Reading Report on the Fundamental Concepts of General Relativity and Torsion Theories

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Abstract

This reading report is a holistic summary and reference sheet for the content studied by myself from January - May 2025 under the guidance of Indranil Das through the URAP Program at Illinois. Chapters 1-5 covers content studied from Hobson, Efstathiou, Lasenby, "General Relativity: An Introduction for Physicists".
Chapter 6 covers Appendix A from Gasperini, "Theory of Gravitational Interactions"

Contents

1 The Spacetime of Special Relativity

The distinction between Newtonian and Relativistic physics can be simplified to the relationship between space and time. Suppose an event that happens at specific coordinates and time in a 3-dimensional space, denoted with (t, x, y, z). Newtonian physics suggests that in two different inertial frames (IFs), ($\ddot{x} = \ddot{y} = \ddot{z} = 0$), while the space coordinates may differ due to the translations, the time coordinate remains absolute. From a Newtonian viewpoint, this is also true for non-IFs (accelerating frames), however, Special Relativity deals only with IFs, and for this section we will not consider any accelerating frames.

The Newtonian view works well for everyday velocities, but begins to break down as the speeds reach significant fractions of the speed of light. For this, Special Relativity provides a more accurate model, imposing the rule that time is **not absolute** in all IFs, and is dependant on the velocity of the frame with relative to another, hence the name "Relativity".

1.1 Inertial Frames

All IFs are non-accelerating and can only differ from each other by a translation, rotation, or movement with constant velocity. It is absolute that **all laws of physics take the same form in all inertial frames.**

Assume two inertial frames, one stationary (S), and one moving with constant velocity v in the x-direction (S'). In order to translate coordinates from S' to S, we must perform a linear transformation:

$$\begin{aligned} t' &= At + Bx \\ x' &= Dt + Ex \\ y' &= y \\ z' &= z \end{aligned}$$

Since $x' = 0$ corresponds to $x = vt$ and $x = 0$ corresponds to $x' = -vt$, we know $D = -Ev = -Av$ so $A = E$:

$$\begin{aligned} t' &= At + Bx \\ x' &= A(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

1.2 Newtonian Geometry

In Newtonian Physics, time is absolute ($t = t'$), so the translations between frames take the form:

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}$$

This is called a *Galilean Transformation.* The velocity $\dot{x}'$ of a particle in S' is $\dot{x} - v$ in S. Since the frames are inertial, $\ddot{x}' = \ddot{x}$. Galilean Transformations preserve an invariant quantity:

$$\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

This is obviously the distance traveled by a particle, and in Newtonian Geometry, is invariant across all IFs.

1.3 Minkowski Geometry

Reaching relativistic speeds, the concept of an absolute time must be abandoned and replaced by a new concept: **The speed of light (c) must be the same in all inertial frames.** The Galilean Transformations no longer apply, and are replaced by *Lorentz Transformations* or *boosts.* These do away with the absoluteness of time, leading to a host of new effects to be discussed later.

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned}$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$. Notice that when $\beta \ll 1$, implying v is much less than c, the Lorentz Transformations simplify to the Galilean Transformations. From this, we can easily say that the Lorentz Transformations are a generalized form of Galilean Transformations, and an even stronger statement, **for inertial frames, Special Relativity is a generalization of Newtonian Physics.**

Now thinking in Minkowski geometry, we find a new invariant quantity:

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

This leads us to the conclusion that instead of being two separate facets of nature, space and time are united in a 4-dimensional continuum, *spacetime*, which operates under the laws of Minkowski Geometry.

1.4 Lorentz Transformations as 4d Rotations

Introducing a *rapidity* parameter:

$$\psi = \tanh^{-1}\beta$$

We can model the relativistic quantities as hyperbolic functions:

$$\begin{aligned} \gamma &= \cosh\psi \\ \gamma\beta &= \sinh\psi \end{aligned}$$

This allows us to write the Lorentz Transformations as 4d "rotations":

$$\begin{aligned} ct' &= ct\cosh\psi - x\sinh\psi \\ x' &= -ct\sinh\psi + x\cosh\psi \\ y' &= y \\ z' &= z \end{aligned}$$

1.5 The Lightcone

In Minkowski Geometry, Δs^2 is an especially important quantity. Consider two events that occur at two distinct points in spacetime. The value of Δs^2 for these two events dictate their type of separation in spacetime.

Firstly, if $\Delta s^2 > 0$, two events are *timelike separated.* This implies that the events have some time separation, as only a nonzero Δt will allow $\Delta s^2 > 0$. Assume event A happens before B in the rest frame. This means that using Lorentz Transformations, there is no IF where event B happens at the same time or before event A, but there is an IF where they happen at the same spacial position.

Inversely, if $\Delta s^2 < 0$ two events are *spacelike separated.* Assume event A happens behind B in the rest frame. Using Lorentz Transformations, there is no IF where event B happens at the same spacial point or behind event A, but there is an IF where they happen at the same time.

Finally, if $\Delta s^2 = 0$ between A and B, the events are *lightlike* or *null separated* and are not affected by any Lorentz Transformation. This is more easily shown on a *spacetime diagram*:

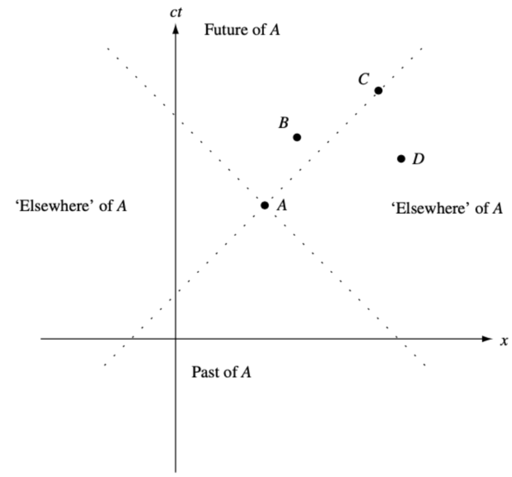

Figure 1: Hobson, fig 1.3

The dotted lines denote the borders of A's *lightcone.* Events that occur in a certain lightcone of A in the rest frame must stay within that lightcone in every other IF. The vertical lightcones are timelike separations and the horizontal ones are spacelike separations. If an event is on the border of a lightcone, it is null separated.

1.6 Spacetime Diagrams

Spacetime Diagrams are especially useful modeling Minkowski Spacetime on a 2d plane. In order to model a Lorentz Transformation, the axes are rotated with the angle $\tan^{-1}(\frac{v}{c})$. The new axes represent the diagram of the new reference frame. The coordinates of the transformed diagram are read parallel to the axes of the transformed diagram.

