Friedman Equations of Motion

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The original form of the equations from the mathematica notebook are given along with the simplifactions based on $V = 0$, $h = 0$, $h^{ij} = 0$, and $f^{ij} = 0$.

Eq.1

$$\begin{aligned}
&- 2r^2 \sin\theta \left(h' - 2nf\left(f^{11} + \phi\right)\vartheta_0'\right) + \tfrac{1}{4} e^{\frac{1}{2}\nu - \lambda} \\
&\Big(e^\lambda \Big(r\Big(r \sin\theta \\
&\Big(- 4h^2 + 2(f^{11} + \phi)(f^{22} + f^{33} + 2\phi) \\
&- 2(f^{12})^2 - 2(f^{13})^2 + (h^{12})^2 + (h^{13})^2 + (h^{23})^2\Big) - 2f^{13}\cos\theta\Big) \\
&- 4\sin\theta\Big) + 4\sin\theta\left(1 + r\nu_r\right)\Big) \\
&= \tfrac{1}{2} r^2 \sin\theta\, e^{-\frac{1}{2}\lambda - \nu}\sqrt{e^{\lambda+\nu}}\left(2V(\vartheta_0)e^\nu + (\vartheta_0')^2\right)
\end{aligned}$$

Simplified:

$$\begin{aligned}
&- 4r^2 \sin\theta\left(nf\phi\vartheta_0'\right) + \frac{e^{\frac{\nu}{2} - \lambda}}{4}\Big(e^\lambda\Big(r\Big(r\sin\theta\big(4\phi^2\big)\Big) - 4\sin\theta\Big) + 4\sin\theta\left(1 + r\nu_r\right)\Big) \\
&= \tfrac{1}{2} r^2 \sin\theta\, e^{-\frac{\lambda}{2} - \nu}\sqrt{e^{\lambda+\nu}}(\vartheta_0')^2
\end{aligned}$$

Finally:

$$-8r^2\left(nf\phi\vartheta_0'\right) + 2e^{\frac{\nu}{2} - \lambda}\Big(e^\lambda\Big(r^2\phi^2 - 1\Big) + r\nu_r + 1\Big) = r^2\, e^{-\frac{\nu}{2}}(\vartheta_0')^2$$

Eq.2

$$\begin{aligned}
&\frac{r}{2}\Big[e^{\frac{\nu}{2}}\Big(r\sin\theta\Big(- 2h\,h^{12} + f^{12}(f^{33} + \phi) - f^{13}f^{23}\Big) - 2f^{23}\cos\theta\Big) \\
&\quad - f^{13}\sin\theta\, e^{\frac{\nu-\lambda}{2}} - r\sin\theta\Big((h^{12})' - 8nff^{12}\vartheta_0'\Big)\Big] = 0
\end{aligned}$$

Simplified:

$$0 = 0$$

Eq.3

$$\frac{r}{2}\Big[e^{\frac{\nu}{2}}\big(r\sin\theta\big(-2h\,h^{13}-f^{12}f^{23}+f^{13}(f^{22}+\phi)\big)+\cos\theta(f^{22}-f^{33})\big)$$
$$+f^{12}\sin\theta\,e^{\frac{\nu-\lambda}{2}}-r\sin\theta\big((h^{13})'-8nff^{13}\vartheta_0'\big)\Big]=0$$

Simplified:

$$0=0$$

Eq.4

$$\frac{1}{4}r\Big[e^{\frac{1}{2}\nu}\big(2h^{12}\cos\theta-r\sin\theta\big[h^{23}\big(2f^{11}+f^{22}+f^{33}+4\phi\big)-f^{12}h^{13}+f^{13}h^{12}\big]\big)$$
$$-8nfr\,h^{23}\sin\theta\,\vartheta_0'+4e^{-\frac{1}{2}\lambda}\sin\theta\,e^{\frac{1}{2}\nu}\,rh(\nu_r+\lambda_\tau)\Big]=0$$

Simplified:

$$0=0$$

Eq.5

$$\frac{1}{4}\sin\theta\Big[e^{\frac{\nu}{2}}f^{13}\,(4+r\nu_r)+2e^{\frac{\lambda}{2}}r$$
$$\Big(e^{\frac{\nu}{2}}\big(2h\,h^{12}+f^{12}(f^{33}+\phi)-f^{13}f^{23}\big)+(h^{12})'+8nff^{12}\vartheta_0'+h^{12}\lambda_\tau\Big)\Big]$$
$$=0$$

