Week 6

Tuesday, 3 September 2024 17:57

First Price Sealed-bid Auction

Two players $v_1, v_2 \sim U[0,1]$

Given : $b_j = c v_j + a$ — unknown. But $v_j \sim U[0,1]$

To prove : $b_i = c v_i + a$

Determine a, c.

utility of Player i.

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

unknown to player i

$$E\pi_i = (v_i - b_i) \times Pr(\text{win}) + 0 \times Pr(\text{not win})$$

$$E\pi_i = (v_i - b_i) \cdot Pr(b_i > b_j)$$

$b_i > b_j \Rightarrow b_i > a + c v_j$

$\Rightarrow v_j < \frac{b_i - a}{c}$

$$E\pi_i = (v_i - b_i) \cdot Pr\left(v_j < \frac{b_i - a}{c}\right)$$

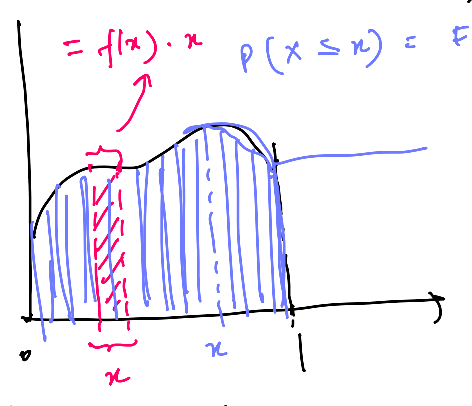
R. V $\sim U[0,1]$

Of the form $F(x) = P(X \le x) = x$ if $x \sim U[0,1]$

$$E\pi_i = (v_i - b_i) \times \left(\frac{b_i - a}{c}\right)$$

$Pr(X \in x)$

$= f(x) \cdot x$ $\quad P(X \le x) = F(x)$



Alternate:

$$E\pi_i = (v_i - b_i) \int_0^{\frac{b_i - a}{c}} dv_j$$

$$E\pi_i = (v_i - b_i) \times v_j \Big|_0^{\frac{b_i - a}{c}}$$

$$E\pi_i = (v_i - b_i) \times \left(\frac{b_i - a}{c}\right)$$

$$E\pi_i = \frac{1}{c}\left[v_i b_i + a b_i - b_i^2 - a v_i\right]$$

To maximize $E\pi_i$ for player i.

$$\frac{\partial E\pi_i}{\partial b_i} = \frac{1}{c}\left[v_i + a - 2b_i\right] = 0$$

$$\Rightarrow b_i = \frac{a + v_i}{2}$$

we need to prove that $b_i = a + c v_i$

If $c = 1/2$, $a = a/2 = 0$

we get $b_i = v_i/2$

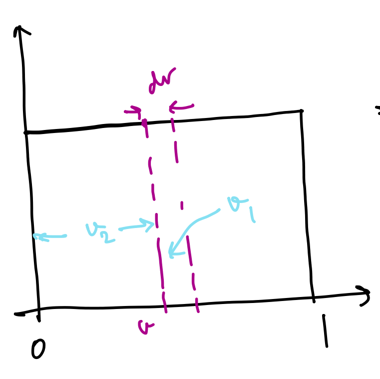
(& $b_j = v_j/2$)

Expected Revenue

ER to the auctioneer.

$ER = \max(b_1, b_2) = \max\left(\frac{v_1}{2}, \frac{v_2}{2}\right)$

$= \frac{1}{2}\max(v_1, v_2)$



If $v_1 > v_2$

$\Rightarrow$ $v_2 \in [0, v]$

and $v_1 \in [v, v + dv]$

$ER = Prob(v_1 > v_2) \times \frac{1}{2}\max(v_1, v_2)$

$+ Prob(v_2 > v_1) \times \frac{1}{2}\max(v_1, v_2)$ — (1)

$f(v_2 \le v)$

$Prob(v_1 > v_2) = Prob(v_2 \in [0, v]) \times Prob(v_1 \in [v, v + dv])$

$= v \cdot dv$ — (2)