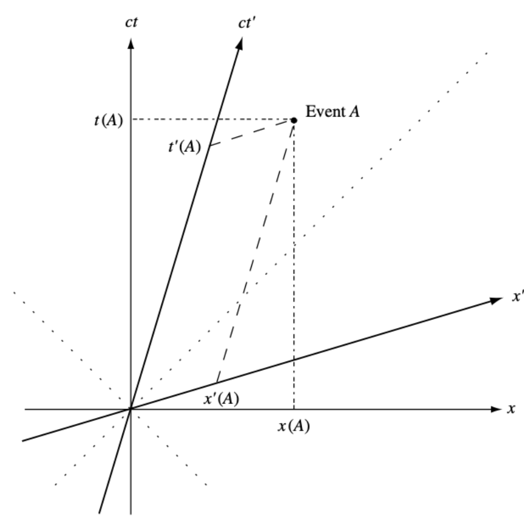

Figure 2: Hobson, fig 1.4

The dotted lines are the lightcone of the origin. Notice how the coordinates of A are read differently based off of which set of axes being used. With this representation, we can think of Minkowski spacetime as being able to stretch and contract around a particle based off of its velocity, much like rubber or a sponge.

1.7 Effects of Special Relativity

When dealing with Minkowski Spacetime, we must define two important variables. *Proper Length* (ℓ_0) is the length of an object in its rest frame, and *Proper Time* (τ) is the speed of time passing for an object in its rest frame. We will find that Lorentz transformations will change the perceived length and passage of time for an object, but **no Lorentz transformation can ever make an object longer than its proper length or experience time faster than its proper time**.

The first effect is *Length Contraction.* Imagine a rod at rest in S'. Proper length of this rod can be shown as $\ell_0 = x'_B - x'_A$. Applying a Lorentz transformation to this equation leaves us with:

$$\ell = \frac{\ell_0}{\gamma}$$

Hence in any other frame, the rod appears to be contracted from its proper length.

The second effect is *Time Dilation.* Suppose in S' there are two clicks separated by a proper time interval $T_0 = t_B - t_A$. Applying a Lorentz transformation:

$$T = \gamma T_0$$

Hence in any other frame, the clicks appear to have a longer time interval between them, or in other words, experience time more slowly.

1.8 Invariant Hyperbolae

When viewing the invariant interval Δs^2 on a Spacetime Diagram, we notice that they form the *invariant hyperbolae.*

$$c^2t^2 - x^2 = c^2t'^2 - x'^2 = \pm 1$$

Any event on a spacetime diagram subject to a Lorentz transformation will remain on its respective invariant hyperbola.

1.9 The Minkowski Spacetime Line Element

Two infinitesimally separated points in Minkowski Spacetime follow the same geometry:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

This is called the *line element.* The invariant interval between any A and B can then be shown as:

$$\Delta s = \int_A^B ds$$

To evaluate this integral, we need a set of equations to show the spacetime path taken.

1.10 Particle Worldlines and Proper Time

A particle worldline is a functional path taken by the particle through spacetime. They describe x, y, and z as functions of t. We can also write this as a 4-vector: $(t(\lambda), x(\lambda), y(\lambda), z(\lambda))$ where λ is any parameter. Usually we use proper time (τ). As expected,

$$d\tau = \frac{dt}{\gamma}$$

Integrating $d\tau$ over a worldline from A to B will yield the elapsed proper time interval.

$$\Delta\tau = \int_A^B d\tau = \int_A^B \frac{dt}{\gamma}$$

Finally, the worldline of a massive particle on the spacetime diagram can be parameterized as follows:

$$\begin{aligned} t &= \gamma\tau \\ x &= v\gamma\tau \\ y &= 0 \\ z &= 0 \end{aligned}$$

1.11 The Doppler Effect

Lorentz transformations can also distort the frequency of waves depending on the frame of the emitter and observer. The following is a formula for the ratio of the observed frequency to the emitted frequency:

$$\frac{f_{obs}}{f_{emit}} = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}$$

1.12 Velocity Addition

Using the differential form of the Lorentz transformations, we can rewrite the velocities of particles in different reference frames.

$$\begin{aligned} u'_x &= \frac{dx'}{dt'} = \frac{u_x - v}{1 - u_x v/c^2} \\ u'_y &= \frac{dy'}{dt'} = \frac{u_y}{\gamma\left(1 - u_x v/c^2\right)} \\ u'_z &= \frac{dz'}{dt'} = \frac{u_z}{\gamma\left(1 - u_x v/c^2\right)} \end{aligned}$$

Furthermore, adding velocities in special relativity along the same direction of motion is simple– just add the rapidity of the velocities together and use the rotational form of Lorentz transformations. Thus, using the formula for rapidity:

$$\begin{aligned} u &= c\tanh\left(\psi_v + \psi_{u'}\right) \\ &= c\frac{\tanh\psi_v + \tanh\psi_{u'}}{1 + \tanh\psi_v \tanh\psi_{u'}} = \frac{u' + v}{1 + u'v/c^2} \end{aligned}$$

1.13 Accelerations

Acceleration in special relativity can be found using a similar method as finding velocity. We find that though acceleration is not invariant, its sign is.

1.14 Event Horizons

A constantly accelerating object would have its worldline on its specific invariant hyperbola. This means that the constantly accelerating object has a sort of ”event horizon” at the edge of the origin’s lightcone. As the object continues to accelerate, other objects around it will seem to slowly freeze in time due to time dilation. The accelerating object will never be able to see past a certain point in time (the event horizon) unless it stops accelerating.

2 Manifolds and Coordinates

Minkowski spacetime is a simple example of a *manifold*, which in a mathematical sense is simply an amorphous collection of points. These points can have set rules and coordinate systems they follow, and usually describe any number of objects or spaces.

2.1 The Concept of a Manifold

In physics, manifolds must be continuous and differentiable at all points. Furthermore, there must also be a way to parameterize the manifold at any point.

2.2 Coordinates

Manifolds can be N-dimensional, with points defined as $\left(x^1, x^2, \ldots, x^N\right)$. For example, in 3D space, we traditionally define points as (x, y, z), but this can also be written as $\left(x^1, x^2, x^3\right)$. Do not confuse this notation with exponentiation.

Coordinate systems for a manifold may be *degenerate*, meaning that one system cannot cover every point in the manifold. For example, take the surface of a sphere. There is no system that can assign every point on a sphere a unique set of coordinates.

