A task that exceeded the technology: early applications of the computer to the lunar three-body problem *

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Abstract: The lunar Three-Body problem is a famously intractable aspect of Newtonian mechanics. The demand for accurate predictions of lunar motion led to practical approximate solutions of great complexity, constituted by trigonometric series with hundreds of terms. Such considerations meant there was demand for high speed machine computation from astronomers during the earliest stages of computer development. One early innovator in this regard was Wallace J. Eckert, a Columbia University professor of astronomer and IBM researcher. His work illustrates some interesting features of the interaction between computers and astronomy.

Keywords: history of astronomy, three body problem, history of computers, W. J. Eckert

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The three body problem in astronomy consists of finding a general expression for the trajectory of three celestial bodies in mutual gravitational attraction. Unlike the two body problem that admits of a ready solution, the three body problem is famously intractable with no definitive solution for the general case and this has made it the focus of a great deal of intellectual effort. The first manifestation of these difficulties was Newton's attempts to use his theory of gravity to derive adequate predictions of the motion of the Moon. The labour of analyzing the Moon's motion under gravity was so difficult that it reputedly gave him headaches, and the explanation of several elements of the Moon's motion eluded Newton.[1] In this talk I will show how when computers were employed in the solution of this problem, changing the scope and goals of efforts. However, the computer alone did not determine the path of method chosen, I will also show how historical practice and human skill and judgement played their role.

In the 18$^{\text{th}}$ and 19$^{\text{th}}$ centuries various equations for the motion of the Moon were developed from Newton's equations via a process of successive approximation to higher terms of a finite series of trigonometric functions. These equations or theories of the Moon, as they were called, achieved greater accuracy and had practical significance for nautical navigation as well as for astronomical purposes.[2] However as solutions became more accurate the work of analysis and algebra required to derive the equations became ever more onerous.

One way of categorizing these theories is dividing literal and numerical theories. The literal method refers to the use of pure algebraic terms (literal terms) in an approximate solution, not substituting in empirically measured numerical terms until the mathematical analysis to derive the solution has been completed. A numerical theory is one where numerical coefficients were substituted during the derivation. Numerical theories allow some simplifications of the derivation, but changing those terms (if they were measured more accurately for example) would require the entire equation to be rederive. On the other hand the latest numerical measure of empirical terms could be instantly substituted into a literal theory without having to rederive the underlying equations.

In the later half of the 19$^{\text{th}}$ century there were two lunar theories both accurate to about 1 second of arc ($1''$). Peter Hansen had developed a lunar theory with the substitution of numerical constants at an early stage publishing the basic theory in 1838 although full predictions would not be available from it until 1857. Charles Delauney on the other hand derived his lunar theory in a completely literal manner working for 20 years and publishing the theory in 1800 pages over two volumes in 1860 and 1867 describing a theory with 800 trigonometric terms. Delauney's theory did not actually take into account complications such as the effects of the planets and a set of predictions from it would not be available until

[1] Szebehely, 1967, p. v; Brewster, 1860, p. 108; Linton, 2004, p. 279
[2] Linton, 2004, p. 298-304

1911.[3]

In the 1880s American George William Hill developed a new approach to Lunar problems hoping to achieve greater accuracy at less effort, previously approximate solutions began by assuming a circular or elliptical orbit for the Moon and correcting from there, instead Hill used another shape the so-called variational orbit as his starting point. Hill recognized that m, the ratio of the mean motions of the Sun and Moon, was the slowest parameter to converge in literal methods and that it would be possible to derive a solution in terms of the variation orbit where m was substituted at the beginning, while other terms could be kept literal.[4]

Hill never finished working out a lunar theory, but his approach and calculations were taken up in two very different directions. French mathematician Henri Poincaré made use of Hill's variation curve as a starting point into his study of the three-body problem. Beginning in the 1880s Poincaré generalized Hill's variation curve and so began characterizing the features of periodic solutions to the three body problem. This work continued for the rest of Poincaré's life and he derived various suggestive theorems and conjectures. An important result was that some orbits with very similar initial conditions could diverge widely and such sensitivity to initial conditions cast doubt on the hopes of predicting celestial motions and particularly to attempts to make statements about qualitative features of the solar system such as the stability of the solar system, a question some thought had been settled by Laplace at the beginning of the 19th century. The exploration of these intricate questions about the qualitative features of the three body problem became an exciting research program for a few scholars following Poincaré. One such follower remarked that: "In my 45 years dedicated to the problem of three bodies I never had a dull moment."[5] Later mathematical researchers in Chaos Theory would suggest an affinity between their own work and Poincaré's work in celestial mechanics.[6]

