Variations on the Action

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I'll start by considering the total action $S_{tot} = S_{EC} + S_m + S_{NY}$ where

$$S_{EC} = -\frac{M_{Pl}^2}{4}\int \epsilon_{abcd} V^a \wedge V^b \wedge R^{cd} \tag{1}$$

$$S_m = \frac{1}{2}\int \mathrm{d}\vartheta \wedge {}^\star \mathrm{d}\vartheta \tag{2}$$

$$S_{NY} = -nf\int \mathrm{d}\vartheta \wedge T^a \wedge V_a \tag{3}$$

S_{EC} is the Einstein-Cartan gravitational action, S_m is the scalar field action, and S_{NY} is the coupling to a Nieh-Yan Form. I'll also define:

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a_c \wedge \omega^{cb}$$
$$T^a = \mathrm{d}V^a + \omega^a_b \wedge V^b$$

The variation of the total action is

$$\delta S_{tot} = \delta S_{EC} + \delta S_m + \delta S_{NY} \tag{4}$$

1 Vielbein

1.1 Einstein-Cartan Action

$$\begin{aligned}
\delta_V S_{EC} &= -\frac{M_{Pl}^2}{4}\int \epsilon_{abcd}(\delta V^a \wedge V^b + V^a \wedge \delta V^b) \wedge R^{cd} \\
&= -\frac{M_{Pl}^2}{4}\int 2\epsilon_{abcd}\delta V^a \wedge V^b \wedge R^{cd} \\
&\qquad \text{(using the antisymmetry of the Levi-Civita Symbol)} \\
&= \boxed{\frac{M_{Pl}^2}{2}\int R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a}
\end{aligned} \tag{5}$$

1.2 Scalar Field Action

$$\begin{aligned}\delta_V S_m &= \frac{1}{2}\int \delta \mathrm{d}\vartheta \wedge {}^\star\mathrm{d}\vartheta + \mathrm{d}\vartheta \wedge \delta^\star \mathrm{d}\vartheta \\ &= \frac{1}{2}\int \mathrm{d}\vartheta \wedge \delta^\star \mathrm{d}\vartheta \qquad \text{since } \delta_V \mathrm{d}\vartheta = 0\end{aligned} \tag{6}$$

Expanding:

$$\begin{aligned}{}^\star\mathrm{d}\vartheta &= {}^\star(\partial_\mu \vartheta \mathrm{d}x^\mu) = {}^\star(\partial_\mu \vartheta V_a^\mu V^a) \\ &= \frac{1}{3!} V_a^\mu \partial_\mu \vartheta \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d\end{aligned}$$

So,

$$\delta_V {}^\star\mathrm{d}\vartheta = \frac{1}{3!}(\delta V_a^\mu)\partial_\mu \vartheta \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d + \frac{1}{3!} V_a^\mu \partial_\mu \vartheta \epsilon^a_{bcd}\, \delta\big(V^b \wedge V^c \wedge V^d\big) \tag{7}$$

Using the identity

$$\begin{aligned}(\delta V_a^\mu) V_\mu^j &= -(\delta V_\mu^j) V_a^\mu \\ (\delta V_a^\mu) &= -(\delta V_\nu^j) V_a^\nu V_j^\mu\end{aligned}$$

and plugging that into eq. (7) and using the antisymmetry of the Levi-Civita Symbol in the second term:

$$\begin{aligned}&\delta_V {}^\star\mathrm{d}\vartheta \\ &= -\frac{1}{3!}(\delta V_\nu^j) V_a^\nu V_j^\mu \partial_\mu \vartheta \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d + \frac{1}{3!} V_a^\mu \partial_\mu \vartheta (3) \epsilon^a_{bcd}\, \delta V^b \wedge V^c \wedge V^d \\ &= -\frac{1}{3!}(\delta V_\nu^j) V_a^\nu \partial_j \vartheta \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d + \frac{1}{2}\partial_a \vartheta \epsilon^a_{bcd}\, \delta V^b \wedge V^c \wedge V^d\end{aligned}$$

Since we are in an orthonormal frame defined by Vielbeins, we can also say that $\partial_a \vartheta \epsilon^a_{bcd} = \partial^a \vartheta \epsilon_{abcd}$. Using this in the second term:

$$\begin{aligned}\delta_V {}^\star\mathrm{d}\vartheta &= -\frac{1}{3!}(\delta V_\nu^j) V_a^\nu \partial_j \vartheta \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d + \frac{1}{2}\partial^a \vartheta \epsilon_{abcd}\, \delta V^b \wedge V^c \wedge V^d \\ &= -(\delta V_\nu^j) V_a^\nu \partial_j \vartheta (\frac{1}{3!}\epsilon^a_{bcd} V^b \wedge V^c \wedge V^d) + \partial^a \vartheta (\frac{1}{2}\epsilon_{abcd} V^c \wedge V^d) \wedge \delta V^b\end{aligned}$$