2.3 Curves and Surfaces

Curves on a manifold can be described by parametric equations as functions of some parameter u.

$$x^a = x^a(u) \qquad (a = 1, 2, \ldots, N)$$

It is also possible to define surfaces within the manifold. The surfaces must have M dimensions where $M < N$. They can be parameterized as follows:

$$x^a = x^a\left(u^1, u^2, \ldots, u^M\right) \qquad (a = 1, 2, \ldots, N)$$

Specifically, surfaces with $M = N - 1$ dimensions are known as *hypersurfaces*, and they can be written as one equation,

$$f\left(x^1, x^2, \ldots, x^N\right) = 0$$

2.4 Coordinate Transformations

Coordinates in any manifold can be transformed using functions of old points.

$$x'^a = x'^a\left(x^1, x^2, \ldots, x^N\right)$$

Differentiations of new coordinates can also be found with respect to the old coordinates:

$$\left[\frac{\partial x'^a}{\partial x^b}\right] = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \cdots & \frac{\partial x'^1}{\partial x^N} \\ \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \cdots & \frac{\partial x'^2}{\partial x^N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x'^N}{\partial x^1} & \frac{\partial x'^N}{\partial x^2} & \cdots & \frac{\partial x'^N}{\partial x^N} \end{pmatrix}$$

The determinant of the given transformation matrix is known as the *Jacobian*(J).

$$J = \det\left[\frac{\partial x'^a}{\partial x^b}\right] \qquad J' = \det\left[\frac{\partial x^a}{\partial x'^b}\right]$$

Taking two differentially separated points on a manifold P and Q, such that $P = (x^a)$ and $Q = (x^a + dx^a)$, we can say,

$$dx'^a = \sum_{b=1}^{N} \frac{\partial x'^a}{\partial x^b}\, dx^b$$

2.5 Einstein Summation Convention

In order to simplify the writing of expressions involving summations, we can apply the *Einstein Summation Convention*. It is important not to think of this as a mathematical process, but simply a useful shorthand to simplify the writing of specific expressions.

Simply, if an index appears twice in an expression, once as a superscript and once as a subscript, it is assumed to be summed over from 1 to N.

For example, take the the equation for dx'^a that we just defined. Using summation convention, we can rewrite it.

$$dx'^a = \frac{\partial x'^a}{\partial x^b}\, dx^b$$

Note that a superscript appearing in a denominator is considered a subscript and vice versa.

In this equation, b appears twice in the right expression, once as a superscript in the denominator, and once as a regular superscript. We can assume b is summed over and will call it a *dummy index*. a, however, appears on both sides of the equation, so we can call it a *free index*, meaning that it can take any value from 1 to N.

2.6 Geometry of Manifolds

Understanding a manifold's geometry is essential to be able to visualize its behavior or "shape". Quite simply, the geometry of a manifold is defined by the behavior of a differential distance between two points on the manifold. The differential distance can be defined as any function as long as it is well behaved.

$$ds^2 = f(x^a, dx^a)$$

Notice the similarity to the notation for the invariant interval in Minkowski spacetime. This is not a coincidence. Minkowski spacetime is an example of a manifold, and the invariant interval is its geometry. Another example is 3D space, where its geometry is defined as $ds^2 = {dx^1}^2 + {dx^2}^2 + {dx^3}^2$.

2.7 Riemannian Geometry

While developing General Relativity, we are most interested in *Riemannian* and *pseudo-Riemannian Manifolds*, which have a geometry defined by the form

$$ds^2 = g_{ab}(x)\, dx^a\, dx^b$$

Where $g_{ab}(x)$ are components of the *metric tensor*, which is any matrix that can define every point on the Riemannian Manifold. The metric tensor must vary smoothly with position and must be symmetric.

If ds^2 is always > 0, the manifold is Riemannian. If it can be ≤ 0 at any point, the manifold is pseudo-Riemannian.

2.8 Intrinsic and Extrinsic Geometry

Imagine a 2D Riemannian Manifold, "Flatland", embedded in 3D Euclidean Space. *Intrinsic properties* are properties of the surface itself, and *Extrinsic properties* are properties that depend on how the surface is embedded in a higher dimensional plane.

For a simple visualization, think of the 2D Riemannian manifold as an infinitely thin piece of paper. Any geometry of the 2D manifold that a flatlander can access (point on the paper, direction of travel on the paper) is intrinsic. Any geometry of the 2D manifold that requires the viewpoint of a higher-dimensional human (curve of the paper, its position in space) is extrinsic.

Any action performed on the paper that does not stretch or tear it, such as folding, bending, and crumpling, will change the extrinsic geometry of the paper, but the intrinsic geometry will remain the same. A flatlander with no concept of a third dimension will always see the paper as the exact same no matter how much we try to deform it.

2.9 Non-Euclidean Geometry

Simply put, a manifold is considered *non-Euclidean* if its geometry does not follow the basic form of Euclidean Geometry. An Euclidean manifold of N dimensions is defined as:

$$ds^2 = {dx^1}^2 + {dx^1}^2 + \cdots + {dx^N}^2$$

Examples of this are surfaces of spheres, hemispheres, toruses, etc.

2.10 Lengths, Areas, and Volumes

Given a defined metric function $g_{ab}(x)$ for a Riemannian Manifold, we can calculate lengths, areas, and volumes in that manifold as follows:

$$\begin{aligned} L_{AB} &= \int_A^B ds = \int_A^B |g_{ab}\, dx^a\, dx^b\,|^{\frac{1}{2}} \\ dA &= \sqrt{|g_{11}g_{22}|}\, dx^1\, dx^2 \\ dV &= \sqrt{|g_{11}g_{22}g_{33}|}\, dx^1\, dx^2\, dx^3 \end{aligned}$$

Note that $g_{ab} = 0$ when $a \neq b$.

2.11 Local Cartesian Coordinates

Transforming all of a Riemannian manifold into Euclidean geometry is not possible, but it is possible to consider a point on the manifold and its close neighborhood as roughly Euclidean.

Euclidean geometry is defined as $\delta_{ab}\, dx^a\, dx^b$, and Riemannian geometry is defined as $g_{ab}\, dx^a\, x^b$. To transform from Riemannian to Euclidean in the neighborhood of point P:

$$\begin{gathered} g'_{ab}(P) = \delta_{ab} \\ \frac{\partial g'_{ab}}{\partial x'^c} = 0 \end{gathered}$$

Thus, in the neighborhood of P,

$$g'_{ab}(x) = \delta_{ab} + \mathcal{O}((x' \cdot x'_P)^2)$$

These are called the *Local Cartesian Coordinates* of point P.

2.12 Tangent Spaces

Consider a 2D Riemannian manifold. This curved surface will have a plane that is tangent at every point. This plane is called the *Tangent Space* (T_P) to the manifold at point P. As a more general definition, a Tangent Space is **a space in the neighborhood of a point on a manifold where the line element is Euclidean**.

3 Vector Calculus on Manifolds

Vectors are an extremely important tool while doing physics, and we will find that this is no different for General Relativity.

3.1 Scalar Fields

A *Scalar Field* on manifold $\mathcal{M}$ assigns a value to each point P in $\mathcal{M}$. Obviously, you can define values for points with a function, and if you change coordinates, a different function is needed to show the same values.

