Machine Learning: Assignment 6
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December 2, 2017

1 Problem 1

i)
Now we know that x is both concave and convex. Therefore 3x would also be convex. Similarly $e^{\mathrm{y+z}}$ is also convex. We can convert the minimum function to a maximum by using max(x) = -min(-x). Hence $-min(-x^2, \log(y))$ becomes $max(x^2, -\log(y))$ which is concave. Therefore f(x,y,z) is concave.

ii)
We find the Hessian matrix for the given function. The required partial derivatives are:

$$\frac{\partial f^2}{\partial x^2} = 6yx - 4y \quad \frac{\partial f^2}{\partial x \partial y} = 3x^2 - 4x \quad \frac{\partial f^2}{\partial y^2} = 0 \quad \frac{\partial f^2}{\partial y \partial x} = 3x^2 - 4x \tag{1}$$

Substituting these values in the Hessian matrix H, we get:

$$\begin{bmatrix} 6yx - 4y & 3x^2 - 4x \\ 3x^2 - 4x & 0 \end{bmatrix}$$

Now for H to be positive semidefinite, $\forall$ w $\in \mathbb{R}^2$ the product $wHw^T \geq 0$. Let the vector be [1, 1]. Then wHw^T is:

$$6yx - 4y + 2(3x^2 - 4x)$$

As x,y $\in (-10, 10)$, we pick any value of x, y and show that the product is < 0.
On taking, x=8 and y=-9 we get, the product to be -76. Hence the Hessian matrix is not positive semidefinite. Therefore the f(x,y) is not convex.

iii)
We find the second order derivative of the function.

$$f(x) = \log x + x^3 \quad f'(x) = \frac{1}{x} + 3x^2 \quad f''(x) = -\frac{1}{x^2} + 6x$$

Now as the domain of the function is $(1, \infty)$ we have $-\frac{1}{x^2} < 6x$. Therefore as $f''(x) \geq 0$ the function is convex.

iv)
We again convert the minimum function to a maximum one. f(x) is then equivalent to $max(-2\log 2x, x^2 - 4x + 32)$
Now, $2x$ is convex as x is convex(also concave). Also, as logarithmic function is concave, hence $-2 * \log 2x$ is convex. Similarly, $-4 * x$ is convex and x^2 and the constant 32 are also convex. Therefore $x^2 - 4x + 32$ is convex.
Hence, f(x) is convex

2 Problem 2

Let $f_1(x)$ and $f_2(x)$ be convex functions. Then, by the definition of convexity we have,

$$\lambda f_1(x) + (1-\lambda) f_1(y) \geq f_1(\lambda x + (1-\lambda) y) \quad \forall \quad \lambda \in [0,1] \tag{2}$$

$$\lambda f_2(x) + (1-\lambda) f_2(y) \geq f_2(\lambda x + (1-\lambda) y) \quad \forall \quad \lambda \in [0,1] \tag{3}$$

Let λ in equation (2) and (3) be the same, then on adding the two equations we get,

$$\lambda(f_1(x) + f_2(x)) + (1-\lambda)(f_1(y) + f_2(y)) \geq f_1(\lambda x + (1-\lambda)y) + f_2(\lambda x + (1-\lambda)y) \quad for \quad \lambda \in [0,1]$$

$$\Rightarrow \lambda h(x) + (1-\lambda) h(y) \geq h(\lambda x + (1-\lambda) y) \quad for \quad \lambda \in [0,1]$$

As $\lambda \in$ [0,1] is arbitrary, therefore the result holds for all $\lambda \in$ [0,1]. Hence h is also a convex function. i.e. sum of two convex functions is also convex.

3 Problem 3

We will disprove the given function using an example.
Let $f_1(x) = x$ and $f_2(x) = x^2$. Then the product, $g(x) = f_1(x).f_2(x) = x^3$. Now we know that x is a function that is convex and x^2 is also a convex function. However, x^3 is not a convex function.
Therefore the product of two convex functions is not necessarily a convex function.

4 Problem 4

Let θ^* be a local minimum and β be the global minimum such that, $f(\beta) < f(\theta^*)$ and $\beta \neq \theta^*$.
We will show via contradiction that $f(\beta) = f(\theta^*)$ and $\beta = \theta^*$. Now we already know that

$$f(\beta) < f(\theta^*) \tag{4}$$

From the definition of convexity, we have

$$\lambda f(\beta) + (1-\lambda) f(\theta^*) \geq f(\lambda\beta + (1-\lambda)\theta^*) \;\; \forall \;\; \lambda \in [0,1] \tag{5}$$

As θ^* is the local minimum and the value $f(\lambda\beta + (1-\lambda)\theta^*)$ is in the neighborhood of $f(\theta^*)$,
therefore $f(\lambda\beta + (1-\lambda)\theta^*) > f(\theta^*)$. Hence equation (5) becomes,

$$\lambda f(\beta) + (1-\lambda) f(\theta^*) > f(\theta^*) \quad \Rightarrow \lambda f(\beta) > \lambda f(\theta^*)$$

As $\lambda \geq 0$, therefore we have

$$f(\beta) > f(\theta^*) \tag{6}$$

From equation (4) and (6), we come to a contradiction, hence $f(\beta) = f(\theta^*)$ and $\beta = \theta^*$ i.e a convex function has no local minimum, only a global minimum.