Friedman Equations of Motion after substitution

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After substituting the known form of ϕ from the pre-substitution TEOM notes as well as the other constraints, the Friedman EOMs have been recalculated. $V(\vartheta_0) = 0$ is enforced now. The equations are presented:

Equation 1

$$\begin{aligned}
&\sin\theta\Big(4nfr^2\,\vartheta_0'\,nf\,\vartheta_0'(-e^{-\nu/2}) \\
&\qquad + e^{\nu/2-\lambda}\Big(1 + r\,\nu_r + e^{\lambda}\big(r^2((nf\,\vartheta_0'(-e^{-\nu/2}))^2 - 0^2) - 1\big)\Big)\Big) \\
&= \frac{1}{2}r^2\sin\theta\,e^{-\lambda/2-\nu}\sqrt{e^{\lambda+\nu}}\Big(2V(\vartheta_0)e^{\nu} + (\vartheta_0')^2\Big)
\end{aligned}$$

$$\begin{aligned}
&4n^2f^2r^2\,(\vartheta_0')^2(-e^{-\nu/2}) \\
&\qquad + e^{\nu/2-\lambda}\Big(1 + r\,\nu_r + e^{\lambda}\big(r^2((nf\,\vartheta_0'(-e^{-\nu/2}))^2) - 1\big)\Big) \\
&= \frac{1}{2}r^2e^{-\nu/2}(\vartheta_0')^2
\end{aligned}$$

$$\begin{aligned}
&-8n^2f^2r^2\,(\vartheta_0')^2 \\
&\qquad + 2e^{\nu-\lambda}\Big(1 + r\,\nu_r + e^{\lambda}\big(r^2((nf\,\vartheta_0'(-e^{-\nu/2}))^2) - 1\big)\Big) \\
&= r^2(\vartheta_0')^2
\end{aligned}$$

Focusing specifically on this term:

$$\begin{aligned}
&2e^{\nu-\lambda}\Big(1 + r\,\nu_r + e^{\lambda}\big(r^2((nf\,\vartheta_0'(-e^{-\nu/2}))^2) - 1\big)\Big) \\
&2e^{\nu-\lambda} + 2e^{\nu-\lambda}r\nu_r + 2e^{\nu}\big(r^2n^2f^2\,(\vartheta_0')^2(e^{-\nu}) - 1\big) \\
&2e^{\nu-\lambda} + 2e^{\nu-\lambda}r\nu_r + 2r^2n^2f^2\,(\vartheta_0')^2 - 2e^{\nu}
\end{aligned}$$

Substituting back into the equation:

$$-8n^2f^2r^2(\vartheta_0')^2 + 2e^{\nu-\lambda} + 2e^{\nu-\lambda}r\nu_r + 2r^2n^2f^2\left(\vartheta_0'\right)^2 - 2e^{\nu} = r^2(\vartheta_0')^2$$

$$-6n^2f^2r^2(\vartheta_0')^2 + 2e^{\nu-\lambda} + 2e^{\nu-\lambda}r\nu_r - 2e^{\nu} = r^2(\vartheta_0')^2$$

$$2e^{\nu-\lambda} + 2e^{\nu-\lambda}r\nu_r - 2e^{\nu} = r^2(\vartheta_0')^2 + 6n^2f^2r^2(\vartheta_0')^2$$

$$2e^{\nu}\left(e^{-\lambda} + e^{-\lambda}r\nu_r - 1\right) = r^2\left(1 + 6n^2f^2\right)(\vartheta_0')^2 \tag{1}$$

Equation 2

$$\begin{aligned} e^{-\lambda/2}r\sin\theta\left(e^{\nu/2}r(0)(\nu_r) + \lambda_\tau\right) &= 0 \\ e^{-\lambda/2}r\sin\theta(\lambda_\tau) &= 0 \\ \lambda_\tau &= 0 \end{aligned} \tag{2}$$

This constraint will be applied to the rest of the equations.