3.2 Vector Fields

Points on $\mathcal{M}$ can also have vectors associated with them, creating a *Vector Field* on the manifold. All of the vectors are associated with points on the manifold, but each of them lie in their own tangent space. Therefore, **it does not make physical sense to be able to add vectors in the tangent spaces of two different points on the manifold.**

3.3 Tangent Vectors to a Curve

A curve $\mathcal{C}$ is defined by N parametric equations $x^a(u)$ where u is some parameter. $\mathcal{C}$ has a tangent vector t defined as follows:

$$t = \lim_{\delta u \to 0} \frac{\delta s}{\delta u}$$

where δs is the infinitesimal separation between point P at u and point G at $u + \delta u$.

3.4 Basis Vectors

At each point P we can define a set of *Basis Vectors* $\mathbf{e}_a$ for the tangent space. If the tangent space has dimension N, basis vectors are **a set of N vectors where any vector in the tangent space of P is a linear combination of them**. We can define the local vector field ($\mathbf{v}(x)$) as

$$\mathbf{v}(x) = v^a(x)\mathbf{e}_a(x)$$

where $v^a(x)$ are the *contravariant components* of $\mathbf{v}(x)$ in the basis $\mathbf{e}_a(x)$. Also consider the set of *dual basis vectors* $\mathbf{e}^a(x)$ where

$$\mathbf{v}(x) = v_a(x)\mathbf{e}^a(x)$$

where where $v_a(x)$ are the *covariant components* of $\mathbf{v}(x)$ in the basis $\mathbf{e}^a(x)$. Note that the different positioning of the index enables effective use of Einstein Summation Convention. Another important equation to remember:

$$\mathbf{e}^a(x) \cdot \mathbf{e}_b(x) = \delta^a_b$$

which shows that the basis and dual basis vectors are reciprocal.

One way to show $\mathbf{e}_a$ is through a *coordinate basis*. Assume two points P and G separated from each other by δs or δx^a along each dimension. The basis vectors at P are defined as:

$$\mathbf{e}_a = \lim_{\delta x^a \to 0} \frac{\delta s}{\delta x^a}$$

We can clearly see here that $\mathbf{e}_a$ is the set of tangent vectors at P, which we defined individually a bit before. The set $\mathbf{e}_a$ provides the basis vectors for the tangent space T_P at P. From this, we can tell that P and G coordinates x^a and $x^a + dx^a$ respectively have their infinitesimal separation given by:

$$ds = \mathbf{e}_a(x) \cdot \mathbf{e}_b(x) = g_{ab}(x)$$

and it follows that

$$\mathbf{e}^a(x) \cdot \mathbf{e}^b(x) = g^{ab}(x)$$

From this, we can write the dot product of two vectors as follows,

$$\mathbf{v} \cdot \mathbf{w} = (v^a \mathbf{e}_a) \cdot (w^b \mathbf{e}_b) = g_{ab} v^a w^b$$

Obviously, the dual is also true.

3.5 Raising and Lowering Indices

We already know the metric tensor definition of dot products,

$$\mathbf{v} \cdot \mathbf{w} = g_{ab} v^a w^b$$
$$\mathbf{v} \cdot \mathbf{w} = g^{ab} v_a w_b$$

But we can also write these in terms of the Kronecker delta like so:

$$\begin{aligned}\mathbf{v} \cdot \mathbf{w} &= (v_a \mathbf{e}^a) \cdot (w^b \mathbf{e}_b) \\ &= (\mathbf{e}^a \cdot \mathbf{e}_b) v_a w^b = \delta^a_b v_a w^b\end{aligned}$$

where the dual $\mathbf{v} \cdot \mathbf{w} = \delta^b_a v^a w_b$ is also true.

With the dot product now given in terms of a Kronecker delta, we can do the following:

$$\delta^a_b v_a w^b = v_a w^a$$

The dual is also true. Notice that the Kronecker delta is nonzero only when $a = b$, so we are able to simplify the equation in that way. Notice the implied Summation Convention leads to the conventional definition of a dot product.

We can now do the following:

$$\mathbf{v}\cdot\mathbf{w} = v_a w^a = g^{ab} v_a w_b$$
$$w^a = g^{ab} w_b$$

and the dual,

$$w_a = g_{ab} w^b$$

Clearly, using the metric tensor, we can raise and lower the indices of vector components, e.g. changing them back and forth from their dual. Writing this in terms of basis vectors,

$$\mathbf{e}_a = g_{ab}\mathbf{e}^b \qquad \mathbf{e}^a = g^{ab}\mathbf{e}_b$$

The ability to raise and lower indices is an extremely important property of the metric tensor.

3.6 Coordinate Transformations

Coordinate transformations can be done relatively simply:

$$ds = dx^a\,\mathbf{e}_a = dx'^a\,\mathbf{e}'_a$$

It follows,

$$\mathbf{e}'_a = \frac{\partial x^b}{\partial x'^a}\mathbf{e}_b \qquad \mathbf{e}'^a = \frac{\partial x'^a}{\partial x^b}\mathbf{e}^b$$

3.7 Coordinate-independent Properties

As we know, vectors originating from different points on a manifold are not in the same space, but there are some properties that are preserved. For example, a dot product of two vectors is coordinate-independent no matter the space. Furthermore, the length of a vector is also coordinate-independent and is given as so:

$$|g_{ab}v^a v^b|^{\frac{1}{2}} = |g^{ab}v_a v_b|^{\frac{1}{2}} = |v_a v^a|^{\frac{1}{2}}$$

The angle between two vectors is also coordinate-independent:

$$\cos\theta = \frac{v_a w^a}{|v_b v^b|^{\frac{1}{2}}|w_c w^c|^{\frac{1}{2}}}$$

3.8 The Affine Connection

Consider the following diagram:

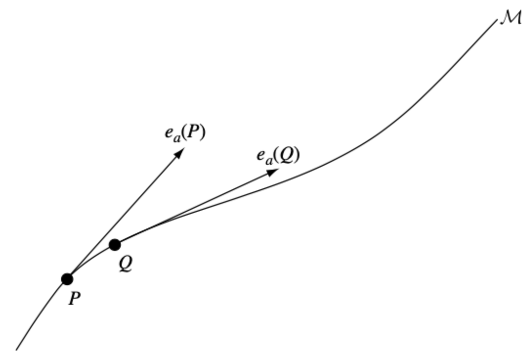

Figure 3: Hobson, fig 3.5

Notice the curvature of the manifold $\mathcal{M}$ and the two distinct points, P and Q. The basis vectors $\mathbf{e}_a(P)$ and $\mathbf{e}_a(Q)$ represent the same vector but are defined in the tangent spaces of two different points. However, when viewed externally, we can see the effect of the tangent spaces. Assuming P and Q are infinitesimally close, basis vectors at Q will also differ infinitesimally from those at P:

$$\mathbf{e}_a(Q) = \mathbf{e}_a(P) + \delta\mathbf{e}_a$$

To transform a vector across tangent spaces, which is also called *transporting* a vector across a manifold, we use the *Affine Connection* (Γ^b_{ac}), denoted by a *Christoffel Symbol of the Second Kind*, which is defined as the derivative of the basis vectors at P:

$$\frac{\partial\mathbf{e}_a}{\partial x^c} = \Gamma^b_{ac}\mathbf{e}_b$$

The Affine Connection tells you how much the b-component of a vector changes when you move along direction x^a, influenced by contributions from its c-component. This can be rewritten simply as:

$$\Gamma^b_{ac} = \mathbf{e}^b\cdot\partial_c\mathbf{e}_a \qquad \partial_c\mathbf{e}^a = -\Gamma^a_{bc}\mathbf{e}^b$$

