

# Deriving the Metric Functions

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## Initial System

The mathematica notebook provides us with the following equations of motion:

$$e^\alpha(1 + r\alpha_r) - e^\beta(e^\alpha + 3n^2 f^2 r^2 \vartheta_{0\tau}^2) = \frac{1}{2} r^2 e^\beta \vartheta_{0\tau}^2 \quad (1)$$

$$e^\alpha(1 - r\beta_r) - e^\beta(e^\alpha - 3n^2 f^2 r^2 \vartheta_{0\tau}^2) = -\frac{1}{2} r^2 e^\beta \vartheta_{0\tau}^2 \quad (2)$$

$$e^\alpha((2 + r\alpha_r)(\alpha_r - \beta_r) + 2r\alpha_{rr}) - 12e^\beta r n^2 f^2 \vartheta_{0\tau}^2 = 2re^\beta \vartheta_{0\tau}^2 \quad (3)$$

$$\beta_\tau = \alpha_\tau = \vartheta_{0r} = 0 \quad (4)$$

So, we can say  $\alpha = \alpha(r)$ ,  $\beta = \beta(r)$ ,  $\vartheta_0 = \vartheta_0(\tau)$ .

Adding (1) and (2) gives:

$$\alpha_r - \beta_r = \frac{2}{r}(e^\beta - 1) \quad (5)$$

and subtracting (1) and (2) gives:

$$\alpha_r + \beta_r = re^{-\alpha+\beta} \vartheta_{0\tau}^2 (1 + 6n^2 f^2) \quad (6)$$

Assume  $u(r) = e^{-\alpha+\beta}$  and  $K(\tau) = \vartheta_{0\tau}^2 (1 + 6n^2 f^2)$ :

$$-\frac{u_r}{u} = \frac{2}{r}(e^\beta - 1) \quad (7)$$

and

$$\alpha_r + \beta_r = ruK \quad (8)$$

Getting  $u_r$  from (9) and  $\beta_r$  from (5) and (8), we can see the system is non-linearly coupled:

$$u_r = -\frac{2u}{r}(e^\beta - 1) \quad (9)$$

$$\beta_r = \frac{1}{2} \left( ruK - \frac{2}{r}(e^\beta - 1) \right) \quad (10)$$

Differentiating (10):

$$\begin{aligned}\beta_{rr} &= \frac{K}{2}(u + ru_r) - \frac{1}{2}\left(-\frac{2}{r^2}(e^\beta - 1) - \frac{1}{r}(\beta_r e^\beta)\right) \\ &= \frac{K(u + ru_r)}{2} + \frac{e^\beta - 1}{r^2} - \frac{\beta_r e^\beta}{r}\end{aligned}$$

Substituting  $u_r$  from (9):

$$\beta_{rr} = \frac{Ku(3 - 2e^\beta)}{2} + \frac{e^\beta - 1}{r^2} - \frac{\beta_r e^\beta}{r}$$

From (10), we see  $Ku = \frac{2\beta_r}{r} + \frac{2(e^\beta - 1)}{r^2}$ :

$$\begin{aligned}\beta_{rr} &= \frac{\beta_r(3 - 2e^\beta)}{r} + \frac{(e^\beta - 1)(3 - 2e^\beta)}{r^2} + \frac{e^\beta - 1}{r^2} - \frac{\beta_r e^\beta}{r} \\ &= -\beta_r \frac{3(e^\beta - 1)}{r} + \frac{-2e^{2\beta} + 6e^\beta - 4}{r^2}\end{aligned}\tag{11}$$

Again, this ODE is unsolvable. Sympy was unable to find a closed-form solution. This only becomes solvable assuming a constant  $\beta$ , or in a torsionless case.

The python file attempts to numerically solve the system posed in (9) and (10) with assumed initial conditions. Observations are written in comments in the code.