Equation 3

$$
\begin{aligned}
&\frac{1}{4}e^{\frac{-\lambda-\nu}{2}}\sin\theta\Big[-e^{\lambda}r\big(\lambda_\tau^2-\lambda_\tau\nu_\tau+2\lambda_{\tau\tau}\big)-4e^{\lambda+\frac{\nu}{2}}r\Big(0\lambda_\tau-4nf\vartheta_0'nf\vartheta_0'(-e^{-\frac{1}{2}\nu})\Big)\\
&\quad+e^{\nu}\Big(-4(0)^2re^{\lambda}+2\nu_r-\lambda_r(2+r\nu_r)\\
&\qquad\qquad+r\Big(\nu_r^2+2\nu_{rr}+4e^{\lambda}(nf\vartheta_0'(-e^{-\frac{1}{2}\nu}))^2\Big)\Big)\Big]\\
&=\frac{1}{2}r\sin\theta e^{-\nu}\sqrt{e^{\lambda+\nu}}\left(2Ve^{\nu}+(\vartheta_0')^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}e^{\frac{1}{2}(-\lambda-\nu)}\Big[-4e^{\lambda+\frac{1}{2}\nu}r\Big(4n^2f^2(\vartheta_0')^2e^{-\frac{1}{2}\nu}\Big)\\
&\quad+e^{\nu}\Big(2\nu_r-\lambda_r(2+r\nu_r)+r\Big(\nu_r^2+2\nu_{rr}+4e^{\lambda}(n^2f^2(\vartheta_0')^2e^{-\nu})\Big)\Big)\Big]\\
&=re^{-\frac{\nu}{2}+\frac{\lambda}{2}}(\vartheta_0')^2
\end{aligned}
$$

$$
\begin{aligned}
&-4e^{\frac{1}{2}\nu}r\Big(4n^2f^2(\vartheta_0')^2e^{-\frac{1}{2}\nu}\Big)\\
&\quad+e^{\nu-\lambda}\Big(2\nu_r-\lambda_r(2+r\nu_r)+r\Big(\nu_r^2+2\nu_{rr}+4e^{\lambda}(n^2f^2(\vartheta_0')^2e^{-\nu})\Big)\Big)\\
&=2r(\vartheta_0')^2
\end{aligned}
$$

$$
\begin{aligned}
&e^{\nu-\lambda}\Big(2\nu_r-\lambda_r(2+r\nu_r)+r\Big(\nu_r^2+2\nu_{rr}+4e^{\lambda}(n^2f^2(\vartheta_0')^2e^{-\nu})\Big)\Big)\\
&=2r(\vartheta_0')^2+16rn^2f^2(\vartheta_0')^2
\end{aligned}
$$

$$
\begin{aligned}
&e^{\nu-\lambda}\Big(2\nu_r-\lambda_r(2+r\nu_r)+r\nu_r^2+2r\nu_{rr}+4re^{\lambda-\nu}n^2f^2(\vartheta_0')^2\Big)\\
&=2r(\vartheta_0')^2+16rn^2f^2(\vartheta_0')^2
\end{aligned}
$$

$$
\frac{1}{2}e^{\nu-\lambda}\Big(2\nu_r-\lambda_r(2+r\nu_r)+r\nu_r^2+2r\nu_{rr}\Big)=r(\vartheta_0')^2+6rn^2f^2(\vartheta_0')^2
$$

$$
e^{\nu-\lambda}\Big(\nu_r-\frac{1}{2}\lambda_r(2+r\nu_r)+\frac{1}{2}r\nu_r^2+r\nu_{rr}\Big)=r\left(1+6n^2f^2\right)(\vartheta_0')^2 \tag{3}
$$

Equation 4

$$
\begin{aligned}
-\frac{1}{2}e^{\frac{1}{2}\nu}r\sin(\theta)\left(\nu_r\left(-e^{-\frac{1}{2}\nu}\vartheta_0' nf\right)+2\left(\frac{1}{2}e^{-\frac{1}{2}\nu}nf\vartheta_0'\nu_r\right)\right)&=0\\
-\nu_r e^{-\frac{1}{2}\nu}\vartheta_0' nf+e^{-\frac{1}{2}\nu}nf\vartheta_0'\nu_r&=0\\
0&=0
\end{aligned}
\tag{4}
$$

Equation 5

Equation 5 is functionally the same structure as Equation 4 and yields:

$$0=0 \tag{5}$$

Equation 6

$$
\begin{aligned}
&\frac{1}{4}e^{\frac{1}{2}(-\lambda-\nu)}\Big[-e^{\lambda}r\big(\lambda_\tau^2-\lambda_\tau\nu_\tau+2\lambda_{\tau\tau}\big)\\
&\quad-4e^{\lambda+\frac{1}{2}\nu}r\big(0\lambda_\tau-4\,nf\,\vartheta_0'\,nf\,\vartheta_0'(-e^{-\frac{1}{2}\nu})\big)\\
&\quad+e^{\nu}\Big(-4(0)^2re^{\lambda}+2\nu_r-\lambda_r\big(2+r\nu_r\big)\\
&\qquad+r\big(\nu_r^2+2\nu_{rr}+4e^{\lambda}(nf\,\vartheta_0'(-e^{-\frac{1}{2}\nu}))^2\big)\Big)\Big]\\
&=\frac{1}{2}re^{-\nu}\sqrt{e^{\lambda+\nu}}\left(2V(\vartheta_0)e^{\nu}+(\vartheta_0')^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}e^{\frac{1}{2}(-\lambda-\nu)}\Big[-4e^{\lambda+\frac{1}{2}\nu}r\big(-4\,nf\,\vartheta_0'\,nf\,\vartheta_0'(-e^{-\frac{1}{2}\nu})\big)\\
&\quad+e^{\nu}\Big(2\nu_r-\lambda_r\big(2+r\nu_r\big)+r\big(\nu_r^2+2\nu_{rr}+4e^{\lambda}(nf\,\vartheta_0'(-e^{-\frac{1}{2}\nu}))^2\big)\Big)\Big]\\
&=re^{-\nu}\sqrt{e^{\lambda+\nu}}(\vartheta_0')^2
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}e^{\frac{1}{2}(-\lambda-\nu)}\Big[-4e^{\lambda+\frac{1}{2}\nu}r\big(-4n^2f^2(\vartheta_0')^2(-e^{-\frac{1}{2}\nu})\big)\\
&\quad+e^{\nu}\Big(2\nu_r-\lambda_r\big(2+r\nu_r\big)+r\big(\nu_r^2+2\nu_{rr}+4e^{\lambda-\nu}n^2f^2(\vartheta_0')^2\big)\Big)\Big]\\
&=re^{(\lambda-\nu)/2}(\vartheta_0')^2
\end{aligned}
$$

$$\frac{1}{2}\Big[-4e^{\frac{1}{2}\nu}r\big(-4n^2f^2(\vartheta_0')^2(-e^{-\frac{1}{2}\nu})\big)$$
$$+e^{\nu-\lambda}\Big(2\nu_r-\lambda_r\big(2+r\nu_r\big)+r\big(\nu_r^2+2\nu_{rr}+4e^{\lambda-\nu}n^2f^2(\vartheta_0')^2\big)\Big)\Big]$$
$$=r(\vartheta_0')^2$$

$$\frac{1}{2}\Big[-16rn^2f^2(\vartheta_0')^2$$
$$+2e^{\nu-\lambda}\nu_r-e^{\nu-\lambda}\lambda_r\big(2+r\nu_r\big)+e^{\nu-\lambda}r\nu_r^2+2e^{\nu-\lambda}r\nu_{rr}+4rn^2f^2(\vartheta_0')^2\Big]$$
$$=r(\vartheta_0')^2$$

$$-6rn^2f^2(\vartheta_0')^2+e^{\nu-\lambda}\nu_r-\frac{1}{2}e^{\nu-\lambda}\lambda_r\big(2+r\nu_r\big)+\frac{1}{2}e^{\nu-\lambda}r\nu_r^2+e^{\nu-\lambda}r\nu_{rr}$$
$$=r(\vartheta_0')^2$$

$$e^{\nu-\lambda}\left(\nu_r-\frac{1}{2}\lambda_r\big(2+r\nu_r\big)+\frac{1}{2}r\nu_r^2+r\nu_{rr}\right)=r(1+6n^2f^2)(\vartheta_0')^2$$

$$e^{\nu-\lambda}\left(\nu_r-\lambda_r-\frac{1}{2}r\nu_r\left(\lambda_r+\nu_r\right)+r\nu_{rr}\right)=r(1+6n^2f^2)(\vartheta_0')^2 \qquad (6)$$

Equation 7

Equation 7 is functionally the same structure as Equation 2 and yields:

$$\lambda_\tau=0 \qquad (7)$$

Equation 8

$$
\begin{aligned}
&e^{\frac{1}{2}(-\lambda-\nu)}\sin\theta\Big[-e^{\lambda}r^2 0\lambda_\tau \\
&\quad -e^{\frac{1}{2}\nu}\Big(-1+r\lambda_r+e^{\lambda}\Big(3r^2\big(0^2-(nf\vartheta_0'(-e^{-\frac{1}{2}\nu}))^2\big)+1\Big)\Big)\Big] \\
&= \frac{1}{2}r^2\sin\theta\, e^{-\frac{3}{2}\nu}\sqrt{e^{\lambda+\nu}}\Big(2Ve^{\nu}-\vartheta_0'^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&2e^{\frac{1}{2}(-\lambda-\nu)}\Big[-e^{\frac{1}{2}\nu}\Big(-1+r\lambda_r+e^{\lambda}\Big(-3r^2(nf\vartheta_0'(-e^{-\frac{1}{2}\nu}))^2+1\Big)\Big)\Big] \\
&= -r^2e^{-\nu+\frac{\lambda}{2}}(\vartheta_0')^2
\end{aligned}
$$