3.9 Transforming the Affine Connection

The Affine Connection can be transformed to another set of coordinates using the following formula:

$$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^f}{\partial x'^b}\frac{\partial x^g}{\partial x'^c}\Gamma^d_{fg} - \frac{\partial x^d}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}\frac{\partial^2 x'^a}{\partial x^d \partial x^f}$$

3.10 Affine Connection and the Metric Tensor

The *Torsion Tensor* (T^b_{ac}) is the difference between the following second order second kind Christoffel

Symbols:

$$\Gamma^b_{ac} - \Gamma^b_{ca} = T^b_{ac}$$

These Christoffel Symbols represent the affine connection across different directions on the manifold. In basic General Relativity, we assume manifolds are *torsionless*, where $T^b_{ac} = 0$. In torsionless manifolds, there is an extremely important relation between the affine connection and the metric tensor:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$$

This is called the *metric connection* and can also be symbolized as $\left\{ {a \atop bc} \right\}$.

We can also define a *Christoffel Symbol of the First Kind* (Γ_{abc}):

$$\Gamma_{abc} = g_{ad}\Gamma^d_{bc}$$

where

$$\Gamma_{abc} = \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc})$$

and

$$\partial_c g_{ab} = \Gamma_{abc} + \Gamma_{bac}$$

3.11 Geodesic Coordinates

In some cases, we would want to obtain a local set of Cartesian Coordinates at a point P. These coordinates are called *Geodesic Coordinates* and must satisfy the following condition:

$$\Gamma'^a_{bc}(P) = 0$$

We can define the geodesic coordinates at P like so:

$$x'^a = x^a - x^a_P + \frac{1}{2}\Gamma^a_{bc}(P)(x^b - x^b_P)(x^c - x^c_P)$$

3.12 Covariant Derivative of a Vector

Consider a vector defined by its contravariant components $\mathbf{v} = v^a \mathbf{e}_a$. We can then say

$$\partial_b \mathbf{v} = (\partial_b v^a)\mathbf{e}_a + v^a(\partial_b \mathbf{e}_a) = (\partial_b v^a + v^c \Gamma^a_{cb})\mathbf{e}_a$$

This is known as the *covariant derivative*. This differs from a regular derivative due to the fact that covariant derivatives take the curvature of the manifold into account. If the manifold is flat, the Affine term vanishes, reducing the covariant derivative into a regular derivative.

We can write the covariant derivative in a more compact form $(\nabla_b v^a)\mathbf{e}_a$ where $\nabla_b v^a = \partial_b v^a + v^c \Gamma^a_{cb}$. The dual, $\nabla_b v_a = \partial_b v_a + v_c \Gamma^c_{ab}$ is also true.

Note that the covariant derivative of a scalar does not depend on the basis vectors, so in that case, it will also be the same as a regular derivative as shown: $\nabla_a \phi = \partial_a \phi$.

3.13 Vector Operators

The 4 basic vector operations are given as follows:

$$\nabla\phi = (\nabla_a \phi)\mathbf{e}^a = (\partial_a \phi)\mathbf{e}^a$$
$$\nabla \cdot \mathbf{v} = \nabla_a v^a = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}v^a\right)$$
$$\nabla^2\phi = \nabla_a \nabla^a \phi = \frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}g^{ab}\partial_b\phi\right)$$
$$\nabla \times \mathbf{v} = \nabla_a v_b - \nabla_b v_a$$

3.14 Intrinsic Derivative of a Vector along a Curve

Some vector fields are only defined on a subspace of a manifold, for example, $\mathbf{v}(u)$ defined only along some curve $x^a(u)$ on a manifold. We can calculate the derivative of the vector field with respect to the parameter u along the curve.

$$\begin{aligned} \frac{d\mathbf{v}}{du} &= \frac{dv^a}{du}\mathbf{e}_a + \Gamma^a_{bc} v^b \frac{dx^c}{du}\mathbf{e}_a \\ &= \left(\frac{dv^a}{du} + \Gamma^a_{bc} v^b \frac{dx^c}{du}\right)\mathbf{e}_a = \frac{Dv^a}{Du}\mathbf{e}_a \end{aligned}$$

The term in parentheses is known as the *intrinsic derivative* of the vector components v^a along a curve $\mathcal{C}$ with parameter u.

3.15 Parallel Transport

If the Affine connection transports vectors while taking into the account the curvature of the manifold, *parallel transport* is transporting the vector across the manifold while keeping it parallel to the original vector itself, or in other words, ignoring the curvature of the manifold. Consider the image:

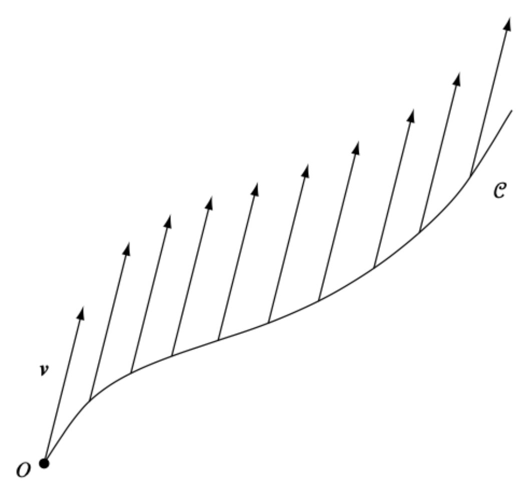

Figure 4: Hobson, fig 3.6

The original vector at point O subject to parallel transport across the manifold creates a field of vectors that are identical to the original vector in its tangent space, but may not be the same vector in their own tangent space. This requires fixing the intrinsic derivative of the vector:

$$\frac{Dv^a}{Du} = 0$$

3.16 Null Curves and Affine Parameters

In pseudo-Reimannian manifolds, there are two types of curves: *Null curves*, which are curves where ds along the curve is 0, and *non-null curves* where $ds \neq 0$.

