

Deriving the Metric Functions

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Initial System

The mathematica notebook provides us with the following equations of motion:

$$e^\alpha(1 + r\alpha_r) - e^\beta(e^\alpha + 3n^2f^2r^2\vartheta_{0\tau}^2) = \frac{1}{2}r^2e^\beta\vartheta_{0\tau}^2 \quad (1)$$

$$e^\alpha(1 - r\beta_r) - e^\beta(e^\alpha - 3n^2f^2r^2\vartheta_{0\tau}^2) = -\frac{1}{2}r^2e^\beta\vartheta_{0\tau}^2 \quad (2)$$

$$e^\alpha((2 + r\alpha_r)(\alpha_r - \beta_r) + 2r\alpha_{rr}) - 12e^\beta rn^2f^2\vartheta_{0\tau}^2 = 2re^\beta\vartheta_{0\tau}^2 \quad (3)$$

$$\beta_\tau = \alpha_\tau = \vartheta_{0r} = 0 \quad (4)$$

So, we can say $\alpha = \alpha(r)$, $\beta = \beta(r)$, $\vartheta_0 = \vartheta_0(\tau)$.

Adding (1) and (2) gives:

$$\alpha_r - \beta_r = \frac{2}{r}(e^\beta - 1) \quad (5)$$

and subtracting (1) and (2) gives:

$$\alpha_r + \beta_r = re^{-\alpha+\beta}\vartheta_{0\tau}^2(1 + 6n^2f^2) \quad (6)$$

Assume $u(r) = e^{-\alpha+\beta}$ and $K(\tau) = \vartheta_{0\tau}^2(1 + 6n^2f^2)$:

$$-\frac{u_r}{u} = \frac{2}{r}(e^\beta - 1) \quad (7)$$

and

$$\alpha_r + \beta_r = ruK \quad (8)$$

Getting u_r from (9) and β_r from (5) and (8), we can see the system is non-linearly coupled:

$$u_r = -\frac{2u}{r}(e^\beta - 1) \quad (9)$$

$$\beta_r = \frac{1}{2} \left(ruK - \frac{2}{r}(e^\beta - 1) \right) \quad (10)$$

Differentiating (10):

$$\begin{aligned}\beta_{rr} &= \frac{K}{2}(u + ru_r) - \frac{1}{2}(-\frac{2}{r^2}(e^\beta - 1) - \frac{1}{r}(\beta_r e^\beta)) \\ &= \frac{K(u + ru_r)}{2} + \frac{e^\beta - 1}{r^2} - \frac{\beta_r e^\beta}{r}\end{aligned}$$

Substituting u_r from (9):

$$\beta_{rr} = \frac{Ku(3 - 2e^\beta)}{2} + \frac{e^\beta - 1}{r^2} - \frac{\beta_r e^\beta}{r}$$

From (10), we see $Ku = \frac{2\beta_r}{r} + \frac{2(e^\beta - 1)}{r^2}$:

$$\begin{aligned}\beta_{rr} &= \frac{\beta_r(3 - 2e^\beta)}{r} + \frac{(e^\beta - 1)(3 - 2e^\beta)}{r^2} + \frac{e^\beta - 1}{r^2} - \frac{\beta_r e^\beta}{r} \\ &= -\beta_r \frac{3(e^\beta - 1)}{r} + \frac{-2e^{2\beta} + 6e^\beta - 4}{r^2}\end{aligned}\tag{11}$$

Again, this ODE is unsolvable. Sympy was unable to find a closed-form solution. This only becomes solvable assuming a constant β , or in a torsionless case.

The python file attempts to numerically solve the system posed in (9) and (10) with assumed initial conditions. Observations are written in comments in the code.