Simplified:

$$0=0$$

Eq.6

$$
\begin{aligned}
\frac{1}{4}e^{-\frac{\lambda+\nu}{2}}\Big[&e^{\frac{\lambda+\nu}{2}}\sin\theta\, f^{23}\,(2+r\nu_r) - e^{\nu}\sin\theta\,(\lambda_r-\nu_r)\,(2+r\nu_r) - 2r(\nu_r)_r \\
&+ e^{\lambda}e^{\nu} \\
&\Big(r\sin\theta\big[-4h^2 + 2(f^{22}+\phi)(f^{11}+f^{33}+2\phi) \\
&- 2(f^{12})^2 - 2(f^{23})^2 + (h^{12})^2 + (h^{13})^2 + (h^{23})^2\big] + 2f^{13}\cos\theta\Big) \\
&- 4e^{\frac{\nu}{2}}r\sin\theta\Big(2h' - 4nf(f^{22}+\phi)\vartheta_0' + h\lambda_\tau\Big) \\
&- r\sin\theta\Big[(\lambda_\tau)_\tau - \lambda_\tau\nu_\tau + 2(\lambda_\tau)_\tau\Big]\Big] \\
&= \frac{1}{2}r\sin\theta\, e^{-\nu}\sqrt{e^{\lambda+\nu}}\,\big(2V(\vartheta_0)e^{\nu} + (\vartheta_0')^2\big)
\end{aligned}
$$

Simplified:

$$
\begin{aligned}
\frac{1}{4}e^{-\frac{\lambda+\nu}{2}}\Big[&-e^{\nu}\sin\theta\,(\lambda_r-\nu_r)\,(2+r\nu_r) \\
&- 2r(\nu_r)_r \\
&+ 4e^{\lambda}e^{\nu}r\phi^2\sin\theta \\
&+ 16e^{\frac{\nu}{2}}rnf\phi\vartheta_0'\sin\theta \\
&- r\sin\theta\Big[(\lambda_\tau)_\tau - \lambda_\tau\nu_\tau + 2(\lambda_\tau)_\tau\Big]\Big] \\
&= \frac{1}{2}r\sin\theta\, e^{-\nu}\sqrt{e^{\lambda+\nu}}\,\big((\vartheta_0')^2\big)
\end{aligned}
$$

Finally:

$$
\begin{aligned}
e^{-\frac{\lambda+\nu}{2}}\Big[&-\frac{e^{\nu}}{2}(\lambda_r-\nu_r)\left(\frac{2}{r}+\nu_r\right) - \frac{(\nu_r)_r}{\sin\theta} \\
&+ 2e^{\lambda}e^{\nu}\phi^2 + 8e^{\frac{\nu}{2}}nf\phi\vartheta_0' - \frac{(\lambda_\tau)_\tau}{2} + \frac{\lambda_\tau\nu_\tau}{2} - (\lambda_\tau)_\tau\Big] = e^{\frac{\lambda-\nu}{2}}(\vartheta_0')^2
\end{aligned}
$$

Eq.7

$$\frac{1}{4}\Big[-e^{\frac{\nu}{2}}\sin\theta\Big(2(f^{11}-f^{33})+r(f^{11}+f^{22}+2\phi)\nu_r\Big)$$
$$+e^{\frac{\lambda}{2}}\Big(-2e^{\frac{\nu}{2}}\big[r\sin\theta\big(2h\,h^{23}-f^{23}(f^{11}+\phi)+f^{12}f^{13}\big)+f^{12}\cos\theta\big]$$
$$-r\sin\theta\big[2(h^{23})'-16nf\,f^{23}\vartheta_0'+h^{23}\lambda_\tau\big]\Big)\Big]=0$$

