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ASSIGNMENT-03

Ques. 1 Given two random variables X and Y and their joint distribution P(X, Y):

	X=2	X=4	X=6
Y= Red (2)	0.19	0.15	0.06
Y= Blue (4)	0.16	0.1	0.14
Y= Green (6)	0.03	0.17	0

Find

1. a) H(X)

For finding marginal entropy of X

$$H(x) = -\Sigma P(x) \log_2 P(x)$$

If $x=2$, $(0.19+0.16+0.03) = 0.38$

$\Rightarrow$ when $x=2$, $P(x) = 0.38$

If $x=4$, $(0.15+0.1+0.17) = 0.42$

$\Rightarrow$ when $x=4$, $P(x) = 0.42$

If $x=6$, $(0.06+0.14+0) = 0.2$

$\Rightarrow$ when $x=6$, $P(x) = 0.2$

For calculating marginal entropy of x,

$$H(x) = -(0.38 \log_2 0.38 + 0.42 \log_2 0.42 + 0.2 \log_2 0.2)$$

$$= -(-0.530452897 - 0.52564628213 - 0.46438562)$$

$$= 0.530452897 + 0.52564628213 + 0.46438562$$

$$= 1.52048479$$

$\boxed{H(x) = 1.5205}$ bits

1.(b) H(Y)

For finding marginal entropy of Y,

$$H(Y) = -\Sigma P(y) \log_2 P(y)$$

If y = Red(2), (0.19 + 0.15 + 0.06) = 0.4

⇒ when y = Red(2), P(y) = 0.4

If y = Blue(4), (0.16 + 0.1 + 0.14) = 0.4

⇒ when y = Blue(4), P(y) = 0.4

If y = Green(6), (0.03 + 0.17 + 0) = 0.2

⇒ when y = Green(6), P(y) = 0.2

For calculating marginal entropy of y,

$$H(y) = -(0.4 \log_2 0.4 + 0.4 \log_2 0.4 + 0.2 \log_2 0.2)$$

$$= -(-0.52877123795 - 0.52877123795 - 0.46438561897)$$

$$= 0.52877123795 + 0.52877123795 + 0.46438561897$$

$$= 1.52192809489$$

$$\boxed{H(y) = 1.5219} \text{ bits}$$

1.(c) $D(X||Y)$

For finding relative Entropy,

$$D(p||q) = \Sigma_x p(x) \log_b (p(x) | q(x))$$

$p(x)$	$q(x)$
0.38	0.4
0.42	0.4
0.2	0.2

$$\Rightarrow D(X||Y) = 0.38 \log_2 \frac{0.38}{0.4} + 0.42 \log_2 \frac{0.42}{0.4} + 0.2 \log_2 \frac{0.2}{0.2}$$

$$= (-0.02812022094 + 0.02956351771 + 0)$$

$$= 0.00144329676$$

$$\boxed{D(X||Y) = 0.0014} \text{ bits}$$

1\. d) $D(y||x)$

For finding relative entropy,

$$D(P||q) = \Sigma_y P(y) \log_b (P(y) / q(y))$$

$P(y)$	$q(y)$
0.4	0.38
0.4	0.42
0.2	0.2

$$D(y||x) = 0.4 \log_2 \frac{0.4}{0.38} + 0.4 \log_2 \frac{0.4}{0.42} + 0.2 \log_2 \frac{0.2}{0.2}$$

$$= 0.0296002 3257 - 0.02815573115 + 0$$

$$= 0.00144450142$$

$$\boxed{D(y||x) = 0.0014 \text{ bits}}$$

1\. (e) $H(X|Y)$

For finding conditional entropy,

$$H(X|Y) = H(x,y) - H(y)$$

Now, for finding joint entropy,

$$H(x,y) = -\Sigma_x \Sigma_y P(x,y) \log_2 P(x,y)$$

$$= -[0.19 \log_2 0.19 + 0.15 \log_2 0.15 + 0.06 \log_2 0.06 + 0.16 \log_2 0.16 + 0.1 \log_2 0.1 + 0.14 \log_2 0.14 + 0.03 \log_2 0.03 + 0.17 \log_2 0.17 + 0 \log_2 0]$$

