PERMUTATION AND COMBINATION

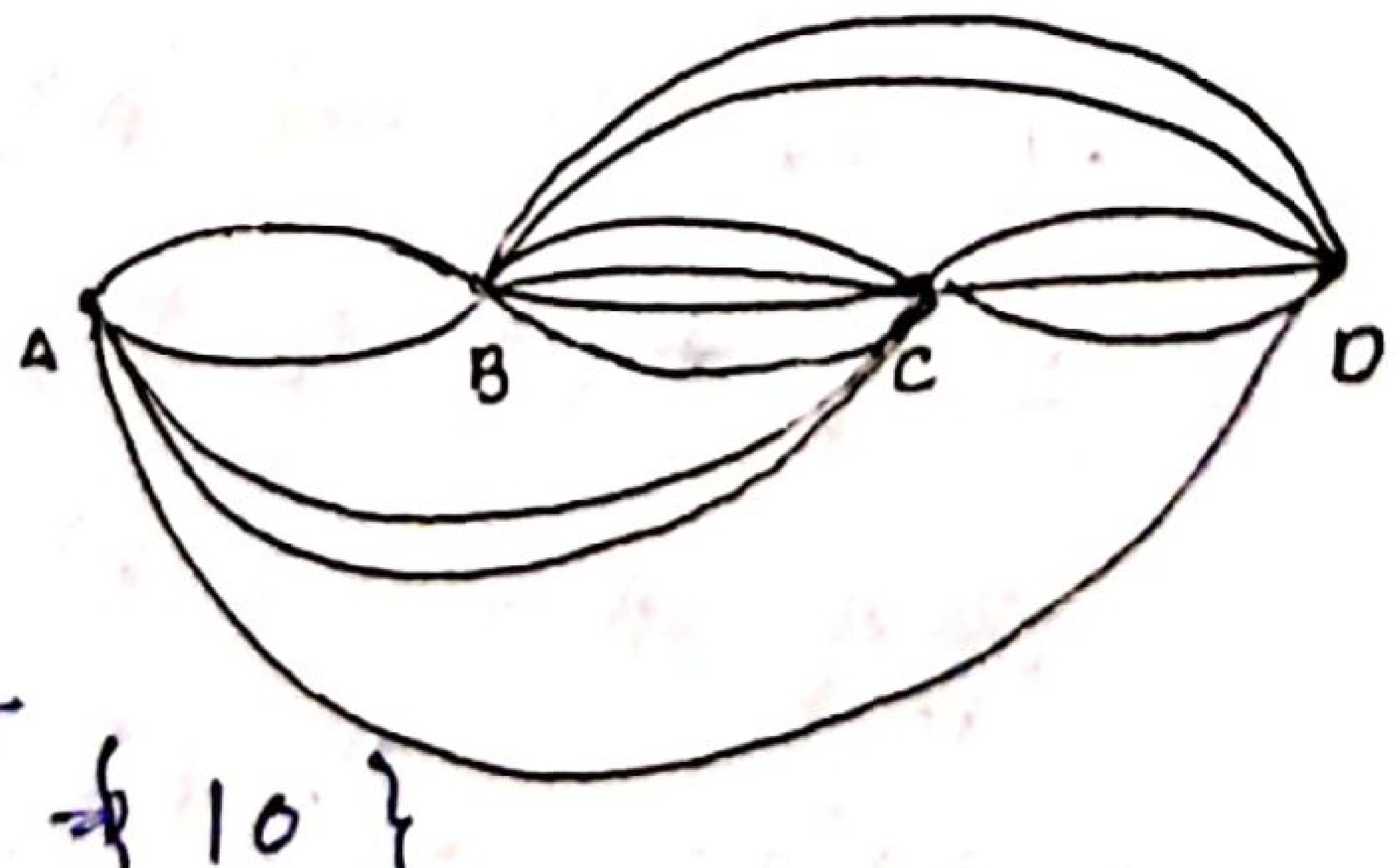
* **Fundamental Principle of Multiplication: (FPM)**

हमारे पास दो काम है पहले काम को करने के लिए 'm' ways और दूसरे काम को करने के 'n' ways है। तो फिर दोनों काम को एक के बाद एक order में करने के $(m \times n)$ ways होंगे। (This can be extended to any number of events)

* **Fundamental Principle of Addition: (FPA)**

हमारे पास दो काम है पहले काम को करने के लिए 'm' ways है और दूसरे को करने के 'n' ways है और यह पहले काम से independent है तो ऐसे में Total no. of ways $(m+n)$ होंगे। (This can be extended to any number of events)

Question:
(movement forward only)



(a) The number of ways of going from city A to city C.

$A \rightarrow C \rightarrow 2$

$A \xrightarrow[2]{} B \xrightarrow[4]{} C \rightarrow 2 \times 4 = 8$ (FPM)

$+ \rightarrow \{10\}$ (FPA)

(b) The number of ways of going from city A to city D via city B

$A \xrightarrow[2]{} B \xrightarrow[2]{} D \rightarrow 2 \times 2 = 4$

$A \xrightarrow[2]{} B \xrightarrow[4]{} C \xrightarrow[3]{} D \rightarrow 2 \times 4 \times 3 = 24$

$\rightarrow \{28\}$

(c) no.s of ways of going from city A to city D

$A \xrightarrow{1} D \rightarrow 1$

$A \xrightarrow[2]{} B \xrightarrow[2]{} D \rightarrow 4$

$A \rightarrow B \rightarrow C \rightarrow D \rightarrow 24$
$2 \times 4 \times 3$

$A \rightarrow C \rightarrow D \rightarrow 6$

$1 + 4 + 24 + 6$

$\Downarrow$

{35 ways}

Q 3 digits numbers using the digits 1, 2, 3, 4, 5 without repetition

$5 \times 4 \times 3 \Rightarrow 60$ ✓

Q Number of words which can be formed from the letters of the word MACHINE if vowels may occupy the

(a) odd Posn.

1 3 5 7

MACHINE

$(4 \times 3 \times 2) \times \underbrace{(4 \times 3 \times 2 \times 1)}_{\text{consonets.}} = 576$

A I E

AIE (consonants) 3

MCHN (vowels) 4

(b) even posn

2 4 6

$(3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)$

A I E

जिसकी संख्या कम होती है हम उससे ~~couting~~ counting करना Prefer करते है।

Q Find Rank of word TOUGH

G _ _ _ _ = 24 (4 3 2 1)

H _ _ _ _ = 24 (4 3 2 1)

O _ _ _ _ = 24 (4 3 2 1)

T G _ _ _ = 6 (3 2 1)

T H _ _ _ = 6 (3 2 1)

T O G _ _ = 2 (2 1)

T O H _ _ = 2 (2 1)

TOUGH = 1

Rank = 24+24+24+6+6+2+2+1
= 89 word

* Permutations:

* ^nP_r denotes no. of arrangements of r things out of n-different things

eg. 4 out of 5 boys to be arranged on _ _ _ _ 4 places $= {}^5P_4$

* Cambinations:

^nC_r denotes no. of selections of r objects out n different objects. eg: 10 optional subject $\Rightarrow {}^{10}C_5$
↳ 5 subject

* Factorial Notation:

The continued product of first n-natural numbers is called a factorial of n. It is denoted by $n!$ or $\underline{|n}$ is defined as.

$$n! = 1\cdot 2\cdot 3 \cdots (n-2)\cdot(n-1)\cdot n$$
$$= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$
$$= n\cdot(n-1)!$$
$$= n\cdot(n-1)\cdot(n-2)! \text{ and so on}$$

$6! = 6\times5\times4\times3\times2\times1 = 720$
$5! = 5\times4\times3\times2\times1 = 120$
$4! = 4\times3\times2\times1 = 24$
$3! = 3\times2\times1 = 6$
$2! = 2\times1 = 2$
$1! = 1 = 1$
$0! = 1 = 1$

Learn & Remember

$n! = n\cdot(n-1)!$
put $(n=1)$
$\rightarrow 1! = 1\cdot(0)!$
$\rightarrow 0! = 1$

Number of Permutations of 'n' different things taken all at a time $= n!$

* Value of the symbol ^nP_r:

r things out of $a_1, a_2, a_3, a_4 \cdots a_n \rightarrow$ n different things

$\Rightarrow$ to be arranged on

①	②	③	④	एक कम	----	(r)	
—	—	—	—	—	----	—	— r places
n	$(n-1)$	$(n-2)$	$(n-3)$		----	$(n-(r-1))$	

By FPM No: of arranging of r things out of n different things $= n(n-1)(n-2)\cdots(n-r+1)$

But no: of arrangement of r things out n diff. things is

$$^nP_r = \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\times2\times1}{(n-r)(n-r-1)\cdots 3\times2\times1}$$

$$\Rightarrow \boxed{^nP_r = \frac{n!}{(n-r)!}}$$ Hence Proved

$$^nP_r = \frac{n!}{(n-r)!} \quad \text{where } r \leq n,\ r \in W,\ n \in N$$

* **Value of the Symbol ^nC_r**

^nC_r denotes no: of selections of r objects out n different objects.

$$^nC_r = \frac{n!}{r! \times (n-r)!} \quad \text{where } r \le n,\ r \in W,\ n \in N$$

r things out of $a_1, a_2, a_3, \ldots, a_n \rightarrow n$ diff. things

to be arranged on r places ---------- $\rightarrow$ r places.

arrangement of r-different things

$$^nC_r \times r! = {}^nP_r$$

Select r things out n diff. things

$$\Rightarrow \left\{ {}^nC_r = \frac{^nP_r}{r!} = \frac{n!}{r!\,(n-r)!} \right\}$$

QED

2. **Shortcut value of ^nC_r**

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

$\rightarrow$ $$^5C_3 = \frac{5!}{3!\cdot 2!} = \frac{5\times4\times3\times\cancel{2\times1}}{3\times2\times1\cdot\cancel{2\times1}} = \frac{5\times4\times3}{3!}$$

3 factors

$\rightarrow$ 3!

$\rightarrow$ $$^6C_4 = \frac{6!}{4!\,2!} = \frac{6\times5\times4\times3\times\cancel{2\times1}}{4\times3\times2\times1\cdot\cancel{2\times1}} = \frac{6\times5\times4\times3}{4!}$$

4 factors

$\rightarrow$ 4 factorial.

$$\therefore\ {}^nC_r = \frac{n(n-1)(n-2)\ldots \text{ upto } r\text{- factors}}{r!}$$

* $^nP_0 = 1$, $\quad ^nP_n = n!$

* $^nC_r = {}^nC_{n-r}$, $\quad ^nC_0 = 1$, $\quad ^nC_n = 1$

* $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

{ छोटे वाले में indexes में 1 add करदे जैसे: $(n+1,\ r-1+1) = (n+1,\ r)$ }

* $^nC_x = {}^nC_y \Rightarrow \boxed{x = y}$ OR $\boxed{x + y = n}$

* $^nP_r = r!\cdot {}^nC_r$

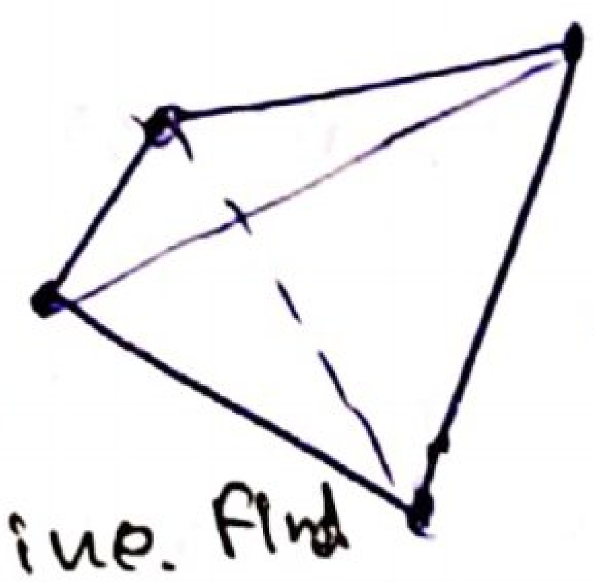
* $\cancel{(2n!)}$

* $(2n)! = 2^n \cdot n!\,(1\cdot3\cdot5 \cdots (2n-1))$
$= n!\,(2\cdot6\cdot10 \cdots (4n-2))$

Proof

$$(2n)! = 2n(2n-1)(2n-2)\cdots3\cdot2\cdot1$$
$$= \underbrace{(2n\cdot(2n-2)(2n-4)\cdots4\cdot2)}_{n\text{ factors}}\underbrace{(1\cdot3\cdot5\cdots(2n-1))}_{n\text{ factors}}$$
$$= 2^n\,(n(n-1)(n-2)\cdots2\cdot1)(1\cdot3\cdot5\cdots(2n-1))$$
$$\boxed{(2n)! = 2^n\,n!\,(1\cdot3\cdot5\cdots(2n-1))}$$
$$(2n)! = 2^n\,n!\,(1\cdot3\cdot5\cdots(2n-1))$$
$$\boxed{(2n)! = n!\,(2\cdot6\cdot10\cdots(4n-2))} \quad \{QED\}$$

Q There are n point in a plane, no three are collinear.
Find
(a). Number of St. lines by joining these points $= {}^nC_2$
(b) Number of triangles $= {}^nC_3$
(c) Number of diagonals in a polygon having n sides.
$= ({}^nC_2 - n)$



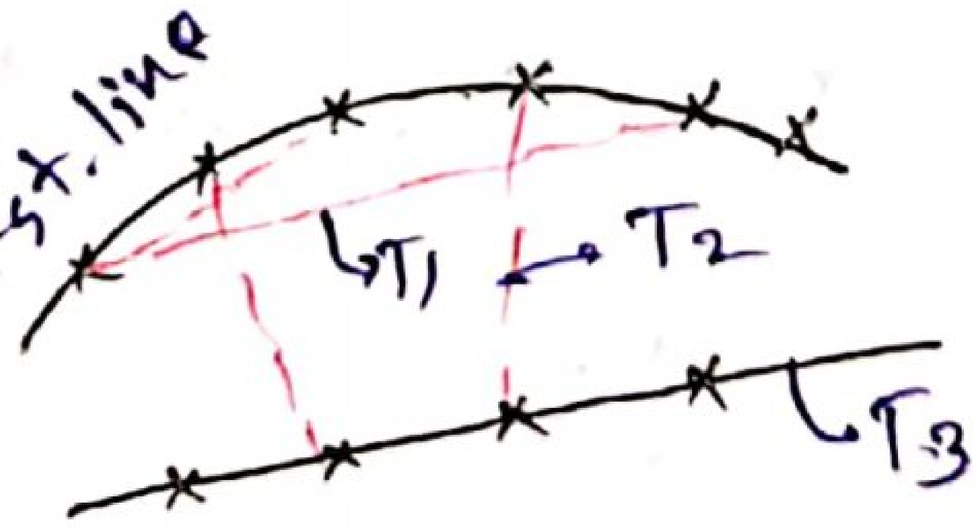
→ no. of diagonals
→ ${}^4C_2 = \frac{4\times3}{2} = 6$
$6 - 4 = 2$