Simplified:

$$\frac{1}{4}\Big[-e^{\frac{\nu}{2}}\sin\theta(2r\phi\nu_r)\Big]=0$$

Finally:

$$-\frac{r\phi\nu_r}{2}e^{\frac{\nu}{2}}\sin\theta=0$$

Eq.8

$$\frac{1}{4}\sin\theta\Big[-2e^{\frac{\nu}{2}}h^{12}\,(1+r\nu_r)$$
$$+e^{\frac{\lambda}{2}}r\Big(e^{\frac{\nu}{2}}\big[h^{13}(f^{11}+2f^{22}+f^{33}+4\phi)-f^{12}h^{23}-f^{23}h^{12}\big]$$
$$+8nf\,h^{13}\,\vartheta_0'-f^{13}\lambda_\tau\Big)\Big]=0$$

Simplified:

$$0=0$$

Eq.9

$$\frac{1}{4}\Big[-e^{\frac{\nu}{2}}f^{12}\,(4+r\nu_r)$$
$$+2e^{\frac{\lambda}{2}}r\Big(e^{\frac{\nu}{2}}\big(2h\,h^{13}-f^{12}f^{23}+f^{13}(f^{22}+\phi)\big)$$
$$+(h^{13})'+8nf\,f^{13}\vartheta_0'+h^{13}\lambda_\tau\Big)\Big]=0$$

Simplified:

$$0=0$$

Eq.10

$$\frac{1}{4}\Big[e^{\frac{\nu}{2}}\Big(2(f^{11}-f^{22})+r(f^{11}+f^{33}+2\phi)\nu_r\Big)$$
$$+\,e^{\frac{\lambda}{2}}r\Big(-2e^{\frac{\nu}{2}}\Big(-2h\,h^{23}-f^{23}(f^{11}+\phi)+f^{12}f^{13}\Big)$$
$$+\,2(h^{23})'+16nff^{23}\vartheta_0'+h^{23}\lambda_\tau\Big)\Big]=0$$

Simplified:

$$\frac{1}{4}\Big[e^{\frac{\nu}{2}}\Big(r(2\phi)\nu_r\Big)\Big]=0$$

Finally:

$$\frac{r\phi\nu_r}{2}e^{\frac{\nu}{2}}=0$$

Eq.11

$$\frac{1}{4}e^{-\frac{\lambda+\nu}{2}}\Big[e^{\nu}r\Big(-e^{\lambda}$$
$$\Big(4h^2-2(f^{33}+\phi)(f^{11}+f^{22}+2\phi)$$
$$+\,2(f^{13})^2+2(f^{23})^2-(h^{12})^2-(h^{13})^2-(h^{23})^2\Big)$$
$$-\,2\lambda_r+2\nu_r-e^{\frac{\lambda}{2}}f^{23}\,(2+r\nu_r)$$
$$+\,r\Big(-\lambda_r\nu_r+(\nu_r)^2+2(\nu_r)_r\Big)\Big)$$
$$-\,4e^{\lambda+\frac{\nu}{2}}r\Big(2h'-4nf(f^{33}+\phi)\vartheta_0'+h\lambda_\tau\Big)$$
$$-\,e^{\lambda}r\Big[\,(\lambda_\tau)^2-\lambda_\tau\nu_\tau+2(\lambda_\tau)_\tau\Big]\Big]$$
$$=\frac{1}{2}re^{-\nu}\sqrt{e^{\lambda+\nu}}\left(2V(\vartheta_0)e^{\nu}+(\vartheta_0')^2\right)$$

Simplified:

$$\frac{1}{4}e^{-\frac{\lambda+\nu}{2}}\Big[e^{\nu}\Big(-4e^{\lambda}\phi^2-2\lambda_r+2\nu_r+r(-\lambda_r\nu_r+(\nu_r)^2+2(\nu_r)_r)\Big)$$
$$+16nf\phi e^{\lambda+\frac{\nu}{2}}\vartheta_0'$$
$$-e^{\lambda}((\lambda_\tau)^2-\lambda_\tau\nu_\tau+2(\lambda_\tau)_\tau)\Big]$$
$$=\frac{1}{2}e^{\frac{\lambda-\nu}{2}}(\vartheta_0')^2$$

Finally:

$$e^{-\frac{\lambda+\nu}{2}}\Big[e^{\nu}\Big(-2e^{\lambda}\phi^2-\lambda_r+\nu_r+\frac{r}{2}(-\lambda_r\nu_r+(\nu_r)^2+2(\nu_r)_r)\Big)$$
$$+8nf\phi e^{\lambda+\frac{\nu}{2}}\vartheta_0'-\frac{e^{\lambda}}{2}((\lambda_\tau)^2-\lambda_\tau\nu_\tau+2(\lambda_\tau)_\tau)\Big]=e^{\frac{\lambda-\nu}{2}}(\vartheta_0')^2$$

