Assignment - I

Problem 1

Given $y_i \sim N(\mu, \sigma^2)$ for $i \in \{1, 2, 3, \dots N\}$
and $y \in R$.

$$P(y_1, y_2, y_3 \dots y_N) = P(Y_1 = y_1, Y_2 = y_2, Y_n = y_n)$$

$$= P(y_1) P(y_2) P(y_3) \dots P(y_n)$$

$[\because P(x, y) = P(x) P(y)$ when x, y are independent variables$]$

$$= \prod_{i=1}^{N} N(y_i \mid \mu, \sigma^2)$$

$$\boxed{\therefore P(y_1, y_2, y_3, \dots y_n) = \prod_{i=1}^{N} N(y_i \mid \mu, \sigma^2)}$$

Problem 2

Given $Y \sim N(\beta X, \sigma^2)$

$\beta \sim N(\mu, \tau^2)$

By Sampling, we get

$Y = \beta X + \varepsilon^{y}$ where $\varepsilon^{y} \sim N(0, \sigma^2)$ —— ①

$\beta = \mu + \varepsilon^{\mu}$ where $\varepsilon^{\mu} \sim N(0, \tau^2)$ —— ②

Now, we substitute β in ① using ①

$$Y = (\mu + \varepsilon^{\mu}) X + \varepsilon^{y}$$

$$= \mu X + \varepsilon^{\mu} X + \varepsilon^{y}$$

We consider a Z where $Z = \varepsilon^{\mu} X + \varepsilon^{y}$

where $\varepsilon^{\mu} X \sim N(0, \tau^2 X^T X)$ and $\varepsilon^{y} \sim N(0, \sigma^2)$

$\therefore \varepsilon^{\mu} X + \varepsilon^{y} = Z \sim N(0, \sigma^2 + \tau^2 X^T X)$

by Affline Transformation Property

$Y = \mu X + Z$

$Y = N(\mu X, \sigma^2 + \tau^2 X^T X)$

$\therefore$ Marginal Distribution $P(Y \mid \sigma^2, \mu, \tau^2)$ is

$$\boxed{P(Y \mid \sigma^2, \mu, \tau^2) = N(\mu X, \sigma^2 + \tau^2 X^T X)}$$

Problem 3

Given Linear Model,

$y_i = \alpha + \beta x_i + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$

Joe predicts his productivity to be negative when Xiang plays his music very loudly.

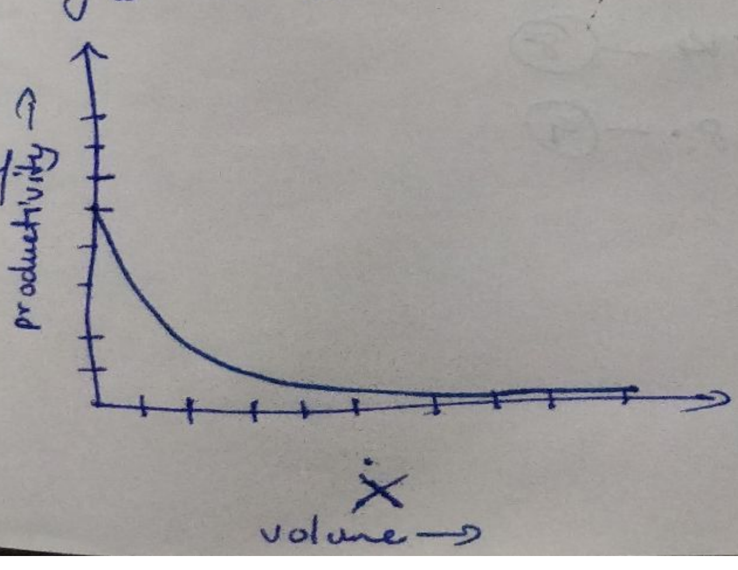
The Solution to this problem is we have to log-transform the equation above then Productivity is always positive.

$$\log(y_i + 0.1) = \alpha + \beta x_i + \epsilon_i$$

[$\because$ we add 0.1 to avoid negative infinity from zeros] log 0 is undefined

$$y_i = e^{\alpha + \beta x_i + \epsilon_i} + 0.1$$

$$y_i = e^{\alpha - 3 x_i + \epsilon_i} + 0.1 \text{ when } \beta = -3$$



Problem - 4

Easiest solution to this problem is by Matrix Multiplication, Consider the Matrix X

$$X = \begin{bmatrix} x & y & z \\ p & q & r \\ a & b & c \end{bmatrix}$$

∴ For Matrix Multiplication, two matrices should be of the form $m \times n$ and $n \times p$, the output matrix is $m \times p$ form