(d) 10 points in a plane No three being collinear except 4 which are in the same line. Find
(a) no: of st. line

M1
${}^{10}C_2 - {}^4C_2 + 1$ → collinear st. line

${}^{10}C_2$ ↓ Total no. of St. line from 10 points

${}^4C_2$ ↓ Total no. of St. line from 4 points



M-2

${}^6C_2 + {}^6C_1 \times {}^4C_1 + 1$ → Type 3

${}^6C_2$ ↙ no. of st line from non collinear points Type1

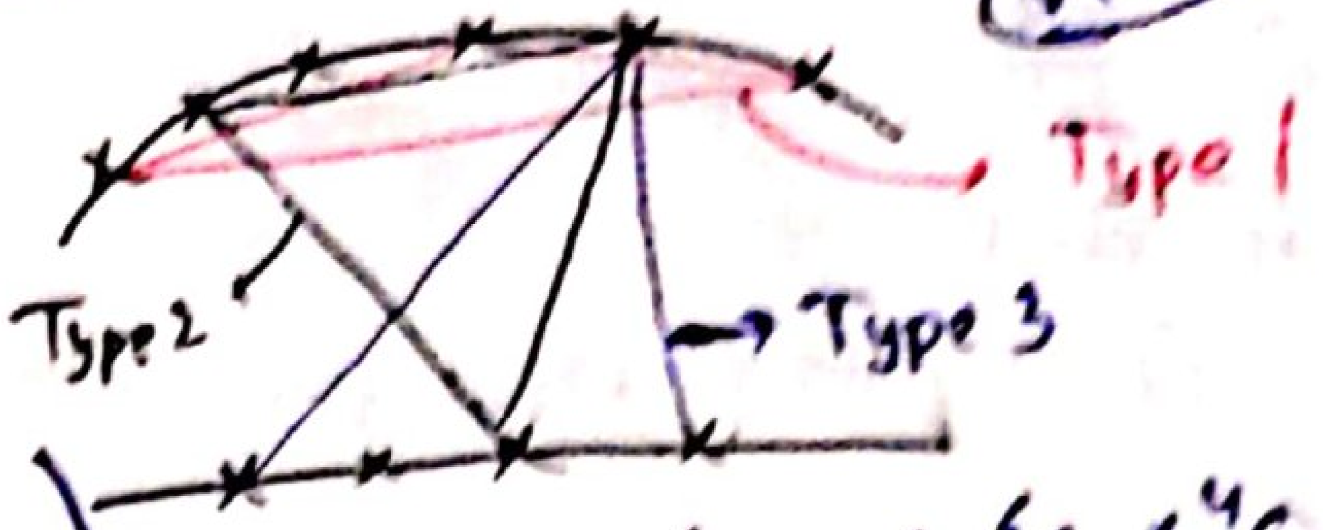
${}^6C_1 \times {}^4C_1$ ↓ no: st. line from co-llinear points + from non-co-linear points Type2

(b) number of triangles

M-1 $^{10}C_3 - {^4C_3}$

(Total no. of triangle from 10 points) (triangles from 4 points)

M-2



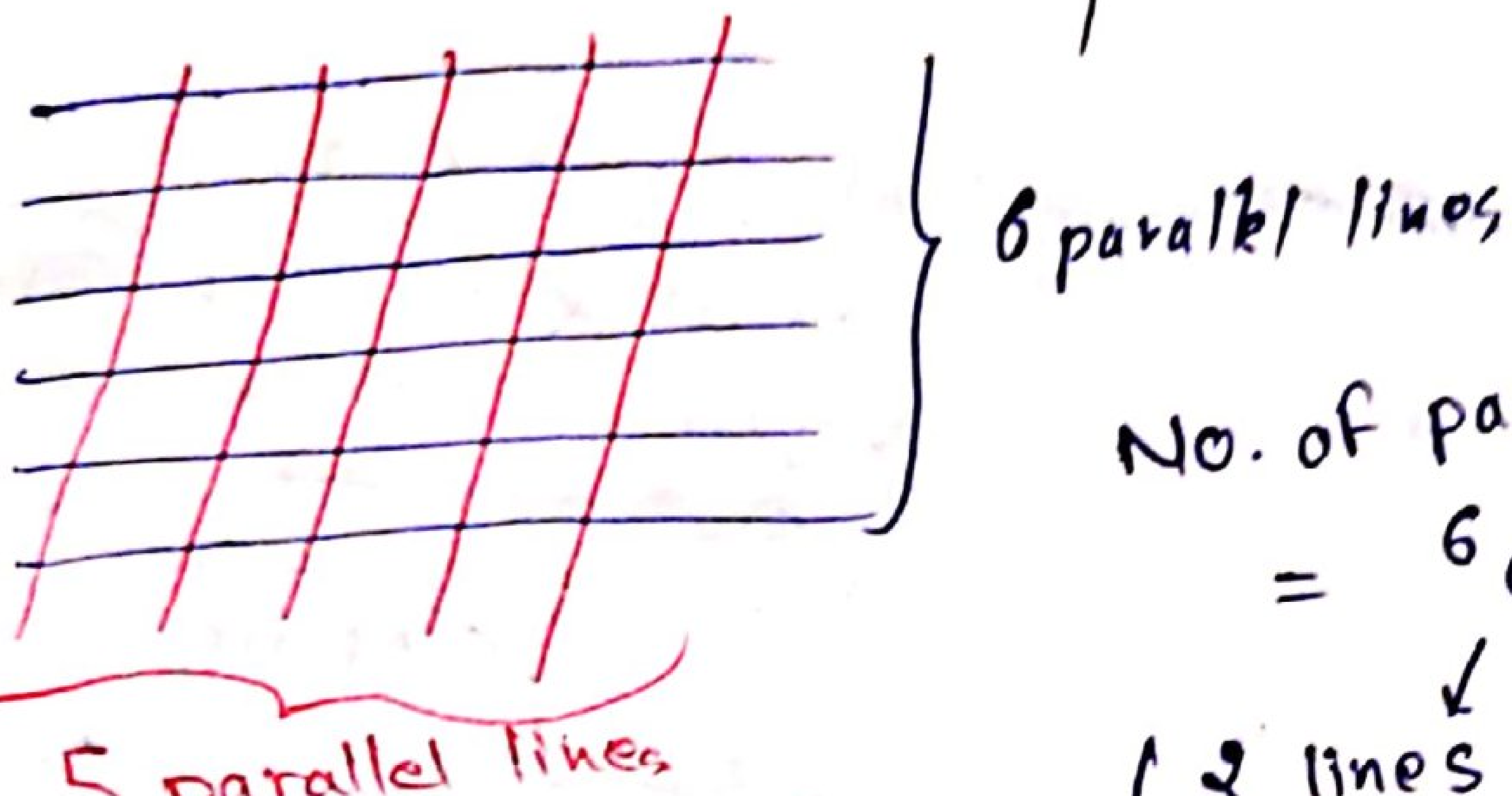
$^6C_3 + {^6C_2} \times {^4C_1} + {^6C_1} \times {^4C_2}$

(triangle from non-collinear point) Type 1

(triangle from collinear & non-collinear but base of Δ on non collinear points) Type 2

(Δs from co-linear & non-linear but base of Δ on co-linear points) Type 3

Q 6 parallel lines which are cut by another set of 5 parallel lines. Numbers of parallelogram



No. of parallelogram

$= {^6C_2} \times {^5C_2}$

(2 lines 6 parallel lines में से) (2 lines 5 parallel lines में से)

MATHEMATICAL GYAAN

Given $a+b+c=1$ and $a,b,c>0$, find the minimum value of $a^2+2b^2+c^2$

$\vec{v}_1 = a\hat{i} + \sqrt{2}\hat{j} + c\hat{k}$

$\vec{v}_2 = \hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \hat{k}$

By C.S inequality

$(\vec{v}_1 \cdot \vec{v}_2)^2 \leq |v_1|^2 \cdot |\vec{v}_2|^2$

$\therefore (a+b+c)^2 \leq (a^2+2b^2+c^2)(+\frac{1}{2}+1)$

$\Rightarrow 1 \leq (a^2+2b^2+c^2) \cdot \frac{5}{2}$

$\Rightarrow a^2+2b^2+c^2 \geq \frac{2}{5}$

Minimum value of $a^2+2b^2+c^2 = \frac{2}{5}$ Ans.

* Sir GAP METHOD और STRING METHOD कहाँ USE करते है,
जहाँ चीजों को हमेशा अलग-अलग रखना हो वहाँ USE करेंगे GAP METHOD और जहाँ चीजों को हमेशा एक साथ रखना हो वहाँ USE करेंगे STRING METHOD

Q How many ways can 7 girls be arranged in a line so that two girls named
(a) Gulabo and Sitabo always come together
(b) Gulabo and Sitabo never occur together

(a) (Gulabo + Sitabo), G_1, G_2, G_3, G_4, G_5 {STRING METHOD}
1 2 3 4 5 6

No. of arrangement:

$2! \times 6!$

2! → gulabo & Sitabo can arrange among themselves

6! → arrangement of 5 girls + 1 unit

जिनको हमेशा साथ रखना उनका एक Unit या Single बना लेंगे

(b) Gulabo, Sitabo

{GAP METHOD}

$\times G_1 \times G_2 \times G_3 \times G_4 \times G_5 \times$

Gaps

जिनको एक साथ नहीं लाना उनको सबसे पहले अलग कर लेंगे। उसके बाद बचे हुओं को arrange करेंगे फिर उनके बीच बची खालि जगह में जिनको अलग किया था। उनको Arrange करेंगे।

M-① $5! \times {}^6C_2 \times 2!$

5! → (arrangement of 5 girls)

${}^6C_2$ → 6 gaps में से दो GAP choose किए

2! → Gulabo & Sitabo को arrange

M-② Total Arrangement − Arrangements in which Gulabo & Sitabo sit together

$\Rightarrow 7! - 2! \times 6!$

Q 4 boys and 4 girls are to be seated in a Line, find:

(a) Number of ways they can be seated so that No two girls are together (OR girls are separated)

4 Girls , 4 Boys.

× B_1 × B_2 × B_3 × B_4 × ~~B_5 ×~~

$4! \times {}^5C_4 \times 4!$

$4!$ → Arrange 4 Boys with GAPS

${}^5C_4$ → Select 4 Gaps for 4 girls

$4!$ → Arrange 4 girls on 4 selected gaps.

(b) Not all the girls are together or at least one girl is separated from the rest of the girls

Total arrangement − Arrangement with all girls together

$8! - 4! \times 5!$

1. 2 3 4 5

$B_1, B_2, B_3, B_4, (G_1 G_2 G_3 G_4)$

$4! \times 5!$

(C) 'Boys and Girls are alternate' or boys as well as girls are separated or each boy follows the girl and each girl follows the boy.

No. of arrangements with Boys & Girls alternate

$B_1 \times B_2 \times B_3 \times B_4 \times + G_1 \times G_2 \times G_3 \times G_4 \times$

(B G B G B G B G
G B G B G B G B)

$4! \times 4! + 4! \times 4!$

$\Rightarrow 2 \times 4! \times 4!$

× B_1 × B_2 × B_3 × B_4 ×

$4! \times {}^5C_4 \times 4!$

Boy, Select GAP, Girls

यह इसलिए गलत हुआ क्योंकि जब हमने GAP SELECT किए तो क्या पता B_1 B_2 के GAP में कोई Girl आए ही ना !

(d) If they are 4 married couples then the number of ways if each couple is together.

$(H_1 W_1), (H_2 W_2), (H_3 W_3), (H_4 W_4)$

$\underbrace{2! \times 2! \times 2! \times 2!}_{\text{each Packets}} \times 4!$

$4!$ → Arrangement of 4 unit

~~(d) If they are 4 married couples then the no. of ways if each couple is together.~~

(e) Boy and girl are alternate and a particular boy & girl are never adjacent to each other in any arrangement.

3 Boys + 3 Girls + 1 Kallu + 1 kaali

M-1 3 Boys + 3 Girls + 1 Kallu + 1 kaali

Total alternate arrangements — Alternate arrangements in which Kallu Kaali sit together.

$= 2\times4!\times4!$ — × B × G × B × G × B × G ×

(Kallu, kaali / kaali, kallu)

$2\times3!\times3!\times{}^7C_1\times1$

(arrangement of 3B & 3 Girls alternate) (7 GAPS में से 1 GAP select) (इस एक GAP में Kallu, Kaali एक ही तरीके Arrange)

M-11

3 Boys + 1 Kallu   3 Girls + 1 kaali

(kaali इन-इन Pos^n पर आ सकती है!)

$B_1 \times B_2 \times$ Kallu $\times B_3 \times$ + kallu $\times B_1 \times B_2 \times B_3 \times$.

$3C_1\times3!$ (Kallu इन-इन Pos^n पर आ सकता है!)

