Deriving the Metric Functions

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Initial System

The mathematica notebook provides us with the following equations of motion:

$$e^{\alpha}(1 + r\alpha_r) - e^{\beta}(e^{\alpha} + 3n^2 f^2 r^2 \vartheta_{0\tau}^2) = \frac{1}{2} r^2 e^{\beta} \vartheta_{0\tau}^2 \tag{1}$$

$$e^{\alpha}(1 - r\beta_r) - e^{\beta}(e^{\alpha} - 3n^2 f^2 r^2 \vartheta_{0\tau}^2) = -\frac{1}{2} r^2 e^{\beta} \vartheta_{0\tau}^2 \tag{2}$$

$$e^{\alpha}\left((2 + r\alpha_r)(\alpha_r - \beta_r) + 2r\alpha_{rr}\right) - 12e^{\beta} r n^2 f^2 \vartheta_{0\tau}^2 = 2re^{\beta}\vartheta_{0\tau}^2 \tag{3}$$

$$\beta_\tau = \alpha_\tau = \vartheta_{0r} = 0 \tag{4}$$

So, we can say $\alpha = \alpha(r)$, $\beta = \beta(r)$, $\vartheta_0 = \vartheta_0(\tau)$.

Adding (1) and (2) gives:

$$\alpha_r - \beta_r = \frac{2}{r}(e^{\beta} - 1) \tag{5}$$

and subtracting (1) and (2) gives:

$$\alpha_r + \beta_r = re^{-\alpha+\beta}\vartheta_{0\tau}^2(1 + 6n^2 f^2) \tag{6}$$

Assume $u(r) = e^{-\alpha+\beta}$ and $K(\tau) = \vartheta_{0\tau}^2(1 + 6n^2 f^2)$:

$$-\frac{u_r}{u} = \frac{2}{r}(e^{\beta} - 1) \tag{7}$$

and

$$\alpha_r + \beta_r = ruK \tag{8}$$

Getting u_r from (9) and β_r from (5) and (8), we can see the system is non-linearly coupled:

$$u_r = -\frac{2u}{r}(e^{\beta} - 1) \tag{9}$$

$$\beta_r = \frac{1}{2}\left(ruK - \frac{2}{r}(e^{\beta} - 1)\right) \tag{10}$$

Torsionless Case

In the torsionless case, $\vartheta_0 = 0$, so in turn $K(\tau) = 0$. Using (5) and (6):

$$\alpha_r - \beta_r = \frac{2}{r}(e^\beta - 1) \tag{11}$$

$$\alpha_r = -\beta_r \tag{12}$$

so,

$$\begin{aligned}
-2\frac{\mathrm{d}\beta}{\mathrm{d}r} &= \frac{2}{r}(e^\beta - 1) \\
-\int \frac{1}{e^\beta - 1}\,\mathrm{d}\beta &= \int \frac{1}{r}\,\mathrm{d}r \\
-(\ln|e^\beta - 1| - \ln|e^\beta|) &= \ln|r| + C \\
-\ln|\frac{e^\beta - 1}{e^\beta}| &= \ln|r| + C \\
\ln|1 - e^{-\beta}| &= -\ln|r| + C \\
1 - e^{-\beta} &= Cr^{-1} \\
e^{-\beta} &= 1 - Cr^{-1}
\end{aligned}$$

$$e^\beta = \left(1 - \frac{C}{r}\right)^{-1} \tag{13}$$

Integrating (12),

$$\alpha = -\beta + D \tag{14}$$

$$\alpha = -\beta \text{ (with a coordinate redefinition)} \tag{15}$$

$$e^\alpha = e^{-\beta} \tag{16}$$

$$e^\alpha = \left(1 - \frac{C}{r}\right) \tag{17}$$

Plugging this into the General Schwarzchild metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{C}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{C}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2 \tag{18}$$

Expecting the Newtonian limit at large r, we find $C = 2M$ when $G = 1$, giving us the Schwarzchild metric.