British mathematician and astronomer E. W. Brown, eschewed such general qualitative questions and continued Hill's project to produce a more accurate lunar theory. The result of Brown's work was both a theory of the Moon published in 1908 after 14 years of work and in 1919 a set of tables that soon replaced Hansen's and Delauney's theory as the source of lunar positions for national almanacs. The equations of Brown's theory consisted of more than 1400 trigonometric terms, and Brown's tables specified numerical constants for the equations, but also took various short cuts combining, approximating or neglecting terms in order to speed calculation and allow the terms to be calculated from 180 tables. The tables consisted of 180 tables over 3 volumes and 660 pages in all. The publication of these tables

[3] Linton, 2004, p. 409-412; Brown, 1919, p. v-vi

[4] Wilson, 2010, p.55-68

[5] Szebehely, 1990, p. vi

[6] Linton, 2004, p. 415-432

was sponsored by Yale and Brown became a professor at Yale into the bargain.[7] These elaborate preparations are typical of what was necessary to turn the algebraic lunar theories into a complete method of predicting lunar positions as seen with Hansen's and Delauney's theories.

Even once derived a complete lunar theory required a great deal of labour to produce a prediction. According to L. J. Comrie who supervised the production of astronomical and navigational tables for the British government in the 1920s and 30s, two people supplied with Brown's tables could perform the necessary calculations just about fast enough to keep up with the Moon. The heavy labour involved meant there would be a great benefit in automating the process. In order to produce a table of predicted Lunar positions at Noon and Midnight GMT, Comrie took steps to automate the calculation of values from the table using accounting machines from the British Tabulating company.[8]

Wallace Eckert was Brown's doctoral student at Yale and a professor at Columbia university in the 1930s. He carried out computational astronomy using IBM accounting machines. This included work on numerical integration of asteroid orbits. Numerical integration of an orbit involves repeated extrapolation of a body's position from a starting point, each extrapolation is over a short period of time and each step is dependent on the last, unlike an algebraic approximation (such as the various theories of the Moon) it requires a great deal of arithmetic and does not easily reveal qualitative features of the motion (such as the periods of repeated motions). Also is only good for the period of time calculated and extending the period covered requires more arithmetic. For these reasons numerical integration was eschewed before the 20$^{\text{th}}$ century except for very specialized applications that did not easily yield a solution in the form of a trigonometric series. Calculating machines like IBM's accounting machines offered the hope of making numerical integration more practical and common. Eckert led in these developments thanks to the support of IBM who provided the machines free of charge to him.[9]

In the mid 1930s Eckert also undertook an attempt with Brown to improve upon Brown's Lunar theory. The basis of the project to improve Brown's theory was the work of Sir George Bidel Airy, a 19$^{\text{th}}$ century Astronomer Royal of England. Airy proposed deriving an improved Lunar theory by substituting an existing approximate theory for the motion of the Moon into the differential equations that govern the Moon's motion. In so far as the approximation was correct it would obey the equalities of the differential equations, but deviations would remain as residuals. The residuals could then be used both as a measure of the mathematical accuracy of the approximate solution and as the basis to derive an improved solution. The

[7] Schlesinger and Brouwer 1941, p. 245-249; Comrie 1932, p. 694; Brouwer 1939, p. 302

[8] Comrie, 1932, p. 694

[9] Tropp, 1978, p. 128-129, Eckert, 1935

improved solution would be derived via treating the residuals as parameters against which to fit a new set of expressions that would negate the residuals when used to modify the original solution, an exercise in linear algebra comparable to curve fitting.[10]

Airy did not have the resources to carry out the massive work required by this solution and gave up after discovering an error in his calculations. Brown had carried out a survey of all the methods of Lunar theory in preparation for his own solution. He published his survey of the history of lunar theory in 1896. Through this work Brown was aware of Airy's attempt and thus proposed that he and Eckert carry out the calculations to test his work and achieve some slight improvement in Lunar theory.[11] They hoped that with IBM's calculating machines they could succeed where Airy failed.