$({}^3C_1\times3!\times{}^2C_1\times3! + 1\times3!\times{}^3C_1\times3!)\times2$

(Kallu) — Kallu के लिए जगह

(3 Boys)

(Select one ticked (✓) GAP for Kaali)

3 Girls

(one way for kallu)

3 Boys

3 girls

(Select 1 GAP from ticked (✓) GAP for Kaali)

अगर Girls पहले है तो Kallu, Kaali
अगर Boy पहले है Kaali, Kallu
ऐसा इसलिए है ताकि Alternate Boy, Girl का Arrangement ना टूटे।

यहाँ '2' इस लिए आया same Process अब Kaali को रखकर भी करेंगे जैसे Kallu को रखकर किया

Q MOTHER PROBLEM

A train having 12 stations enroute (on the route) has to be stopped at 4 stations. Number of ways it can be stopped if no two of the stopping stations are consecutive

S S S S S S S S S S S S

S S S S S S S

ऐसा लगता (महसूस) होता है की 4 stations इस 8 stations के बीच GAPS में से ही आए हो क्योंकि two stations को consecutive ना होने के लिए बीच में एक stations का आना जरूरी है जहाँ पर Train नहीं Ruki

∴ × S × S × S × S × S × S × S × S ×

$\Rightarrow {}^9C_4 \rightarrow$ 9 GAPS में से 4 GAPS select करो चार GAPS जहाँ Train Ruki थी!

m-stations train to be stopped at r-stations such that no two are consecutive

ANS ⇒ ${}^{m-r+1}C_r$

Q Find the no. of ways in which 3 distinct no:s can be selected from the set $\{3^1, 3^2, 3^3, \ldots, 3^{100}, 3^{101}\}$ so that they form a G.P

$3^1, 3^4, 3^7$ OR $3^7, 3^4, 3^1$

Fix होते ही अपने आप Fix हो जाएगा!

a, b, c are in G.P $b^2 = ac$

$3^a, 3^b, 3^c$ are in G.P

$(3^b)^2 = 3^a \cdot 3^c \Rightarrow 3^{2b} = 3^{a+c}$

$\rightarrow 2b = a+c \rightarrow b = \frac{a+c}{2} \rightarrow$ a+c should be even

power 1, 2, 3, ---, 101

51 odd 50 Even

C-1 a+c = even: a = odd, b = odd

C-2 a+c = even: a = even, b = even

${}^{51}C_2 + {}^{50}C_2$ Ans.

Q Find number of 4 digits divisible by 4 which can be formed by using the digits 1, 2, 3, 4, 5, 6 if

(a) Repetition of digits is not allowed

(b) Repetition is of digits is allowed

4 से Divisible होने के लिए Last की ~~[illegible]~~ Two digits 4 से Divide होनी चाहिए

(a) 1, 2, 3, 4, 5, 6

$\square\,\square\,\square\,\square$

$4 \times 3 \times 8$

Last two digit can be

12, 16, 24, 32, 36, 52, 56, 64

1 2 3 4 5 6 7 8

$\Rightarrow 4\times3\times8 = 96$

(b) $\square\,\square\,\square\,\square$

$6\times 6 \times 9$

Last two digit can be

12, 16, 24, 32, 36, 44, 52, 56, 64

1 2 3 4 5 6 7 8 9

अबकी बार 44 भी Include हो जाएगा

$\Rightarrow 6\times6\times9 = 324$

Q The number of integers greater than 6000 that can be formed, using the digits 3, 5, 6, 7 and 8 without repetition is:

4 Digit

(6, 7, 8) $3\times 4 \times 3 \times 2 = 72$

5 Digit

$5\times 4\times 3\times 2\times 1 = 120$

$72 + 120 = 192$ ANS

Q The number of four-digit numbers strictly greater than 4321 that can formed using the digits 0, 1, 2, 3, 4, 5 (repetition of digits is allowed) is:

Four Digit No: > 4321

* 432_ 4 (ways) (2, 3, 4, 5) = 4
* 43_ _ → $3\times6 = 18$ (3 × 6)
* 4_ _ _ = 72 (2× 6 × 6)
* 5_ _ _ = 216 (6 6 6)

216
94
———
310 Ans

Q No: of four digit even numbers that can be made using digits 0, 2, 3, 4, 5, 6, & 7
(a) if repetition is allowed (b) if repetition is not allowed.

(a)

$6 \times 7 \times 7 \times 4 = 1176$

(b) Case 1: 0 at End

| | | | 0 |

$6 \times 5 \times 4 \times 1 = \underline{120}$

Case 2: 0 is not Not Present in no.

$3 \times 4 \times 5 \times 3 (2, 4, 6) = \underline{180}$

Case 3: 0 is present but not at End

5 4 3 (2, 4, 6)

For 0 → 2 ways
Last place → 3 ways

No: of No:s $= 4 \times 5 \times 3 \times 2 = \underline{120}$

FINAL ANS $= 120 + 180 + 120$
$= \underline{420}$

| | 6 | 3 | 4 |

3 5 6 3 (2, 4, 6)

(zero नही आ सकता)

0, 2, ~~3~~, ~~4~~, 5, ~~6~~, 7

(अगर हम center मे 0 नही लेते)

| | 3 | 0 | 4 |

4 5 6 3

~~0~~, 2, ~~3~~, ~~4~~, 5, 6, 7

(अब इसमे 4 तरीके आगए)

अगर zero ~~कही~~ बहार आगया तब इस confusion से बचने के लिए cases बनाए

Why CASES

क्योंकि zero को हम first Posn पर नही रख सकते परंतु zero Last End Point पर आ सकता है और 0 के Last मे आने से No: even बनता है। अगर हम zero को neglect करके ~~cases बनाएगे तो~~ counting करेंगे तो हमारी कुछ even संख्याए रह जाएंगी।

OTHER REASON

| | | 0 |

अगर zero हम Last मे ठहरा देते है तो कोई Problem नही फिर हम आगे कोई भी Non zero +ve No: ठहरा सकते।

| 1 | 2 | |

अगर zero इधर नहीं ठहराते तो आगे 1-Place पर zero आएगा या 2-Place पर zero आएगा तो बताना confusing है

Q 4 shoes are selected at random out of 5 different pair of shoes, find number of ways such that selected shoes has

(a) Exactly one pair (b) No complete pair.

(a) P_1 $(J_1 J_2)$ P_2 $(J_3 J_4)$ P_3 $(J_5 J_6)$ P_4 $(J_7 J_8)$ P_5 $(J_9 J_{10})$

M-I

$${}^5C_3 \times {}^3C_1 \times {}^2C_1 \times {}^2C_1 = 10 \times 3 \times 2 \times 2 = \boxed{120}$$

(5 pair में से 3 pair select) (Selected 3 pair में से एक pair select) (एक Pair के दो Juto(Shoes) में से एक Shoe) (अलग Pair के दो Shoes में से एक Shoe)

M-II

$${}^5C_1 \times \frac{{}^8C_1 \times {}^6C_1}{2} = \frac{240}{2} = 120$$

(5 pair में से 1 pair select) (8 shoes में से एक shoe) (6 shoes में से एक shoes)

$J_1 J_2 \cancel{J_3} J_4 J_7 J_8 J_9 J_{10}$

$J_1 J_2 J_3 J_4 \cancel{J_7} J_8 J_9 J_{10}$

(अगर Left से आएगें तो J_4, J_8 आ सकता है और Right से आएगें तो J_4, J_8 फिर से आ सकते है जोकि दो बार count हो गया, इसलिए 2 से Divide करेंगे ताकि दो बार count ना करें)

(इसमें एक shoe जिसका पहले वाले के साथ Pair बनता था उसको निकाल दिया)

(b) $${}^5C_4 \times {}^2C_1 \times {}^2C_1 \times {}^2C_1 \times {}^2C_1$$

(5 pair में से 4 pair Select) (P_1 में से 1-shoe) (P_2 में से 1-shoe) (P_3 में से 1-shoe) (P_4 में से 1-shoe)

(let P_1, P_2, P_3, P_4)

Q The no: of 5 digit no:s such that the sum of their digits even is

P_1	P_2	P_3	P_4	P_5
9	10	10	10	5
(1, 2, 3, ..., 9)	(0, 1, 2, 3, ..., 9)	(0, 1, 2, 3, ..., 9)	(0, 1, 2, 3, ..., 9)	

$9 \times 10 \times 10 \times 10 = 5 = \underline{45000}$

अगर $P_1+P_2+P_3+P_4$ को Add करके

Even → Last Place even → 5 (0, 2, 4, 6, 8)

Odd → Last Place odd → 5 (1, 3, 5, 7, 9)

Q If $(100)! = 2^m \cdot I$ where I is an odd integer then find m. Find the number of cyphers (zero) at the end of 100!

$100! = 2^m \cdot I$

$4! = 24 = 2^3 \cdot 3$ ↳ odd

2 one times	2, 4, 6, 8, 10 ---- 100	= 50	$= \left[\frac{100}{2}\right]$
2 two times	4, 8, 12, 16, -----, 100	= 25	$= \left[\frac{100}{2^2}\right]$
2 three times	8, 16, 24, -----, 96	= 12	$= \left[\frac{100}{2^3}\right]$
2 four times	16, 32, 48, 64, 80, 96	= 6	$= \left[\frac{100}{2^4}\right]$
2 five times	32, 64, 96	= 3	$= \left[\frac{100}{2^5}\right]$
2 six times	64	= 1	$= \left[\frac{100}{2^6}\right]$

$m = \underline{97}$

No. of cyphers in 100! → 2 → more in number; 5 → less in number

[जो less हो उससे decide करते है]

No: of 5s in 100

$= \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \left[\frac{100}{5^3}\right] + \dots$

$= 20 + 4 + 0 = 24$

No. of cyphers at end of 100! = 24

MAX POWER OF A PRIME p in n! is $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots$

Last digit जो भी हो तो वो किस Prime no: का multiple है यह देखा जाता है जैसे last में zero $5 \times 2 = 10$ करने पर आता है इसलिए हमने 2 & 5 को देखा इधर अगर 14 आता तो 7×2 को देखते

* sir grouping का use कहाँ करते है।

इसका use लोगों के बीच different चीजों को distribute करने के लिए किया जाता है या इसका use different चीजों के groups बनाने के लिए किया जाता है।

eg 6 Books

Kallu (4) Lallu (2)

6 Books → 2 Bundle/groups → 4 Book, 2 Book

$$^nC_r = \frac{n!}{r!(n-r)!}$$

(Select 4 Book From 6 Books) → $^6C_4 \times {}^2C_2$ ← (Bachi हुई 2 Books में से दोनों Book select)

$$\Rightarrow \frac{6!}{4!\,2!} \times \frac{2!}{2!\cdot 0!} = \frac{6!}{2!\cdot 2!}$$

eg 9 Book (diff)

Kallu (4) Lallu (3) Sallu (2)

$^9C_4 \times {}^5C_3 \times {}^2C_2$ → (Select बची हुई 2 Books)

(Select 4 Book From 9) (Select 3 Books From बची हुई 5)

$$\Rightarrow \frac{9!}{4!\,5!} \times \frac{5!}{3!\,2!} \times 1 \Rightarrow \frac{9!}{4!\,3!\,2!} = \frac{(4+3+2)!}{4!\,3!\,2!}$$

eg 15 Book (diff)

6 7 2

$$\Rightarrow \frac{15!}{6!\,7!\,2!} = \frac{(6+7+2)!}{6!\,7!\,2!}$$

* (m+n+p) diff Books no. of ways to form three group of size m, n & p $= \dfrac{(m+n+p)!}{m!\,n!\,p!}$ {m, n, p are all differen.}

* If these groups are to be distributed among three persons $= \dfrac{(m+n+p)!}{m!\,n!\,p!} \times 3!$

Q 4 Books (ABCD)

2 2

$$\Rightarrow \frac{4!}{2!2!} = \frac{24}{4} = 6$$

Banega	Bachega
1) AB	CD
2) CD	AB
3) BC	AD
4) AD	BC
5) AC	BD
6) BD	AC

2!

$$\Rightarrow \frac{4!}{(2!2!)2!} = \text{No: of ways of groups of size two each}$$

यें 2! इसलिए आया को क्योंकि इसमें (AB)(CD) & (CD)(AB) same ही है यानि हमने एक ही case को 2! times ज्यादा calculate कर लिया

Q 15 Books

3 3 2 2 4 1

2बार 2बार

$$\Rightarrow \frac{15!}{3!3!\,2!\,2!2!\,2!\,4!\,1!}$$

Q 25 Books divided into 8 peoples

5 2 2 2 3 3 4 4.

3 Baar 2 Baar 2 बार

$$\Rightarrow \frac{25!}{5!\,2!2!2!\,3!\,3!3!\,2!\,4!4!\,2!}$$

* if 3n different things are to be divided into 3 group of size n each. then no: of ways

$$3n \to n,\ n,\ n \qquad \frac{(3n)!}{n!n!n!\,3!}$$

* if above 3 groups are also to distributed among 3 people

$$\text{no. of ways} = \frac{(3n)!}{n!n!n!\ 3!} \times 3!$$

Q 6 different books are to be distributed b/w R | S | G. if each child gets at least one book

6 Book

$$1\ 2\ 3 \rightarrow \left(\frac{6!}{1!\,2!\,3!}\right) \times 3!$$

$+$

$$2\ 2\ 2 \rightarrow \left(\frac{6!}{(2!)^3\, 3!}\right) \times 3!$$

$+$

$$4\ 1\ 1 \rightarrow \left(\frac{6!}{4!\,(1!)^2\, 2!}\right) \times 3!$$

(group formation) (Arrangement)

Q No: of ways in which 8 persons can be seated in 3 different taxies each having 3 seats for passangers and duly numbered if

(a) Internal arrangement of persons inside the taxi in immaterial.