$$
2e^{-\lambda+\nu}\Big(-1+r\lambda_r+e^{\lambda}\Big(-3r^2n^2f^2(\vartheta_0')^2e^{-\nu}+1\Big)\Big)=r^2(\vartheta_0')^2
$$

$$
2e^{-\lambda+\nu}\Big(-1+r\lambda_r-3r^2n^2f^2(\vartheta_0')^2e^{\lambda-\nu}+e^{\lambda}\Big)=r^2(\vartheta_0')^2
$$

$$
2e^{-\lambda+\nu}\Big(-1+r\lambda_r+e^{\lambda}\Big)=r^2(\vartheta_0')^2+6r^2n^2f^2(\vartheta_0')^2
$$

$$
2e^{\nu-\lambda}\Big(-1+r\lambda_r+e^{\lambda}\Big)=r^2(1+6n^2f^2)(\vartheta_0')^2 \tag{8}
$$

List of Equations

$$2e^{\nu}\left(e^{-\lambda}+e^{-\lambda}r\nu_r-1\right)=r^2(1+6n^2f^2)(\vartheta_0')^2 \tag{9}$$

$$\lambda_\tau=0 \tag{10}$$

$$e^{\nu-\lambda}\Big(\nu_r-\frac{1}{2}\lambda_r(2+r\nu_r)+\frac{1}{2}r\nu_r^2+r\nu_{rr}\Big)=r(1+6n^2f^2)(\vartheta_0')^2 \tag{11}$$

$$e^{\nu-\lambda}\left(\nu_r-\lambda_r-\frac{1}{2}r\nu_r\left(\lambda_r+\nu_r\right)+r\nu_{rr}\right)=r(1+6n^2f^2)(\vartheta_0')^2 \tag{12}$$

$$2e^{\nu-\lambda}\Big(-1+r\lambda_r+e^{\lambda}\Big)=r^2(1+6n^2f^2)(\vartheta_0')^2 \tag{13}$$

Equations 9 and 13 can be equated to each other:

$$\begin{aligned}
2e^{\nu}\left(e^{-\lambda}+e^{-\lambda}r\nu_r-1\right)&=2e^{\nu-\lambda}\Big(-1+r\lambda_r+e^{\lambda}\Big)\\
e^{-\lambda}+e^{-\lambda}r\nu_r-1&=e^{-\lambda}\Big(-1+r\lambda_r+e^{\lambda}\Big)\\
e^{-\lambda}+e^{-\lambda}r\nu_r-1&=-e^{-\lambda}+e^{-\lambda}r\lambda_r\\
2e^{-\lambda}+e^{-\lambda}r\nu_r-e^{-\lambda}r\lambda_r&=1\\
e^{-\lambda}\left(2+r\nu_r-r\lambda_r\right)&=1\\
2+r\left(\nu_r-\lambda_r\right)&=e^{\lambda}
\end{aligned}$$

$$\nu_r-\lambda_r=\frac{e^{\lambda}-2}{r} \tag{14}$$

Equations 11 and 12 can be equated to each other:

$$
\begin{aligned}
e^{\nu-\lambda}\Big(\nu_r - \frac{\lambda_r}{2}(2+r\nu_r) + \frac{r}{2}\nu_r^2 + r\nu_{rr}\Big) &= e^{\nu-\lambda}(\nu_r - \lambda_r - \frac{r}{2}\nu_r(\lambda_r+\nu_r) + r\nu_{rr}) \\
\nu_r - \frac{\lambda_r}{2}(2+r\nu_r) + \frac{r}{2}\nu_r^2 + r\nu_{rr} &= \nu_r - \lambda_r - \frac{r}{2}\nu_r(\lambda_r+\nu_r) + r\nu_{rr} \\
-\frac{\lambda_r}{2}(2+r\nu_r) + \frac{r}{2}\nu_r^2 &= -\lambda_r - \frac{r}{2}\nu_r(\lambda_r+\nu_r) \\
-\lambda_r - \frac{r}{2}\lambda_r\nu_r + \frac{r}{2}\nu_r^2 &= -\lambda_r - \frac{r}{2}\nu_r\lambda_r - \frac{r}{2}\nu_r^2 \\
\frac{r}{2}\nu_r^2 &= -\frac{r}{2}\nu_r^2 \\
r\nu_r^2 &= 0
\end{aligned}
$$

$$
\nu_r = 0 \tag{15}
$$

Applying this to Equation 14 we get the following differential equation:

$$
-\lambda_r = \frac{e^\lambda - 2}{r} \tag{16}
$$

Final Equations

$$
-\lambda_r = \frac{e^\lambda - 2}{r} \tag{17}
$$

$$
\nu_r = 0 \tag{18}
$$

$$
\lambda_\tau = 0 \tag{19}
$$