CS345 Theoretical Assignment 1

Ayush Agarwal, 13180

M.Arunothia, 13378

Contents

1 Non-Dominated Points **2**
- 1.1 Overview 2
- 1.2 Pseudo-Code 2
- 1.3 Time Complexity 2

2 Open Rectangle Query **3**
- 2.1 Data Structure Design : 3
- 2.2 Algorithm : 4
- 2.3 Pseudo Code : 4
- 2.4 Space Complexity : 4
- 2.5 Time Complexity : 5
  - 2.5.1 Query Time : 5
  - 2.5.2 Pre-processing Time : 5

3 Constraint of each commando **5**
- 3.1 Overview 5
- 3.2 Algorithm 5
- 3.3 Pseudo Code 6
- 3.4 Time Complexity 6
- 3.5 Proof of Correctness 6
  - 3.5.1 What is to be proved? or Claim 6
  - 3.5.2 Proof by Induction 6
  - 3.5.3 Base Cases 7
  - 3.5.4 Hypothesis 7
  - 3.5.5 Inductive Step 7

1 Non-Dominated Points

1.1 Overview

Given a set of coordinates P, we create list of each layer in the following manner. First sort the coordinates based on Y-coordinate in descending order. Then maintain an array A of size n. Start with the first coordinate from the sorted array P(of all coordinates). This point will be a non-dominated point and will be a part of layer 1. Update the first index of A with the x-coordinate of this point. Now take the second point from P. If its x coordinate is greater than the x-coordinate of earlier point, it means that it will be part of layer 1. If so then add it to layer 1 and update the layer 1's index in A. Otherwise it will be in second layer, so add it in layer 2 and update the layer 2's index in A with it's x coordinate. Repeat the above procedure for all points.

1.2 Pseudo-Code

Non-Dominated-points(P)
{
    $P \longrightarrow reverse\_sort(P)$ //sort in descending order of Y
    $Layer[n]; A[n]$
    $A[0] = P[0].x$
    $Layer[1].push()$
    $i = 1; right = 1$
    $while(i < P.length())$
        $point = P[i]$
        $index = binary\_search\_predecessor(A, 0, right, point)$
                // returns the predecessor's index
        $Layer[index].push(point)$
        $A[index] = point.x$
        $If(index > right)right + +$
    $returnLayer$

}

1.3 Time Complexity

Sorting step takes $O(nlogn)$ time, followed by binary_search for each point which takes $logn$ time per point. While iterates for all the points and in each iteration binary_search is invoked, thus the loop takes $n * logn$ time. Overall algorithm takes time
$O(nlogn) + O(nlogn) = O(nlogn)$

2 Open Rectangle Query

2.1 Data Structure Design :

Given an array 'a' of 'n' coordinate points, we construct a Binary Search Tree (BST) call it 'data' in the following manner.

- Sort the array 'a' w.r.t the x-coordinates of the points. Call this sorted array 'b'.
- Divide 'b' into $\frac{n}{Log[n]}$ parts, starting from the beginning. Index each of the part incrementally from 1 to $\frac{n}{Log[n]}$.
- Construct BST 'data' with $\frac{n}{Log[n]} = N$ nodes from 'b' using the above indexing for the comparisons.
- Now, we have a BST 'data' with 'N' nodes augmented with an array of $Log[n]$ size at every node. Sort this array at every node on basis of y-coordinates of the points.
- This completes the description of augmented BST 'data'.

