Deriving the Metric Functions

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Initial System

The mathematica notebook provides us with the following equations of motion:

$$e^{\alpha}(1+r\alpha_r) - e^{\beta}(e^{\alpha} + 3n^2 f^2 r^2 \vartheta_{0\tau}^2) = \frac{1}{2} r^2 e^{\beta} \vartheta_{0\tau}^2 \tag{1}$$

$$e^{\alpha}(1-r\beta_r) - e^{\beta}(e^{\alpha} - 3n^2 f^2 r^2 \vartheta_{0\tau}^2) = -\frac{1}{2} r^2 e^{\beta} \vartheta_{0\tau}^2 \tag{2}$$

$$e^{\alpha}\left((2+r\alpha_r)(\alpha_r - \beta_r) + 2r\alpha_{rr}\right) - 12e^{\beta} r n^2 f^2 \vartheta_{0\tau}^2 = 2re^{\beta}\vartheta_{0\tau}^2 \tag{3}$$

$$\beta_\tau = \alpha_\tau = \vartheta_{0r} = 0 \tag{4}$$

So, we can say $\alpha = \alpha(r)$, $\beta = \beta(r)$, $\vartheta_0 = \vartheta_0(\tau)$.

Adding (1) and (2) gives:

$$\alpha_r - \beta_r = \frac{2}{r}(e^{\beta} - 1) \tag{5}$$

and subtracting (1) and (2) gives:

$$\alpha_r + \beta_r = re^{-\alpha+\beta}\vartheta_{0\tau}^2(1 + 6n^2 f^2) \tag{6}$$

Assume $u(r) = e^{-\alpha+\beta}$ and $K(\tau) = \vartheta_{0\tau}^2(1+6n^2f^2)$:

$$-\frac{u_r}{u} = \frac{2}{r}(e^{\beta} - 1) \tag{7}$$

and

$$\alpha_r + \beta_r = ruK \tag{8}$$

Getting u_r from (9) and β_r from (5) and (8), we can see the system is nonlinearly coupled:

$$u_r = -\frac{2u}{r}(e^{\beta} - 1) \tag{9}$$

$$\beta_r = \frac{1}{2}\left(ruK - \frac{2}{r}(e^{\beta} - 1)\right) \tag{10}$$

Differentiating (10):

$$\begin{aligned}\beta_{rr} &= \frac{K}{2}(u + ru_r) - \frac{1}{2}(-\frac{2}{r^2}(e^\beta - 1) - \frac{1}{r}(\beta_r e^\beta)) \\ &= \frac{K(u + ru_r)}{2} + \frac{e^\beta - 1}{r^2} - \frac{\beta_r e^\beta}{r}\end{aligned}$$

Substituting u_r from (9):

$$\beta_{rr} = \frac{Ku(3 - 2e^\beta)}{2} + \frac{e^\beta - 1}{r^2} - \frac{\beta_r e^\beta}{r}$$

From (10), we see $Ku = \frac{2\beta_r}{r} + \frac{2(e^\beta - 1)}{r^2}$:

$$\begin{aligned}\beta_{rr} &= \frac{\beta_r(3 - 2e^\beta)}{r} + \frac{(e^\beta - 1)(3 - 2e^\beta)}{r^2} + \frac{e^\beta - 1}{r^2} - \frac{\beta_r e^\beta}{r} \\ &= -\beta_r \frac{3(e^\beta - 1)}{r} + \frac{-2e^{2\beta} + 6e^\beta - 4}{r^2} \qquad (11)\end{aligned}$$

Again, this ODE is unsolvable. Sympy was unable to find a closed-form solution. This only becomes solvable assuming a constant β, or in a torsionless case.

The python file attempts to numerically solve the system posed in (9) and (10) with assumed initial conditions. Observations are written in comments in the code.

Solving with Ansatz

We can define an ansatz $e^\alpha = 1 - \frac{C}{r}$ where C is some constant. α_r is then $\left(1 - \frac{C}{r}\right)^{-1} \frac{C}{r^2} = \frac{C}{r(r-C)}$. With this, (5) can be rewritten:

$$\begin{aligned}\frac{C}{r(r - C)} - \beta_r &= \frac{2}{r}(e^\beta - 1) \\ \beta_r &= \frac{C}{r(r - C)} - \frac{2}{r}(e^\beta - 1)\end{aligned}$$

Let $y(r) = e^{\beta}$ and $\beta_r = \frac{y'}{y}$:

$$\begin{aligned}
\frac{y'}{y} &= \frac{C}{r(r-C)} - \frac{2}{r}(y-1) \\
y' &= -\frac{2}{r}y^2 + \left(\frac{C}{r(r-C)} + \frac{2}{r}\right) y \\
y' &= -\frac{2}{r}y^2 + \frac{2r-C}{r(r-C)}y \\
\frac{y'}{y^2} &= -\frac{2}{r} + \frac{2r-C}{r(r-C)}\frac{1}{y}
\end{aligned}$$

