Variations on the Action

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I'll start by considering the total action $S_{tot} = S_{EC} + S_m + S_{NY}$ where

$$S_{EC} = -\frac{M_{Pl}^2}{4}\int \epsilon_{abcd} V^a \wedge V^b \wedge R^{cd} \tag{1}$$

$$S_m = \frac{1}{2}\int \mathrm{d}\phi \wedge {}^\star\mathrm{d}\phi \tag{2}$$

$$S_{NY} = -nf\int \mathrm{d}\phi \wedge T^a \wedge V_a \tag{3}$$

S_{EC} is the Einstein-Cartan gravitational action, S_m is the scalar field , and S_{NY} is the coupling to a Nieh-Yan Form. I'll also define:

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a_c \wedge \omega^{cb}$$
$$T^a = \mathrm{d}V^a + \omega^a_b \wedge V^b$$

1 Variation with respect to the Vielbein

$$\delta_V S_{tot} = \delta_V S_{EC} + \delta_V S_m + \delta_V S_{NY} \tag{4}$$

1.1 Einstein-Cartan Action

$$\begin{aligned}
\delta_V S_{EC} &= -\frac{M_{Pl}^2}{4}\int \epsilon_{abcd}(\delta V^a \wedge V^b + V^a \wedge \delta V^b) \wedge R^{cd} \\
&= -\frac{M_{Pl}^2}{4}\int 2\epsilon_{abcd}\delta V^a \wedge V^b \wedge R^{cd} \\
&\text{(using the antisymmetry of the Levi-Civita Symbol)} \\
&= \frac{M_{Pl}^2}{2}\int R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a \\
&= \boxed{\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a}
\end{aligned} \tag{5}$$

1.2 Scalar Field

$$\begin{aligned}\delta_V S_m &= \frac{1}{2}\int \delta \mathrm{d}\phi \wedge {}^\star \mathrm{d}\phi + \mathrm{d}\phi \wedge \delta^\star \mathrm{d}\phi \\ &= \frac{1}{2}\int \mathrm{d}\phi \wedge \delta^\star \mathrm{d}\phi \qquad \text{since } \delta_V \mathrm{d}\phi = 0 \end{aligned} \tag{6}$$

Expanding:

$$\begin{aligned}{}^\star \mathrm{d}\phi &= {}^\star(\partial_\mu \phi \mathrm{d}x^\mu) = {}^\star(\partial_\mu \phi V_a^\mu V^a) \\ &= \frac{1}{3!} V_a^\mu \partial_\mu \phi \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d\end{aligned}$$

So,

$$\delta_V {}^\star \mathrm{d}\phi = \frac{1}{3!}(\delta V_a^\mu)\partial_\mu \phi \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d + \frac{1}{3!} V_a^\mu \partial_\mu \phi \epsilon^a_{bcd}\, \delta\big(V^b \wedge V^c \wedge V^d\big) \tag{7}$$

Using the identity

$$\begin{aligned}(\delta V_a^\mu) V_\mu^j &= -(\delta V_\mu^j) V_a^\mu \\ (\delta V_a^\mu) &= -(\delta V_\nu^j) V_a^\nu V_j^\mu\end{aligned}$$

and plugging that into eq. (7) and using the antisymmetry of the Levi-Civita Symbol in the second term:

$$\begin{aligned}&\delta_V {}^\star \mathrm{d}\phi \\ &= -\frac{1}{3!}(\delta V_\nu^j) V_a^\nu V_j^\mu \partial_\mu \phi \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d + \frac{1}{3!} V_a^\mu \partial_\mu \phi (3) \epsilon^a_{bcd}\, \delta V^b \wedge V^c \wedge V^d \\ &= -\frac{1}{3!}(\delta V_\nu^j) V_a^\nu \partial_j \phi \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d + \frac{1}{2}\partial_a \phi \epsilon^a_{bcd}\, \delta V^b \wedge V^c \wedge V^d\end{aligned}$$