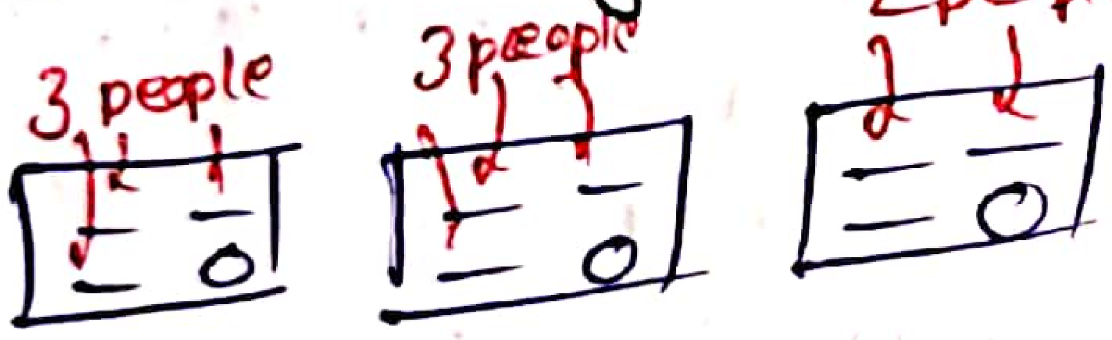
8 (people) → 3, 3, 2

$$\left(\frac{8!}{3!\,3!\,2!\,2!}\right) \times 3!$$

(grouping) (Arrangement cars)

(b) Internal arrangement also matters

M-I



$$8 \to 3, 3, 2 \quad \left\{\left(\frac{8!}{3!\cdot 3!\cdot 2!\cdot 2!}\right) \times 3!\right\} \times 3! \times 3! \times {}^3C_2 \times 2! = 9!$$

(grouping & arrangement distribution)

(पहली car में 3 seats पर 3 person)

(दूसरी car में 3 seats पर 3 person)

(तीसरी car में 3 seats में से 2 seat select)

(दो लोगों का Arrange उन दो seats पर)

M-II

$${}^9C_8 \times 8! = 9!$$

(Select Total 9 seats में से 8 seats)

(Arrangement of 8 persons in 8 seats)

⇒ To arrange aaabc in a line find no. of ways (extra)

Sol^n let no. of arrangements be x.

x arrangement
- aaabc ⟶ $a_1 a_2 a_3 b c$ ⟶ 3! arrangements
- abaca ⟶ $a_1 b a_2 c a_3$ ⟶ 3! arrangements
- abaac ⟶ $a_1 b a_2 a_3 c$ ⟶ 3! arrangements
- ⋮

यहाँ पर same object को Different objects में convert कर लिया जिसे हमे पता चले अगर ये Different objects हो तो अभी के Arrangement से कितने ज्यादा Arrange. थे Diff^r objects में.

$\therefore x \times 3! = 5!$

$\Rightarrow \left\{ x = \frac{5!}{3!} \right\}$

n things ⟶ p-alike, q-alike, r-different

$$\text{no. of arrangements} = \frac{n!}{p! \cdot q!}$$

eg. MATHEMATICS ⟶ $\frac{(11)!}{2! \cdot 2! \cdot 2!}$

Q 21 White (W) and 19 Black (B) balls are arranged in a Line (balls of the same colour alike), find the number of arrangement if Black balls are separated.

21 white 19 Black

× W × W × W × - - - - - × W × No. of GAPS = 22

ANS =

$$1 \times {}^{22}C_{19} \times 1$$

- 1 : No. of ways to arrangements for white ball
- ${}^{22}C_{19}$: Select 19 GAPS for Black Balls out of 22 GAPS
- 1 : ways to arrange Black Balls in selected 19 GAPS

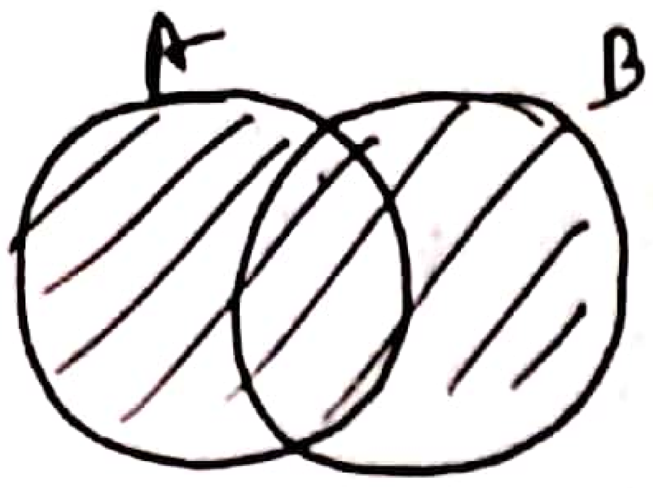
Q Find the number of ~~ways~~ words ~~formed~~ by taking 5 Letters of the word INDEPENDENCE at a time.

{ ① E's = 4 ; ② N's = 3 ; ③ D's = 2 ; ④ I's = 1 ; ⑤ P's = 1 ; ⑥ C's = 1 }

Category	Selection	Arrangements	No. of words
* All 5 diff^r	$^6C_5$	$5!$	$^6C_5 \times 5! = 720$
* 3 diff^r + 2 Alike	$^3C_1 \times {^5C_3}$	$\frac{5!}{2!}$	$^3C_1 \times {^5C_3} \times \frac{5!}{2!} = 1800$
	(E, N, D हमको दो Alike दे सकते) (बाकि 5 में से तीन Diff^r words)		+
* 3 Alike + 2 Diff^r	$^2C_1 \times {^5C_2}$	$\frac{5!}{3!}$	$^2C_1 \times {^5C_2} \times \frac{5!}{3!} = 400$
	(E, N हमको 3 Alike एकसाथ दे सकते) (select 2 Diff^r words from left 5 category)		+
* 2 Alike + 2 other Alike + 1 diff^r	$^3C_2 \times {^4C_1}$	$\frac{5!}{2!\,2!}$	$^3C_1 \times {^4C_1} \times \frac{5!}{(2!)^2} = 360$
	(E, N, D में से दो अलग-अलग Pairs) (Select 1 Diff^r word from left 4 category)		+
* 3 Alike + 2 other Alike	$^2C_1 \times {^2C_1}$	$\frac{5!}{3!\,2!}$	$^2C_1 \times {^2C_1} \times \frac{5!}{3!\,2!} = 40$
	(E, N से 3 Alike) (E/N, D से 2 Alike)		+
* 4 Alike + 1 Diff^r	$1 \times {^5C_1}$	$\frac{5!}{4!}$	$^5C_1 \times \frac{5!}{4!} = 25$
	(only E से 4 Alike) (select 1 Diff^r word from left 5 category)		

Total No. of words = 3345

* Principle of Inclusion & Exclusion :

* 

$$\underbrace{n(A \cup B)}_{\text{union (A or B)}} = n(A) + n(B) - \underbrace{n(A \cap B)}_{\text{intersection (A and B)}}$$

* 

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

* $n(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)$

nC_2 = No. of intersections two at a time
nC_3 = No. of intersection three at a time

$$= \underbrace{n(A_1) + n(A_2) + n(A_3) + \cdots + n(A_n)}_{\text{Sabka individual sum}} - \underbrace{n(A_1 \cap A_2) - n(A_2 \cap A_3) - \cdots - n(A_n \cap A_1)}_{\text{subtract intersection of two at a time}}$$

$$+ \underbrace{n(A_1 \cap A_2 \cap A_3) + n(A_2 \cap A_3 \cap A_4)}_{\text{sum intersection of three at a time}} - \cdots - \begin{pmatrix}\text{Interection} \\ \text{of 4}\end{pmatrix} + \begin{pmatrix}\text{intese.} \\ \text{of 5}\end{pmatrix} - \cdots$$

Q GOOGLE, Find No: of words which can be formed by using all it's letters such that no two alike letters occur together

Strategy :

(1) we first find all words in which either 2 G or 2 O's are together

(2) Total words − No! of words found above

let G = denotes words in which 2 G's occurs together

O = denotes words in which 2 O's occur together

$$n(G \cup O) = n(G) + n(O) - n(G \cap O)$$

$$n(G \cup O) = n(G) + n(O) - n(G \cap O)$$

$\boxed{GG}$ OOLE $\qquad$ $\boxed{OO}$ GGLE $\qquad$ $\boxed{GG}\boxed{OO}$ LE

$$\frac{5!}{2!} + \frac{5!}{2!} - 4! \Rightarrow 60 + 60 - 24$$

$$= 96$$

Now total words in GOOGLE $= \dfrac{6!}{2! \cdot 2!} = 180$

$\therefore$ No: of words in which Neither two O nor two G are together $= 180 - 96 = \boxed{84}$

M-2

$\times G \times G \times L \times E \times \quad - \times \boxed{GG} \times L \times E \times$

$$\frac{4!}{2!} \times {}^5C_2 \times 1 \quad - \quad 3! \times {}^4C_2 \times 1$$

(Arrangements of 4 words) (Select 2 GAPS FROM 5 GAPS) (Arrange O's in 2 GAPS) (Arrangement of 3 words) (Select 2 GAPS FROM ~~GAP~~ 4 GAPS) (Arrange O's in 2 GAPS)

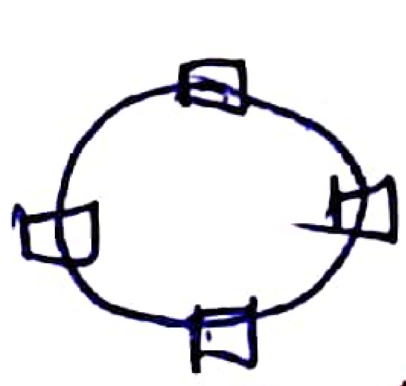
$= 12 \times 10 - 6 \times 6$

$= 120 - 36$

$= 84$

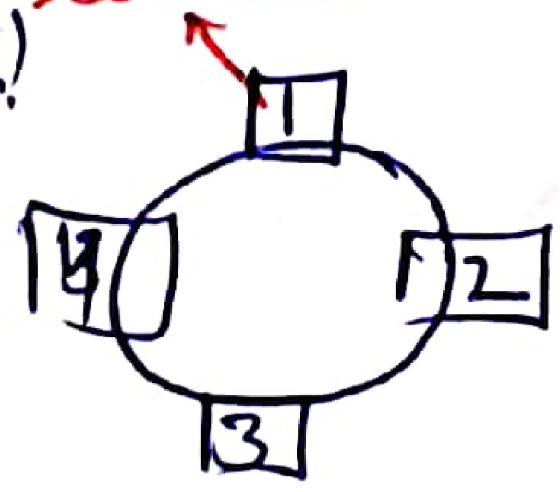
* If n-different things are arranged around a circle then no: of arrangements $= \dfrac{n!}{n} = (n-1)!$

ये ऐसा इसलिए होता है क्योंकि Last Rotation के बाद Arrangement पहले जैसे हो जाती है अगर Arrangement मे Rotation (Last) के बाद Farak पड़े तो वो Linear arrangement की ही बराबर होगी।



→ No. of Arrangement of 4 peoples $= (4-1)! = 3!$

Seat No.



→ No. of arrangements $= 4!$

*

∴ n-different things are to arranged in a NECKLACE /GARLAND no. of arrangements

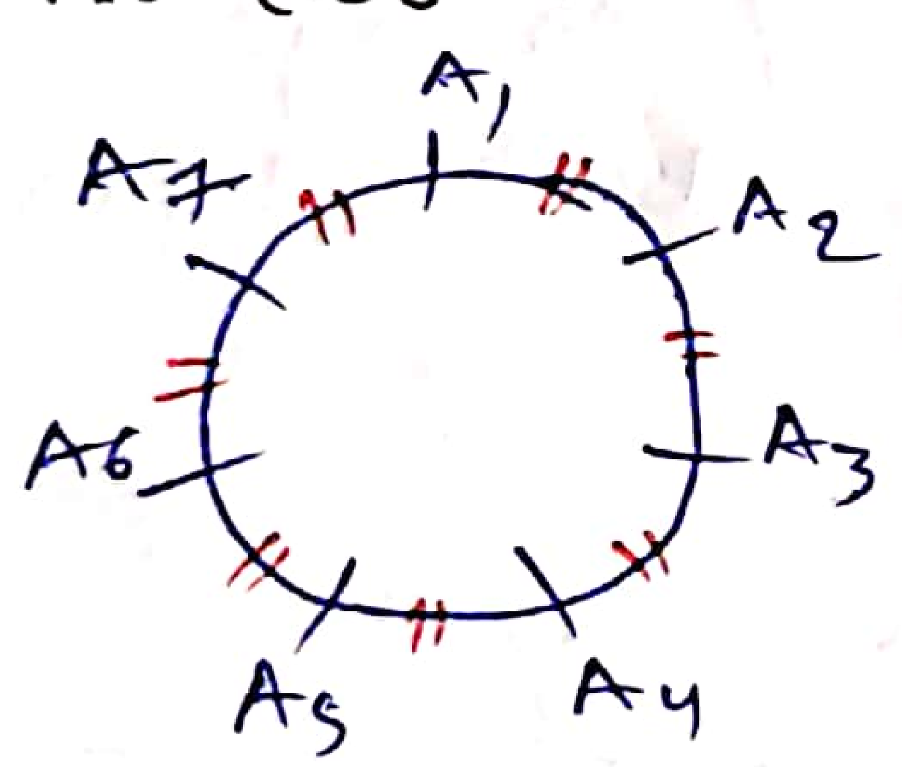
OR

n-different things are arranged round a circle then no: of arrangements in which all shall not have SAME NEIGHBOURS $= \frac{(n-1)!}{2}$

* circle mai no: of gaps = no: of objects

* Line mai no: of gaps = no: of objects + 1

Q No: of ways in which 7 Americans and 7 British people can be seated on a ~~to~~ round table so that no two Americans are consecutive.