Eq.12

$$\frac{1}{4}\Big[-2e^{\frac{\nu}{2}}\,h^{13}\,(1+r\nu_r)$$
$$+e^{\frac{\lambda}{2}}r\Big(-e^{\frac{\nu}{2}}\big[h^{12}(f^{11}+f^{22}+2f^{33}+4\phi)+f^{13}h^{23}-f^{23}h^{13}\big]$$
$$-8nf\,h^{12}\vartheta_0'+f^{12}\lambda_\tau\Big)\Big]=0$$

Simplified:

$$0=0$$

Eq.13

$$\frac{1}{2}e^{-\frac{\nu}{2}}\Big[re^{\frac{\lambda+\nu}{2}}\Big(r\sin\theta\big(-f^{12}h^{13}+f^{13}h^{12}-h^{23}(f^{22}+f^{33}+2\phi)\big)-h^{12}\cos\theta\Big)$$
$$-2r\sin\theta\lambda_\tau\Big]=0$$

Simplify:

$$\frac{1}{2}e^{-\frac{\nu}{2}}\Big[-2r\sin\theta\lambda_\tau\Big]=0$$

Finally:

$$-r\lambda_\tau e^{-\frac{\nu}{2}}\sin\theta=0$$

Eq.14

$$\frac{1}{2}r\sin\theta\Big[re^{\frac{\lambda}{2}}\big(h^{13}(f^{11}+f^{33}+2\phi)+f^{12}h^{23}+f^{23}h^{12}\big)+h^{12}\Big]=0$$

Simplify:

$$0=0$$

Eq.15

$$\frac{1}{2}r\sin\theta\Big[h^{13}-re^{\frac{\lambda}{2}}\big(h^{12}(f^{11}+f^{22}+2\phi)-f^{13}h^{23}+f^{23}h^{13}\big)\Big]=0$$

Simplify:

$$0=0$$

Eq.16

$$\begin{aligned}\frac{1}{4}e^{-\frac{\lambda+\nu}{2}}\sin\theta\Big[e^{\frac{\nu}{2}}\Big(4\\ &-e^{\lambda}\Big(r^2\big(12h^2-8\phi(f^{11}+f^{22}+f^{33})-4f^{33}(f^{11}+f^{22})\\ &\quad-4f^{11}f^{22}+4(f^{12})^2+4(f^{13})^2+4(f^{23})^2\\ &\quad+(h^{12})^2+(h^{13})^2+(h^{23})^2-12\phi^2\big)+4\Big)-4r\lambda_r\Big)\\ &-4e^{\lambda}r^2h\,\lambda_\tau\Big]\\ &=\frac{1}{2}r^2\sin\theta\,e^{-\frac{3}{2}\nu}\sqrt{e^{\lambda+\nu}}\Big(2V(\vartheta_0)e^{\nu}-(\vartheta_0')^2\Big)\end{aligned}$$

Simplify:

$$\begin{aligned}\frac{1}{4}e^{-\frac{\lambda+\nu}{2}}\sin\theta\Big[e^{\frac{\nu}{2}}\Big(4-e^{\lambda}\Big(r^2\big(-12\phi^2\big)+4\Big)-4r\lambda_r\Big)\Big]\\ =\frac{1}{2}r^2\sin\theta\,e^{-\frac{3}{2}\nu}\sqrt{e^{\lambda+\nu}}\Big(-(\vartheta_0')^2\Big)\end{aligned}$$

Finally:

$$2\Big(e^{\lambda}(3r^2\phi^2-1)-r\lambda_r+1\Big)=-r^2e^{\lambda-\nu}(\vartheta_0')^2$$