Any parameters that vary along the curve, for example the length for non-null curves and some parameter u for null curves, are called the *affine parameters.*

3.17 Geodesics

A *geodesic* is a curve on a manifold that satisfies two properties: its tangent vector always points in the same direction, and is is the shortest path between two points on the manifold. We can define a geodesic as the set of parametric functions $x^a(u)$ with some fixed direction tangent vector $t(u)$. Fixing this tangent vector gives us

$$\frac{Dt^a}{Du} = \lambda(u)t^a$$

where $\lambda(u)$ is some function. This equation means that the tangent vector only changes in a way proportional to itself, which ensures that it remains parallel to itself along the curve. Since we know that

$$t^a = \frac{dx^a}{du}$$

we can rewrite the equation:

$$\frac{D}{Du}(\frac{dx^a}{du}) = \lambda(u)\frac{dx^a}{du}$$

If the curve is parameterized in a way that $\lambda(u) = 0$, then u is an affine parameter and the geodesic equation becomes:

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

3.18 Stationary Property

The length (L) along a curve in any manifold is defined:

$$L = \int_A^B ds = \int_A^B |g_{ab}\dot{x}^a\dot{x}^b|^{\frac{1}{2}} du$$

If a curve $x^a(u)$ is non-null, we can test to see if it is a geodesic if its $\delta L = 0$ when the curve is perturbed by some value $\delta x^a(u)$.

3.19 Lagrangian Geodesic Procedure

We can find the setup through Lagrangian mechanics as well. If we consider the Lagrangian for any manifold to be $g_{ab}\dot{x}^a\dot{x}^b$, e.g. no other forces acting on some particle on the manifold, we can plug that into the Euler-Lagrange equation to find the geodesic formula once again:

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$$

which is the same equation we found for the geodesic previously. This equation is valid for both null and non null geodesics.

3.20 Alternate form of the Geodesic Equation

Another way to write the geodesic equation is in terms of the derivative of the components of the tangent vector:

$$\dot{t}_a = \frac{1}{2}(\partial_a g_{cd})t^c t^d$$

4 Tensor Calculus on Manifolds

There are an infinite number of coordinate systems we can use to define any sort of manifold, and as a result, our equations need to be formulated in a way that they represent the same scalar and vector fields no matter the coordinate system in use. The idea of such fields can be generalized as quantities known as tensors.

4.1 Tensor Fields on Manifolds

Consider a vector field defined for every point on a manifold. That means for every point on the manifold, there is a vector $\mathbf{t}$ defined. If you take the dot product of field $\mathbf{t}$ with another arbitrary field $\mathbf{v}$, you will have a scalar field defined on every point of the manifold.

Think of this as a linear function that takes an input of vectors and produces a scalar. This linear function can be defined like so:

$$\mathbf{t}(\mathbf{v}) = \mathbf{t} \cdot \mathbf{v}$$

Generally, a *tensor* is a linear map of some number of vectors to a real number, with the tensor's *rank* being the number of vectors it takes as arguments. By this logic, it is also true that regular vectors are actually rank-1 tensors, and scalar functions are zero-rank tensors.

An important tensor is the *metric tensor*, which is defined:

$$\mathbf{g}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$$

A good way to imagine tensors are through arrays. A tensor can be shown as an n-dimensional array, where n is the rank of the tensor.

Tensors also have the following useful property:

$$\mathbf{t}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{t}(\mathbf{u}) + \beta\mathbf{t}(\mathbf{v})$$

This can be generalized to higher dimensions by expanding each argument individually:

$$\begin{aligned}
&\mathbf{t}(\alpha\mathbf{u} + \beta\mathbf{v}, \gamma\mathbf{v} + \epsilon\mathbf{z}) \\
&= \alpha\mathbf{t}(\mathbf{u}, \gamma\mathbf{v} + \epsilon\mathbf{z}) + \beta\mathbf{t}(\mathbf{v}, \gamma\mathbf{v} + \epsilon\mathbf{z}) \\
&= \alpha\gamma\mathbf{t}(\mathbf{u}, \mathbf{v}) + \alpha\epsilon\mathbf{t}(\mathbf{u}, \mathbf{z}) + \beta\gamma\mathbf{t}(\mathbf{v}, \mathbf{v}) + \beta\epsilon\mathbf{t}(\mathbf{v}, \mathbf{z})
\end{aligned}$$

4.2 Components of Tensors

Basis vectors can be used to find the components of tensors:

$$\begin{aligned}
\mathbf{t}(\mathbf{e}_a) &= t_a & \mathbf{t}(\mathbf{e}^a) &= t^a \\
\mathbf{t}(\mathbf{e}_a, \mathbf{e}_b) &= t_{ab} & \mathbf{t}(\mathbf{e}^a, \mathbf{e}^b) &= t^{ab} \\
\mathbf{t}(\mathbf{e}_a, \mathbf{e}^b) &= t_a^b & \mathbf{t}(\mathbf{e}^a, \mathbf{e}_b) &= t_b^a
\end{aligned}$$

Furthermore, when using vectors in component form, the dummy indexes can be changed from subscript to superscript without changing the result, and vice versa:

$$\mathbf{t}(\mathbf{u}, \mathbf{v}) = t_{ab}u^a v^b = t^{ab}u_a v_b = t_b^a u_a v^b = t_a^b u^a v_b$$

4.3 Symmetries of Tensors

Tensors are *symmetric* when $t_{ab} = t_{ba}$, and *antisymmetric* when $t_{ab} = -t_{ba}$.

Using this definition, a tensor can be rewritten like so:

$$t_{ab} = \frac{1}{2}(t_{ab} + t_{ba}) + \frac{1}{2}(t_{ab} - t_{ba})$$

There is a notation for this for rank-N tensors:

$$\begin{aligned}
t_{(ab\ldots c)} &= \frac{1}{N!}(\text{sum of all indice permutations}) \\
t_{[ab\ldots c]} &= \frac{1}{N!}(\text{alternating sum of all permutations})
\end{aligned}$$

4.4 The Metric Tensor

The metric tensor happens to have a special property regarding its mixed components:

$$g_b^a = \delta_b^a \qquad g_a^b = \delta_a^b$$

This is due to the definition of the metric tensor. When plugging in one covariant and one contravariant basis vector to the metric tensor, we get an equation along the lines of $\mathbf{e}^m \cdot \mathbf{e}_n$ which equals to δ_n^m.