Finally, using the definition of a dual, we can simplify the terms in parenthesis:

$$\begin{aligned}
\delta_V {}^\star \mathrm{d}\vartheta &= -(\delta V^j_\nu) V^\nu_a \partial_j \vartheta ({}^\star V^a) + \partial^a \vartheta^\star (V_a \wedge V_b) \wedge \delta V^b \\
&= -\partial_j \vartheta (\delta V^j_a)({}^\star V^a) + \partial^a \vartheta \, \delta V^b \wedge {}^\star (V_a \wedge V_b) \\
&= -\partial_j \vartheta^\star (\delta V^j_a V^a) + \partial^a \vartheta \, \delta V^b \wedge {}^\star (V_a \wedge V_b) \\
&= -\partial_j \vartheta^\star (\delta V^j) + \partial^a \vartheta \, \delta V^b \wedge {}^\star (V_a \wedge V_b)
\end{aligned}$$

Relabeling indices, we are left with

$$\delta_V {}^\star \mathrm{d}\vartheta = \partial^a \vartheta \, \delta V^b \wedge {}^\star (V_a \wedge V_b) - \partial_b \vartheta^\star (\delta V^b) \tag{8}$$

Plugging this back into eq. (6):

$$\begin{aligned}
\delta_V S_m &= \frac{1}{2} \int \mathrm{d}\vartheta \wedge \left(\partial^a \vartheta \, \delta V^b \wedge {}^\star (V_a \wedge V_b) - \partial_b \vartheta^\star (\delta V^b) \right) \\
&= \frac{1}{2} \int \partial^a \vartheta \mathrm{d}\vartheta \wedge \delta V^b \wedge {}^\star (V_a \wedge V_b) - \partial_b \vartheta \mathrm{d}\vartheta \wedge {}^\star (\delta V^b)
\end{aligned}$$

Using the identity $A \wedge {}^\star B = B \wedge {}^\star A$ when A and B have the same dimension in the second term:

$$\begin{aligned}
\delta_V S_m &= \frac{1}{2} \int \partial^a \vartheta \mathrm{d}\vartheta \wedge {}^\star (V_a \wedge V_b) \wedge \delta V^b - \partial_b \vartheta (\delta V^b) \wedge {}^\star \mathrm{d}\vartheta \\
&= \boxed{\frac{1}{2} \int \left(\partial^a \vartheta \mathrm{d}\vartheta \wedge {}^\star (V_a \wedge V_b) \wedge \delta V^b + \partial_b \vartheta^\star \mathrm{d}\vartheta \wedge \delta V^b \right)}
\end{aligned} \tag{9}$$

1.3 Nieh-Yan Form

$$\delta_V S_{NY} = -nf \int \mathrm{d}\vartheta \wedge (\delta T^a \wedge V_a + T^a \wedge \delta V_a)$$

Expanding:

$$\begin{aligned}
&\delta_V S_{NY} \\
&= -nf \int \mathrm{d}\vartheta \wedge ((\delta\, \mathrm{d}V^a + \omega^a_b \wedge \delta V^b) \wedge V_a + (\mathrm{d}V^a + \omega^a_b \wedge V^b) \wedge \delta V_a) \\
&= -nf \int \mathrm{d}\vartheta \wedge (\delta\, \mathrm{d}V^a \wedge V_a + \omega^a_b \wedge \delta V^b \wedge V_a + \mathrm{d}V^a \wedge \delta V_a + \omega^a_b \wedge V^b \wedge \delta V_a) \\
&= -nf \int \mathrm{d}\vartheta \wedge (\delta\, \mathrm{d}V^a \wedge V_a + \mathrm{d}V^a \wedge \delta V_a + \omega^a_b \wedge (\delta V^b \wedge V_a + V^b \wedge \delta V_a))
\end{aligned} \tag{10}$$