We know that m and p are equal 3 from output

For Multiplication to be done, $n = 3$

$-x + 5y + z = 26$ — ①

$-5x - 4y + 4z = -15$ — ②

$+3x - 3z = 7$ — ③

$-p + 5q + r = 25$ — ④

$-5p - 4q + 4r = 3$ — ⑤

$+3p - 3r = -9$ — ⑥

$-a + 5b + c = 45$ — ⑦

$-5a - 4b + 4c = -14$ — ⑧

$3a - 3c = -8$ — ⑨

Solving ①, ② ③ we get x, y, z

$x = -17$; $y = \frac{17}{3}$; $z = \frac{-58}{3}$

Solving ④, ⑤, ⑥ we get p, q, r

$p = \frac{-43}{5}$; $q = \frac{22}{5}$ $r = \frac{-28}{5}$

Solving ⑦, ⑧, ⑨, we get a, b, c

$a = \frac{-46}{5}$; $b = \frac{127}{15}$; $c = \frac{-98}{15}$

$$X = \begin{pmatrix} x & y & z \\ p & q & r \\ a & b & c \end{pmatrix} = \begin{bmatrix} -17 & \frac{17}{3} & -\frac{58}{3} \\ \frac{-43}{5} & \frac{22}{5} & \frac{-28}{5} \\ \frac{-46}{5} & \frac{127}{15} & \frac{-98}{15} \end{bmatrix}$$

$$X = \begin{bmatrix} -17 & 5.66 & -19.33 \\ -8.60 & 4.40 & -5.6 \\ -9.2 & 8.466 & -6.533 \end{bmatrix}$$

Another solution, that is not easy but is fast

Let $A = \begin{bmatrix} 26 & -15 & 7 \\ 25 & 3 & -9 \\ 45 & -14 & -8 \end{bmatrix}$

$B = \begin{bmatrix} -1 & -5 & 3 \\ 5 & -4 & 0 \\ 1 & 4 & -3 \end{bmatrix}$

Then Equation becomes,

$A = XB$

Multiplying B^{-1} on both sides

$AB^{-1} = XBB^{-1}$

$AB^{-1} = XI$

$AB^{-1} = X$

$\boxed{\therefore \; X = AB^{-1}}$

We now find the B^{-1} using Minors, Co-factors and Adjugate Method.

$\text{Determinant}(B) = -1\left[\begin{vmatrix} -4 & 0 \\ 4 & -3 \end{vmatrix}\right] + 5\left[\begin{vmatrix} 5 & 0 \\ 1 & -3 \end{vmatrix}\right] + 3\left[\begin{vmatrix} 5 & -4 \\ 1 & 4 \end{vmatrix}\right]$

$= -1(12-0) + 5(-15-0) + 3(20+4)$

$\text{Det}(B) = -12 - 75 + 3 \times 24 = -15$

Matrix of Minors

We take the transpose of B that B^T and get matrix of Minors

$$B^T = \begin{bmatrix} -1 & 5 & 1 \\ -5 & -4 & 4 \\ 3 & 0 & -3 \end{bmatrix}$$

Minors

$$= \begin{vmatrix} -4 & 4 \\ 0 & -3 \end{vmatrix} \quad \begin{vmatrix} -5 & 4 \\ 3 & -3 \end{vmatrix} \quad \begin{vmatrix} -5 & -4 \\ 3 & 0 \end{vmatrix} \quad \begin{vmatrix} 5 & 1 \\ 0 & -3 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 1 \\ 3 & -3 \end{vmatrix} \quad \begin{vmatrix} -1 & 5 \\ 3 & 0 \end{vmatrix} \quad \begin{vmatrix} 5 & 1 \\ -4 & 4 \end{vmatrix} \quad \begin{vmatrix} -1 & 1 \\ -5 & 4 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 5 \\ -5 & -4 \end{vmatrix}$$

Substitute the values of Minors in corresponding positions in the matrix

$$\begin{bmatrix} 12 & 3 & 12 \\ -15 & 0 & -15 \\ -24 & +1 & 29 \end{bmatrix}$$

Now we get matrix of co-factors by applying checkerboard of minuses

$$\begin{bmatrix} 12 & 3 & 12 \\ -15 & 0 & -15 \\ -24 & +1 & 29 \end{bmatrix} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\mathrm{Cof}(B) = \begin{bmatrix} 12 & -3 & 12 \\ 15 & 0 & 15 \\ 24 & -1 & 29 \end{bmatrix}$$

Now, we need to get Adjugate Matrix,
Since we had done the transpose before, $\mathrm{Adj}(B) = \mathrm{Cof}(B)$

$$\mathrm{Adj}(B) = \begin{bmatrix} 12 & -3 & 12 \\ 15 & 0 & 15 \\ 24 & -1 & 29 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\det(B)} * \mathrm{Adj}(B)$$

$$B^{-1} = \frac{-1}{15}\begin{bmatrix} 12 & -3 & 12 \\ 15 & 0 & 15 \\ 24 & -1 & 29 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \frac{12}{15} & \frac{-3}{15} & \frac{12}{15} \\ \frac{24}{15} & \frac{-1}{15} & \frac{29}{15} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} -0.8 & 0.2 & -0.8 \\ -1 & 0 & -1 \\ -1.6 & +0.066 & -1.933 \end{bmatrix}$$