4.5 Raising and Lowering Indices

Using the metric tensor's ability to raise and lower the indices of vectors, we can define the following rule for the same process on tensors:

$$t_{abc} = g_{cd}t^d ab$$

Taking this even further:

$$t_{ab}^d = g_{ae}g^{df}t_{bf}^e$$

4.6 Mapping Tensors into Tensors

Consider a tensor $\mathbf{t}(\cdot, \cdot, \mathbf{u})$. Technically, this is a rank-3 tensor. However, in this case, we have fixed one of the arguments to be $\mathbf{u}$, leaving only two unknown arguments. This effectively makes it a second rank tensor, and we can write it as so:

$$t_{abc}u^c = s_{ab}$$

4.7 Elementary Operations with Tensors

Addition and subtraction:

$$t_{ab} + r_{ab} = s_{ab}$$
$$t_{ab} - r_{ab} = d_{ab}$$

Scalar multiplication:

$$\alpha \cdot \mathbf{t}(\mathbf{e}_a, \mathbf{e}_b) = \alpha t_{ab}$$

Outer product, or *tensor product* produces a tensor of higher rank. The outer product of a rank-M and rank-N tensor produces a rank-$(M+N)$ tensor. A simple example is an outer product of two vectors, or rank-1 tensors:

$$(\mathbf{u} \otimes \mathbf{v})(\mathbf{e}_a, \mathbf{e}_b) = u_a v_b$$

and for higher ranks:

$$h^a_{bc} = t^a_b s_c$$

Contraction of a tensor is performed by summing over a covariant and contravariant argument of a tensor, resulting in a tensor of lower rank:

$$\mathbf{q}(\cdot) = \mathbf{h}(\mathbf{e}^a, \cdot, \mathbf{e}_a)$$

and in terms of its components:

$$q_b = h^a_{ba}$$

Inner products are contractions put together with outer products:

$$q_b = t^a_b s_a$$

4.8 Tensors as Geometrical Objects

Tensors can be thought of as a linear combination of *tensor bases* just like vectors are a linear combination of basis vectors. In fact, tensors are just a generalization of vectors. Consider the derivation:

$$\begin{gathered}(\mathbf{e}_a \otimes \mathbf{e}_b)(\mathbf{e}^c, \mathbf{e}^d) = \mathbf{e}_a(\mathbf{e}^c)\mathbf{e}_b(\mathbf{e}^d) = \delta^c_a \delta^d_b \\ \text{so, } t^{ab}(\mathbf{e}_a \otimes \mathbf{e}_b)(\mathbf{e}^c, \mathbf{e}^d) = t^{ab}\delta^c_a\delta^d_b = t^{cd} = \mathbf{t}(\mathbf{e}^c, \mathbf{e}^d) \\ \therefore \mathbf{t} = t^{ab}(\mathbf{e}_a \otimes \mathbf{e}_b)\end{gathered}$$

Clearly, $\mathbf{t}$ is being expressed as a linear combination of basis vectors.

4.9 Tensors and Coordinate Transformations

The general rule for transforming the components of a tensor to a different set of coordinates is as follows:

$$t'^{c}_{ab} = \frac{\partial x^d}{\partial x'^a}\frac{\partial x^e}{\partial x'^b}\frac{\partial x'^c}{\partial x^f} t^f_{de}$$

Note that the contravariant indices of t' are in the denominator of their respective derivatives and the covariant indices are in the numerator. The rule can have many forms based on this pattern.

4.10 Tensor Equations

Usually, it is far more convenient to work with the components of a tensor rather than the geometrical entity itself. A much used convention is to confuse the tensor object with its components.

Tensors are especially important in physics due to the fact that if $t_{ab} = s_{ab}$ in one coordinate system, $t'_{ab} = s'_{ab}$ in any other.

Basically, **if a tensor equation is true in one coordinate system, it is true in all coordinate systems.**

4.11 Covariant Derivative of a Tensor

Covariant derivatives of a tensor have a the following form based on the tensor components:

$$\begin{aligned}\nabla_c t^{ab} &= \partial_c t^{ab} + \Gamma^a_{dc} t^{db} + \Gamma^b_{dc} t^{ad} \\ \nabla_c t^a_b &= \partial_c t^a_b + \Gamma^a_{dc} t^d_b - \Gamma^d_{bc} t^a_d \\ \nabla_c t_{ab} &= \partial_c t_{ab} - \Gamma^d_{ac} t_{db} - \Gamma^d_{bc} t_{ad}\end{aligned}$$

4.12 Intrinsic Derivative of a Tensor

Some tensors are only defined along a curve based on some parameter u. Such tensors can be shown as follows:

$$\mathbf{t}(u) = t^{ab}(u)\mathbf{e}_a(u) \otimes \mathbf{e}_b(u)$$

Taking the derivative along the curve,

$$\begin{aligned}\frac{d\mathbf{t}}{du} &= \frac{dt^{ab}}{du}\mathbf{e}_a \otimes \mathbf{e}_b \\ &\quad + t^{ab}\frac{dx^c}{du}\frac{\partial \mathbf{e}_a}{\partial x^c} \otimes \mathbf{e}_b + t^{ab}\mathbf{e}_a \otimes \frac{dx^c}{du}\frac{\partial \mathbf{e}_b}{\partial x^c} \\ &= \left(\frac{dt^{ab}}{du} + \Gamma^a_{bc} t^{ab}\frac{dx^c}{du} + \Gamma^b_{dc} t^{ad}\frac{dx^c}{du}\right)\mathbf{e}_a \otimes \mathbf{e}_b\end{aligned}$$

we find the definition of the intrinsic derivative of a tensor:

$$\frac{Dt^{ab}}{Du} = \frac{dt^{ab}}{du} + \Gamma^a_{bc} t^{ab}\frac{dx^c}{du} + \Gamma^b_{dc} t^{ad}\frac{dx^c}{du}$$

5 The Equivalence Principle and Spacetime Curvature

Using the mathematical framework we just outlined, we can formulate a Relativistic Theory of Gravity, or in more simple terms, General Relativity.

5.1 Newtonian Gravity

In Newtonian physics, the force of gravity is defined:

$$\mathbf{F}_G = m\mathbf{g} = -m\nabla\Phi$$

where the Gravitational Potential

$$\nabla^2\Phi = 4\pi G\rho$$

Newtonian theory does not specify any time dependence on gravity and its potential, making it completely inconsistent with special relativity.

5.2 The Equivalence Principle

Think of a free-falling elevator in Earth's gravitational field. In a short amount of time, any objects on the elevator will appear to be weightless, since everything has the same acceleration relative to the free-falling elevator.

Over time, one will notice tidal forces from the inhomogeneity of Earth's G-field. Two particles floating weightlessly on opposite sides of the elevator will gradually drift towards the center of the elevator. In fact, over time, all objects in the elevator will drift towards the center.