Focusing on the term with the spin connection:

$$\begin{aligned}
&\omega^a_b \wedge (\delta V^b \wedge V_a + V^b \wedge \delta V_a) \\
&= \omega^a_b \wedge \delta V^b \wedge V_a + \omega^a_b \wedge V^b \wedge \delta V_a \\
&= \omega^a_b \wedge \delta V^b \wedge \eta_{ac} V^c + \omega^a_b \wedge V^b \wedge \eta_{ac}\, \delta V^c \\
&= \eta_{ac}\omega^a_b \wedge \delta V^b \wedge V^c + \eta_{ac}\omega^a_b \wedge V^b \wedge \delta V^c \\
&= \omega_{bc} \wedge \delta V^b \wedge V^c + \omega_{bc} \wedge V^b \wedge \delta V^c
\end{aligned} \tag{11}$$

Focusing on the second term:

$$\begin{aligned}
&\omega_{bc} \wedge V^b \wedge \delta V^c \\
&= \omega_{cb} \wedge V^c \wedge \delta V^b \qquad \text{(swapped indices)} \\
&= -\omega_{cb} \wedge \delta V^b \wedge V^c \\
&= \omega_{bc} \wedge \delta V^b \wedge V^c \\
&\text{(using the antisymmetry of the spin connection)}
\end{aligned}$$

Plugging this back into eq. (11):

$$\omega_{bc} \wedge \delta V^b \wedge V^c + \omega_{bc} \wedge \delta V^b \wedge V^c = 2\omega_{bc} \wedge \delta V^b \wedge V^c$$

Plugging this back into eq.(10):

$$\delta_V S_{NY} = -nf \int \mathrm{d}\vartheta \wedge (\delta\, \mathrm{d}V^a \wedge V_a + \mathrm{d}V^a \wedge \delta V_a + 2\omega_{bc} \wedge \delta V^b \wedge V^c) \tag{12}$$

Now, focusing on $\delta\,\mathrm{d}V^a \wedge V_a + \mathrm{d}V^a \wedge \delta V_a$. We can use the Leibniz Rule:

$$
\begin{aligned}
\mathrm{d}(\delta V^a \wedge V_a) &= \mathrm{d}\,\delta V^a \wedge V_a + (-1)\,\delta V^a \wedge \mathrm{d}V_a \\
&= \mathrm{d}\,\delta V^a \wedge V_a - \delta V^a \wedge \mathrm{d}V_a \\
&= \mathrm{d}\,\delta V^a \wedge V_a - \mathrm{d}V_a \wedge \delta V^a \\
&= \mathrm{d}\,\delta V^a \wedge V_a - \mathrm{d}V^a \wedge \delta V_a \\
\mathrm{d}(\delta V^a \wedge V_a) + \mathrm{d}V^a \wedge \delta V_a &= \delta\,\mathrm{d}V^a \wedge V_a
\end{aligned}
$$

Plugging this into eq. (12):

$$
\begin{aligned}
&\delta_V S_{NY} \\
&= -nf \int \mathrm{d}\vartheta \wedge (\mathrm{d}(\delta V^a \wedge V_a) + \mathrm{d}V^a \wedge \delta V_a + \mathrm{d}V^a \wedge \delta V_a + 2\omega_{bc} \wedge \delta V^b \wedge V^c) \\
&= -nf \int \mathrm{d}\vartheta \wedge (\mathrm{d}(\delta V^a \wedge V_a) + 2\mathrm{d}V^a \wedge \delta V_a + 2\omega_{bc} \wedge \delta V^b \wedge V^c)
\end{aligned}
$$

The first term ends up being a boundary term due to, so we can drop the term. Therefore:

$$
\begin{aligned}
\delta_V S_{NY} &= -2nf \int \mathrm{d}\vartheta \wedge (\mathrm{d}V^a \wedge \delta V_a + \omega_{bc} \wedge \delta V^b \wedge V^c) \\
&= -2nf \int \mathrm{d}\vartheta \wedge \mathrm{d}V^a \wedge \delta V_a + \mathrm{d}\vartheta \wedge \omega_{bc} \wedge \delta V^b \wedge V^c \\
&= -2nf \int \mathrm{d}\vartheta \wedge \mathrm{d}V_a \wedge \delta V^a - \mathrm{d}\vartheta \wedge \omega_{bc} \wedge V^c \wedge \delta V^b \\
&= -2nf \int \mathrm{d}\vartheta \wedge \mathrm{d}V_a \wedge \delta V^a - \mathrm{d}\vartheta \wedge \omega_{cb} \wedge V^b \wedge \delta V^c \\
&= \boxed{-2nf \int \mathrm{d}\vartheta \wedge \mathrm{d}V_a \wedge \delta V^a + \mathrm{d}\vartheta \wedge \omega_{bc} \wedge V^b \wedge \delta V^c}
\end{aligned}
\tag{13}
$$

