Vielbein of the General Schwarzchild Metric

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The General Schwarzchild Metric:

$$\mathrm{d}s^2 = -e^{\nu(t,r)}\,\mathrm{d}t^2 + e^{\lambda(t,r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2 \tag{1}$$

Note the the trace of the Minkowski metric is $(-,+,+,+)$. This gives the following Metric Tensor and its inverse:

$$g_{\mu\nu} = \begin{pmatrix} -e^{\nu(t,r)} & 0 & 0 & 0 \\ 0 & e^{\lambda(t,r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2}$$

$$g^{\mu\nu} = \begin{pmatrix} -e^{-\nu(t,r)} & 0 & 0 & 0 \\ 0 & e^{-\lambda(t,r)} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{3}$$

Calculating the Vielbeins:

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu \tag{4}$$

Since the metric is diagonal, the equation can be simplified by saying $a = b$. I will also constrain the Vielbein to being a diagonal matrix, which allows me to simplify further by saying $\mu = \nu$. This also gives a correspondance between local and Schwarzchild coordinates, where $a = 0, 1, 2, 3$ correspond to $\mu = t, r, \theta, \varphi$ respectively in the Vielbein. Anything else is 0.

So I am left with:

$$g_{\mu\mu} = \eta_{aa}(e^a_\mu)^2 \tag{5}$$

Solving this explicitly:

$$\begin{aligned} g_{tt} &= (-1)(e^0_t)^2 = -e^{\nu(t,r)} & \qquad e^0_t &= e^{\frac{\nu(t,r)}{2}} \\ g_{rr} &= (e^1_r)^2 = e^{\lambda(t,r)} & e^1_r &= e^{\frac{\lambda(t,r)}{2}} \\ g_{\theta\theta} &= (e^2_\theta)^2 = r^2 & e^2_\theta &= r \\ g_{\varphi\varphi} &= (e^3_\varphi)^2 = r^2\sin^2\theta & e^3_\varphi &= r\sin\theta \end{aligned}$$

Therefore, the Vielbeins:

$$e^a_\mu = \begin{pmatrix} e^{\frac{\nu(t,r)}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\lambda(t,r)}{2}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}$$

$$e^\mu_a = g^{\mu\nu}\eta_{ab}e^b_\nu = \begin{pmatrix} e^{\frac{-\nu(t,r)}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{-\lambda(t,r)}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}$$

$$e^{a\mu} = g^{\mu\nu}e^a_\nu = \begin{pmatrix} -e^{\frac{-\nu(t,r)}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{-\lambda(t,r)}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}$$