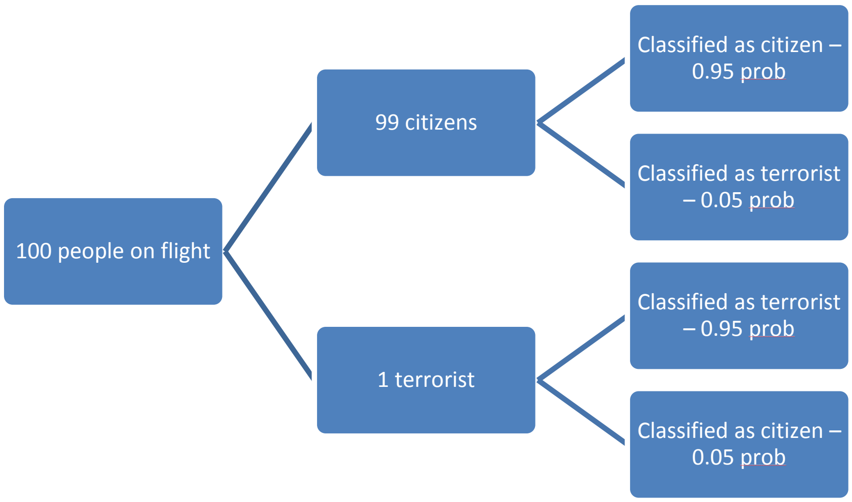
Solutions to Worksheet 2 **Harshita Agarwala**
Probability Theory

Answer 1)
The problem can first be classified as:



Now let A be the event of the person being an actual terrorist and B be the event that a person has been classified as a terrorist.
Now we need to find, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Now, $P(A \cap B)$ = 0.01 and P(B) =(0.95*99 + 1)/100 ~ 6/100 = 0.06
Therefore, P(A|B) = 1/6 or 0.167

Answer 2)
We again use conditional probability to solve: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
Here, A is the event that both balls in the box are red (i.e HH was flipped in coin toss) and B is the even that 3 times red balls are drawn.
Now, the as the coin is flipped twice the possible outcomes are: {HH,HT,TH,TT}
All these events have a 0.25 probability of occurring

Therefore, $P(A \cap B)$ = P(A).P(B|A) = 0.25 * 1 = 0.25

And P(B) = P(HH)*P(B|HH) + P(TH)*P(B|TH) + P(HT)*P(B|HT)
= 0.25*1 + 0.25 + 0.25*($\frac{1}{2} * \frac{1}{2} * \frac{1}{2}$) + 0.25*($\frac{1}{2} * \frac{1}{2} * \frac{1}{2}$)
= 0.3125
Therefore, P(A|B) = 0.25 / 0.3125 = 0.8

Answer 3)
Let x be the no of trials in which success occurs
Then, P(X=x) $= \frac{1}{2^x}$
Then, E[X] = $\sum_x x.P(x)$ = $\sum_x \frac{x}{2^x}$ = $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots$
$= 1 + \frac{1}{2}(3y^2 + 4y^3 + \cdots)$ where y = ½ -------------------- **1**
Now we know via sum of infinite geometric series, $1 + k + k^2 + \cdots = \frac{1}{1-k}$, $0 < k < 1$
Differentiating both sides with respect to k we get, $1 + 2k + 3k^2 + \cdots = \frac{1}{(1-k)^2}$
Now, $\frac{1}{2}(3k^2 + 4k^3 + \cdots) = \frac{1}{2}(\frac{1}{(1-k)^2} - 1 - 2k)$
For k = 1/2 , we have in equation **1,**
E[X] = 1+1 = 2

Therefore, the expected number of trials in which we get the first head is 2.
This means that the expected number of tails is 1
And similarly the expected number of heads is 1

We can directly use the expectation of a negative binomial distribution:
$\mathrm{P(X = x)} = \binom{x-1}{k-1}\rho^k(1-\rho)^{x-k}$
Here, x is the number of trials at which kth success occurs and ρ is the probability of success
The mean or the expectation of this distribution is $\frac{k}{\rho}$
Here we take k = 1 and $\rho = 1/2$
Therefore E[X] = 2 which implies that first head occurs at the 2nd trial and therefore expectation of tails is 1 and heads is also 1

