CS345 Theoretical Assignment 4

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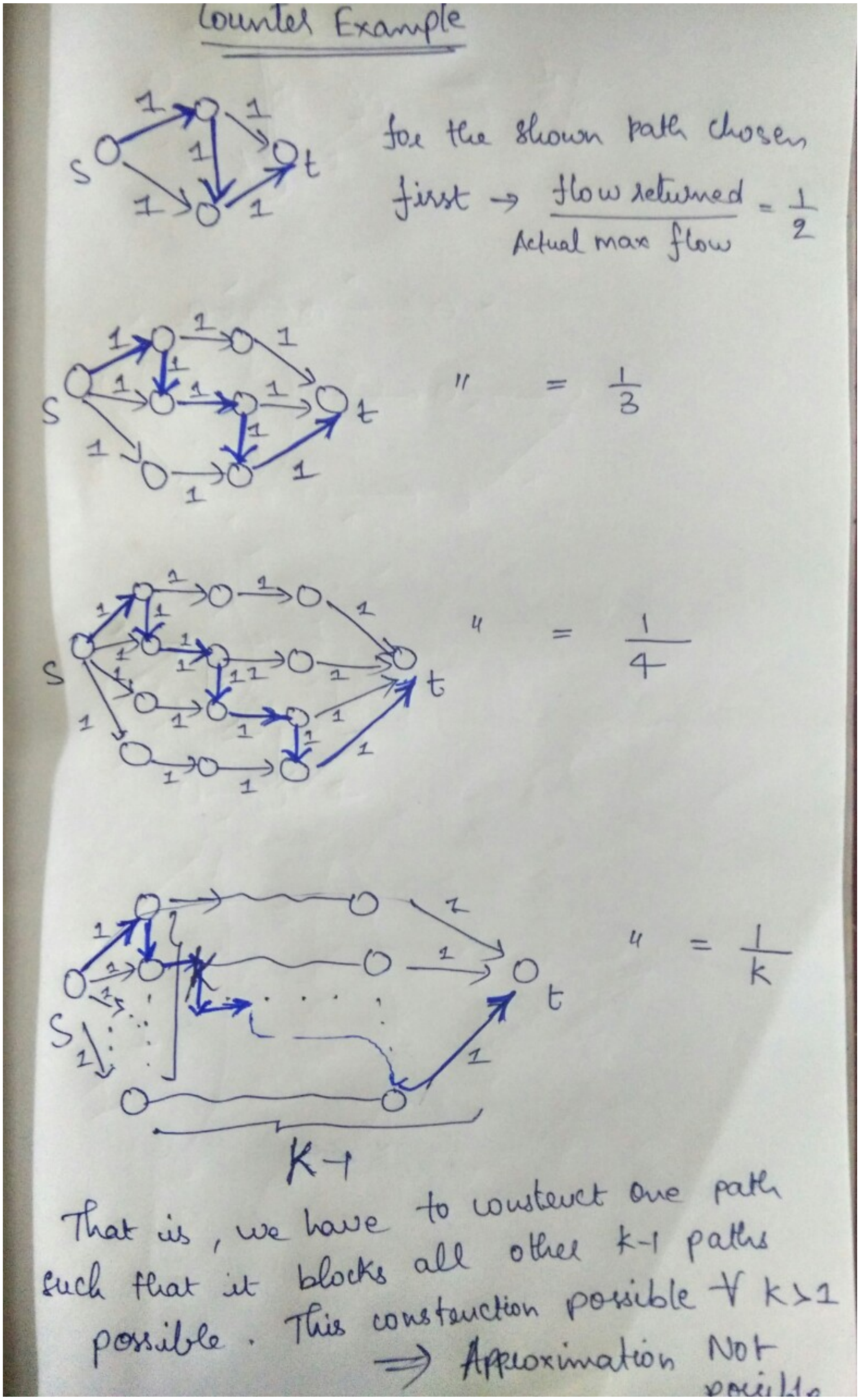
Contents