Let $u(r) = \frac{1}{y} = e^{-\beta}$. Then, $u'(r) = -\frac{y'}{y^2}$:

$$\begin{aligned}
-u' &= -\frac{2}{r} + \frac{2r-C}{r(r-C)}u \\
u' + \frac{2r-C}{r(r-C)}u &= \frac{2}{r}
\end{aligned}$$

This ODE can be solved using the integrating factor method. First, set our integrating factor:

$$\begin{aligned}
\mu(r) &= \exp\left(\int \frac{2r-C}{r(r-C)}dr\right) \\
&= \exp(\ln r + \ln|r-C|) \\
&= r(r-C)
\end{aligned}$$

Using the integrating factor, the ODE can be written as:

$$\begin{aligned}
(\mu(r)u)' &= \frac{2}{r}\mu(r) \\
(\mu(r)u)' &= 2(r-C) \\
\mu(r)u &= \int 2(r-C)dr \\
ur(r-C) &= 2\left(\frac{1}{2}r^2 - Cr + D\right) \\
u &= \frac{r^2 - 2Cr + D}{r(r-C)}
\end{aligned}$$

Remembering that $u = e^{-\beta}$:

$$e^{\beta} = \frac{r(r-C)}{r^2 - 2Cr + D} \tag{12}$$

This gives us equations for e^{α} and e^{β}:

$$e^{\alpha} = 1 - \frac{C}{r} \qquad e^{\beta} = \frac{r(r-C)}{r^2 - 2Cr + D}$$

Klein-Gordon Equation

We also have the Klein-Gordon equation:

$$\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \vartheta\right) = \frac{nf}{2} \partial_\mu \left(T^a_{\nu\rho} V_{a\sigma}\right) \varepsilon^{\mu\nu\rho\sigma}$$

For this, mathematica gives us:

$$\begin{aligned} &e^{\frac{3}{2}\alpha}(4 + r\alpha_r - r\beta_r)(\vartheta_{0r}) + 2e^{\frac{3}{2}\alpha} r\vartheta_{0rr} \\ &\quad + e^{\frac{1}{2}\beta}(e^{\frac{1}{2}(\alpha+\beta)} + 6e^{\frac{1}{2}(\alpha+\beta)} n^2 f^2) r((\alpha_\tau - \beta_\tau)(\vartheta_{0\tau}) - 2\vartheta_{0\tau\tau}) = 0 \end{aligned}$$

$$\begin{aligned} &e^{\frac{1}{2}\beta}(e^{\frac{1}{2}(\alpha+\beta)} + 6e^{\frac{1}{2}(\alpha+\beta)} n^2 f^2) r(-2\vartheta_{0\tau\tau}) = 0 \\ &- e^{\frac{1}{2}\alpha+\beta}(1 + 6n^2 f^2) r(2\vartheta_{0\tau\tau}) = 0 \end{aligned}$$

$$-2re^{\frac{1}{2}\alpha+\beta}\vartheta_{0\tau\tau}(1 + 6n^2 f^2) = 0 \tag{13}$$

Torsionless Case

In the torsionless case, $\vartheta_0 = 0$, so in turn $K(\tau) = 0$. Using (5) and (6):

$$\alpha_r - \beta_r = \frac{2}{r}(e^{\beta} - 1) \tag{14}$$

$$\alpha_r = -\beta_r \tag{15}$$

so,

$$
\begin{aligned}
-2\frac{\mathrm{d}\beta}{\mathrm{d}r} &= \frac{2}{r}(e^{\beta}-1) \\
-\int \frac{1}{e^{\beta}-1}\,\mathrm{d}\beta &= \int \frac{1}{r}\,\mathrm{d}r \\
-(\ln|e^{\beta}-1| - \ln|e^{\beta}|) &= \ln|r| + C \\
-\ln|\frac{e^{\beta}-1}{e^{\beta}}| &= \ln|r| + C \\
\ln|1-e^{-\beta}| &= -\ln|r| + C \\
1-e^{-\beta} &= Cr^{-1} \\
e^{-\beta} &= 1 - Cr^{-1}
\end{aligned}
$$

$$
e^{\beta} = \left(1 - \frac{C}{r}\right)^{-1} \tag{16}
$$

Integrating (15),

$$
\begin{aligned}
\alpha &= -\beta + D \\
\alpha &= -\beta \text{ (with a coordinate redefinition)} \\
e^{\alpha} &= e^{-\beta} \\
e^{\alpha} &= \left(1 - \frac{C}{r}\right) \qquad (17)
\end{aligned}
$$

Plugging this into the General Schwarzchild metric:

$$
\mathrm{d}s^2 = -\left(1-\frac{C}{r}\right)\mathrm{d}t^2 + \left(1-\frac{C}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2 \tag{18}
$$

Expecting the Newtonian limit at large r, we find $C = 2M$ when $G = 1$, giving us the Schwarzchild metric.