Now we have to calculate AB^{-1}

$$AB^{-1} = \begin{bmatrix} 26 & -15 & 7 \\ 25 & 3 & -9 \\ 45 & -14 & -8 \end{bmatrix}\begin{bmatrix} -0.8 & 0.2 & -0.8 \\ -1 & 0 & -1 \\ -1.6 & +0.066 & -1.933 \end{bmatrix}$$

$$= \begin{bmatrix} 26\times-0.8-15\times-1\ 7\times-1.6 & 26\times0.2+7\times0.0667 & 26\times-0.8-15\times-1+7\times1.933 \\ 25\times-0.8+3\times-1\ -9\times-1.6 & 25\times0.2-9\times0.0667 & 25\times-0.8+3\times-1-9\times-1.933 \\ 45\times-0.8-14\times-1-8\times1.6 & 45\times0.2-8\times0.0667 & 45\times-0.8-14\times-1-8\times-1.933 \end{bmatrix}$$

$$= \begin{bmatrix} -17 & 5.6669 & -19.33 \\ -8.6 & 4.4 & -5.603 \\ -9.2 & 8.4664 & -6.536 \end{bmatrix}$$

So, with both solutions we got the same answers but ~~ans~~ how we got it is different that is different approaches were involved to the problem. One was easy and other was fast.

Problem 5

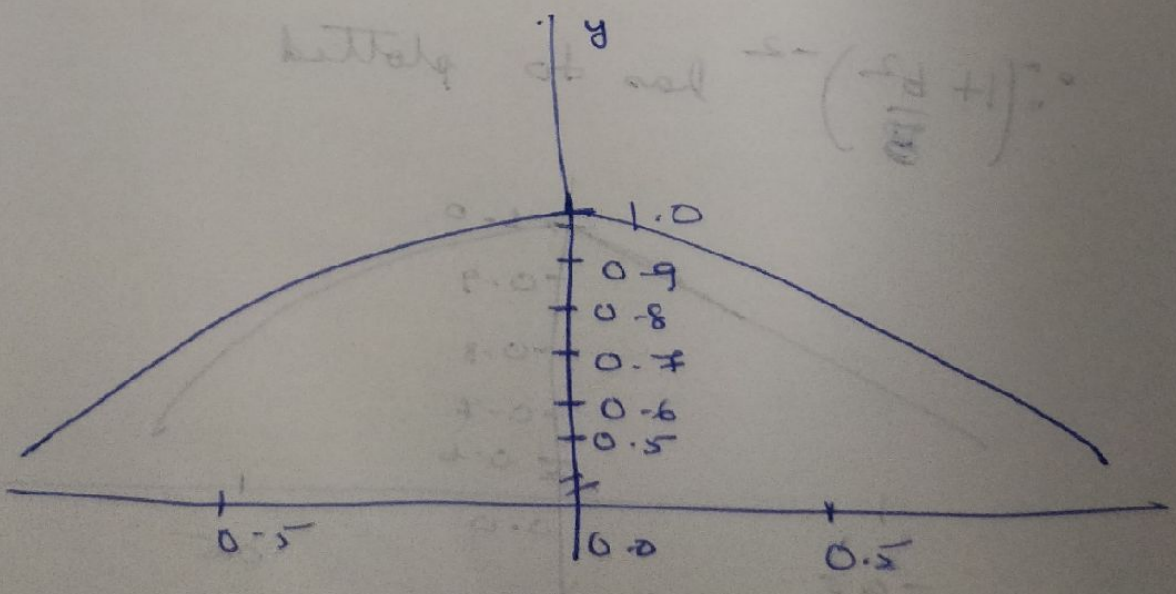
Given $\arg\min_b \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\sqrt{\upsilon\pi}\,\Gamma(\upsilon/2)} \left(1+\frac{b^2}{\upsilon}\right)^{-\frac{\upsilon+1}{2}}$

The minimum value of β 'b' has to found here, hence any change in value of $\Gamma(x)$ or change in 'x' does not affect 'b'

Therefore we can consider only

$$\arg\min_b \left(1+\frac{b^2}{\upsilon}\right)^{-\frac{\upsilon+1}{2}}$$

Let us plot this fn for $\upsilon = 3$ ~~that~~ is $\left(1+\frac{b^2}{3}\right)^{-2}$ function.



Here, this function has maximum at $b=0$ or $(0,1)$ but reaches minimum only at $-\infty, \infty$.

Hence its minimum cannot be found.

Therefore the laptop running R program is crashing as R program is unable to find the minimum point and continues to increase the value of x with stepsize but does not get to the minimum point or the minimum value or reach the minimum value. This is the case for any value of v that is we get the same graph for any value v.