1 Any Guarantee of our First-Attempt Algorithm

1.1 Counter Example

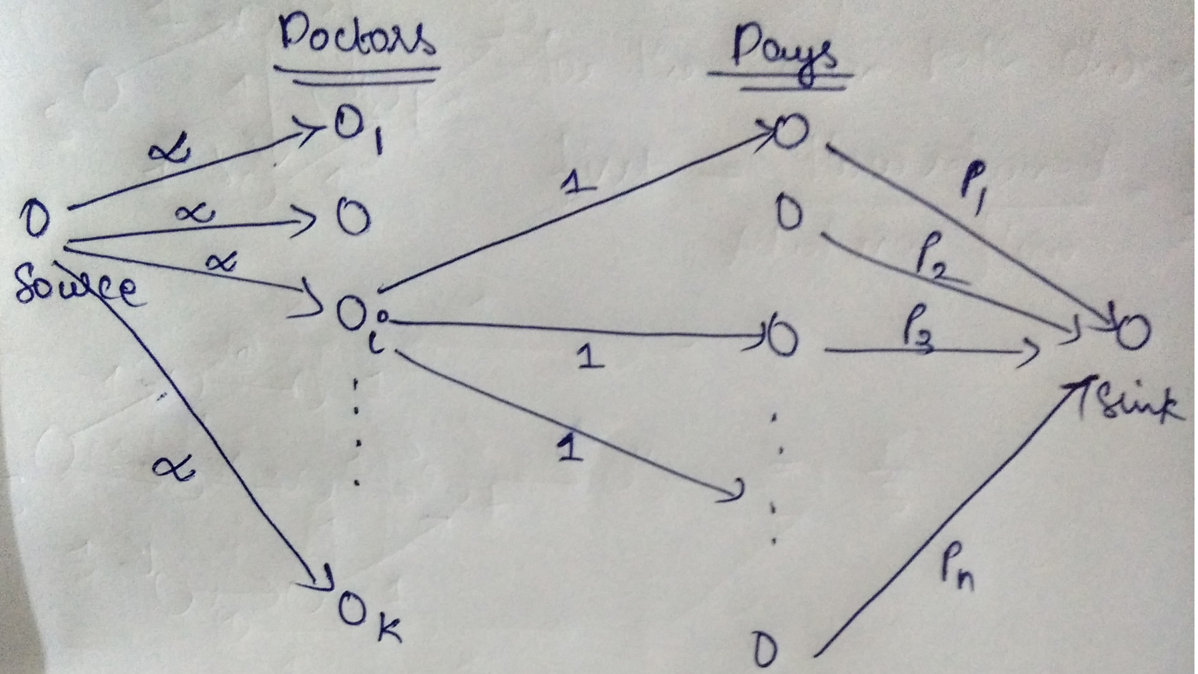


2 A max flow application

2.1 Without Extra Constraint

2.1.1 Overview

A max flow Application



Doctors[i] to Days[j] there is an edge iff j lies in the list L_i.

All these edges inserted between Doctors & Days hold capacity 1.

$L_1', \ldots, L_k'$ exists iff Max flow of above graph is $\sum_{i=1}^{n} p_i$.

2.1.2 Notations

p_i denotes the exact number of doctors required on day i
L_i denotes the list of days where doctor i is available
L'_i denotes the list of days that doctor i has to work to produce the required match. Note, $L'_i \subseteq L_i$
$D = \sum_{i=1}^{n} p_i$
n = Number of days in total
k = Number of Doctors in total

2.1.3 Claim

Construction of L'_is is possible if and only if the max-flow in the source-sink graph constructed (in image) is D.

2.1.4 Proof - Part 1

Given L'_is list for all doctors, show that the max flow of source-sink graph shown above is D
Construct a flow f of the source-sink graph as follows.
$f(source, Doctor[i])$ is $|L'_i|$ - satisfies capacity constraint as these edges had infinite capacity
$f(Doctor[i], Day[j])$ is 1 if j lies in L'_i, otherwise is 0 - satisfies capacity constraint
$f(Day[j], sink)$ is p_j - satisfies capacity constraint
Flow Conservation is ensured as it is given to us that L'_is of such definitions exist.
Hence, the above is a valid flow.
As the $cut - capacity$ between $sink$ and the rest of the graph is D, by $min - cut - max - flow$ theorem, f should be a max flow of the source-sink graph. Hence, proved.

