeta function

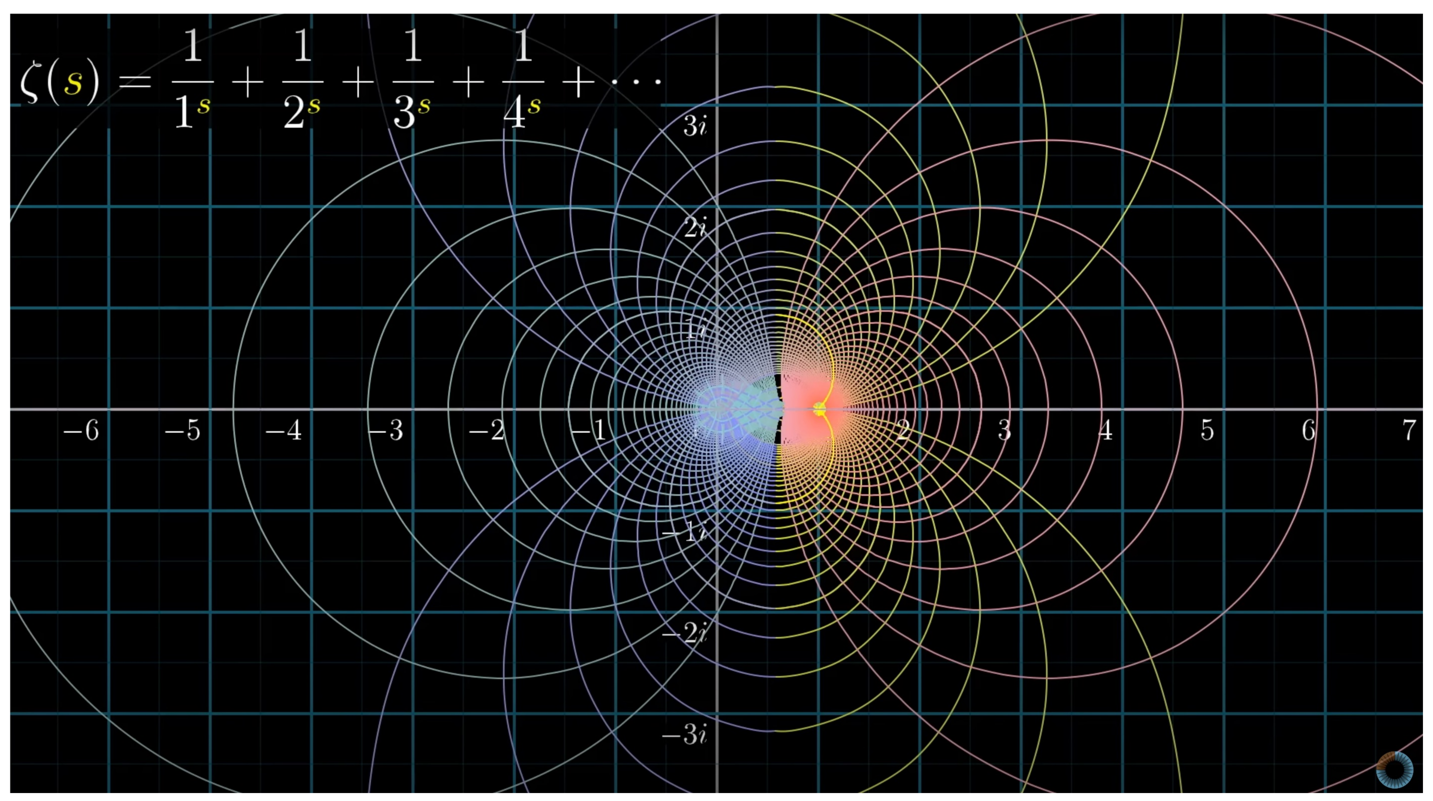
One of the most important functions in number theory is the Riemann Zeta Function. We define ζ as a complex function by

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

The series only converges for $\Re(s) > 1$, but we can fix this! Analytic continuation:







Why it matters

Because those pictures were pretty.
But also, prime numbers!