2 Spin Connection

2.1 Einstein-Cartan Action

$$\begin{aligned}
\delta_\omega S_{EC} &= -\frac{M_{Pl}^2}{4}\int \epsilon_{abcd} V^a \wedge V^b \wedge \delta R^{cd} \\
&= -\frac{M_{Pl}^2}{4}\int \epsilon_{abcd} V^a \wedge V^b \wedge (\mathrm{d}\,\delta\omega^{cd} + \delta\omega^c_f \wedge \omega^{fd} + \omega^c_f \wedge \delta\omega^{fd}) \\
&= -\frac{M_{Pl}^2}{4}\int V^a \wedge V^b \wedge (\epsilon_{abcd}\,\mathrm{d}\,\delta\omega^{cd} + \epsilon_{abcd}\,\delta\omega^c_f \wedge \omega^{fd} + \epsilon_{abcd}\omega^c_f \wedge \delta\omega^{fd})
\end{aligned} \tag{14}$$

Simplifying:

$$\begin{aligned}
&\epsilon_{abcd}\,\delta\omega^c_f \wedge \omega^{fd} + \epsilon_{abcd}\omega^c_f \wedge \delta\omega^{fd} \\
&\quad = \epsilon_{abcd}\,\delta\omega^c_f \wedge \eta^{fi}\omega^d_i + \epsilon_{abcd}\omega^c_f \wedge \eta^{fi}\,\delta\omega^d_i \\
&\quad = \epsilon_{abcd}\eta^{fi}\,\delta\omega^c_f \wedge \omega^d_i - \epsilon_{abcd}\eta^{fi}\,\delta\omega^d_i \wedge \omega^c_f \\
&\quad \text{(swapping indices } c \leftrightarrow d \text{ and } f \leftrightarrow i\text{)} \\
&\quad = \epsilon_{abcd}\eta^{fi}\,\delta\omega^c_f \wedge \omega^d_i - \epsilon_{abdc}\eta^{if}\,\delta\omega^c_f \wedge \omega^d_i \\
&\quad = \epsilon_{abcd}\eta^{fi}\,\delta\omega^c_f \wedge \omega^d_i + \epsilon_{abcd}\eta^{fi}\,\delta\omega^c_f \wedge \omega^d_i \\
&\quad = 2\epsilon_{abcd}\eta^{fi}\,\delta\omega^c_f \wedge \omega^d_i \\
&\quad = 2\epsilon_{abcd}\,\delta\omega^c_f \wedge \omega^{fd}
\end{aligned}$$

Plugging this into eq. (14):

$$\delta_\omega S_{EC} = -\frac{M_{Pl}^2}{4}\int V^a \wedge V^b \wedge (\epsilon_{abcd}\,\mathrm{d}\,\delta\omega^{cd} + 2\epsilon_{abcd}\,\delta\omega^c_f \wedge \omega^{fd})$$

We can drop the total derivative:

$$\begin{aligned}
\delta_\omega S_{EC} &= -\frac{M_{Pl}^2}{2}\int \epsilon_{abcd} V^a \wedge V^b \wedge \delta\omega^c_f \wedge \omega^{fd} \\
&= \boxed{\frac{M_{Pl}^2}{2}\int \epsilon_{abcd} V^a \wedge V^b \wedge \omega^{fd} \wedge \delta\omega^c_f}
\end{aligned} \tag{15}$$

2.2 Scalar Field Action

$\delta_\omega S_m$ is trivially 0.

2.3 Nieh-Yan Form

$$
\begin{aligned}
\delta_\omega S_{NY} &= -nf \int \mathrm{d}\vartheta \wedge \delta T^a \wedge V_a \\
&= -nf \int \mathrm{d}\vartheta \wedge (\delta\omega^a_b \wedge V^b) \wedge V_a \\
&= -nf \int \mathrm{d}\vartheta \wedge \delta\omega^a_b \wedge \eta^{bc} V_c \wedge V_a \\
&= -nf \int \mathrm{d}\vartheta \wedge \eta^{bc}\, \delta\omega^a_b \wedge V_c \wedge V_a \\
&= -nf \int \mathrm{d}\vartheta \wedge \delta\omega^{ac} \wedge V_c \wedge V_a
\end{aligned}
$$

Antisymmetrizing:

$$
\begin{aligned}
\delta_\omega S_{NY} &= -\frac{nf}{2} \int \mathrm{d}\vartheta \wedge \delta\omega^{ac} \wedge V_c \wedge V_a \\
&= \boxed{\frac{nf}{2} \int \mathrm{d}\vartheta \wedge V_a \wedge V_c \wedge \delta\omega^{ac}} \qquad (16)
\end{aligned}
$$

I am antisymmetrizing here but not when varying S_{EC}. This is because in eq. (15), the spin connections are being summed over with each other and the Levi-Civita Symbol, both of which are antisymmetric. In eq. (16), however, the spin connection is being summed over with two Vielbeins which are not antisymmetric, hence the $\frac{1}{2}$ factor must be added.