2.1.5 Proof - Part 2

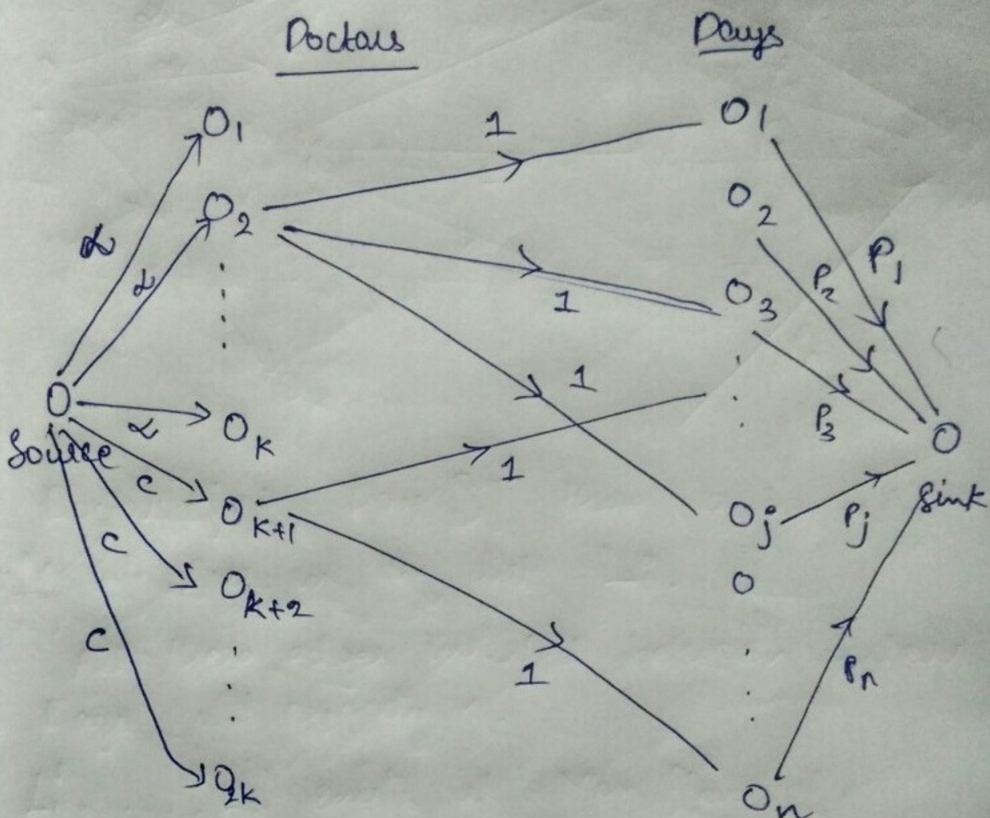
Given the max flow of source-sink graph to be D, show that L'_is list for all doctors exist
Let f be the integral max-flow of the source-sink graph. (Note: Integral flow exists was proved in class)
Construct L'_i as follows.
If $f(Doctor[i], Day[j])$ is 1 then add j to L'_i otherwise do nothing.
Note: $f(Doctor[i], Day[j])$ can be only 0 or 1 by integral flow property.
The L'_is thus constructed are valid as $value(f) = D$ implying every day j has got the exact number of doctors wanted (p_j). Moreover, $L'_i \subseteq L_i$ is ensured from the construction of the source-sink graph itself. Hence, proved.

2.2 With Extra Constraint

2.2.1 Overview

A max flow Application

With Constraint



$$\text{capacity}(\text{Source}, \text{Doctor}[i]) = \begin{cases} \alpha, & 1 \le i \le k \\ c, & k+1 \le i \le 2k \end{cases}$$

$$\text{Capacity}(\text{Doctor}[i], \text{Days}[j]) = \begin{cases} 1, & 1 \le i \le k \ \& \ j \in L_i \\ 0, & 1 \le i \le k \ \& \ j \notin L_i \\ 1, & k+1 \le i \le 2k \ \& \ j \notin L_{\{i-k\}} \\ 0, & k+1 \le i \le 2k \ \& \ j \in L_{\{i-k\}} \end{cases}$$

Capacity $= 0 \Rightarrow$ No edge.

2.2.2 Notations

p_i denotes the exact number of doctors required on day i
L_i denotes the list of days where doctor i is available
L_i' denotes the list of days that doctor i has to work to produce the required match.
$K_i = L_i \cap L_i'$
$C_i = L_i' - K_i$
$D = \sum_{i=1}^{n} p_i$
n = Number of days in total
k = Number of Doctors in total

2.2.3 Claim

Construction of L_i's is possible if and only if the max-flow in the source-sink graph constructed (in image) is D.

2.2.4 Proof - Part 1

Given L_i's list for all doctors, show that the max flow of source-sink graph shown above is D
Construct a flow f of the source-sink graph as follows.
If $i \leq k$ then
$f(source, Doctor[i])$ is $|K_i|$ - satisfies capacity constraint as these edges had infinite capacity
$f(Doctor[i], Day[j])$ is 1 if j lies in K_i, otherwise is 0 - satisfies capacity constraint
If $i > k$ then
$f(source, Doctor[i])$ is $|C_i|$ - satisfies capacity constraint by definition of L_i's existence.
$f(Doctor[i], Day[j])$ is 1 if j lies in C_i, otherwise is 0 - satisfies capacity constraint

$f(Day[j], sink)$ is p_j - satisfies capacity constraint

Flow Conservation is ensured as it is given to us that L_i's of such definitions exist.
Hence, the above is a valid flow.
As the $cut-capacity$ between $sink$ and the rest of the graph is D, by $min-cut-max-flow$ theorem, f should be a max flow of the source-sink graph. Hence, proved.

2.2.5 Proof - Part 2

Given the max flow of source-sink graph to be D, show that L_i's list for all doctors exist
Let f be the integral max-flow of the source-sink graph. (Note: Integral flow exists was proved in class)
Construct L_i' as follows.
If $f(Doctor[i], Day[j])$ or $f(Doctor[i+k], Day[j])$ is 1 then add j to L_i' otherwise do nothing.
Note: $f(Doctor[i], Day[j])$ can be only 0 or 1 by integral flow property.
The L_i's thus constructed are valid as $value(f) = D$ implying every day j has got the exact number of doctors wanted (p_j). Moreover, both $f(Doctor[i], Day[j])$ and $f(Doctor[i+k], Day[j])$ will not be together 1 from the construction of source-sink graph itself. Hence, proved.