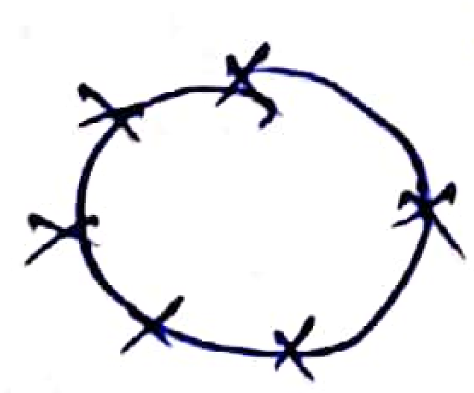
ANS: $6! \times 7!$

(Arrangements of 7 British)

(Arrangements of 7 Americans b/w Britishes)

Q Out of 10 flowers of Different Colours how many different garlands can made if each garland consists of 6 flower of different colour

10 DIFF^r flowers



ANS ⇒ $^{10}C_6 \times \frac{(6-1)!}{2}$

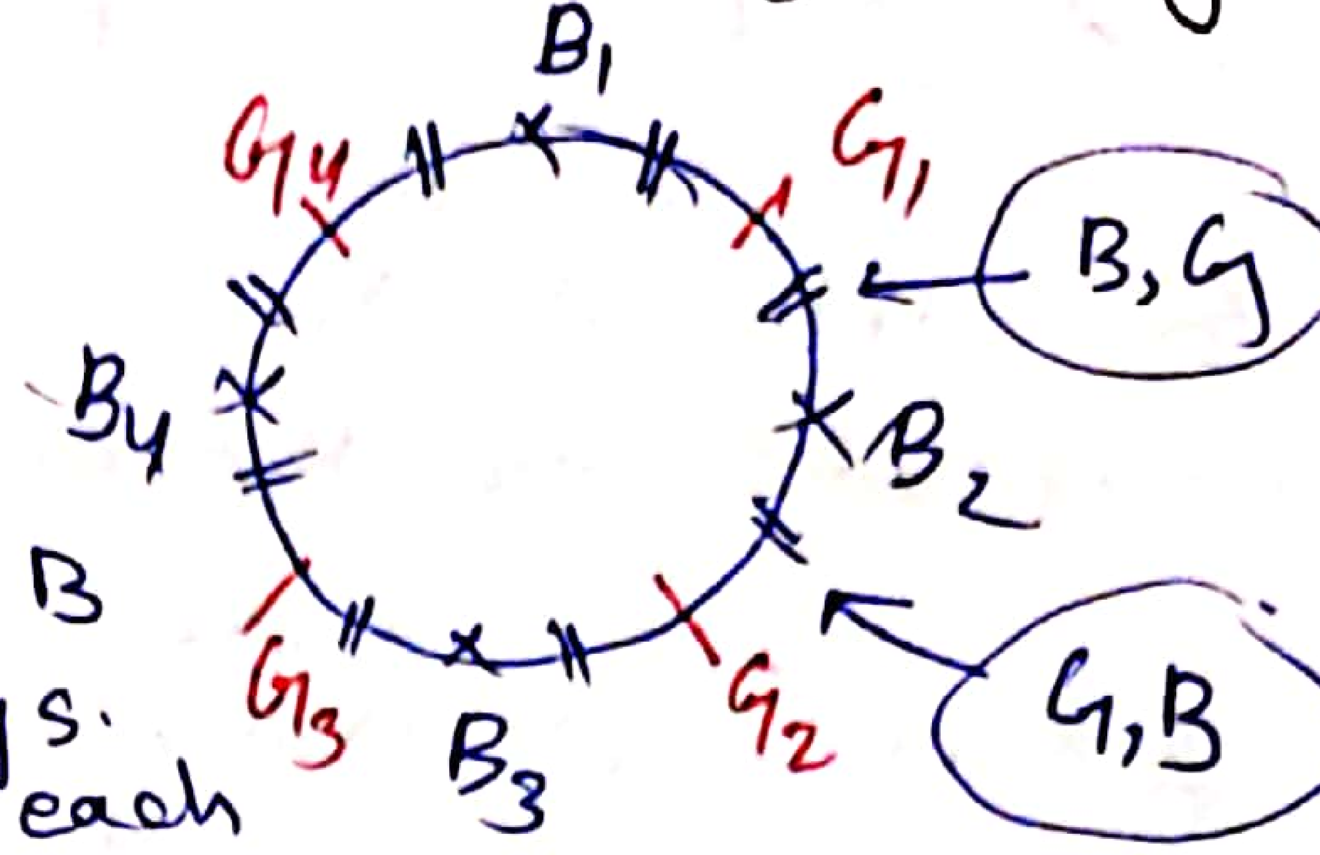
(Select 6 flowers out of 10 DIFF^r flowers)

(Arrangements of 6 flowers in GARLAD)

Q No: of ways in which 5B and 5G can be seated on a circle alternately if a particular B and G are never adjacent to each other in any arrangement.

<u>M-I</u> 4 Boys + 1 Ravan

4 Girls + 1 sita

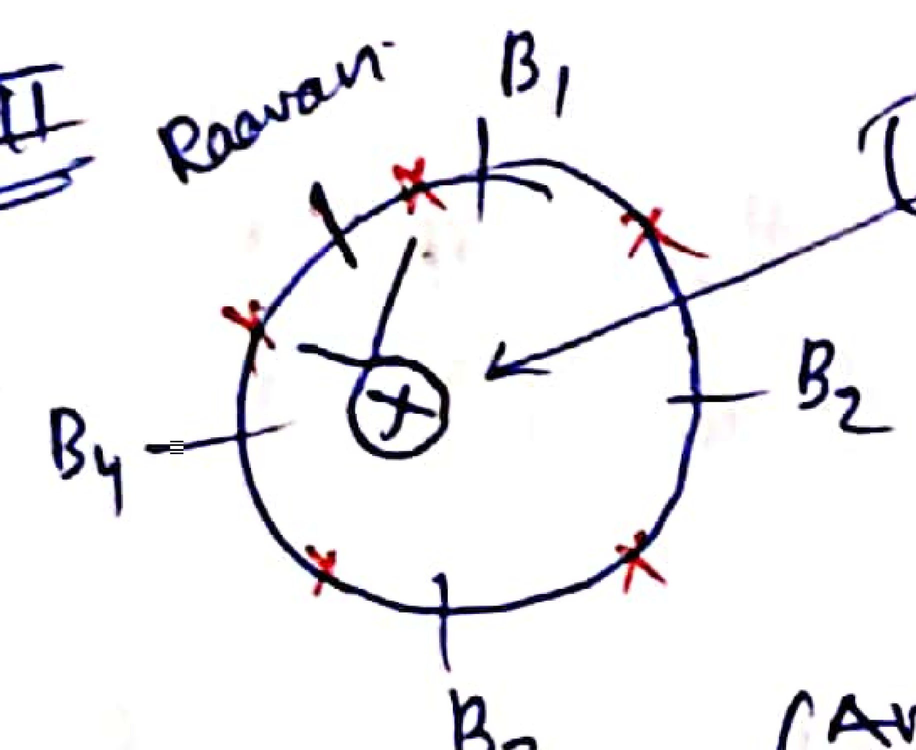


Total alternate arrangements − particular B & G always adjacent to each other

$$\Rightarrow 4! \times 5! \quad - \quad 3! \times 4! \times {}^{8}C_1 \times 1$$

(Arrangment of 5 Boy) (Arrangemt of 5 girls) (Arrange 4 Boys around circle) (Arrange 4 girls around circle) (Select one seat from 8 GAPS) (या तो G पहले आएगा या Boy)

<u>M-II</u>



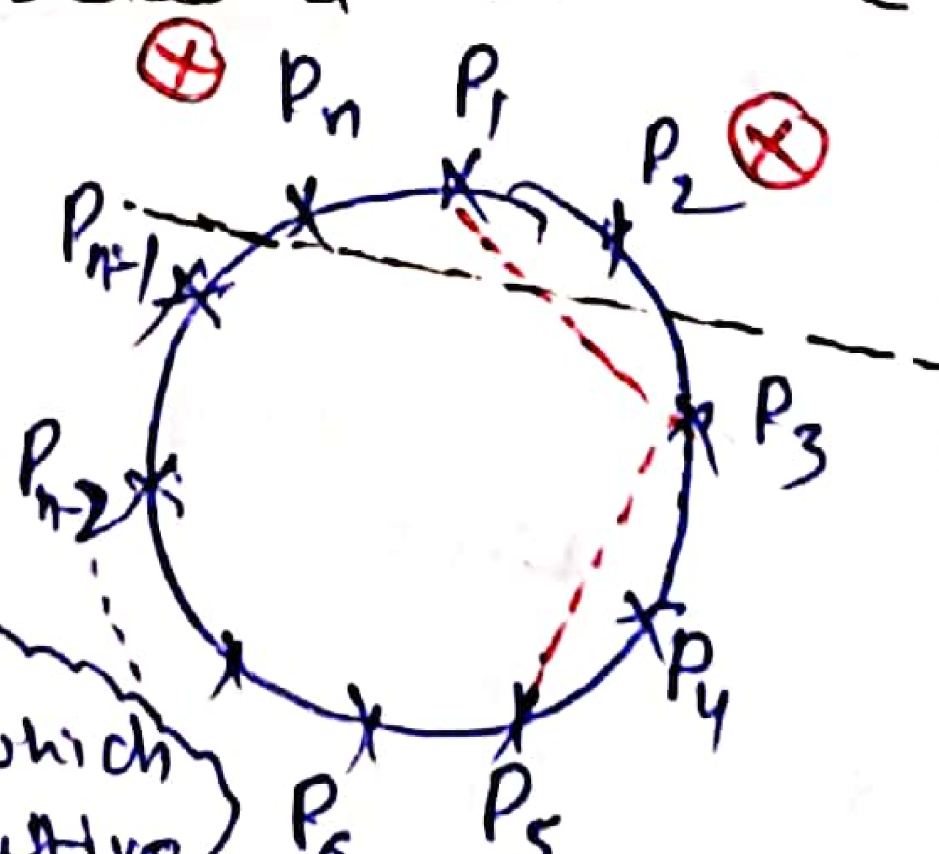
(sita इन-इन place पर नहीं आ सकती)

$$4! \times {}^{3}C_1 \times 4!$$

(Arrange 5 Boys) (Select 1 GAP out of 3 gaps जिसमे sita आ सकती है) (Arrange Baaki 4 girls).

Q n people $A_1, A_2, \ldots, A_n$ sitting on a circle. No: of ways in which 3 people can be selected if no two of ~~two~~ them are consecutive

<u>M-I</u>



To select 3 persons such that no two are consecutive

$$\underbrace{P_{n-1}, P_{n-2}, \ldots\ldots P_4, P_3}_{(n-1)\text{ people}}$$

(Select two peoples which are not consecutive)

$$\frac{{}^{n}C_1 \times {}^{n-4}C_2}{3}$$

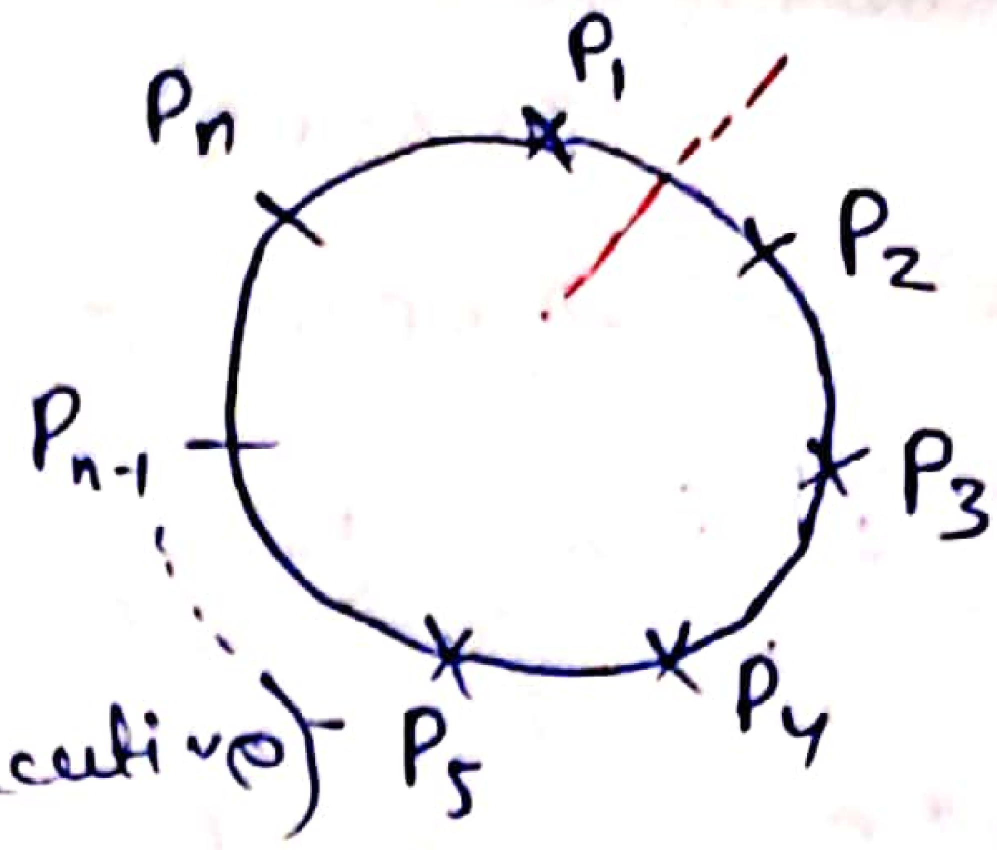
(Select one person)

Select two non-consecutive out of (n−3) people

→ each case counted three time

क्योंकि P_1 (Direct select), P_1 (select P_3), P_1 (select P_5) करने पर भी P_1 आ सकता

M-II



$\underbrace{P_1, \overbrace{P_n, P_{n-1} \ldots P_4, P_3}^{(n-4)\text{ people}}, P_2}_{n\text{-people}}$

(select 3 when no two are consecutive) → Mother Problem. $\left({}^{n-r+1}C_r\right)$

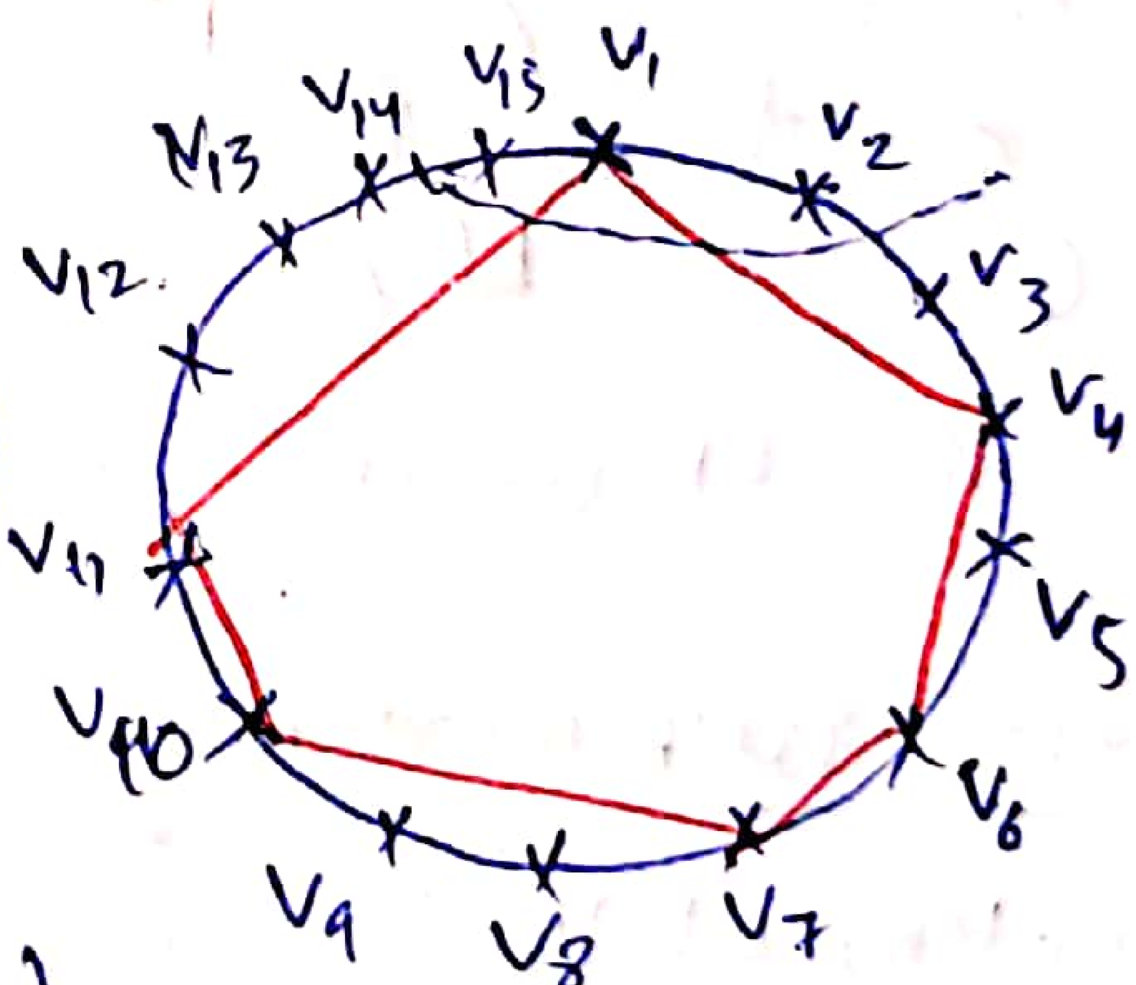
$\Rightarrow {}^{n-2}C_3$ — (cases when P_1 & P_2 are selected ↑ इस Time P_n & P_3 को काम के नहीं होंगे।)

(Select 3 from n when no two are consecutive)

${}^{n-4}C_1$

(P_1, P_2 select करने के बाद & P_n, P_3 को हटाने के बाद $(n-4)$ people में से एक Select)

Q How many hexagons can be selected by joining the vertices of a quindecagon (15 sides) if none of the side of hexagon is also the side of the 15-gon.