List of Equations

$$-8r^2\,(nf\phi\vartheta_0') + 2e^{\frac{\nu}{2}-\lambda}\Big(e^{\lambda}\big(r^2\phi^2-1\big) + r\nu_r + 1\Big) = r^2\,e^{-\frac{\nu}{2}}(\vartheta_0')^2 \tag{1}$$

$$\begin{aligned} e^{-\frac{\lambda+\nu}{2}}\Big[&-\frac{e^{\nu}}{2}\,(\lambda_r-\nu_r)\left(\frac{2}{r}+\nu_r\right) - \frac{(\nu_r)_r}{\sin\theta} \\ &+ 2e^{\lambda}e^{\nu}\phi^2 + 8e^{\frac{\nu}{2}}nf\phi\vartheta_0' - \frac{(\lambda_\tau)_\tau}{2} + \frac{\lambda_\tau\nu_\tau}{2} - (\lambda_\tau)_\tau\Big] = e^{\frac{\lambda-\nu}{2}}(\vartheta_0')^2 \end{aligned} \tag{2}$$

$$-\frac{r\phi\nu_r}{2}e^{\frac{\nu}{2}}\sin\theta = 0 \tag{3}$$

$$\frac{r\phi\nu_r}{2}e^{\frac{\nu}{2}} = 0 \tag{4}$$

$$\begin{aligned} e^{-\frac{\lambda+\nu}{2}}\Big[&e^{\nu}\Big(-2e^{\lambda}\phi^2 - \lambda_r + \nu_r + \frac{r}{2}(-\lambda_r\nu_r + (\nu_r)^2 + 2(\nu_r)_r)\Big) \\ &+ 8nf\phi e^{\lambda+\frac{\nu}{2}}\vartheta_0' - \frac{e^{\lambda}}{2}((\lambda_\tau)^2 - \lambda_\tau\nu_\tau + 2(\lambda_\tau)_\tau)\Big] = e^{\frac{\lambda-\nu}{2}}(\vartheta_0')^2 \end{aligned} \tag{5}$$

$$-r\lambda_\tau e^{-\frac{\nu}{2}}\sin\theta = 0 \tag{6}$$

$$2\Big(e^{\lambda}(3r^2\phi^2-1) - r\lambda_r + 1\Big) = -r^2e^{\lambda-\nu}(\vartheta_0')^2 \tag{7}$$

Looking at eq.3 and eq.4, we know exponential and sinusoidal terms cannot equal zero. Furthermore, r and ϕ are coordinates. Therefore, both equations yield:

$$\nu_r = 0 \tag{8}$$

In the same vein, eq.6 yields:

$$\lambda_\tau = 0 \tag{9}$$

These constraints are consistent with the General Schwarzchild metric:

$$\mathrm{d}s^2 = -e^{\nu(t,r)}\,\mathrm{d}t^2 + e^{\lambda(t,r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$$

Simplifying the equations even further, we are left with:

$$-8r^2\left(nf\phi\vartheta_0'\right) + 2e^{\frac{\nu}{2}-\lambda}\Big(e^{\lambda}\big(r^2\phi^2 - 1\big) + 1\Big) = r^2\,e^{-\frac{\nu}{2}}(\vartheta_0')^2 \tag{10}$$

$$e^{-\frac{\lambda+\nu}{2}}\Big[-\frac{e^{\nu}\lambda_r}{r} + 2e^{\lambda}e^{\nu}\phi^2 + 8e^{\frac{\nu}{2}}nf\phi\vartheta_0'\Big] = e^{\frac{\lambda-\nu}{2}}(\vartheta_0')^2 \tag{11}$$

$$e^{-\frac{\lambda+\nu}{2}}\Big[e^{\nu}\Big(-2e^{\lambda}\phi^2 - \lambda_r\Big) + 8nf\phi e^{\lambda+\frac{\nu}{2}}\vartheta_0'\Big] = e^{\frac{\lambda-\nu}{2}}(\vartheta_0')^2 \tag{12}$$

$$2\Big(e^{\lambda}(3r^2\phi^2 - 1) - r\lambda_r + 1\Big) = -r^2e^{\lambda-\nu}(\vartheta_0')^2 \tag{13}$$

