Mathematical Optimization for Economic Applications

Lecture 2

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Topics

- Recap
- Unconstrained Optimization
- Constrained Optimization

What is Optimization?

- The process of choosing the best option from a set of alternatives.
- The most *feasible* and *desirable* way of doing a task.
- For eg, you love eating chocolates. But you cannot eat chocolates for breakfast, lunch, and dinner. There is a budget constraint and it is not good for your health. So, you eat as much as your pocket and your stomach will allow to maximize your "utility."

Economic Applications

- We study the economics of ***scarcity***.
- If everything is available in abundance and no damage accrues from any production or consumption, this economics will not apply.
- But since that does not happen, we need to **optimize**, and make informed decisions to maximize or minimize certain outcomes.

Economic Applications

- Utility Maximization (Microeconomics)
- Profit Maximization (Microeconomics)
- Resource Allocation (Macroeconomics)
- Payoff Maximization in multi-player setting (Game Theory)
- Intertemporal Payoff Maximization (All)

Types of Optimization

- Single variable. Eg, profit maximization where profit π is $\pi(q)$ where q is quantity produced.
- Multi-variable. Eg, consumer utility maximization.
- Constrained. Eg, budget constraint in utility maximization.
- Unconstrained. Eg, profit maximization for production of single good.
- Single agent. Eg, a firm, a consumer.
- Multi-agent. Eg, firms competing in a market.

Types of Optimization

- Linear.
- Non-linear.
- Static or not varying with time.
- Dynamic or time-varying.

Method of Optimization

A few things form the basic construct of an optimization problem. These are:

- What type of optimization is it?
- What are you choosing—the choice variables?
- What is the domain for choice variables?
- What is the **objective function**?
- Are there any constraints?

Method of Optimization: Example 1

A simple example in profit maximization of a single firm with a single good with price P, quantity produced q, total cost cq and inverse demand function $P(q)$. Here $q \geq 0$.

Eg: Profit Maximization

$$\max_{q} \pi(q) = P \cdot q - c \cdot q = P(q) \cdot q - c \cdot q$$

This is a single variable, single agent, static, unconstrained maximization problem.

Method of Optimization: Example 2

We can also sometimes minimize costs $C(q)$, instead of maximizing profit.

Here $q \geq 0$.

Eg: Cost Minimization

$$\min_{q} C(q) = \alpha q^2 - \beta q + c_0$$

This is a single variable, single agent, static, unconstrained minimization problem.

Method of Optimization: Example 3

A consumer choosing between two goods x and y, priced at P_x and P_y subject to a budget constraint of $B = P_x \cdot x + P_y \cdot y$.

Here $x, y \geq 0$.

Utility is given by $U(x, y) = f(x, y)$

Eg: Utility Maximization

$$\max_{x,y} U(x, y) = f(x, y)$$

$$s.t. \quad B = P_x \cdot x + P_y \cdot y$$

This is a 2-variable, single agent, static, constrained maximization problem.

Unconstrained Optimization

Consider the fictional case where cost of manufacturing a good is constant $= k$. The quantity of the good produced is q. The price of the good is given by inverse market demand function $\mathbf{P(q) = 2q^2 - 17q + 40}$.

What type of optimization will we do here?

Unconstrained Optimization

Follow the steps of optimization:

Unconstrained Optimization

Follow the steps of optimization:

1. Type: Maximization problem

Unconstrained Optimization

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2. Choice variable: quantity q

Unconstrained Optimization

Follow the steps of optimization:

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2. Choice variable: quantity q
3. Domain: $q \geq 0$

Unconstrained Optimization

Follow the steps of optimization:

1. Type: Maximization problem
2. Choice variable: quantity q
3. Domain: $q \geq 0$
4. Objective function: $\boldsymbol{\pi(q)} = \mathbf{P}(\mathbf{q}) \cdot \mathbf{q} - \mathbf{k}$

Unconstrained Optimization

Follow the steps of optimization:

1. Type: Maximization problem
2. Choice variable: quantity q
3. Domain: $q \geq 0$
4. Objective function: $\boldsymbol{\pi(q)} = \mathbf{P}(\mathbf{q}) \cdot \mathbf{q} - \mathbf{k}$
5. Constraints: none

Unconstrained Optimization

$$\boldsymbol{\pi(q) = P(q) \cdot q - k = 2q^3 - 17q^2 + 40q - k}$$

$$\max_{\boldsymbol{q}} \pi(\boldsymbol{q}) = 2q^3 - 17q^2 + 40q - k \quad s.t. \quad q \geq 0$$

Unconstrained Optimization

We use differential calculus to find the maxima of the function $\pi(q)$. So,

$$\frac{d\pi}{dq} = 6q^2 - 34q + 40 = (q-4)(6q-10) = 0$$

Therefore, $q = 4$ or $q = 5/3$.

$$\frac{d^2\pi}{dq^2} = 12q - 34$$

Second derivative is negative for $q = 5/3$.

Therefore, $q = 5/3$ gives maxima.

Unconstrained Optimization

Question:

Suppose the inverse demand function faced by a firm for a single good is given by:

$$P(q) = 32 - 2q$$

The cost function is given by $C(q) = 2q^2$.

$q \geq 0$

Find the quantity q for which profit is maximized for the firm.

What is the maximum profit the firm can achieve?

Constrained Maximization: Utility

A consumer choosing between two goods x and y, priced at P_x and P_y subject to a budget constraint of $B = P_x \cdot x + P_y \cdot y$.

Here $x, y \geq 0$.

Utility is given by $U(x, y) = f(x, y)$

$$\max_{x,y} U(x, y) = f(x, y)$$

$$s.t. \quad B = P_x \cdot x + P_y \cdot y$$

Constrained Maximization: Utility

- If $f(x,y) = xy$ and $x + 3y = 24$, how do we solve it?

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Constrained Maximization: Utility

- If $\boldsymbol{f(x,y) = xy}$ and $\boldsymbol{x + 3y = 24}$, how do we solve it?
- Steps:
  1. Type: Maximization
  2. Choice variables: x, y

Constrained Maximization: Utility

- If $\boldsymbol{f(x,y) = xy}$ and $\boldsymbol{x + 3y = 24}$, how do we solve it?
- Steps:
  1. Type: Maximization
  2. Choice variables: x, y
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Constrained Maximization: Utility

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  1. Type: Maximization
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  4. Objective fn: $f(x, y) = xy$

Constrained Maximization: Utility

- If $\boldsymbol{f(x, y) = xy}$ and $\boldsymbol{x + 3y = 24}$, how do we solve it?
- Steps:
  1. Type: Maximization
  2. Choice variables: x, y
  3. Domain: $x, y \geq 0$
  4. Objective fn: $f(x, y) = xy$
  5. Constraint: $x + 3y = 24$

Simplification to Unconstrained Optimization

Instead of using the *Lagrange* method for constrained optimization, we can also substitute x in terms of y. So,

$$x = 24 - 3y$$

So, the problem becomes:

Utility Maximization

$$\max_y U(y) = 24y - 3y^2$$

So, we get $(x, y) = (0, 8), (24, 0)$.

Constrained Optimization

Question:

A consumer faces the choice between buying chocolate and icecream. Per unit price of chocolate is ₹10 and per unit price of icecream is ₹20. The total budget of the consumer is ₹400. If the consumer consumes x units of chocolate and y units of icecream, her utility is given by $\boldsymbol{U(x, y) = xy}$.

How many chocolates and icecreams can she consume to maximize her utility?

Reference Reading

1. *Essential Mathematics for Economic Analysis (5e)* by Sydsaeter, Hammond, Strom and Carvajal.
2. *Fundamental Methods of Mathematical Economics (4e)* by Chiang and Wainwright.
3. *Mathematics for Economists* by Simon and Blume.

Thank you!

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