$$= -[-0.4552264485 - 0.41054483912 - 0.24353362134 - 0.42301699036 - 0.33219280948 - 0.39711017748 - 0.15176681067 - 0.43458686924 + 0]$$

$$H(x,y) = 2.8479798558 \text{ bits}$$

$$\boxed{H(x,y) = 2.8480 \text{ bits}}$$

Now, calculating conditional entropy,

$$H(X|Y) = 2.8480 - 1.5219$$ (taking value of $H(y)$ from 1.(b))

$$\boxed{H(X|y) = 1.3261 \text{ bits}}$$

1.(f) $H(y|x)$

For finding conditional entropy,

$$H(y|x) = H(x,y) - H(x)$$

$$= 2.8480 - 1.5205$$ (taking value of $H(x)$ from 1.(a) & $H(x,y)$ from 1.(e))

~~H(y|x)~~ $\boxed{H(y|x) = 1.3275 \text{ bits}}$

1.(g) H(x,y)

For calculating Joint entropy,

$$H(x,y) = -\Sigma_x \Sigma_y P(x,y) \log_2 P(x,y)$$

$$= -[0.19 \log_2 0.19 + 0.15 \log_2 0.15 + 0.06 \log_2 0.06 + 0.16 \log_2 0.16 + 0.1 \log_2 0.1 + 0.14 \log_2 0.14 + 0.03 \log_2 0.03 + 0.17 \log_2 0.17 + 0 \log_2 0]$$

$$= -[-2.84797998558]$$

$$= 2.84797998558$$

$$\boxed{H(x,y) = 2.8480 \text{ bits}}$$

1.h) H(y) − H(y|x)

$H(y) - H(y|x) = 1.5219 - 1.3275$ (taking value of H(y) from 1.(b) & H(y|x) from 1.(f))

$$\boxed{H(y) - H(y|x) = 0.1944 \text{ bits}}$$

1. i) $I(x;y)$

For calculating mutual information of x & y,

$I(x;y) = H(y) - H(y|x)$

$I(x;y) = 1.5219 - 1.3275$ (taking value of $H(y)$ from 1.(b) & $H(y|x)$ from 1.(f))

$\boxed{I(x;y) = 0.1944 \text{ bits}}$

OR

$I(x;y) = H(x) - H(x|y)$

$= 1.5205 - 1.3261$ (taking value of $H(x)$ from 1.(a) & $H(x|y)$ from 1.(e))

$\boxed{I(x;y) = 0.1944 \text{ bits}}$

OR

$I(x;y) = H(x) + H(y) - H(y,x)$

$= 1.5205 + 1.5219 - 2.8480$ (taking value of $H(x)$ from 1.a), $H(y)$ from 1.b) & $H(y,x)$ from 1.g))

$\boxed{I(x;y) = 0.1944 \text{ bits}}$

1. i) If x is the number of wheels on a vehicle and y is the color of the vehicle, what does $I(x;y)$ tell us? (Let's say we can observe the number of wheels easily but not the color)

i) $I(x;y) = 0.1944$ bits

~~$I(x;y)$ tells us about the uncertainty of x i.e. number of wheels on a vehicle~~

~~$I(x;y)$ is the amount of~~

$I(x;y)$ tells us about the uncertainty of x i.e. the number of wheels on a vehicle if we already know y i.e. the color of the vehicle and vice-a-versa.
As given in the question, the number of wheels are easily observable but not the colour of the vehicle, therefore, colour of vehicle (y) is uncertain.
We can say, $I(x;y)$ tells us about the uncertainty of y i.e. the color of vehicle if we already know x i.e. number of wheels on a vehicle.

$X \rightarrow$ The number of wheels on a vehicle
$Y \rightarrow$ Color of the vehicle.

Ques. 2 Using a standard (balanced) die, the following sequence of numbers is rolled: 1, 3, 5, 2, 6. Compute the amount of information in bits for this event (i.e. the minimal amount of information required to store this sequence)

Ans. 2 Given, using standard die numbers rolled ⇒ 1, 3, 5, 2, 6

Since the die is balanced, probability of getting 1, 3, 5, 2, 6 is $\frac{1}{6}$ each.