The first part of Airy's method involves taking derivatives of hundreds of trigonometric terms (ie sine functions) and carrying out various multiplications and gathering together the like terms to give the residuals. Brown and Eckert recognized that the huge task of book keeping the thousands of terms could be done by punched card machines. Eckert completed this part of the work and was able to confirm to Brown the basic conformity of his theory with Newton's laws before Brown's death in 1938. However no new solution was ever derived from these calculations as Eckert became busy with other work. Eckert left Columbia University in 1940 to become director of the U.S. Naval Observatory's Nautical Almanac Office in Washington. Eckert returned to New York in 1945 as director of the IBM office of pure research and director of IBM's Thomas J. Watson Scientific Computing Laboratory associated with Columbia university.[12]

Eckert would continue to take the automation of lunar calculations to new heights. By the 1930s empirical accuracy of Brown's Lunar theory had been well established and helped confirm the conclusion that the Earth's rotation is not constant but varies. This meant solar time had inaccuracies detectable due to discrepancies in the timing of events such as eclipses. This led to an initiative by the International Astronomical Union to establish ephemeris time as the new standard of astronomical timekeeping, using the motion of the Moon as the reference point. In aid of this Wallace J. Eckert carried out a set of calculations at the Watson laboratory on IBM's brand new Selective Sequence Electronic Calculator in 1948, finding the position of the Moon directly from Brown's theory without approximation. This large scale electronic calculator could derive a position in 8 minutes instead of hours at far greater accuracy and was one of the first machines to combine electronic calculation speed with the ability to automatically carry out long chains of instructions.[13]

Eckert revived the idea of using Airy's method in about 1957, at this point he was aided

[10] Airy, 1886, p. vi-viii

[11] Airy, 1888, p. 2; Brown, 1960; Brown, 1938, p. 785-788

[12] Eckert, 1940, p. 97-98; Brouwer, 1939, p. 305; Brennan, 1971, p. 11-13

[13] Sadler and Clemence, 1954, p. vii-viii; Eckert *et al.*, 1954, p. 347-350

by his long time assistant at the Watson Laboratory, astronomer Rebecca Jones. The initial work was performed on the IBM 650 a relatively modest machine by the standards of the day. Finding the solution to this large system of equations proved more difficult than Eckert had anticipated. Jones soon left the laboratory and Eckert recruited Harry F. Smith Jr. a graduate student at Columbia in mathematics and Watson laboratory staff member. Smith was an expert on computation, but not astronomy. Through his career Eckert several times employed assistants who lacked specialized astronomy training and training them in the necessary elements in order to carry out calculations and machine work. Eckert and Smith continued working with the IBM 650 deriving residuals and one improved solution with some work done on the more powerful IBM 704. The IBM 650 had a high speed table feature that they found useful in the evaluation of the differential equations, able to hold at one time 1500 of the 30 000 terms produced during multiplication. The first improved solution involved 3000 residuals making for a system of over 3 000 equations with 3000 unknowns to be solved.[14]

The essential method of solution involved the standard linear algebra technique of treating the system of equations as a matrix. A system of linear equations can be solved directly by manipulating such a matrix. Computers were well suited to carrying out matrix operations in their fast indexed memory. However a 3 500 by 3 500 matrix contains about twelve million terms far more than could fit into the working memory of a computer in the 1960s.

Eckert and Smith could divide the terms involved into those which could generate small divisors and therefore large effects on the solution and other terms whose contribution was smaller. They devised a plan to solve the small divisor terms via the direct method of matrix inversion, the others would be solved by Southwell Relaxation technique.[15]

Relaxation techniques achieve a solution by trial and error, substituting a potential solution into one of the equations, deriving the effects and then plugging another trial solution into the next equation. Southwell's method modified this by seeking out the largest terms for trial solution first. Southwell's method was original designed as a way to speed hand computation of this sort of problem.[16] Originally it seemed inappropriate for computer methods because of the time searching would take over arithmetic processes.