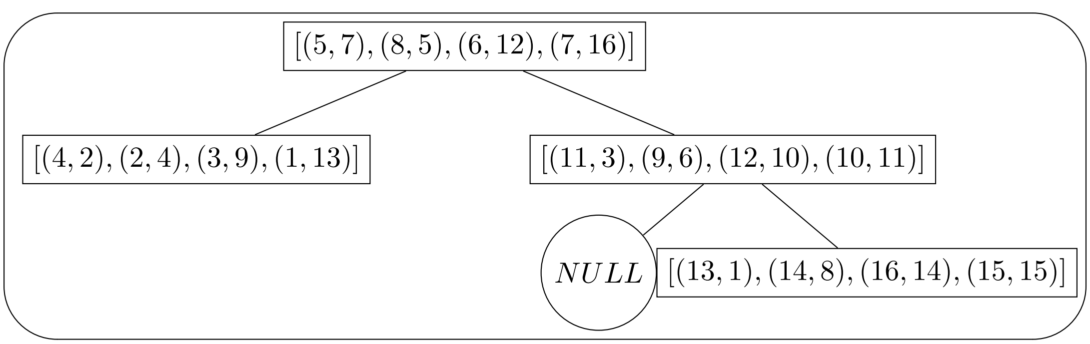
Given Array 'a'

(13,1)	(4,2)	(11,3)	(2,4)	(8,5)	(9,6)	(5,7)	(14,8)	(3,9)	(12,10)	(10,11)	(6,12)	(1,13)	(16,14)	(15,15)	(7,16)

Sorted Array 'b' based on x coordinates

(1,13)	(2,4)	(3,9)	(4,2)	(5,7)	(6,12)	(7,16)	(8,5)	(9,6)	(10,11)	(11,3)	(12,10)	(13,1)	(14,8)	(15,15)	(16,14)

BST 'data' constructed for this example



2.2 Algorithm :

STEP 1 : Start

STEP 2 : If($x_2 - x_1 < 2 * (Log[n])$, traverse elements in this range of x and return the points satisfying $y > y_{bottom}$.
else Initialise variables node_i to the x value of nearest node ahead of x_1 and node_j to the x value of nearest node behind x_2.

STEP 3 : Find the elements satisfying $y > y_{bottom}$ in the x range x_1 to node_i and in x range node_j to x_2, and report them.

STEP 4 : Find the elements satisfying $y > y_{bottom}$ in the x range node_i to node_j and report them.

STEP 5 : Stop.

2.3 Pseudo Code :

Report_points(x_1, x_2, y_bottom)
{
if($x_2 - x_1 < 2 * (Log[n])$)
Locate the required x range in the BST.
Report elements between that x range satisfying $y > y_{bottom}$ using binary search

else if(x_1 and x_2 exists in data points)
Locate the required x range in the BST.
Report elements between that x range satisfying $y > y_{bottom}$ using binary search

else
node_i $\longrightarrow$ x value of the nearest node ahead of x_1
node_j $\longrightarrow$ x value of the nearest node before x_2
report_i = Report_points(x_1, node_i, y_{bottom})
report_j = Report_points(node_j, x_2, y_{bottom})
report_rest = Report_points(nodeı, nodeȷ, y_{bottom})
}

2.4 Space Complexity :

The data structure we invented, is a BST of size N*(augmentation size). Therefore, space used is $N * Log[n]$= n. (Refer Sub section Data Structure Design). Implying space complexity is O(n).

2.5 Time Complexity :

2.5.1 Query Time :

2.5.2 Pre-processing Time :

- The first sort based on x coordinates requires O(n*Log[n]).
-
-
-

3 Constraint of each commando

3.1 Overview

This problem is approached using the divide and conquer strategy. As the square dimension is in powers of 2, we can divide the square we get in every iteration into 4 squares of equal size. Let 'P' = p(x,y) be the coordinates of Prime Minster's cabin. Define a recursive function Report(Square,n,P) which works by divide and conquer. Where 'Square' gives the square boundaries and 'n' gives its side length.

3.2 Algorithm

- Start.
- **if(**$n < 2$**)** return with reporting "No points"
- **else if(**$n == 2$**)** return position of commando as the one diametrically opposite to P in the square. Orientation will be facing P so that he can cover the L-shape leaving out Prime minister's office.
- **else** divide the square into four equal squares of side length n/2. P_1, P_2, P_3 and P_4 are estimated as follows. The quadrant that has P currently will retain it as its P_1 and this square is called $Square_1$. The other three quadrants will have their P_i to be the point that is diametrically opposite to the only corner that they have of the bigger square. where $i \in \{2, 3, 4\}$.
- Report(n,P,Square) = Report(n/2,P_1,$Square_1$) + Report(n/2,P_2,$Square_2$) + Report(n/2,P_3,$Square_3$) + Report(n/2,P_4,$Square_4$) + the commando position and orientation who can cover P_2, P_3 and P_4.
- Stop.