6 vertices to be selected For Hexagon. No two should be consecutive

$$\frac{{}^{15}C_1 \times {}^{8}C_5}{6}$$

(each cases counted 6 times)

(select 1 vertices out 15)

(After selection of 1 vertice then select 5 vertices from remaining 12 vertices. such that No two should be consecutive)

$\underbrace{V_{14}, V_{13}, V_{12}, \ldots, V_3}_{12\text{ vertices}}$

Select बची हुई 5 vertices जिनमें दो consecutive ना हो।
(Mother Problem.)

$\left\{{}^{12-5+1}C_5 = {}^{8}C_5\right\}$

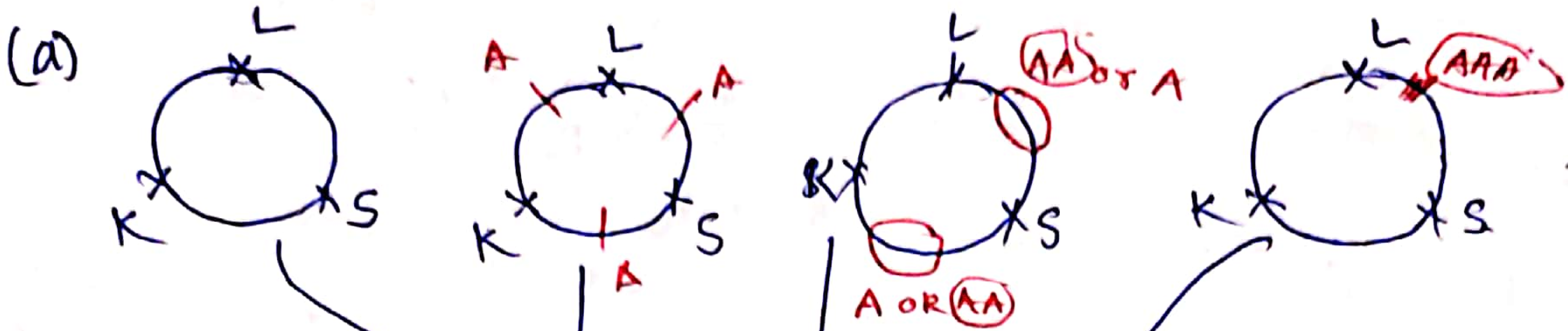
Q Find number of circular permutations of ~~(a)~~ –

(a) The word ALASKA

LSK AAA

~~(b) The word ELEMENT~~

(a)



Alsaa na sochna!!

ALASKA

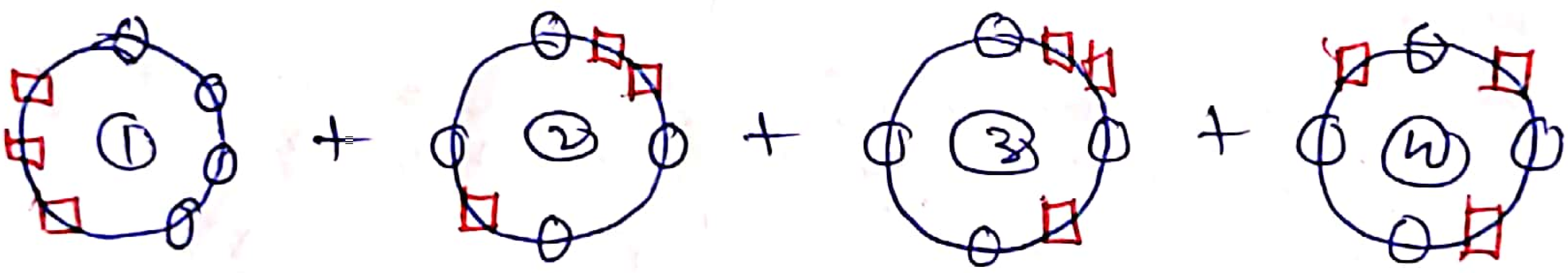
$= \frac{5!}{3!} = \frac{120}{6} = 20$

$$2!\{1 + {}^3C_2 \times 2! + {}^3C_1 \times 1\} = 20 \text{ ways}$$

Q Find number of garlands that can be formed with 4 identical roses and 3 identical lillies

4 identical roses, 3 identical lillies

○○○○ □□□



$\Rightarrow$ 4 ways {Garland उल्टा सीधा Same होगा जिसकी वजह से Arrangement नहीं होगी इधर}

* TOTAL NO: OF COMBINATIONS:

(Means selecting at least one out of n things)

CASE 1: When things are all different: n diffrent things

Total No: of combinations = selecting at least one thing out of n

$$= {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n - {}^nC_0$$

$$= 2^n - 1$$

CASE 2: When all are same / Alike

n alike things = At least one to be selected

only one selected = 1 way
only two selected = 1 way
only three selected = 1 way
⋮
only n selected = 1 way

Total No: of combinations = n

CASE III : when some are alike of one type, some are alike of another type rest all different.

$n = p + q + r$

(a-like of 1st type) (alike of 2nd type) (all diff^{t})

(दो case जिसमें no selection)

Total no: of combinations

$$= (p+1)(q+1)\left[{}^rC_0 + {}^rC_1 + {}^rC_2 + \dots + {}^rC_r\right] - 1$$

$$= \left[(p+1)(q+1)\,2^r\right] - 1$$

eg: Atleast one pakoda to be selected

5 Aaloo + 7 Gobhi + 3 Diff^r → paneer, onion, palak

(ये $^3C_1$ से start नहीं होगा। इसका मतलब ये हो जाएगा के एक Diff^r pakoda select करना पड़ेगा)

$$6 \text{ ways} \times 8 \text{ ways} \times \left({}^3C_0 \times {}^3C_1 \times {}^3C_2 + {}^3C_3\right) - 1$$

(0, 1, 2, 3, 4, 5) (0, 1, 2, 3, 4, 5, 6, 7)

(possibility when nothing is selected)

ANS: $(5+1)(7+1)({}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3) - 1$

Q Out of 2 cocas, 3 Mangoes and 4 Apples how many diff^r ~~selected~~ selections of fruits can be made if each selection has (i) atleast one fruit
(ii) atleast one fruit of every species.

in following cases:

(a) CASE I: fruit of the same species are alike.

(b) CASE II: Fruits of the species are different

(a) (i) $(2+1)(3+1)(4+1) - 1 = 59$

(ii) $(2)\cdot(3)\cdot(4) = 24$

(b) (i) Atleast one fruit

$$\Rightarrow ({}^2C_0 + {}^2C_1 + {}^2C_2)\cdot({}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3)({}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) - 4$$

$$= 2^2 \cdot 2^3 \cdot 2^4 - 1 = 2^9 - 1$$

(ii) Atleast one fruit of each species

$$\Rightarrow ({}^2C_1 + {}^2C_2)({}^3C_1 + {}^3C_2 + {}^3C_3)({}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) = (2^2-1)(2^3-1)(2^4-1)$$

$$= 3 \cdot 7 \cdot 15 = 315$$

* $N = 72 = 2^3 3^2$

factors/divisors $= 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72 = 12$ divisors

$2^0 3^0 \quad 2^1 3^0 \quad 2^0 3^1 \quad 2^2 3^0 \quad 2^1 3^1 \quad 2^3 \cdot 3^1$

No. of divisors $= (3+1)(2+1)$

* $N = p_1^{a_1}, p_2^{a_2} \cdot p_3^{a_3} \ldots p_n^{a_n}$ where $p_1, p_2, \ldots p_n \in$ Prime No:s

$a_1, a_2, \ldots a_n \in$ Natural No:s

No: of divisors of $n = (a_1+1)(a_2+1) \ldots (a_n+1)$

* $N = 72 = 2^3 \cdot 3^2$ is to be resolved in to product of two factors

1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72 [Even No: of factors]

$72 = 1\times72, 2\times36, 3\times24, 4\times18, 6\times12, 8\times9$

No. of ways in which 72 can be reasons as product of 2 factors $= \frac{(3+1)(2+1)}{2}$

* $N = 36 = 2^2 \cdot 3^2$

No: of divisors $= (2+1)(2+1) = 9$

$36 = 1, 2, 3, 4, 6, 9, 12, 18, 36$

No: of divisors of a perfect square is always odd

$36 = 1\times36, 2\times18, 12\times3, 4\times9, \boxed{6\times6}$

No: of ways 36 can be resolved as product of two factors

$36 = \frac{(2+1)(2+1)-1}{2} + \underset{(6\times6)}{1} = \frac{(2+1)(2+1)+1}{2}$

* $N = p_1^{a_1} \cdot p_2^{a_2} \ldots p_n^{a_n}$ $p_i \in$ prime, $a_i \in N$

No: of ways in which N can split/resolved as a product of two factors.

$$= \begin{cases} \dfrac{(a_1+1)(a_2+1)\ldots(a_n+1)}{2}, & \text{if N is not perfect Square} \\[2ex] \dfrac{(a_1+1)(a_2+1)\ldots(a_n+1)+1}{2}, & \text{if N is a perfect Square} \end{cases}$$

* $N = 72 = 2^3 \cdot 3^2 = f_1 \times f_2$

relatively prime, i.e they have no common factor i.e HCF = 1

$72 = 1\times 72,\ 2\times 36,\ 3\times 24,\ 4\times 18,\ 6\times 12,\ 8\times 9$

$72 = 2^3 \cdot 3^2$

2 options go in f_1 or go in f_2

f_1	$\times$	f_2
2^3		3^2
3^2		2^3
$2^3\times 3^2$		$2^0 3^0$
$2^0 3^0$		$2^3 3^2$

2 options go in f_1 or go in f_2

No. of ways $= \frac{2\times 2}{2} = 2^{2-1}$

$N = 2^3\cdot 5^2\cdot 7^3$

(a) $f_1\times f_2 = \frac{(3+1)(2+1)(3+1)}{2}$

(b) $f_1\times f_2$ relatively prime $= \frac{2\times 2\times 2}{2} = 2^{3-1}$

* If $N = P_1^{a_1}\cdot P_2^{a_2}\cdots P_n^{a_n}$ is to be resolved as a product of two factors $f_1\times f_2$ which are relatively prime/co-prime $= \frac{\overbrace{2\times 2\times\cdots\times 2}^{n\text{-times}}}{2} = 2^{n-1}$

Q Consider the number $N = 75600\ (2^4\cdot 3^3\cdot 5^2\cdot 7^1)$

(i) No: of devisors $\Rightarrow (4+1)(3+1)(2+1)(1+1) = 120$

(2) No: of proper divisors → does not include 1 & no: itself

$(4+1)(3+1)(2+1)(1+1) - 2 = 118$

(3) No: of odd devisors

{No 2 should be taken} $(3+1)(2+1)(1+1) = 24$

(4) Number of even divisors

$(4)(3+1)(2+1)(1+1) = 96$ OR $\underset{\text{(total)}}{120} - \underset{\text{(odd divisors)}}{24} = 96$

Atleast one 2 should be taken.

$N = 75600\ (2^4 \cdot 3^3 \cdot 5^2 \cdot 7)$

(5) Number of divisors divisible by 5 → Atleast one 5 should be taken.