Smith and Eckert found that they needed to include a third process of elimination because of interactions between the terms solved by the two techniques. The large matrix had to be subdivided based on method of solution and then subdivided again more arbitrarily in order to yield matrices of between 300-500 terms that can could be fit in the working memory of the computer for relatively quick solution.[17]

[14] Eckert, 1958, p. 416-417; Eckert and Smith, 1961; Eckert and Smith, 1976, p. 235-236.
[15] Smith, 1965, p. 30-41
[16] Hoffman, 2001, p. 64-66
[17] Smith, 1965, p. 42-44, 72-77

The initial work on the IBM 650 and 704 had been done to 11 digits of precision. Eckert and Smith began working on the transistorized IBM 7090 in 1961 and sought to increase the precision and check the dependence of the solution on the precision. The solution done on the IBM 650 became a baseline and independent test on later calculations which were done with different coding schemes. Once again inserting Brown's expressions into the equations of motion this time with a precision of between 12 and 17 digits yielded over 6 000 residual terms. This improved theory was itself fed back into the equations to yield new residuals and derive an even more precise theory with some elements calculated to 18 digits of precision. After various trials a final solution was derived from a set of over 9 000 residuals. The final solution was carried out in the late 1960s by Smith on various IBM 7094, including ones at the IBM research center, Columbia university, Imperial College London, and at Lyngby in Denmark.[18]

The evaluation of these large matrices required a great deal of work and ingenuity, few standard algorithms were available. Smith carried out the coding on the 7090 and 7094 using the Share Operating System (SOS), 7090 SORT, and the SHARE-Compiler-Assembler-Translator (SCAT) programming system. He derived over 20 programs consisting of over 50 000 words of instructions. Many of the programs had to be run multiple times with minor variation and Smith found this easier to achieve with the features of the SCAT system that allowed modification of existing programs by the addition of more programming cards at the end. SOS and SCAT were less popular than the language FORTRAN for scientific work and so Smith's use of them here is an interesting example of using these systems in demanding technical work. These programming systems were viewed as demanding more skill from the end user and suggest Smith's connection to IBM and technical focus.[19]

Since the problem involved large amounts of data stored on magnetic tape, this made the problem slow taking 50 hours to find the residuals and solve the matrix on a 7094 and Smith estimated the total run time of the various solutions at several hundred hours. Also, some terms were never satisfactorily eliminated limiting the precision of the final theory.[20]

Eckert had hoped to improve upon the accuracy of Brown's theory by two orders of magnitude, since Brown's theory was understood to be accurate to about $0''.01$ this would be $0''.0001$ (5×10^{-9}). The Eckert-Smith solution fell slightly short of this goal. On the other had it far exceeded the modest goals of the original project devised by Brown and Eckert in the 1930s. Also, while the bulk of the work had been done by 1965 the publication of the final analysis of results, a large monograph, stretched into the 1970s after Eckert's death. Still the major correction to Brown's theory implied by this work had already been drawn

[18] Eckert and Smith 1976, p. 235-238, 248; Smith, 2007

[19] Smith, 1965, p. 114-115; Akera, 2007, p. 270

[20] Smith, 1965, p. 114-115, 132; Eckert and Smith, 1976, p. 189

out, tested against observation and published by 1965.[21]

Smith and Eckert argued that the solution tested both the convergence of series used to approximate the Moon's motion and the linear algebra methods used, neither of which had been proven with full mathematical rigor. Along with the various innovations in the project to use Airy method were some significant conservative elements, including the way in which the integer values of the arguments of sine functions were encoded into machine readable form, a scheme first described by Eckert for his work in the 1930s. The herculean nature of this task was recognized by Eckert's contemporaries. Eckert's obituary in the journal *Celestial Mechanics* noted of this project "the task evidently exceeded... the state of the technology, yet it was carried to completion."[22]

As the calculations of Airy's method wound down Eckert continued to work on Lunar theory. First he and several assistants reconverted Brown's solution from the rectangular coordinates it was devised in into the polar coordinates an Earth bound observer uses.[23] Brown had done this at reduced precision and Eckert sought to wring from Brown's work ever last bit of precision.