$Prob(v_2 > v_1) = Prob(v_1 \in [0, v]) \times Prob(v_2 \in [v, v + dv])$

$= v \cdot dv$ — (3)

From eqn (1), (2), (3)

$ER = v \cdot dv \cdot \frac{1}{2}\max(v_1, v_2) + v\,dv \cdot \frac{1}{2}\max(v_1, v_2)$

$ER = 2v\,dv \cdot \frac{1}{2} \cdot \max(v_1, v_2)$

$ER = v\,dv \cdot v = v^2 dv$

Total expected revenue over $[0,1]$ is :

$$E\pi = \int_0^1 v^2 dv = \frac{v^3}{3}\Big|_0^1 = \frac{1}{3}$$

N-Player First Price Sealed Bid Auction

N players

valuation of any player i is $v_i \sim U[0,1]$ $\forall i$

Utility of player i

$$u_i = \begin{cases} v_i - b_i & b_i > b_j \ \forall j \ne i \\ 0 & b_i < b_j \ \forall j \ne i \end{cases}$$

$\underbrace{j = 1, 2, \ldots i-1, i+1, \ldots N}_{N-1 \text{ bids}}$

We are given $b_j = a + c v_j$

$$\pi_i = (v_i - b_i) \times Pr(\text{win}) + 0 \times Pr(\text{not win})$$

$$\pi_i = (v_i - b_i) \times Pr(b_i > b_j \ \forall j \ne i)$$

$$\pi_i = (v_i - b_i) \times \left(Pr(b_i > b_j)\right)^{n-1}$$

Now $b_i > b_j \Rightarrow b_i > a + c v_j$

$\Rightarrow v_j < \frac{b_i - a}{c}$

$$\pi_i = (v_i - b_i) \times \left(\underbrace{Pr\left(v_j < \frac{b_i - a}{c}\right)}_{\text{CDF}}\right)^{n-1}$$

$$\pi_i = (v_i - b_i) \times \left(\frac{b_i - a}{c}\right)^{n-1}$$

Player i will maximize π_i wrt b_i.

$$\therefore \frac{\partial \pi_i}{\partial b_i} = (-1)\left(\frac{b_i - a}{c}\right)^{n-1} + (v_i - b_i)\left(\frac{n-1}{c}\left(\frac{b_i - a}{c}\right)^{n-2}\right) = 0$$

$$\Rightarrow \left(\frac{b_i - a}{c}\right)^{n-2}\left[-\left(\frac{b_i - a}{c}\right) + \frac{n-1}{c}(v_i - b_i)\right] = 0$$

$$\Rightarrow \left(\frac{b_i - a}{c}\right)^{n-2} \cdot \frac{1}{c}\left[a - b_i + (n-1)v_i - (n-1)b_i\right] = 0$$

$\Rightarrow -nb_i + a + (n-1)v_i = 0$

$\Rightarrow a + (n-1)v_i = nb_i$

$\Rightarrow b_i = \frac{a}{n} + \left(\frac{n-1}{n}\right)v_i$

It's of the form $b_j = a + c v_j$

If $a = a/n = 0$

& $c = \frac{n-1}{n}$

$$b_i = \left(1 - \frac{1}{n}\right)v_i$$

Or $b_i = a$

Second Price Sealed Bid Auction

N players with valuation $v_i \sim F_i$ $\forall i$

Bidders bid simultaneously $b_1, b_2, \ldots b_N$

Utility of player i :

$$u_i(b_i, b_{-i}) = \begin{cases} v_i - b_j & b_i > b_j \ \forall j \ne i \\ 0 & b_i < b_j \ \forall j \ne i \end{cases}$$

F_j

#1 $v_i > b_i > b_j$

Player i wins

$u_i = v_i - b_j > 0$

#2 $b_i > v_i > b_j$

Player i win

& $u_i = v_i - b_j > 0$

If $b_i = v_i$, $u_i = v_i - b_j > 0$

#3 $b_i > b_j > v_i$

Player i wins.

$u_i = v_i - b_j < 0$

strictly dominated strategy for player i.

#4 $v_i > b_j > b_i$

Player i loses.

$\therefore u_i = 0$