$\Rightarrow (4+1)(3+1)(2)(1+1) = 80$

(6) Divisors divisible by 14

Atleast one 2 & one 7 should be taken

$(4)(3+1)(2+1)(1) = 48$

(7) Divisors divisible by 9

Atleast two 3 should be taken

$(4+1)(2)(2+1)(1+1) = 60$

either two 3

OR

All three

(8) Divisors of Type $4p-2$, $p \in N$

इसका मतलब हुआ Ke divisor of type $2(2p-1)$

$\underbrace{\text{even} \quad \text{odd}}_{\text{Divisor with Exatly one 2}}$

$\Rightarrow (1)(3+1)(2+1)(1+1) = 24$

For 2 only one way

i.e take only one 2

(9) Number of divisors divisible by 10

Atleast one 2 & Atleast one 5

$(4)(3+1)(2)(1+1) = 64$

(10) sum of all the divisors

$\Rightarrow (2^0+2^1+2^2+2^3+2^4) \cdot (3^0+3^1+3^2+3^3)(5^0+5^1+5^2)(7^0+7^1)$

(11) Sum of all the even divisors

$\Rightarrow (2^1+2^2+2^3+2^4)(3^0+3^1+3^2+3^3)(5^0+5^1+5^2)(7^0+7^1)$

(12) sum of all divisors divisible by 6 ⇒ neglect 2^0 & 3^0

$\rightarrow (2^1+2^2+2^3+2^4)(3^1+3^2+3^3)(5^0+5^1+5^2)(7^0+7^1)$

(13) sum of divisors divisible by 20 ⇒ neglect $2^0, 2^1, 5^0$

$\Rightarrow (2^2+2^3+2^4)(3^0+3^1+3^2+3^3)(5^1+5^2)(7^0+7^1)$

Concept: carry over / Haasil

$$\begin{array}{r} {}^{1}\,{}^{2} \\ 267 \\ 139 \\ 245 \\ \hline 651 \end{array}$$

$$\begin{array}{ccc} 2 & 6 & 7 \\ 1 & 3 & 9 \\ 2 & 4 & 5 \\ \downarrow & \downarrow & \downarrow \\ 5 & 13 & 21 \end{array}$$

sum of UNIT PLACE $\times 10^0$
+
Ten Place sum $\times 10^1$
Hundred Place sum $\times 10^2$

*** Summation of Numbers:**

$267 + 139 + 245$
$= 5 \times 10^2 + 13 \times 10^1 + 21 \times 10^0$
$= 500 + 130 + 21$
$= \underline{651}$

Q Sum of all the numbers greater than 10000 formed by the digits 1, 3, 5, 7, 9 no digit being repeated.

No: of Numbers $= 5! = 120$

M-1

Ten Thousand, Thousand, Hundred, Tens, unit place

| | | | | 1 | $= 24$ NO:S
$1 \times 2 \times 3 \times 4$

| | | | | 3 | $= 24$ NO:S
$1 \times 2 \times 3 \times 4$

| | | | | 5 | $= 24$ NO:S

| | | | | 7 | $= 24$ NO:S

| | | | | 9 | $= 24$ NO:S

sum of units place $= 1 \times 24 + 3 \times 24 + 5 \times 24 + 7 \times 24 + 9 \times 24$
$= (1+3+5+7+9) \times 24$

sum of tens place $= 1 \times 24 + 3 \times 24 + 5 \times 24 + 7 \times 24 + 9 \times 24$
$= (1+3+5+7+9) \times 24$

sum of Hundred place $= (1+3+5+7+9) \times 24$

sum of Thousandth place $= (1+3+5+7+9) \times 24$

sum of Ten Thousandth place $= (1+3+5+7+9) \times 24$

$\therefore$ sum of All 120 NO:S
$= (1+3+5+7+9) \times 24 \times 10^0 + (1+3+7+9) \times 24 \times 10^1 + (1+3+7+5+9) \times 24 \times 10^2$ ------
$= (1+3+5+7+9) \times 24 \left[10^0 + 10^1 + 10^2 + 10^3 + 10^4\right]$
$= 25 \times 24 \times 11111 = \underline{666600}$

M-2

$$\begin{array}{cccccc} & 66 & 66 & 66 & 60 & \\ & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & \vdots & \vdots & \vdots & \vdots & \vdots \\ + & \times & \times & \times & \times & \times \\ \hline & 666 & 6 & 6 & 0 & 0 \end{array}\Bigg\}\ 120 \text{ No's}$$

$(1+3+5+7+9)\times 24$

$\Rightarrow 25\times 24 = 600$

$$\begin{array}{r} 600 \\ +60 \\ \hline 660 \end{array} \qquad \begin{array}{r} 600 \\ +66 \\ \hline 666 \end{array}$$

M-3

1, 3, 5, 7, 9

$S = 13579 + 13597 + 13759 + \cdots + 97351 + 97513 + 97531$

$S = 97531 + 97513 + 97351 + \cdots + 13759 + 13597 + 13579$

$2S = 111110 + 111110 + 111110 + \cdots$ up to 120 times

$2S = 120 \times 111110$

$\Rightarrow S = 60 \times 111110 = 6666600$

* Dearrangement :-

n Letters are to be ~~sent~~ kept in n directed envolopes no: of ways in which they can be placed if none of the letter goes into its own envelope is.

$$= n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots + (-1)^n \frac{1}{n!}\right]$$

Proof:

Letters: $L_1, L_2, \ldots L_n$

Envolopes: $E_1, E_2, \ldots E_n$

A_1 = No: of ways letter ① goes into its own envelope

A_2 = No: of ways letter ② goes into its own envelope

$\vdots$

A_n = No: of ways letter ⓝ goes into its own envelope

$$n(A_1\cup A_2\cup A_3\cup\dots\cup A_n)$$

$$=\overbrace{n(A_1)+n(A_2)+\dots+n(A_n)}^{{}^nC_1}-\overbrace{\left[n(A_1\cap A_2)+n(A_1\cap A_3)\dots\right]}^{{}^nC_2}$$

L_1 L_2 L_3 L_4 .... L_n
E_1 E_2 E_3 E_4 .... E_n $\quad \underbrace{}_{(n-1)}$

L_1 L_2 L_3 L_4 ..... L_n
E_1 E_2 E_3 E_4 --- E_n $\quad \underbrace{}_{(n-2)}$

$$+\overbrace{\left[n(A_1\cap A_2\cap A_3)+\dots\right]}^{{}^nC_3}\dots\dots+(-1)^{n+1}\overbrace{\left[n(A_1\cap A_2\cap A_3\cap\dots\cap A_n)\right]}^{{}^nC_n}$$

L_1 L_2 L_3 L_4 --- L_n
E_1 E_2 E_3 E_4 --- E_n $\quad \underbrace{}_{(n-3)}$

L_1 L_2 L_3 L_4 --- L_n
E_1 E_2 E_3 E_4 --- E_n

$$=(n-1)!\times n-(n-2)!\cdot{}^nC_2+(n-3)!\cdot{}^nC_3-(n-4)!\cdot{}^nC_4$$
$$-----(-1)^{n+1}\cdot 1$$

$n(A_1\cup A_2\cup A_3\cup\dots\cup A_n)$ = no. of ways in which 1st letter ~~2nd~~ or 2nd letter OR 3rd letter OR ---- ---- OR n^{th} letter goes in to its own envelope

$$=(n-1)!\,{}^nC_1-(n-2)!\,{}^nC_2+(n-3)!\cdot{}^nC_3-(n-4)!\cdot{}^nC_4+\dots+(-1)^{n+1}\cdot 1$$

$$=(n-1)!\,n-(n-2)!\,\frac{n(n-1)}{2!}+(n-3)!\,\frac{n(n-1)(n-2)}{3!}-\frac{(n-4)!\,n(n-1)(n-2)(n-3)}{4!}$$
$$+\dots\dots+(-1)^{n+1}\cdot 1$$

$$=n!-\frac{n!}{2!}+\frac{n!}{3!}-\frac{n!}{4!}+\dots\dots+(-1)^{n+1}\frac{n!}{n!}$$

No. of dearrangements of n-letters

$$=n!-n(A_1\cup A_2\cup A_3\cup\dots\cup A_n)$$
$$=\cancel{n!}-\left(\cancel{n!}-\frac{n!}{2!}+\frac{n!}{3!}-\frac{n!}{4!}+\dots+(-1)^{n+1}\cdot\frac{n!}{n!}\right)$$
$$=n!\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\dots-\frac{(-1)^{n+1}}{n!}\right]$$

$$-(-1)^n(-1)^1=+(-1)^n$$

$$=n!\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\dots+\frac{(-1)^n}{n!}\right]$$

$$\boxed{D_n=n!\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\dots+\frac{(-1)^n}{n!}\right]}$$

Q In How many ways 4 Letters can be kept in 4 directed envelopes if none of the Letter goes into its own envelope.

$$D_4 = 4!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$$
$$= 24\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)$$
$$= 12 - 4 + 1$$
$$= 9 \text{ ways}$$

* $D_2 = 1$
* $D_3 = 3!\left(\frac{1}{2!} - \frac{1}{3!}\right) = 2$
* $D_4 = 4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$
* $D_5 = 5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$

concept :-

coefficient of $x^p y^q z^r$ in the expansion of

$$(x+y+z)^{p+q+r} = \frac{(p+q+r)!}{p!\,q!\,r!}$$

eg: coefficient of $x^5 y^4 z^3$ in $(x+y+z)^{12}$ is $\frac{12!}{5!\,4!\,3!} \times 1$

x, y, z का (specific no. milega)

(2) coefficient of $x^6 y^3 z^3$ in $(x+y+z)^{12}$

is $\frac{12!}{6!\,3!\,3!\,2!} \times 2!$ ⟶ y & z आपस में Arrange हो सकते।

same group size (3 & 3)

Diff Books

$$(x+y)^{3} = 1\cdot x^3 + 1\cdot y^3 + 3x^2y + 3xy^2$$

Kallu Lallu

Books

$$\frac{3!}{2!\,1!} = 3 \qquad \frac{3!}{1!\,2!} = 3$$

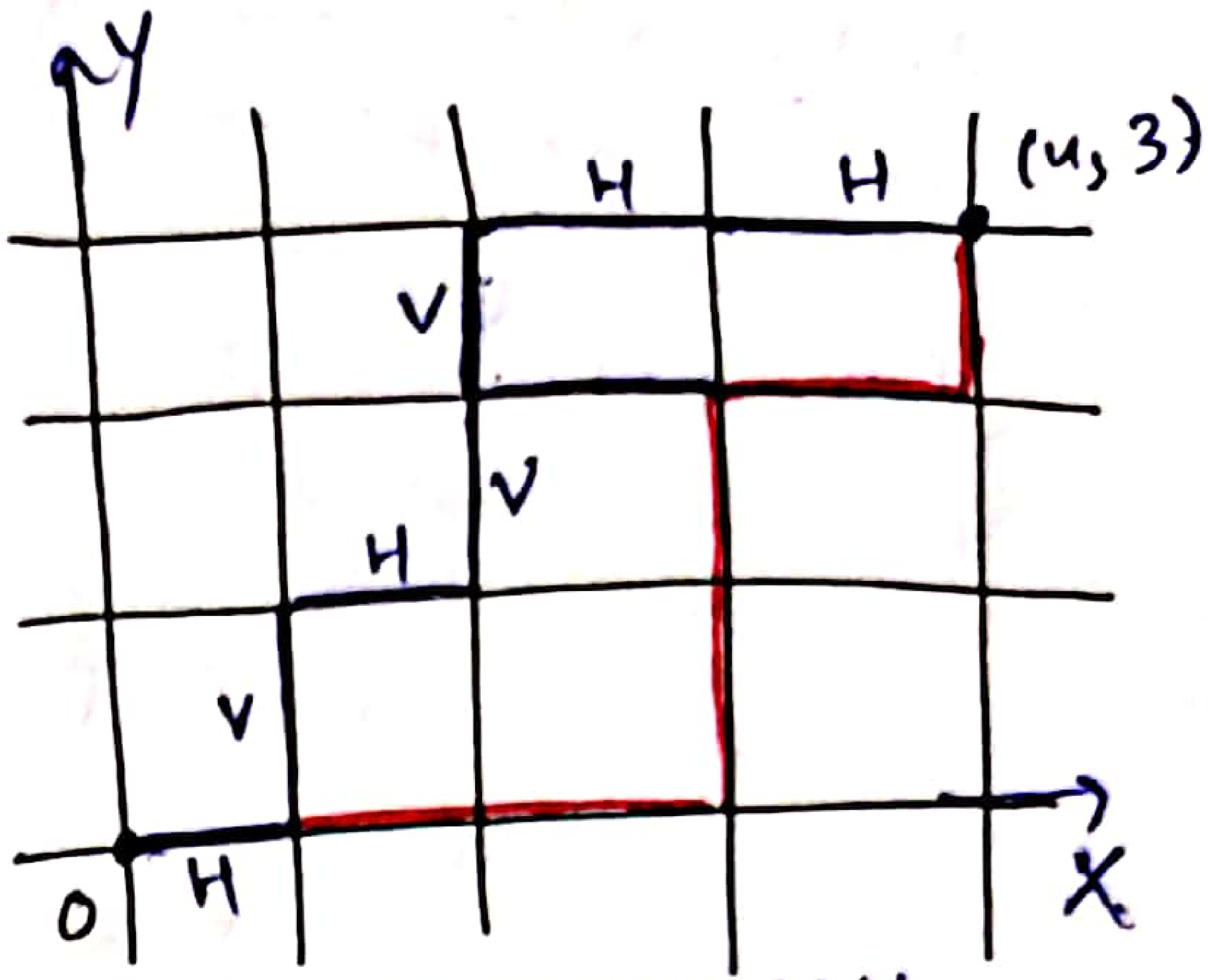
Kallu Lallu Kallu Lallu

$(x+y+z)^{10}$ coeff of $x^3y^5z^2 = \frac{10!}{3!\,5!\,2!}$

coeff of $x^3y^3z^4 = \frac{10!}{3!\,3!\,4!\,2!} \times 2!$

* GRID PROBLEMS :

Complete cartesian plane partitioned by drawing lines equidistand apart like the lines || to x and y axis on a chess board.