At about the same time Eckert conceived his most ambitious project, to carry out a new derivation of Lunar theory in the method of Brown, but taking it to new levels of accuracy. Unlike with Airy's method it was not possible to build on past success, because when Brown carried out his derivation various physical parameters were inserted, terms were rounded off or only taken to a few terms and so on. Indeed the proper parameters of Brown's equation and the origin of numerical factors could be unclear in places.[24] Therefore to achieve new accuracy required starting from scratch with the Hill variation curve. It was for this reason that no one had attempted to check the calculations or improve on Brown's theory by reproducing it.

In 1940 Eckert had commented on the attempt to use Airy's method: "when used in connection with a good literal theory it [Airy's method] gives result which could be obtained by the literal method alone only at the cost of tremendous effort."[25] Twenty-five years on and Eckert proposed to undertake just such a "tremendous effort".

Eckert's proposal had several novel elements beyond simply pursuing higher precision in the Hill-Brown method. In the earlier method several elements of the forces, were solved separately from the rest of the three body problem and added as separate perturbations.

[21] Eckert and Smith, 1976, p. 196-197, 262; Eckert and Smith, 1966, p. 243; Eckert and Smith, 1976, p. 189; Klock and Scott, 1965, p. 335

[22] Smith, 1965, p. 44; Eckert and Smith, 1976, p. 195-196; Eckert and Smith, 1976, p. 235; Eckert, 1940, p. 66-67, 100; In Memoriam, 1972, p. 3

[23] Eckert *et al*, 1966, p. 314

[24] Eckert and Eckert 1967, 1304-1307; Woolard, 1959

[25] Eckert, 1940, p. 98

Eckert felt that: "Aesthetically, it is desirable to solve the entire "main problem" by a single unified method."[26] Another aesthetic goal was to consider all terms up to the seventh order rather than truncating or continuing on the more *ad hoc* basis that had traditionally been used.[27]

For the purposes of comparison, he proposed solving the equations for multiple values of some parameters. Eckert argued this would allow interpolation of the solution to different values of these parameters with high precision without having to recalculate the whole solution. Among other comparisons Eckert hoped to compare the results to the Eckert-Smith solution and thus mutually confirm the accuracy and mathematical soundness of the two solutions.[28]

The work on the new solution was undertaken by various assistants including Eckert's wife Dorothy (herself a trained astronomer), M. Judith Walker and chiefly staff member Sarah Bellesheim. Bellesheim was Eckert's chief assistant and wrote the computer code for the project. Like Smith before her, she had no training in astronomy before beginning this project, but learned the necessary techniques on the job. Machines used included the humble IBM 1620, and the IBM 360/ 91. The code was written chiefly in FORTRAN, but some of the IBM 1620 work was done in the Symbolic Programming System (SPS). Work had to be carried out to 24 digits of precision requiring Bellesheim to code triple precision routines in assembly language, suggesting an extraordinary level of precision being demanded.[29]

The work continued for five years after Eckert's death in 1971. During this period Martin Gutzwiller of the IBM Laboratory in Yorktown Heights supervised Bellesheim as she worked part-time completing it. During the 1970s various other literal series had been developed by various astronomers using various computer methods, often using high-speed computers. One of the most accurate of these was the *Épheéméride Lunaire Parisienne* (ELP). The ELP was the work of Michelle Chapront-Touzé based on some initial work by her husband Jean Chapront and L. Mangeney in 1969. The approach combined a more traditional derivation of an approximate solution with a series of successive approximation not unlike Airy's method. Chapront-Touzé developed and refined the method over the course of the 1970s achieving a final form in 1980. Gutzwiller compared the Eckert-Bellesheim effort with some newer solutions. In the comparison he found that Eckert's plan of calculating all terms up to the seventh order and no more had left certain significant terms uncalculated.[30]

Given these new developments Gutzwiller decided not to work out the final details of

[26]Eckert, 1973, 66. Note the "main problem" here is the three body problem of Earth-Moon-Sun.
[27]Eckert and Bellesheim, 1976, p. 42
[28]Eckert and Eckert, 1967, p. 1299; Eckert, 1973, p. 66
[29]Eckert and Eckert, 1967, p. 1305, 1307; Eckert and Bellesheim, 1976, p. 43-55
[30]Gutzwiller, 1976, p. [iii]; Chapront-Touzé and Chapront, 2000, p. 38-40; Gutzwiller, 1979, p. 891-899