Simplifying eq.10:

$$\begin{aligned}
-8r^2\left(nf\phi\vartheta_0'\right) + 2e^{\frac{\nu}{2}-\lambda}\Big(e^{\lambda}\big(r^2\phi^2 - 1\big) + 1\Big) &= r^2\,e^{-\frac{\nu}{2}}(\vartheta_0')^2 \\
2e^{\frac{\nu}{2}-\lambda}\Big(e^{\lambda}\big(r^2\phi^2 - 1\big) + 1\Big) &= r^2\,e^{-\frac{\nu}{2}}(\vartheta_0')^2 + 8r^2\left(nf\phi\vartheta_0'\right) \\
2e^{\frac{\nu}{2}-\lambda}\Big(e^{\lambda}\big(\phi^2 - \frac{1}{r^2}\big) + \frac{1}{r^2}\Big) &= e^{-\frac{\nu}{2}}(\vartheta_0')^2 + 8nf\phi\vartheta_0'
\end{aligned}$$

Next, eq.11 and eq.12 can be equated:

$$e^{-\frac{\lambda+\nu}{2}}\Big[-\frac{e^{\nu}\lambda_r}{r}+2e^{\lambda}e^{\nu}\phi^2+8e^{\frac{\nu}{2}}nf\phi\vartheta_0'\Big]= \\ e^{-\frac{\lambda+\nu}{2}}\Big[e^{\nu}\Big(-2e^{\lambda}\phi^2-\lambda_r\Big)+8nf\phi e^{\lambda+\frac{\nu}{2}}\vartheta_0'\Big]$$

$$\begin{aligned}
-\frac{e^{\nu}\lambda_r}{r}+2e^{\lambda}e^{\nu}\phi^2+8e^{\frac{\nu}{2}}nf\phi\vartheta_0' &= e^{\nu}\Big(-2e^{\lambda}\phi^2-\lambda_r\Big)+8nf\phi e^{\lambda+\frac{\nu}{2}}\vartheta_0' \\
-\frac{e^{\nu}\lambda_r}{r}+2e^{\lambda}e^{\nu}\phi^2+8e^{\frac{\nu}{2}}nf\phi\vartheta_0' &= -2e^{\nu}e^{\lambda}\phi^2-e^{\nu}\lambda_r+8nf\phi e^{\lambda+\frac{\nu}{2}}\vartheta_0' \\
-e^{\nu}\lambda_r\left(\frac{1}{r}-1\right)+4e^{\lambda+\nu}\phi^2 &= 8nf\phi\vartheta_0'\left(e^{\lambda+\frac{\nu}{2}}-e^{\frac{\nu}{2}}\right) \\
\lambda_r\left(\frac{1}{r}-1\right) &= 4e^{\lambda}\phi^2-8nf\phi\vartheta_0'\frac{\left(e^{\lambda+\frac{\nu}{2}}-e^{\frac{\nu}{2}}\right)}{e^{\nu}}
\end{aligned}$$

Simplifying the exponents:

$$\begin{aligned}
\lambda_r\left(\frac{1}{r}-1\right) &= 4e^{\lambda}\phi^2-8nf\phi\vartheta_0'\left(e^{\lambda-\frac{\nu}{2}}-e^{-\frac{\nu}{2}}\right) \\
\lambda_r\left(\frac{1}{r}-1\right) &= 4e^{\lambda}\phi^2-8nf\phi\vartheta_0'e^{-\frac{\nu}{2}}\left(e^{\lambda}-1\right)
\end{aligned}$$

Moving on to eq.13:

$$\begin{aligned}
2\Big(e^{\lambda}(3r^2\phi^2-1)-r\lambda_r+1\Big) &= -r^2e^{\lambda-\nu}(\vartheta_0')^2 \\
e^{\lambda}(3r^2\phi^2-1)-r\lambda_r+1 &= -\frac{1}{2}r^2e^{\lambda-\nu}(\vartheta_0')^2
\end{aligned}$$

Therefore, the final equations of motion are:

$$2e^{\frac{\nu}{2}-\lambda}\Big(e^{\lambda}\big(\phi^2-\frac{1}{r^2}\big)+\frac{1}{r^2}\Big)=e^{-\frac{\nu}{2}}(\vartheta_0')^2+8nf\phi\vartheta_0' \tag{14}$$

$$\lambda_r\left(\frac{1}{r}-1\right)=4e^{\lambda}\phi^2-8nf\phi\vartheta_0'e^{-\frac{\nu}{2}}\left(e^{\lambda}-1\right) \tag{15}$$

$$e^{\lambda}(3r^2\phi^2-1)-r\lambda_r+1=-\frac{1}{2}r^2e^{\lambda-\nu}(\vartheta_0')^2 \tag{16}$$