Torsion-Induced Corrections to the Schwarzschild Metric from Nieh-Yan Modified Gravity

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Abstract

We investigate black hole solutions in Einstein-Cartan gravity with a Nieh-Yan coupling to a scalar field. By evaluating the equations of motion for a Schwarzschild-like ansatz, we derive a torsionful correction to the radial component of the metric. The resulting modification depends on the scalar field's time derivative, introducing a constant rescaling to the radial term while preserving the Schwarzschild form in the absence of torsion.

1 Actions

We consider total action $S_{tot} = S_{EC} + S_m + S_{NY}$ composed of the following terms: the Einstein-Cartan gravitational action

$$S_{EC} = -\frac{M_{Pl}^2}{4} \int \varepsilon_{abcd} V^a \wedge V^b \wedge R^{cd}, \tag{1}$$

the scalar field action

$$S_m = \frac{1}{2} \int \mathrm{d}\vartheta \wedge {}^\star \mathrm{d}\vartheta, \tag{2}$$

and the Nieh-Yan coupling

$$S_{NY} = -nf \int \mathrm{d}\vartheta \wedge T^a \wedge V_a. \tag{3}$$

The curvature and torsion two forms are defined

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \tag{4}$$

$$T^a = \mathrm{d}V^a + \omega^a{}_b \wedge V^b. \tag{5}$$

1.1 Equations of Motion

Varying with respect to the Vielbein yields the Einstein equations

$$\frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \varepsilon_{abcd} - 2nf(\mathrm{d}\vartheta \wedge \mathrm{d}V_a + \mathrm{d}\vartheta \wedge \omega_{ba} \wedge V^b) = \tau_D, \tag{6}$$

where

$$\tau_D = -\frac{1}{2}(\partial^b \vartheta \mathrm{d}\vartheta \wedge {}^\star(V_b \wedge V_a) + \partial_a \vartheta {}^\star \mathrm{d}\vartheta)$$

Varying with respect to the spin connection yields the torsion constraint

$$\varepsilon_{abcd} T^c \wedge V^d = -\frac{2nf}{M_{Pl}^2} \mathrm{d}\vartheta \wedge V_a \wedge V_b \tag{7}$$

Varying with respect to the scalar yields the Klein-Gordon equation

$$\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \vartheta\right) = \frac{nf}{2} \partial_\mu \left(T_{\nu\rho}{}^a V_{a\sigma}\right) \varepsilon^{\mu\nu\rho\sigma} \tag{8}$$

When evaluating the Equations of Motion, we set $M_{Pl} = 1$.

1.2 Spin Connection

The spin connection is decomposed into torsion-free and torsion-induced parts

$$\omega^{ab} = \bar{\omega}^{ab} + \tilde{\omega}^{ab} \tag{9}$$

where

$$\tilde{\omega}^{0b} = 0 \qquad \tilde{\omega}^{ab} = \phi(\tau)\varepsilon^{ab}{}_{k}V^{k}$$

2 Application to Black Holes

To model a static, spherically symmetric black hole, we adopt the generalized Schwarzschild metric

$$\mathrm{d}s^2 = -e^{\alpha(\tau,r)}\,\mathrm{d}\tau^2 + e^{\beta(\tau,r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\,. \tag{10}$$

This has the following metric tensor and inverse:

$$g_{\mu\nu} = \begin{pmatrix} -e^{\alpha(\tau,r)} & 0 & 0 & 0 \\ 0 & e^{\beta(\tau,r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{11}$$

$$g^{\mu\nu} = \begin{pmatrix} -e^{-\alpha(\tau,r)} & 0 & 0 & 0 \\ 0 & e^{-\beta(\tau,r)} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{12}$$

The Vielbeins are

$$V^a{}_\mu = \begin{pmatrix} e^{\frac{\alpha(\tau,r)}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\beta(\tau,r)}{2}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}$$

$$V_a{}^\mu = g^{\mu\nu}\eta_{ab}V^b{}_\nu = \begin{pmatrix} e^{\frac{-\alpha(\tau,r)}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{-\beta(\tau,r)}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}$$

$$V^{a\mu} = g^{\mu\nu}V^a{}_\nu = \begin{pmatrix} -e^{\frac{-\alpha(\tau,r)}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{-\beta(\tau,r)}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}$$

2.1 Torsion Constraint

Evaluating the Torsion constraint with the general Schwarzchild metric and full spin connection we find

$$\phi(\tau) = -nfe^{-\frac{\alpha}{2}}\vartheta_\tau \tag{13}$$

2.2 Einstein Equations

Evaluating the Einstein equations with the torsion-full spin connection yields the following differential equations:

$$e^\alpha(1 + r\alpha_r) - e^\beta(e^\alpha + 3n^2f^2r^2\vartheta_\tau^2) = \frac{1}{2}r^2e^\beta\vartheta_\tau^2 \tag{14}$$

$$e^\alpha(1 - r\beta_r) - e^\beta(e^\alpha - 3n^2f^2r^2\vartheta_\tau^2) = -\frac{1}{2}r^2e^\beta\vartheta_\tau^2 \tag{15}$$

$$e^\alpha\left((2 + r\alpha_r)(\alpha_r - \beta_r) + 2r\alpha_{rr}\right) - 12e^\beta rn^2f^2\vartheta_\tau^2 = 2re^\beta\vartheta_\tau^2 \tag{16}$$

$$\beta_\tau = \alpha_\tau = \vartheta_r = 0 \tag{17}$$

So, we can say $\alpha = \alpha(r)$, $\beta = \beta(r)$, $\vartheta_0 = \vartheta_0(\tau)$. We introduce the standard Schwarzschild ansatz for e^α:

$$e^\alpha = a = 1 - \frac{C}{r} \qquad \alpha_r = a^{-1}\frac{C}{r^2} \tag{18}$$

Evaluating eq.14 with the ansatz gives us

$$e^\beta = \frac{1}{a}\left(1 + \frac{r^2\vartheta_\tau^2\left(\frac{1}{2} + 3n^2f^2\right)}{a}\right)^{-1}$$

$$e^\beta = \left(1 - \frac{C}{r}\right)^{-1}\left(1 + \frac{r^2\vartheta_\tau^2\left(\frac{1}{2} + 3n^2f^2\right)}{1 - \frac{C}{r}}\right)^{-1} \tag{19}$$

This corresponds to the Schwarzschild radial term modified by a torsion-dependent correction.

2.3 Klein Gordon Equation

Evaluating the Klein Gordon Equation yields

$$-2re^{\frac{1}{2}\alpha+\beta}\vartheta_{\tau\tau}(1 + 6n^2f^2) = 0 \tag{20}$$

which requires the scalar field's time derivative ϑ_τ to be constant.

3 Torsionful Metric

The torsionful metric for a black hole then takes the form

$$\mathrm{d}s^2 = -\left(1 - \frac{C}{r}\right)\mathrm{d}\tau^2 + \left(1 - \frac{C}{r}\right)^{-1}\mathcal{T}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2 \tag{21}$$

where $\mathcal{T}$ is the torsional correction term

$$\mathcal{T} = \left(1 + \frac{r^2\vartheta_\tau^2\left(\frac{1}{2} + 3n^2f^2\right)}{1 - \frac{C}{r}}\right)^{-1}$$

Because ϑ_τ is constant, the torsional correction $\mathcal{T}$ remains constant, effectively rescaling the radial metric component.