Answer 4)
We have, $E[X] = \int_{-\infty}^{\infty} x.f(x)dx$

$$= \int_{-\infty}^{a} x.f(x)dx + \int_{a}^{b} x.f(x)dx + \int_{b}^{-\infty} x.f(x)dx$$

$$= \int_{a}^{b} x.f(x)dx = \frac{1}{b-a}\int_{a}^{b} x\,dx = \frac{1}{2(b-a)}\,[x^2]_a^b = \frac{(b^2 - a^2)}{2(b-a)}$$

$$= \frac{(b+a)}{2}$$

and $E[X^2] = \int_{-\infty}^{\infty} x^2.f(x)dx$

$$= \frac{1}{b-a}\int_{a}^{b} x^2\,dx = \frac{1}{3(b-a)}\,[x^3]_a^b = \frac{(b^3 - a^3)}{3(b-a)} = \frac{(b^2 + a^2 + ab)}{3}$$

Now we know, $Var(X) = E[X^2] - (E[X])^2 = \frac{(b^2 + a^2 + ab)}{3} - \frac{(b+a)^2}{4}$

$$= \frac{(b^2 + a^2 - 2ab)}{12} = \frac{(b-a)^2}{12}$$

Therefore, $\boldsymbol{E[X] = \frac{(b+a)}{2}}$ **and** $\boldsymbol{Var(X) = \frac{(b-a)^2}{12}}$

Answer 5)

We have, $E_Y\left[\, E_{X|Y}[X]\right] = E_Y[\int_{-\infty}^{\infty} x.f(x|y)dx] = E_Y[\int_{-\infty}^{\infty} x.\frac{f(x,y)}{h(y)}dx]$

$$= \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} x.\frac{f(x,y)}{h(y)}dx)\,.h(y)\,dy$$

$$= \int_{-\infty}^{\infty} x.\int_{-\infty}^{\infty}\frac{f(x,y)}{h(y)}h(y)dydx$$

$$= \int_{-\infty}^{\infty} x.(\int_{-\infty}^{\infty} f(x,y)dy)dx = \int_{-\infty}^{\infty} x.g(x)dx$$

$$= E_X[X]$$

where h(y) and g(x) are the marginal probability density functions of Y and X respectively and f(x|y) is the conditional probability of X given Y=y. Also, f(x,y) is the joint probability density function of X and Y.

Therefore, $\boldsymbol{E_Y\left[\, E_{X|Y}[X]\right] = E_X[X]}$ ----------- **(2)**

Now, $E_Y[Var_{X|Y}(X)] = E_Y[E_{X|Y}[X^2] - \left(E_{X|Y}[X]\right)^2]$

$= E_X[X^2] - E_Y\left[\,(E_{X|Y}[X])^2\right]$ (using equation **2)** ----------- **(3)**

Also, $Var_Y\left(E_{X|Y}[X]\right) = E_Y\left[\,(E_{X|Y}[X])^2\right] - (E_Y\left[\,E_{X|Y}[X]\right])^2 = E_Y\left[\,(E_{X|Y}[X])^2\right] - (E_X[X])^2$ --- **(4)**

Adding **(3)** and **(4)** we get,

$$\boldsymbol{E_Y[Var_{X|Y}(X)] + Var_Y\left(E_{X|Y}[X]\right) = E_X[X^2] - (E_X[X])^2 = Var_X(X)}$$

Hence, proved

Answer 6)

Let $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$

and μ be the mean and σ^2be the variance of each random variable X_i , i = 1,2,... n

Then, $E[Y] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n} * n\mu = \mu = E[X_i]$ for all i = 1,2, ... n

Also, $Var[Y] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2}{n}$

Now, let ε be any small positive real number.

On applying Chebyshev's inequality on the variable Y

We have, $P(|Y - E[Y]| > \varepsilon) \le \frac{Var(Y)}{\varepsilon^2} = \frac{\sigma^2/n}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$

Now, as $n \to \infty$, $\frac{\sigma^2}{n\varepsilon^2} \to 0$

Therefore we have, $\boldsymbol{P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - E[X_i]\right| > \varepsilon\right) \to 0 \qquad as\ n \to \infty}$

Hence proved.