Horizontal steps = 4

Vertical steps = 3

No: of Paths = $\frac{7!}{4! \cdot 3!}$

path 1 : H V H V V H H

path 2 : H H H V V H V

$(0,0) \longrightarrow (m,n)$

No: of path = $\frac{(m+n)!}{m!\, n!}$

$(m_1, n_1) \longrightarrow (m_2, n_2)$

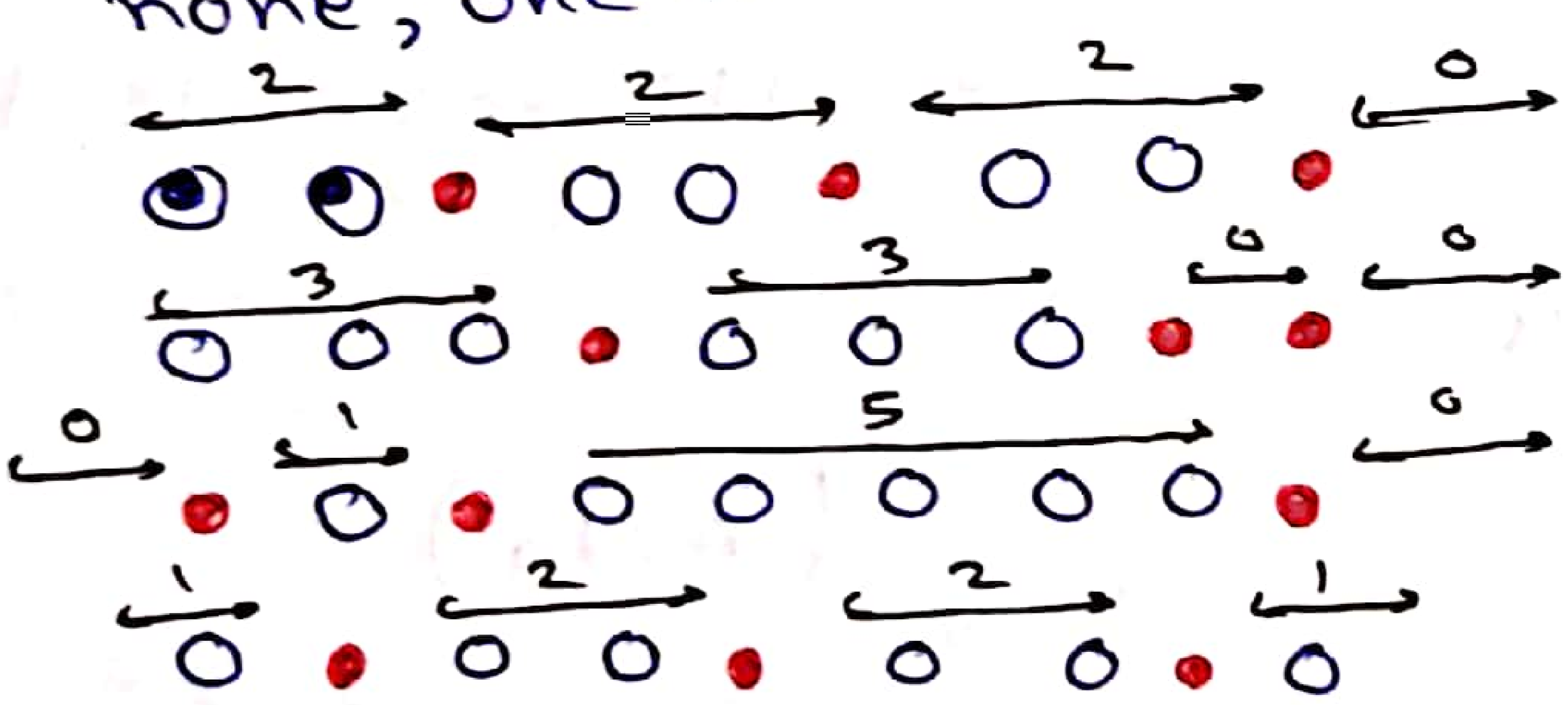
$H = m_2 - m_1$

$V = n_2 - n_1$

No: of paths = $\frac{[(m_2-m_1)+(n_2-n_1)]!}{(m_2-m_1)!\,(n_2-n_1)!}$

* Distribution of Alike Objects (Beggar Method) :-

Ex: In How many different ways 6 identical coins can be distributed b/w 4 persons each person can receive none, one or more coins.



2220

3300

0150

1221

6 – coins + 3 red coins

identical identical

$\Rightarrow \frac{9!}{3! \cdot 6!} = \frac{9 \times 8 \times 7}{6}$

= 84 ways.

eg: 10 identical coins → 5 Beggars

10 coins + 4 red coins → 4 के बीच GAP पाँच बनते हैं और Beggar भी पाँच हैं।

$\frac{14!}{10!\,4!} =$

$\Rightarrow {}^{14}C_4 \Rightarrow {}^{10+5-1}C_{5-1}$

coins

Beggar

CONCEPT:

No: of ways in which n identical coins can be distributed among p persons if each person receiving none, one or more coins $= \dfrac{(n+p-1)!}{n!\,(p-1)!}$

$\underbrace{OOOO----O}_{n\text{ identical coins}}$ $\quad \underbrace{\bullet\bullet\bullet\bullet.....\bullet}_{(p-1)\text{ fake coins}}$ $= {}^{n+p-1}C_{p-1}$

$$\boxed{{}^{n + p - 1}C_{p-1}}$$

coin — n; Beggar — p

Q In how many ways 30 marks be allotted to 8 questions if atleast 2 marks are to be given to each question, assuming the marks are to be allotted in non negative integral values

30 marks

give 2 marks to each

$\underbrace{Q1\ Q2\ Q3\ Q4\ Q5\ Q6\ Q7\ Q8}_{8\,(\text{Beggar})}$

$\Rightarrow 30 - 2\times 8$

$\Rightarrow$ 14 mark left (coin)

$\Rightarrow {}^{(\text{coin})+(\text{Beggar})-1}C_{(\text{Beggar})-1}$

$\Rightarrow {}^{14+8-1}C_{8-1} = {}^{21}C_7$ (ANS)

Q No: of ways in which 'K' identical balls can be distributed in 'p' different boxes if no box remains empty.

K identical ball

$\downarrow$ give 1 to each

P Box (No Box should be empty)

$\Downarrow$

$(K-P)$ Balls / coins left

No: of Distributions $= {}^{K-P+P-1}C_{P-1}$

$= \left\{ {}^{K-1}C_{P-1} \right\}$ ANS.

Q(i) Number of Natural solⁿ of the equation $x+y+z=102$

1 to each

$$x+y+z=102.$$

coins left = 99

Beggars = 3

$$^{99+3-1}C_{3-1} = {}^{101}C_2 = \underline{5050}$$

(ii) $x+y+z=102$ find no: of non-(+ve) integral solⁿ.

means a variable can have value = 0

$$^{102+3-1}C_{3-1}$$

$$\rightarrow \{{}^{104}C_2\} \text{ ANS.}$$

Miracubus concept of False Beggar:

$$x+y+z+t \leq 30$$

$x+y+z+t=30$

$x+y+z+t=29$

$x+y+z+t=\cancel{28}\,4$

$x+y+z+t=3$

$x+y+z+t=2$

$x+y+z+t=1$

$x+y+z+t=0$

$x+y+z+t=0 \quad {}^{0+4-1}C_{4-1} = {}^3C_3$

$x+y+z+t=1 \quad {}^{1+4-1}C_{4-1} = {}^4C_3$

$x+y+z+t=2 \quad {}^{2+4-1}C_{4-1} = {}^5C_3$

$x+y+z+t=3$

$x+y+z+t=29 \quad {}^{29+4-1}C_{4-1} = {}^{32}C_3$

$x+y+z+t=30 \quad {}^{30+4-1}C_{4-1} = {}^{33}C_3$

Total no: of solⁿ $= {}^3C_3 + {}^4C_3 + {}^5C_3 + {}^6C_3 + \dots + {}^{32}C_3 + {}^{33}C_3$

${}^4C_4$

$= {}^4C_4 + {}^4C_3 + {}^5C_3 + {}^6C_3 + \dots + {}^{32}C_3 + {}^{33}C_3$

${}^5C_4 \quad {}^6C_4 \quad {}^7C_4$ ---- so on.

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

$= {}^{34}C_4$

Kraantikari Tarikaa !!

Beggar, coins

$$x+y+z+t \leqslant 30$$

$$x+y+z+t+u = 30$$

↳ False Beggar

No: of Solutions $= {}^{30+5-1}C_{5-1}$

$= {}^{34}C_4$

Q Find the no: of solutions of the following equations under given conditions

(1) $x+y+z=20$, $x,y,z \in N$

1 to each.

$x+y+z=20$ → ANS $= {}^{17+3-1}C_{3-1} = {}^{19}C_2$

coin left $= 17$

(2) $x+y+z+w=21$ where $x,y,z,t \in I$ and $x \geqslant 2, y \geqslant 2, z \geqslant 3, w \geqslant 0$

3, 2, 2

coin Left $= 21-2-2-3 = 14$

ANS ⇒ ${}^{14+4-1}C_{4-1} = {}^{17}C_3$

(3) $x+y+z+w=21$ where $x,y,z,t \in I$ and $x \geqslant 4, y \geqslant -3, z \geqslant -2, w \geqslant -2$

↓

$y \geqslant -2$

−2 to y, −2 to w

$x+y+z+w=21$

4 to x, −3 to z

coins left $= 21-4+2+3+2$

$= 24$

ANS: ${}^{24+4-1}C_{4-1} = {}^{27}C_3$

(4) $x+y+z+w=22$ where x,y,z,t are odd numbers

$x = 2x'+1$, $x' \geqslant 0$

$y = 2y'+1$, $y' \geqslant 0$

$z = 2z'+1$, $z' \geqslant 0$

$w = 2w'+1$, $w' \geqslant 0$

$2w' + 2x' + 2y' + 2z' + 4 = 22$

$x'+y'+z'+w' = 9$

ANS: ${}^{9+4-1}C_{4-1}$

→ $\{{}^{12}C_3\}$

(5) $30 < x+y+z+t \leq 50$ where $x, y, z, t \in W$

$30 < x+y+z+t \leq 50$

$x+y+z+t > 30$

we solve

$x+y+z+t \leq 30$

$x+y+z+t+u = 30$

No: of Sol^n: ${}^{30+5-1}C_{5-1} = {}^{34}C_4$

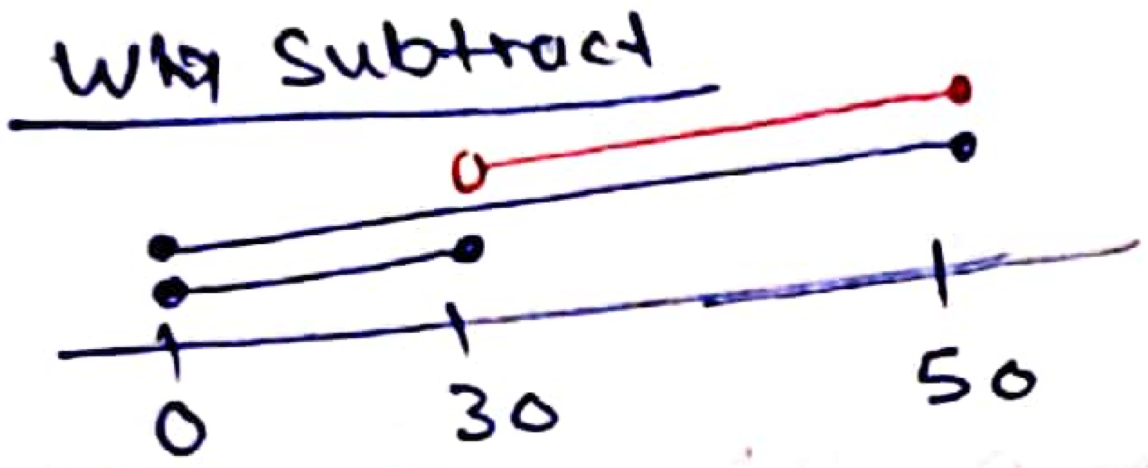
$x+y+z+t \leq 50$, $x, y, z, t \in W$

$x+y+z+t+u = 50$

No: of Sol^n: ${}^{50+5-1}C_{5-1} = {}^{54}C_4$

$\therefore$ No: of Sol^n $= {}^{54}C_4 - {}^{34}C_4$

Why Subtract



(6) $x+y+z = 20$ $x, y, z \in N$ are all diff^r

$x \geq 1$,

$y = x + t_1$; $t_1 \geq 1$

$z = x + t_1 + t_2$; $t_2 \geq 1$

$x + x + t_1 + x + t_1 + t_2 = 20$

$\Rightarrow 3x + 2t_1 + t_2 = 20$ ✓

$x \neq y$, $y \neq z$ $\not\Rightarrow$ $x \neq z$

$2 \neq 3$, $3 \neq 2$ $\not\Rightarrow$ $2 \neq 2$

If $\underline{x=1}$

$2t_1 + t_2 = 17 \Rightarrow t_2 = 17 - 2t_1$ $\Rightarrow$ {8 sol^n}

$t_1 = 1, 2, 3, \ldots, 8$

if $\underline{x=2}$

$2t_1 + t_2 = 14 \Rightarrow t_2 = 14 - 2t_1$ $\Rightarrow$ {6 sol^n}

$t_1 = 1, 2, 3, 4 \ldots 6$

if $\underline{x=3}$

$2t_1 + t_2 = 11$

$t_2 = 11 - 2t_1 \Rightarrow t_1 = 1, 2, 3, 4, 5$ $\Rightarrow$ {5 sol^n}

if $\underline{x=4}$ $2t_1 + t_2 = 8 \Rightarrow t_2 = 8 - 2t_1$

$t_1 = 1, 2, 3,$ $\Rightarrow$ {3 sol^n}

if $\underline{x=5}$ $2t_1 + t_2 = 5 \Rightarrow t_2 = 5 - 2t_1$

$t_1 = 1, 2$ $\rightarrow$ {2 sol^n}

Total Sol^n's $= 24$ sol^n

$x+y+z = 20$

ANS $\Rightarrow$ $24 \times 3!$

(Arrangment of x, y, z)

Q Find the number of Positive integral solutions of $x \cdot y \cdot z = 24$.

$$x \cdot y \cdot z = 2^{3} \cdot 3^{1}$$

2^3 will be distributed among x, y, z

No: of ways to distribute 3 two's among x, y, z ; $a_1 + a_2 + a_3 = 3 \Rightarrow {}^{3+3-1}C_{3-1} = {}^{5}C_2 = 10$

No: of ways to distribute 1 three among x, y, z ; $b_1 + b_2 + b_3 = 1 \Rightarrow {}^{1+3-1}C_{3-1} = {}^{3}C_2 = 3$

Total = 10 × 3

Number	Rule	Example.
2	Last ~~minute~~ digit must be either 0, 2, 4, 6, 8	28, 570
3	Sum of digit must be divisible by 3	147; 1+4+7 = 12
4	No: formed by last 2 digits must be divisible by 4	144, 1096
5	Last digit must either be 0 or 5	20, 5275
6	The no: must be divisible by both 2 and 3	1044; 1+0+4+4 = 9
8	No: formed by last 3 digits must be divisible by 8	248, 1640
9	Sum of digits must be divisible by 9	675; 6+7+5=18
10	The last digit must be 0	30, 250
11	Difference of sum of digits at odd places and sum of digits at even places must either 0 or divisible by 11	4059; 5+4=9 9−9=0