Since we are in an orthonormal frame defined by Vielbeins, we can also say that $\partial_a \phi \epsilon^a_{bcd} = \partial^a \phi \epsilon_{abcd}$. Using this in the second term:

$$\begin{aligned}\delta_V {}^\star \mathrm{d}\phi &= -\frac{1}{3!}(\delta V_\nu^j) V_a^\nu \partial_j \phi \epsilon^a_{bcd} V^b \wedge V^c \wedge V^d + \frac{1}{2}\partial^a \phi \epsilon_{abcd}\, \delta V^b \wedge V^c \wedge V^d \\ &= -(\delta V_\nu^j) V_a^\nu \partial_j \phi (\frac{1}{3!}\epsilon^a_{bcd} V^b \wedge V^c \wedge V^d) + \partial^a \phi (\frac{1}{2}\epsilon_{abcd} V^c \wedge V^d) \wedge \delta V^b\end{aligned}$$

Finally, using the definition of a dual, we can simplify the terms in parenthesis:

$$\begin{aligned}
\delta_V {}^\star \mathrm{d}\phi &= -(\delta V^j_\nu) V^\nu_a \partial_j \phi ({}^\star V^a) + \partial^a \phi^\star (V_a \wedge V_b) \wedge \delta V^b \\
&= -\partial_j \phi (\delta V^j_a)({}^\star V^a) + \partial^a \phi \, \delta V^b \wedge {}^\star (V_a \wedge V_b) \\
&= -\partial_j \phi^\star (\delta V^j_a V^a) + \partial^a \phi \, \delta V^b \wedge {}^\star (V_a \wedge V_b) \\
&= -\partial_j \phi^\star (\delta V^j) + \partial^a \phi \, \delta V^b \wedge {}^\star (V_a \wedge V_b)
\end{aligned}$$

Relabeling indices, we are left with

$$\delta_V {}^\star \mathrm{d}\phi = \partial^a \phi \, \delta V^b \wedge {}^\star (V_a \wedge V_b) - \partial_b \phi^\star (\delta V^b) \tag{8}$$

Plugging this back into eq. (6):

$$\begin{aligned}
\delta_V S_m &= \frac{1}{2} \int \mathrm{d}\phi \wedge (\partial^a \phi \, \delta V^b \wedge {}^\star (V_a \wedge V_b) - \partial_b \phi^\star (\delta V^b)) \\
&= \frac{1}{2} \int \partial^a \phi \mathrm{d}\phi \wedge \delta V^b \wedge {}^\star (V_a \wedge V_b) - \partial_b \phi \mathrm{d}\phi \wedge {}^\star (\delta V^b)
\end{aligned}$$

Using the identity $A \wedge {}^\star B = B \wedge {}^\star A$ when A and B have the same dimension in the second term:

$$\begin{aligned}
\delta_V S_m &= \frac{1}{2} \int \partial^a \phi \mathrm{d}\phi \wedge {}^\star (V_a \wedge V_b) \wedge \delta V^b - \partial_b \phi (\delta V^b) \wedge {}^\star \mathrm{d}\phi \\
&= \boxed{\int \frac{1}{2} (\partial^a \phi \mathrm{d}\phi \wedge {}^\star (V_a \wedge V_b) \wedge \delta V^b + \partial_b \phi^\star \mathrm{d}\phi \wedge \delta V^b)}
\end{aligned} \tag{9}$$

1.3 Nieh-Yan Form

$$\delta_V S_{NY} = -nf \int \mathrm{d}\phi \wedge (\delta T^a \wedge V_a + T^a \wedge \delta V_a)$$