3.3 Pseudo Code

```
Report(n,P)
{
    if(n < 2)
        print "No Commandos required". return

    else if(n == 2)
        Return Position = diametrically opposite point to P.
        and Orientation = direction facing P. return

    else
        P_1 = P and Square_1 is the corresponding Square.
        Find Square_i and P_i as defined above.where i ∈ {2,3,4}.
```

Report(n,P,Square) = Report(n/2,P_1,$Square_1$) + Report(n/2,P_2,$Square_2$) + Report(n/2,P_3,$Square_3$) + Report(n/2,P_4,$Square_4$) + the commando position and orientation who can cover P_2, P_3 and P_4.

```
    return
}
```

3.4 Time Complexity

$$T(n) = 4 * T(n/2) + a$$
$$= 4^{log(n)} + constant$$
$$= O(n^2)$$

3.5 Proof of Correctness

3.5.1 What is to be proved? or Claim

That given any square of side length 'n' (a power of 2) and Prime Minister's office P we can exhaust the rest of the square with non overlapping pieces of L-shaped tiles (L-shape containes 3 unit sqaures and hence is equivalent to the given problem).

3.5.2 Proof by Induction

Induction is carried out on the length of the square.

3.5.3 Base Cases

- $(n < 2)$ - No commandos needed.

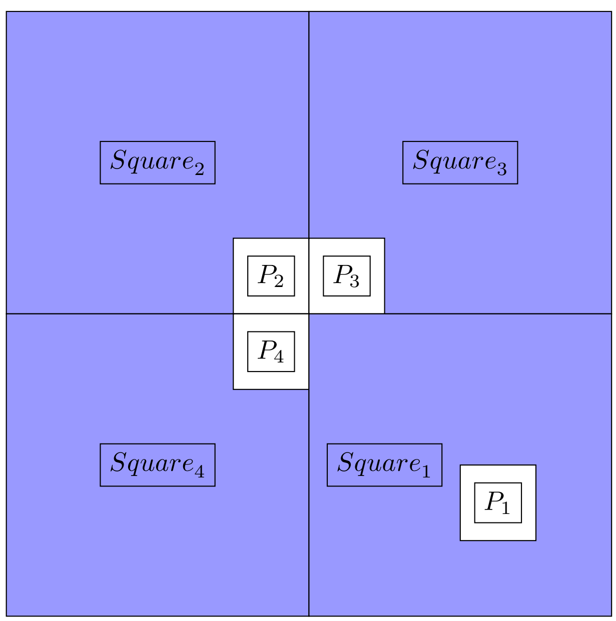
- $(n == 2)$ - Only one commando needed and he should be placed so as to cover 'L' shape that leaves out Prime Minister's cabin.

3.5.4 Hypothesis

Let $n = 2^k$.
Let us assume that the Claim is true for all $i <= k$ where $k > 2$ and both are integers.

3.5.5 Inductive Step

Let us prove that claim for $k + 1$ is true.
The square of size 2^{k+1} can be broken into four squares of size 2^k. As the claim holds for sizes $<= k$, these 4 squares can be exhausted in the required way, (i.e) PM cabin anywhere we chose it to be and the rest exhausted with non overlapping 'L' shapes.



Then chose, the PM cabin's as shown above. This way just by adding one more commando, on p_2 facing towards $Square_1$, we can exhaust our

required full square!
Thus, proved by induction.
Note, this has to be an optimal solution as we are ensuring that every box is guarded by just one commando (this is because of the combine step which does not disturb any guarded box and the base case for which the claim holds).