The information with an element whose probability of occurrence is P is given by

$$\cancel{I = -\log(P)} \Rightarrow I = \log(1/P)$$

To calculate bits base = 2

$$\text{Probability} = \frac{1}{6},\ A_1, A_3, A_5, A_2, A_6$$

$$I = I_{A_1} + I_{A_3} + I_{A_5} + I_{A_2} + I_{A_6}$$

$$= \log_2\left(\frac{1}{1/6}\right) + \log_2\left(\frac{1}{1/6}\right) + \log_2\left(\frac{1}{1/6}\right) + \log\left(\frac{1}{1/6}\right) + \log\left(\frac{1}{1/6}\right)$$

$$= \log_2(6) + \log_2(6) + \log_2(6) + \log_2(6) + \log_2(6)$$

$$= \log_2(6^5)$$

$$= 5 \log_2 6$$

$$= 5 \times 2.584962501$$

$I = 12.9248125$ bits

$$\boxed{I = 12.9248 \text{ bits}}$$

$\Rightarrow$ Hence amount of bits are "12.9248"

Ques.3 Consider two die, one die being balanced, i.e. the probability of each of the 6 numbers is equal, and one which is loaded where the probability of 1 is 0.4, the probability of 6 is 0.1, and the probability of the other numbers are equal. Compute the entropy an event produced by each of the two dice.

Ans-3 Given, 1 die is ~~loaded~~ balanced

$\Rightarrow$ probability $= \frac{1}{6}$ each

For loaded die,

probability of 1 is 0.4

probability of 6 is 0.1

$$P(1) + P(6) = 0.4 + 0.1 = 0.5$$

we know, sum of all probability is 1 and given probability of other four members are equal.

$\Rightarrow$ Sum of probability of $2, 3, 4, 5 = 0.5$

$$\Rightarrow P(2) = P(3) = P(4) = P(5) = 0.125$$

The entropy (E) on event produced by a balanced die is

$$E = -\Sigma_x P(x) \log_b P(x)$$

$$= -\left[\frac{1}{6}\log_2\frac{1}{6} + \frac{1}{6}\log_2\frac{1}{6} + \frac{1}{6}\log_2\frac{1}{6} + \frac{1}{6}\log_2\frac{1}{6} + \frac{1}{6}\log_2\frac{1}{6} + \frac{1}{6}\log_2\frac{1}{6}\right]$$

$$= -\left[6 \times \frac{1}{6} \times \log_2\left(\frac{1}{6}\right)\right]$$

$$= -(-2.584962501)$$

$$= 2.584962501$$

$$\boxed{E = 2.5850 \text{ bits}}$$

The Entropy (E) on event produced by loaded die is

$$E = -\Sigma_x P(x) \log_b (P(x))$$

$$= -[0.4 \times \log_2(0.4) + 0.1\log_2(0.1) + 0.125\log_2(0.125) + 0.125\log_2(0.125) + 0.125\log_2(0.125) + 0.125\log_2(0.125)]$$

$$= -[-0.528771238 - 0.3321928095 + 4\times(-0.375)]$$

$$= 0.528771238 + 0.3321928095 + 1.5$$

$$= 2.360964048$$

$$\boxed{E = 2.3610 \text{ bits}}$$

Ques. 4 The entropy of a probability distribution is 5.8 bits. Is this enough information to calculate how many Hartleys the entropy of the same distribution is? If it can be calculated, how many hartleys is it? How would the answer change if we were asking about nat-s?

Ans. 4 One hartley $= \log_2 10$ bits $= 3.321928095$ bits

1 bit $= \frac{1}{3.321928095}$ Hartley

1 bit $= 0.3010299957$ Hartley

Given, Entropy $= 5.8$ bits

This information is enough to calculate Hartleys

Entropy $= 5.8$ bits $= 5.8 \times 0.3010299957$ Hartleys

Entropy $= 1.745973975$ Hartleys

$\boxed{\text{Entropy} = 1.7460 \text{ Hartleys}}$

Also, 1 Hartley $= 2.302585093$ nates

Entropy $= 1.7460$ Hartleys $= 1.7460 \times 2.302585093$ nats

Entropy $= 4.020253648$ nats

$\boxed{\text{Entropy} = 4.0203 \text{ nats}}$