Vielbein of the Schwarzchild Metric

7/22/2025

The Schwarzschild Metric using natural units (G = 1 and c = 1):

$$ds^{2} = -\left(1 - \frac{2M}{r}\right)dt^{2} + \left(1 - \frac{2M}{r}\right)^{-1}dr^{2} + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta d\phi^{2}$$

This gives the following metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0\\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0\\ 0 & 0 & r^2 & 0\\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

and it's inverse

$$g^{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 & 0\\ 0 & 1 - \frac{2M}{r} & 0 & 0\\ 0 & 0 & \frac{1}{r^2} & 0\\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

Using this, I can calculate the forms of the Vielbein $e^a_\mu,\,e^\mu_a,\,$ and $e^{a\mu}.$

Sanity check: The Vielbein is simply a set of basis vectors for each point of the manifold that express our metric. It will satisfy the following equation:

$$g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu}$$
 $a, b = 0, 1, 2, 3$ $\mu, \nu = t, r, \theta, \phi$

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ and is the Minkowski metric. Since the metric is diagonal, the equation can be simplified by saying a = b. I will also constrain the Vielbein to being a diagonal matrix, so $e^a_{\mu} = \text{diag}(e^0_t, e^1_r, e^2_{\theta}, e^3_{\phi})$, which allows me to simplify further by saying $\mu = \nu$. This also gives a correspondance between local and Schwarzschild coordinates, where a = 0, 1, 2, 3 correspond to $\mu = t, r, \theta, \phi$ respectively in the Vielbein. Anything else is 0.

So, I am left with the equation:

$$g_{\mu\mu} = \eta_{aa} (e^a_\mu)^2$$

Solving this explicitly:

$$g_{tt} = -\left(1 - \frac{2M}{r}\right) = -(e_t^0)^2 \qquad \to e_t^0 = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}$$

$$g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} = (e_r^1)^2 \qquad \to e_r^1 = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$$

$$g_{\theta\theta} = r^2 = (e_{\theta}^2)^2 \qquad \to e_{\theta}^2 = r$$

$$g_{\phi\phi} = r^2 \sin^2 \theta = (e_{\phi}^3)^2 \qquad \to e_{\phi}^3 = r \sin \theta$$

So, the Vielbein:

$$e_{\mu}^{a} = \begin{pmatrix} \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} & 0 & 0 & 0\\ 0 & \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} & 0 & 0\\ 0 & 0 & r & 0\\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}$$

$$e_a^{\mu} = g^{\mu\nu} \eta_{ab} e_{\nu}^b = \begin{pmatrix} \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} & 0 & 0 & 0\\ 0 & \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} & 0 & 0\\ 0 & 0 & \frac{1}{r} & 0\\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}$$

$$e^{a\mu} = g^{\mu\nu}e^a_\nu = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} & 0 & 0 & 0\\ 0 & \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} & 0 & 0\\ 0 & 0 & \frac{1}{r} & 0\\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}$$