Expanding:

$$\begin{aligned}
&\delta_V S_{NY} \\
&= -nf \int \mathrm{d}\phi \wedge ((\delta\, \mathrm{d}V^a + \omega^a_b \wedge \delta V^b) \wedge V_a + (\mathrm{d}V^a + \omega^a_b \wedge V^b) \wedge \delta V_a) \\
&= -nf \int \mathrm{d}\phi \wedge (\delta\, \mathrm{d}V^a \wedge V_a + \omega^a_b \wedge \delta V^b \wedge V_a + \mathrm{d}V^a \wedge \delta V_a + \omega^a_b \wedge V^b \wedge \delta V_a) \\
&= -nf \int \mathrm{d}\phi \wedge (\delta\, \mathrm{d}V^a \wedge V_a + \mathrm{d}V^a \wedge \delta V_a + \omega^a_b \wedge (\delta V^b \wedge V_a + V^b \wedge \delta V_a))
\end{aligned} \tag{10}$$

Focusing on the term with the spin connection:

$$\begin{aligned}
&\omega^a_b \wedge (\delta V^b \wedge V_a + V^b \wedge \delta V_a) \\
&= \omega^a_b \wedge \delta V^b \wedge V_a + \omega^a_b \wedge V^b \wedge \delta V_a \\
&= \omega^a_b \wedge \delta V^b \wedge \eta_{ac} V^c + \omega^a_b \wedge V^b \wedge \eta_{ac}\, \delta V^c \\
&= \eta_{ac} \omega^a_b \wedge \delta V^b \wedge V^c + \eta_{ac} \omega^a_b \wedge V^b \wedge \delta V^c \\
&= \omega_{bc} \wedge \delta V^b \wedge V^c + \omega_{bc} \wedge V^b \wedge \delta V^c
\end{aligned} \tag{11}$$

Focusing on the second term:

$$\begin{aligned}
&\omega_{bc} \wedge V^b \wedge \delta V^c \\
&= \omega_{cb} \wedge V^c \wedge \delta V^b \qquad \text{(swapped indices)} \\
&= -\omega_{cb} \wedge \delta V^b \wedge V^c \\
&= \omega_{bc} \wedge \delta V^b \wedge V^c \\
&\text{(using the antisymmetry of the spin connection)}
\end{aligned}$$

Plugging this back into eq. (11):

$$\omega_{bc} \wedge \delta V^b \wedge V^c + \omega_{bc} \wedge \delta V^b \wedge V^c = 2\omega_{bc} \wedge \delta V^b \wedge V^c$$

Plugging this back into eq.(10):

$$\delta_V S_{NY} = -nf \int \mathrm{d}\phi \wedge (\delta\, \mathrm{d}V^a \wedge V_a + \mathrm{d}V^a \wedge \delta V_a + 2\omega_{bc} \wedge \delta V^b \wedge V^c) \tag{12}$$

Now, focusing on $\delta\, \mathrm{d}V^a \wedge V_a + \mathrm{d}V^a \wedge \delta V_a$. We can use the Leibniz Rule:

$$\begin{aligned}
\mathrm{d}(\delta V^a \wedge V_a) &= \mathrm{d}\, \delta V^a \wedge V_a + (-1)\, \delta V^a \wedge \mathrm{d}V_a \\
&= \mathrm{d}\, \delta V^a \wedge V_a - \delta V^a \wedge \mathrm{d}V_a \\
&= \mathrm{d}\, \delta V^a \wedge V_a + \mathrm{d}V_a \wedge \delta V^a \\
&= \mathrm{d}\, \delta V^a \wedge V_a + \mathrm{d}V^a \wedge \delta V_a \\
&= \delta\, \mathrm{d}V^a \wedge V_a + \mathrm{d}V^a \wedge \delta V_a
\end{aligned}$$

which is the exact expression. Plugging this into eq. (12):

$$\delta_V S_{NY} = -nf \int \mathrm{d}\phi \wedge \left(\mathrm{d}(\delta V^a \wedge V_a) + 2\omega_{bc} \wedge \delta V^b \wedge V^c\right)$$