Vielbein of the Schwarzchild Metric

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The Schwarzchild Metric using natural units ($G = 1$ and $c = 1$):

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$$

This gives the following metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and it's inverse

$$g^{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & 1 - \frac{2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

Using this, I can calculate the forms of the Vielbein e^a_μ, e^μ_a, and $e^{a\mu}$.

Sanity check: The Vielbein is simply a set of basis vectors for each point of the manifold that express our metric. It will satisfy the following equation:

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu \qquad a, b = 0, 1, 2, 3 \quad \mu, \nu = t, r, \theta, \varphi$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ and is the Minkowski metric. Since the metric is diagonal, the equation can be simplified by saying $a = b$. I will also constrain the Vielbein to being a diagonal matrix, so $e^a_\mu = \mathrm{diag}(e^0_t, e^1_r, e^2_\theta, e^3_\varphi)$, which allows me to simplify further by saying $\mu = \nu$. This also gives a correspondance between local and Schwarzchild coordinates, where $a = 0, 1, 2, 3$ correspond to $\mu = t, r, \theta, \varphi$ respectively in the Vielbein. Anything else is 0.

So, I am left with the equation:

$$g_{\mu\mu} = \eta_{aa}(e^a_\mu)^2$$

Solving this explicitly:

$$
\begin{aligned}
g_{tt} &= -\left(1-\frac{2M}{r}\right) = -(e^0_t)^2 && \rightarrow \quad e^0_t = \left(1-\frac{2M}{r}\right)^{\frac{1}{2}} \\
g_{rr} &= \left(1-\frac{2M}{r}\right)^{-1} = (e^1_r)^2 && \rightarrow \quad e^1_r = \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} \\
g_{\theta\theta} &= r^2 = (e^2_\theta)^2 && \rightarrow \quad e^2_\theta = r \\
g_{\varphi\varphi} &= r^2\sin^2\theta = (e^3_\varphi)^2 && \rightarrow \quad e^3_\varphi = r\sin\theta
\end{aligned}
$$

So, the Vielbein:

$$
e^a_\mu = \begin{pmatrix} \left(1-\frac{2M}{r}\right)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}
$$

$$
e^\mu_a = g^{\mu\nu}\eta_{ab}e^b_\nu = \begin{pmatrix} \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}
$$

$$
e^{a\mu} = g^{\mu\nu}e^a_\nu = \begin{pmatrix} -\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}
$$