Spin Connection Check

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Metric and Vielbeins from "Vielbein of the General Schwarzchild Metric" (8/7/2025).

Writing the metric in coordinates $x^\mu = (r, \theta, \phi, \tau)$:

$$g_{\mu\nu} = \begin{pmatrix} e^{\lambda(\tau,r)} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -e^{\nu(\tau,r)} \end{pmatrix}$$

Needed Vielbeins in the same coordinates:

$$e^A{}_\mu = \begin{pmatrix} e^{\lambda/2} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r\sin\theta & 0 \\ 0 & 0 & 0 & e^{\nu/2} \end{pmatrix}$$

$$e_A{}^\mu = \begin{pmatrix} e^{-\lambda/2} & 0 & 0 & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r\sin\theta} & 0 \\ 0 & 0 & 0 & e^{-\nu/2} \end{pmatrix}$$

Equation for the spin connection:

$$\omega_\nu{}^{AB} = e^A{}_\mu \left(\partial_\nu e^{B\mu} + \Gamma^\mu{}_{\sigma\nu}\, e^{B\sigma}\right)$$

So:

$$\omega_r{}^{11} = e^1{}_r \left(\partial_r e^{1r} + \Gamma^r{}_{rr} e^{1r}\right)$$

where

$$e^1{}_r = e^{\lambda/2}$$

$$e^{1r} = e^{-\lambda/2}$$

$\Gamma^r{}_{rr} = \frac{1}{2}\,\partial_r\lambda\,(\tau, r)$ from the EOM notebook

Solving components:

$$\partial_r e^{1r} = -\tfrac{1}{2}\,\lambda_r\,e^{-\lambda/2}$$
$$\Gamma^r{}_{rr} e^{1r} = \tfrac{1}{2}\,\lambda_r\,e^{-\lambda/2}.$$

Finally:

$${\omega_r}^{11} = e^{\lambda/2}\left(-\tfrac{1}{2}\lambda_r e^{-\lambda/2} + \tfrac{1}{2}\lambda_r e^{-\lambda/2}\right) = \boxed{0}$$