Eckert's theory or publish the work. Instead after publishing his 1979 comparions, he teamed with astronomer Dieter Schmidt to produce a new theory along the lines of the Hill-Brown method, which was published in full in 1986. However for practical reasons Gutzwiller and Schmidt returned to *ad hoc* criterion to decide what terms to include and exclude and they did not attempt to calculate solutions for different values of the parameters. Also they were free to take advantage of standardized linear algebra software.[31]

Eckert's final work was in many ways his most ambitious. Thirty years earlier, the project had seemed too onerous to undertake. He clearly had a confidence in the potential for computer aided algebra in lunar theory. However, the methods remained a work in progress, and so he and his assistants were forced to innovate with the materials available. It is interesting that in this work he had taken a more principled approach of including terms on the basis of the character of the solution (the order of the terms) rather than a more *ad hoc* basis (the size of the coefficients) that had traditionally been used by Brown. It suggests a conviction in the power of the later computer technology to allow new possibilities. Despite these efforts, his successors would still find themselves led back to a need for a more *ad hoc* rules by the requirements of computation.

It is significant that one method in Lunar theory Eckert did not explore in any detail was the use of numerical integration. He had worked on numerical integration of asteroids in the 1930s and the outer planets in 1950 and was familiar with the method. The Jet Propulsion Laboratory (JPL), who calculated celestial motions for the US Space program as part of NASA, would develop numerical integration when in the late 60s they found the accuracy of traditional lunar theory, Brown's work refined by Eckert, to be insufficiently accurate for their purposes. Still much of the work effort of landing men on the Moon depended on refinements of the traditional methods. Numerical integration allowed a better inclusion of the effects of the planets among other things, although it had some drawbacks.[32] Eckert was actually insensitive to these subsidiary issues since he focused his attention and verification on the traditional three body problem aspect of Lunar theory, which had so long fascinated astronomers. The JPL had consulted with Eckert in developing their lunar ephemerides and their researchers included people working on a wide variety of approaches in celestial mechanics.

Aside from its use in space exploration, the JPL numerical integrations became the basis for the American and British Nautical Alamancs in 1984, finally replacing the Hill-Brown theory as the method of calculation, but *Connaissance des Temps* continued to use an more analytical solutions, Michelle Chapront-Touzé's ELP, to produce a complete Lunar

[31] Gutzwiller and Schmidt, 1986, p. 11-12, 31, 37

[32] Eckert, 1935; Clemence *et al.*, 1960; Butrica, 2014, p. 118-122; Mullholand, 1969; see Dallas, 1970 for an example of celestial mechanics research at the JPL at the time.

ephemeris. The accuracy of numerical integration for work on current celestial positions is generally recognized, but researchers like Chapront-Touzé argue analytic methods have the potential for insight into dynamics from the forms of equations and that the equations can produce better positions for long time periods required to estimate the ancient observations of celestial objects.[33]

In Eckert's attempts to solve the lunar 3 body problem on computer, we see how his choices and plans of attack depended not only on his available computer power. Rather his choices made use of the repertoire of different techniques developed by previous astronomers and guided by aesthetics and judgement steeped in this tradition. Also, rather than seeking the latest technology to speed work at times Eckert and his assistants made do with much more basic machines and worked within those limitations. Still, the use of computers meant that whereas one successful lunar theory might be the work of a life time now a researcher might hope to adopt several approaches to the problem and carry them through.

Precisely these historical legacies and contingencies, along with the flourishing of analytic methods in the 1970s would put the lie to Shlomo Sternberg's 1969 lament that thanks to radar astronomy and *brute* numerical integration:

> In a certain sense, the work of the classical astronomers in perturbation theory is no longer relevant for predicting the motion of the planets. [...]In a very real sense, one of the most exalted of human endeavors, going back to the priests of Babylon and before, has been taken over by machine."[34]

People not computers decide how equations will be solved and in doing so they must make full use of the mathematical and intellectual heritage they possess.

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[33] CHAPRONT-TOUZÉ and CHAPRONT, 2000, p. 34, 56
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