

Equations of Motion from the Action

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From my previous notes, I had calculated the Variations on the total action:

$$\begin{aligned}\delta_V S_{tot} &= \frac{M_{Pl}^2}{2} \int R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a \\ &\quad + \frac{1}{2} \int (\partial^a \vartheta d\vartheta \wedge \star(V_a \wedge V_b) \wedge \delta V^b + \partial_b \vartheta^\star d\vartheta \wedge \delta V^b) \\ &\quad - 2nf \int d\vartheta \wedge dV_a \wedge \delta V^a + d\vartheta \wedge \omega_{bc} \wedge V^b \wedge \delta V^c\end{aligned}\quad (1)$$

$$\delta_\omega S_{tot} = \frac{M_{Pl}^2}{2} \int \epsilon_{abcd} V^a \wedge V^b \wedge \omega^{fd} \wedge \delta \omega_f^c + \frac{nf}{2} \int d\vartheta \wedge V_a \wedge V_c \wedge \delta \omega^{ac} \quad (2)$$

$$\delta_\vartheta S_{tot} = - \int \delta \vartheta d^\star d\vartheta + nf \int d(T^a \wedge V_a) \wedge \delta \vartheta \quad (3)$$

1 Vielbein

$$\begin{aligned}\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a + \frac{1}{2} (\partial^a \vartheta d\vartheta \wedge \star(V_a \wedge V_b) \wedge \delta V^b + \partial_b \vartheta^\star d\vartheta \wedge \delta V^b) \\ - 2nf (d\vartheta \wedge dV_a \wedge \delta V^a + d\vartheta \wedge \omega_{bc} \wedge V^b \wedge \delta V^c) = 0\end{aligned}$$

Relabeling indices:

$$\begin{aligned}\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a + \frac{1}{2} (\partial^b \vartheta d\vartheta \wedge \star(V_b \wedge V_a) \wedge \delta V^a + \partial_a \vartheta^\star d\vartheta \wedge \delta V^a) \\ - 2nf (d\vartheta \wedge dV_a \wedge \delta V^a + d\vartheta \wedge \omega_{ba} \wedge V^b \wedge \delta V^a) = 0 \\ \int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a + \frac{1}{2} (\partial^b \vartheta d\vartheta \wedge \star(V_b \wedge V_a) + \partial_a \vartheta^\star d\vartheta) \wedge \delta V^a \\ - 2nf (d\vartheta \wedge dV_a + d\vartheta \wedge \omega_{ba} \wedge V^b) \wedge \delta V^a = 0 \\ \int (\frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} + \frac{1}{2} (\partial^b \vartheta d\vartheta \wedge \star(V_b \wedge V_a) + \partial_a \vartheta^\star d\vartheta) \\ - 2nf (d\vartheta \wedge dV_a + d\vartheta \wedge \omega_{ba} \wedge V^b)) \wedge \delta V^a = 0\end{aligned}$$

Since δV^a is arbitrary:

$$\begin{aligned} \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} + \frac{1}{2} (\partial^b \vartheta d\vartheta \wedge \star (V_b \wedge V_a) + \partial_a \vartheta \star d\vartheta) \\ - 2nf (d\vartheta \wedge dV_a + d\vartheta \wedge \omega_{ba} \wedge V^b) = 0 \end{aligned} \quad (4)$$

so, the Equation of Motion:

$$\begin{aligned} \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} - 2nf (d\vartheta \wedge dV_a + d\vartheta \wedge \omega_{ba} \wedge V^b) \\ = -\frac{1}{2} (\partial^b \vartheta d\vartheta \wedge \star (V_b \wedge V_a) + \partial_a \vartheta \star d\vartheta) \end{aligned} \quad (5)$$

2 Spin Connection

$$\begin{aligned} \int \frac{M_{Pl}^2}{2} \epsilon_{abcd} V^a \wedge V^b \wedge \omega_f^d \wedge \delta \omega_f^c + \int \frac{nf}{2} d\vartheta \wedge V_a \wedge V_c \wedge \delta \omega^{ac} = 0 \\ \int \frac{M_{Pl}^2}{2} \epsilon_{abcd} V^a \wedge V^b \wedge \omega_f^d \wedge \delta \omega^{fc} + \int \frac{nf}{2} d\vartheta \wedge V_a \wedge V_c \wedge \delta \omega^{ac} = 0 \end{aligned}$$

Relabeling indices:

$$\begin{aligned} \int \frac{M_{Pl}^2}{2} \epsilon_{abcd} V^a \wedge V^b \wedge \omega_f^d \wedge \delta \omega^{fc} + \int \frac{nf}{2} d\vartheta \wedge V_f \wedge V_c \wedge \delta \omega^{fc} = 0 \\ \int \left(\frac{M_{Pl}^2}{2} \epsilon_{abcd} V^a \wedge V^b \wedge \omega_f^d + \frac{nf}{2} d\vartheta \wedge V_f \wedge V_c \right) \wedge \delta \omega^{fc} = 0 \end{aligned}$$

So,

$$\frac{M_{Pl}^2}{2} \epsilon_{abcd} V^a \wedge V^b \wedge \omega_f^d + \frac{nf}{2} d\vartheta \wedge V_f \wedge V_c = 0$$

3 Scalar Field

$$- \int \delta \vartheta d\star d\vartheta + nf \int d(T^a \wedge V_a) \wedge \delta \vartheta = 0$$

Simplifying:

$$\int \delta\vartheta (-d^*\mathrm{d}\vartheta + nf\,d(T^a \wedge V_a)) = 0$$

This implies:

$$\begin{aligned} -d^*\mathrm{d}\vartheta + nf\,d(T^a \wedge V_a) &= 0 \\ nf\,d(T^a \wedge V_a) &= d^*\mathrm{d}\vartheta \\ nf\,d(V^a \wedge T_a) &= d^*\mathrm{d}\vartheta \end{aligned}$$

From section 2 of Pollari (in literature):

$$\frac{nf}{2} \partial_\mu (V_\nu^a T_{\rho\sigma a}) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = d^*\mathrm{d}\vartheta$$

Writing this in terms of the D'Alembert operator, which can be written as $\square\vartheta = \delta\vartheta = *d^*\mathrm{d}\vartheta$ since ϑ is a 0-form. Therefore: $d^*\mathrm{d}\vartheta = \square\vartheta^*1$. The equation above is written as:

$$\begin{aligned} \square\vartheta &= \frac{nf}{2} \partial_\mu (V_\nu^a T_{\rho\sigma a})^* (dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma) \\ \square\vartheta &= \frac{nf}{2} \partial_\mu (V_\nu^a T_{\rho\sigma a}) \sqrt{|g|} \varepsilon^{\mu\nu\rho\sigma} \end{aligned}$$

Writing the D'Alembertian in coordinate form:

$$\begin{aligned} \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \vartheta \right) &= \frac{nf}{2} \partial_\mu (V_\nu^a T_{\rho\sigma a}) \sqrt{|g|} \varepsilon^{\mu\nu\rho\sigma} \\ \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \vartheta \right) &= \frac{nf}{2} \partial_\mu (T_{\rho\sigma}^a V_{a\nu}) \sqrt{|g|} \varepsilon^{\mu\nu\rho\sigma} \\ \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \vartheta \right) &= \frac{nf}{2} \partial_\mu (T_{\nu\rho}^a V_{a\sigma}) \sqrt{|g|} \varepsilon^{\mu\nu\rho\sigma} \end{aligned}$$