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

is the Euler Product for $\zeta(s)$.
Define the prime counting function

$$\pi(x) \ = \ \#\{\text{primes} \le x\}.$$

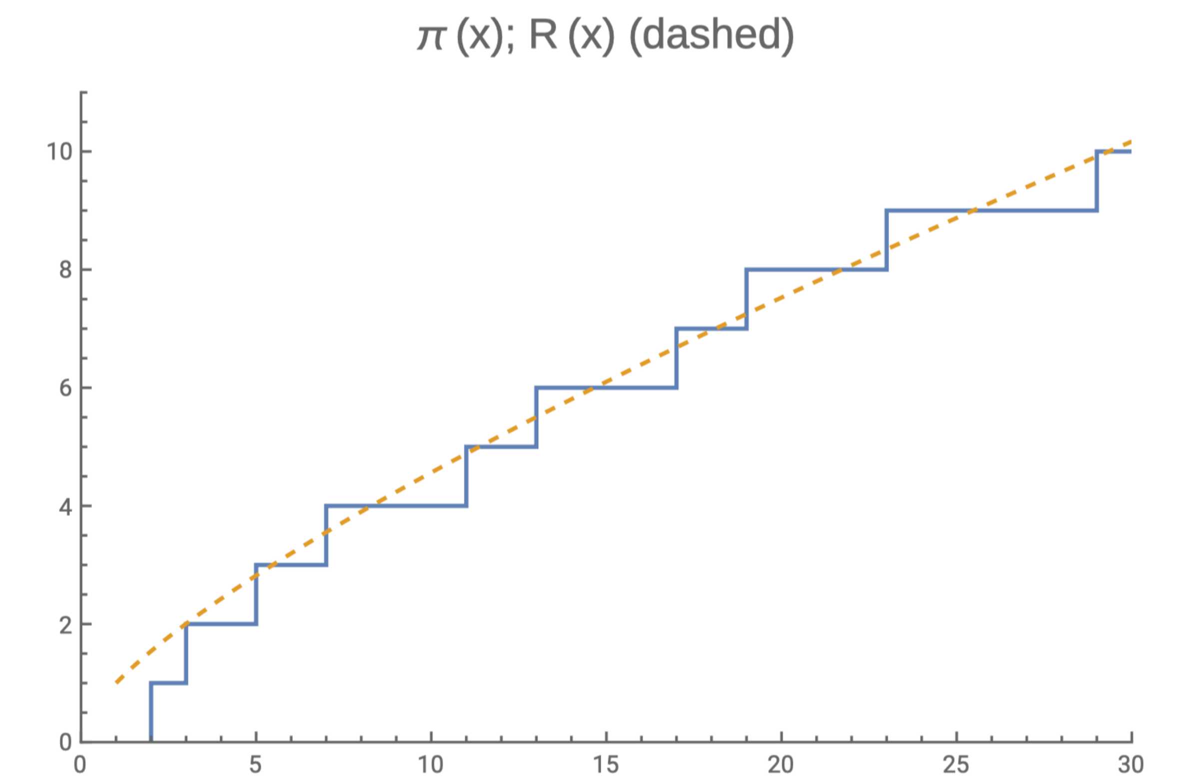
Also, Riemann's prime counting function:

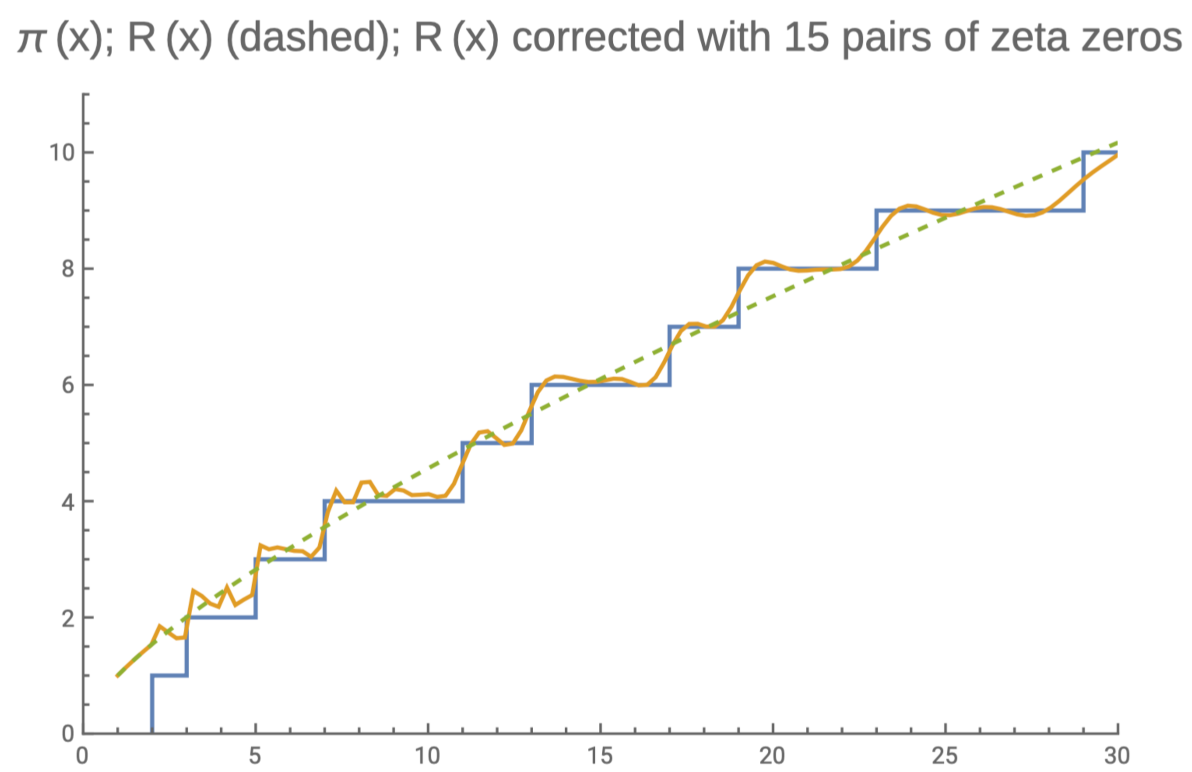
$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \operatorname{li}(x^{1/n}); \quad \operatorname{li}(x) = \int_0^x \frac{dt}{\log t}$$

Riemann showed that

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho}),$$

where ρ indexes zeroes of $\zeta(s)$.





Distribution of low-lying zeroes

We now generalize to $\mathfrak{f}$ ray-class L-functions. Let K be an algebraic field of degree n, $\mathfrak{f}$ a fixed integral ideal in K. Define

$$\mathfrak{I}_{\mathfrak{f}} = \left\{ \frac{\mathfrak{a}}{\mathfrak{b}} \,\middle|\, \mathfrak{a}, \mathfrak{b} \text{ both co-prime to } \mathfrak{f} \right\}.$$

$$\mathfrak{P}_{\mathfrak{f}} = \{(\alpha) \in \mathfrak{I}_{\mathfrak{f}} \mid \alpha \gg 0, \alpha \equiv 1 \bmod^* \mathfrak{f}\}.$$

Let χ be a ray-class character modulo $\mathfrak{f}$, that is, a character of the group $\mathfrak{I}_{\mathfrak{f}}/\mathfrak{P}_{\mathfrak{f}}$. A ray-class L-function with χ is defined by

$$L(s,\chi) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ \mathfrak{a} \neq 0}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

The 1-level density is a statistic for measuring density of low-lying zeroes:

$$D_{1;\mathfrak{f}}(\eta) = \frac{1}{h_{\mathfrak{f},K}} \sum_{\chi \in \widehat{\mathrm{Cl}_{\mathfrak{f}}(K)}} \sum_{\gamma_\chi} \eta\left(\gamma_\chi \frac{\log F}{2\pi}\right).$$

<u>Goal:</u> Obtain good estimates for $D_{1;\mathfrak{f}}(\eta)$.

<u>Results:</u> Successfully found explicit formula for averaged 1-level density, matching expectations from the generalized Ratios Conjecture.

<u>Ongoing research:</u> Attempt to surpass Ratios Conjecture, obtaining a smaller error term.

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