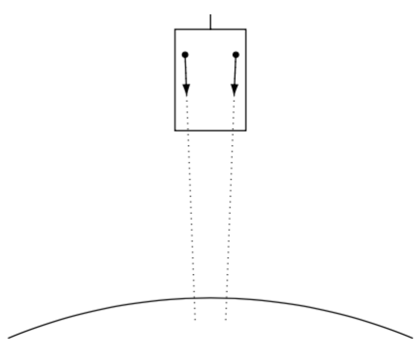

Figure 5: Hobson, fig 7.1

Nevertheless, in a short period of time, the free-falling elevator can be reasonably approximated to be an inertial frame. This leads us to the *Equivalence Principle*: **In a freely-falling non-rotating laboratory occupying a small region of spacetime, the laws of physics are those of special relativity.**

5.3 Gravity as Spacetime Curvature

Einstein proposed a new way of thinking about gravity: **Gravity is not a conventional force, but rather a manifestation of spacetime curvature induced by the presence of matter.**

Therefore, if gravity is no longer regarded as a force, any particles moving under the influence of gravity would have no change in momentum with respect to their proper time. **The worldline of a freely falling particle under gravity is a geodesic in curved spacetime.**

The Equivalence Principle also restricts spacetime to being a Pseudo-Reimannian manifold. The local neighborhood of any event P that occurs in the spacetime manifold must be able to be defined by a coordinate system X^μ such that the line element takes the form $ds^2 \approx \eta_{\mu\nu}dx^\mu dx^\nu$. At point P, the line element is exactly that.

Keep in mind:

$$\eta_{\mu\nu} \approx \begin{pmatrix} \pm1 & 0 & \cdots & 0 \\ 0 & \pm1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm1 \end{pmatrix}$$

In other words, in the vicinity of event P, X^μ represents a local Cartesian inertial frame where special relativity holds. Only a Pseudo-Reimannian manifold can satisfy this constraint. In such a manifold, the general form of the line element is:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

5.4 Local Inertial Coordinates

The curvature of spacetime means that it is impossible to define coordinates where $g_{\mu\nu} = \eta_{\mu\nu}$ for every single point on the manifold. We must use arbitrary coordinates x^μ to label points in spacetime, with it being possible for coordinates to have complex physical meanings. Most of the time, x^0 is the timelike coordinate and $x^{1,2,3}$ are spacelike coordinates, however, there can also be arbitrary null coordinates with no physical meaning at all.

Fortunately, as dictated by the equivalence principle, problems of physical meaning can be overcome by working within a local inertial coordinate system. In fact, at every point in spacetime, there are an infinite number of these coordinate systems, all related to each other through Lorentz transformations.

5.5 Observers in a Curved Spacetime

Observers will travel along their own worldline in a curved spacetime, with the worldline being parameterized by $x^\mu(\tau)$ where τ is their proper time. Each observer will also carry with them a tetrad of orthovectors which form the basis of their local frame $\mathbf{e}_a(\tau)$. Note the relation:

$$\mathbf{e}_a(\tau) \cdot \mathbf{e}_b(\tau) = \eta_{ab}$$

In freely falling frames, the tetrad is simply parallel transported along the worldline of the observer.

5.6 Weak Gravitational Fields and the Newtonian Limit

In the absence of gravity, spacetime would have a Minkowski geometry. In a weak gravitational field, spacetime would be very close to the Minkowski geometry, or only slightly curved. Such a region would have the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$. We find that these small corrections to the metric take the form $h_{00} = \frac{2\Phi}{c^2}$. Therefore, in weak gravitational fields, the metric is as follows:

$$g_{00} = 1 + \frac{2\Phi}{c^2}$$

5.7 Intrinsic Curvature of a Manifold

The curvature of a manifold can also be quantified. If a manifold is flat, there are coordinates X^μ with the following line element:

$$ds^2 = \epsilon_1(dx^1)^2 + \epsilon_2(dx^2)^2 + \cdots + \epsilon_n(dx^n)^2$$

where $\epsilon_a = \pm 1$. However, arbitrary coordinates will not have such an equation. For example, consider the line element for 3d spherical coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

Some ds^2, however, are far too complicated. We need a way to find curvature directly from g_{ab}, independent of the coordinate system.

5.8 The Curvature Tensor

Think of a covariant vector field v_a. It's covariant derivative is given:

$$\nabla_b v_a = \partial_b v_a - \Gamma^d_{ab} v_d$$

Taking the covariant derivative again,

$$\begin{aligned}\nabla_c \nabla_b v_a &= \partial_c(\nabla_b v_a) - \Gamma^e_{ac}\nabla_b v_e - \Gamma^e_{bc}\nabla_e v_a \\ &= \partial_c\partial_b v_a - (\partial_c \Gamma^d_{ab}) v_d - \Gamma^d_{ab}\partial_c v_d \\ &- \Gamma^e_{ac}(\partial_b v_e - \Gamma^d_{eb} v_d) - \Gamma^e_{bc}(\partial_e v_a - \Gamma^d_{ae} v_d)\end{aligned}$$

Swapping the indices b and c gives us the corresponding equation for $\nabla_b \nabla_c v_a$. Subtracting gives:

$$\nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a = R^d_{abc} v_d$$

where

$$R^d_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{ab}\Gamma^d_{ec}$$

R is the *curvature tensor*, also known as the *Riemann Tensor*. For flat manifolds, $R^d_{abc} = 0$.

5.9 Properties of the Curvature Tensor

The curvature tensor has four main properties. First, the raising and lowering of its indices using the metric tensor:

$$R_{abcd} = g_{ae} R^e_{bcd}$$

Secondly, the explicit form of the curvature tensor:

$$\begin{aligned}R_{abcd} = \\ &\frac{1}{2}(\partial_d\partial_a g_{bc} - \partial_d\partial_b g_{ac} + \partial_c\partial_b g_{ad} - \partial_c\partial_a g_{bd}) \\ &- g^{ef}(\Gamma_{eac}\Gamma_{fbd} - \Gamma_{ead}\Gamma_{fbc})\end{aligned}$$

Thirdly, its symmetries:

$$\begin{aligned}R_{abcd} &= -R_{bacd} \\ R_{abcd} &= -R_{abdc} \\ R_{abcd} &= R_{cdab}\end{aligned}$$

Finally, the cyclic property:

$$R_{abcd} + R_{acdb} + R_{abdc} = 0$$

5.10 The Ricci Tensor and Scalar

Contracting the first two indices of the Curvature tensor gives $R^a_{acd} = 0$. Contracting on the first and last indices gives:

$$R^c_{abc} = R_{ab}$$

where R_{ab} is the *Ricci Tensor*. It has the property $R^a_b = R^b_a$. Further contracting the Ricci tensor:

$$R^a_a = g^{ab} R_{ab} = R$$

which is called the *Ricci Scalar*. These together describe the *Einstein Tensor*:

$$G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R$$

where $\nabla_b G^{ab} = 0$. The Einstein tensor is used to describe the curvature of space in the Einstein Field Equations.

6 Understanding Gravitational Actions on Free Scalar Fields

With this, the general overview of the mathematics behind General Relativity is complete. We will now shift to understanding the derivation (A.104-A.112) from Gasperini, "The Theory of Gravitational Interactions", which solves for a gravitational action on a free scalar field. Before covering the explicit derivation, some extra concepts and formalism must be covered.