## Solving with Ansatz

We define an ansatz:

$$e^\alpha = a = 1 - \frac{C}{r} \quad \alpha_r = a^{-1} \frac{C}{r^2}\tag{12}$$

Evaluating (1) with the ansatz:

$$\begin{aligned}
a(1 + \frac{C}{r}a^{-1}) &= e^\beta (\frac{1}{2}r^2\vartheta_\tau^2 + a + 3n^2f^2r^2\vartheta_\tau^2) \\
a + \frac{C}{r} &= e^\beta (\frac{1}{2}r^2\vartheta_\tau^2 + a + 3n^2f^2r^2\vartheta_\tau^2) \\
e^\beta &= (\frac{1}{2}r^2\vartheta_\tau^2 + a + 3n^2f^2r^2\vartheta_\tau^2)^{-1} \\
e^\beta &= (a + r^2\vartheta_\tau^2 (\frac{1}{2} + 3n^2f^2))^{-1} \\
e^\beta &= \frac{1}{a} \left( 1 + \frac{r^2\vartheta_\tau^2 (\frac{1}{2} + 3n^2f^2)}{a} \right)^{-1}
\end{aligned} \tag{13}$$

## Klein-Gordon Equation

We also have the Klein-Gordon equation:

$$\partial_\mu (\sqrt{-g}g^{\mu\nu}\partial_\nu\vartheta) = \frac{nf}{2}\partial_\mu (T_{\nu\rho}^a V_{a\sigma}) \varepsilon^{\mu\nu\rho\sigma}$$

For this, mathematica gives us:

$$\begin{aligned}
&e^{\frac{3}{2}\alpha}(4 + r\alpha_r - r\beta_r)(\vartheta_{0r}) + 2e^{\frac{3}{2}\alpha}r\vartheta_{0rr} \\
&+ e^{\frac{1}{2}\beta}(e^{\frac{1}{2}(\alpha+\beta)} + 6e^{\frac{1}{2}(\alpha+\beta)}n^2f^2)r((\alpha_\tau - \beta_\tau)(\vartheta_{0\tau}) - 2\vartheta_{0\tau\tau}) = 0
\end{aligned}$$

$$\begin{aligned}
&e^{\frac{1}{2}\beta}(e^{\frac{1}{2}(\alpha+\beta)} + 6e^{\frac{1}{2}(\alpha+\beta)}n^2f^2)r(-2\vartheta_{0\tau\tau}) = 0 \\
&- e^{\frac{1}{2}\alpha+\beta}(1 + 6n^2f^2)r(2\vartheta_{0\tau\tau}) = 0
\end{aligned}$$

$$-2re^{\frac{1}{2}\alpha+\beta}\vartheta_{0\tau\tau}(1 + 6n^2f^2) = 0 \tag{14}$$

## Torsionless Case

In the torsionless case,  $\vartheta_0 = 0$ , so in turn  $K(\tau) = 0$ . Using (5) and (6):

$$\alpha_r - \beta_r = \frac{2}{r}(e^\beta - 1) \tag{15}$$

$$\alpha_r = -\beta_r \tag{16}$$

so,

$$\begin{aligned}
-2\frac{d\beta}{dr} &= \frac{2}{r}(e^\beta - 1) \\
-\int \frac{1}{e^\beta - 1} d\beta &= \int \frac{1}{r} dr \\
-(\ln |e^\beta - 1| - \ln |e^\beta|) &= \ln |r| + C \\
-\ln \left| \frac{e^\beta - 1}{e^\beta} \right| &= \ln |r| + C \\
\ln |1 - e^{-\beta}| &= -\ln |r| + C \\
1 - e^{-\beta} &= Cr^{-1} \\
e^{-\beta} &= 1 - Cr^{-1} \\
e^\beta &= \left(1 - \frac{C}{r}\right)^{-1}
\end{aligned} \tag{17}$$

Integrating (15),

$$\begin{aligned}
\alpha &= -\beta + D \\
\alpha &= -\beta \text{ (with a coordinate redefinition)} \\
e^\alpha &= e^{-\beta} \\
e^\alpha &= \left(1 - \frac{C}{r}\right)
\end{aligned} \tag{18}$$

Plugging this into the General Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{C}{r}\right) dt^2 + \left(1 - \frac{C}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \tag{19}$$

Expecting the Newtonian limit at large  $r$ , we find  $C = 2M$  when  $G = 1$ , giving us the Schwarzschild metric.