3 Scalar Field

3.1 Einstein-Cartan Action

$\delta_\vartheta S_{EC}$ is trivially 0.

3.2 Scalar Field Action

$$\begin{aligned}\delta_\vartheta S_m &= \frac{1}{2}\int (\delta\,\mathrm{d}\vartheta \wedge {}^\star\mathrm{d}\vartheta + \mathrm{d}\vartheta \wedge \delta^\star\mathrm{d}\vartheta) \\ &= \frac{1}{2}\int (\mathrm{d}(\delta\vartheta) \wedge {}^\star\mathrm{d}\vartheta + \mathrm{d}\vartheta \wedge {}^\star\mathrm{d}(\delta\vartheta)) \end{aligned} \tag{17}$$

We can simplify using the identity $A \wedge {}^\star B = B \wedge {}^\star A$ when A and B have the same dimension.

$$\begin{aligned}\mathrm{d}(\delta\vartheta) \wedge {}^\star\mathrm{d}\vartheta + \mathrm{d}\vartheta \wedge {}^\star\mathrm{d}(\delta\vartheta) &= \mathrm{d}(\delta\vartheta) \wedge {}^\star\mathrm{d}\vartheta + \mathrm{d}(\delta\vartheta) \wedge {}^\star\mathrm{d}\vartheta \\ &= 2\,\mathrm{d}(\delta\vartheta) \wedge {}^\star\mathrm{d}\vartheta\end{aligned}$$

Plugging this into eq. (17):

$$\delta_\vartheta S_m = \int \mathrm{d}(\delta\vartheta) \wedge {}^\star\mathrm{d}\vartheta \tag{18}$$

Using the Leibniz Rule:

$$\begin{aligned}\mathrm{d}(\delta\vartheta \wedge {}^\star\mathrm{d}\vartheta) &= \mathrm{d}(\delta\vartheta) \wedge {}^\star\mathrm{d}\vartheta + \delta\vartheta \wedge \mathrm{d}^\star\mathrm{d}\vartheta \\ &= \mathrm{d}(\delta\vartheta) \wedge {}^\star\mathrm{d}\vartheta + \delta\vartheta\,\mathrm{d}^\star\mathrm{d}\vartheta\end{aligned}$$

Since $\delta\vartheta$ is a 0-form, a wedge product with it is the same as regular multiplication. So:

$$\mathrm{d}(\delta\vartheta) \wedge {}^\star\mathrm{d}\vartheta = \mathrm{d}(\delta\vartheta \wedge {}^\star\mathrm{d}\vartheta) - \delta\vartheta\,\mathrm{d}^\star\mathrm{d}\vartheta$$

Plugging this into eq. (18) and dropping the total derivative:

$$\delta_\vartheta S_m = \boxed{-\int \delta\vartheta\,\mathrm{d}^\star\mathrm{d}\vartheta} \tag{19}$$

3.3 Nieh-Yan Form

$$\delta_\vartheta S_{NY} = -nf\int \mathrm{d}(\delta\vartheta) \wedge T^a \wedge V_a \tag{20}$$

Using the Leibniz Rule:

$$\begin{aligned}\mathrm{d}(\delta\vartheta \wedge T^a \wedge V_a) &= \mathrm{d}\,\delta\vartheta \wedge T^a \wedge V_a + \delta\vartheta \wedge \mathrm{d}(T^a \wedge V_a) \\ \mathrm{d}(\delta\vartheta \wedge T^a \wedge V_a) - \delta\vartheta \wedge \mathrm{d}(T^a \wedge V_a) &= \mathrm{d}\,\delta\vartheta \wedge T^a \wedge V_a\end{aligned}$$

Plugging this into eq. (20) and dropping the total derivative:

$$\begin{aligned} \delta_\vartheta S_{NY} &= -nf \int - \delta\vartheta \wedge \mathrm{d}(T^a \wedge V_a) \\ &= nf \int \delta\vartheta \wedge \mathrm{d}(T^a \wedge V_a) \\ &= \boxed{nf \int \mathrm{d}(T^a \wedge V_a) \wedge \delta\vartheta} \end{aligned} \tag{21}$$