Solving with Ansatz

We define an ansatz:

$$e^\alpha = a = 1 - \frac{C}{r} \quad \alpha_r = a^{-1} \frac{C}{r^2}\tag{12}$$

Evaluating (1) with the ansatz:

$$\begin{aligned}
a(1 + \frac{C}{r}a^{-1}) &= e^\beta(\frac{1}{2}r^2\vartheta_\tau^2 + a + 3n^2f^2r^2\vartheta_\tau^2) \\
a + \frac{C}{r} &= e^\beta(\frac{1}{2}r^2\vartheta_\tau^2 + a + 3n^2f^2r^2\vartheta_\tau^2) \\
e^\beta &= (\frac{1}{2}r^2\vartheta_\tau^2 + a + 3n^2f^2r^2\vartheta_\tau^2)^{-1} \\
e^\beta &= (a + r^2\vartheta_\tau^2(\frac{1}{2} + 3n^2f^2))^{-1} \\
e^\beta &= \frac{1}{a} \left(1 + \frac{r^2\vartheta_\tau^2(\frac{1}{2} + 3n^2f^2)}{a} \right)^{-1}
\end{aligned} \tag{13}$$

Klein-Gordon Equation

We also have the Klein-Gordon equation:

$$\partial_\mu (\sqrt{-g}g^{\mu\nu}\partial_\nu\vartheta) = \frac{nf}{2}\partial_\mu (T_{\nu\rho}^a V_{a\sigma}) \varepsilon^{\mu\nu\rho\sigma}$$

For this, mathematica gives us:

$$\begin{aligned}
&e^{\frac{3}{2}\alpha}(4 + r\alpha_r - r\beta_r)(\vartheta_{0r}) + 2e^{\frac{3}{2}\alpha}r\vartheta_{0rr} \\
&+ e^{\frac{1}{2}\beta}(e^{\frac{1}{2}(\alpha+\beta)} + 6e^{\frac{1}{2}(\alpha+\beta)}n^2f^2)r((\alpha_\tau - \beta_\tau)(\vartheta_{0\tau}) - 2\vartheta_{0\tau\tau}) = 0 \\
&e^{\frac{1}{2}\beta}(e^{\frac{1}{2}(\alpha+\beta)} + 6e^{\frac{1}{2}(\alpha+\beta)}n^2f^2)r(-2\vartheta_{0\tau\tau}) = 0 \\
&- e^{\frac{1}{2}\alpha+\beta}(1 + 6n^2f^2)r(2\vartheta_{0\tau\tau}) = 0 \\
&- 2re^{\frac{1}{2}\alpha+\beta}\vartheta_{0\tau\tau}(1 + 6n^2f^2) = 0
\end{aligned} \tag{14}$$

Torsionless Case

In the torsionless case, $\vartheta_0 = 0$, so in turn $K(\tau) = 0$. Using (5) and (6):

$$\alpha_r - \beta_r = \frac{2}{r}(e^\beta - 1) \tag{15}$$

$$\alpha_r = -\beta_r \tag{16}$$

so,

$$\begin{aligned}
-2 \frac{d\beta}{dr} &= \frac{2}{r}(e^\beta - 1) \\
-\int \frac{1}{e^\beta - 1} d\beta &= \int \frac{1}{r} dr \\
-(\ln |e^\beta - 1| - \ln |e^\beta|) &= \ln |r| + C \\
-\ln \left| \frac{e^\beta - 1}{e^\beta} \right| &= \ln |r| + C \\
\ln |1 - e^{-\beta}| &= -\ln |r| + C \\
1 - e^{-\beta} &= Cr^{-1} \\
e^{-\beta} &= 1 - Cr^{-1} \\
e^\beta &= \left(1 - \frac{C}{r}\right)^{-1}
\end{aligned} \tag{17}$$

Integrating (15),

$$\begin{aligned}
\alpha &= -\beta + D \\
\alpha &= -\beta \text{ (with a coordinate redefinition)} \\
e^\alpha &= e^{-\beta} \\
e^\alpha &= \left(1 - \frac{C}{r}\right)
\end{aligned} \tag{18}$$

Plugging this into the General Schwarzschild metric:

$$ds^2 = - \left(1 - \frac{C}{r}\right) dt^2 + \left(1 - \frac{C}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \tag{19}$$

Expecting the Newtonian limit at large r , we find $C = 2M$ when $G = 1$, giving us the Schwarzschild metric.