Sanity check on Einstein Equation

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We start with the Einstein equation found by varying the total action w.r.t. the Vielbein:

$$\begin{aligned}\frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} - 2nf(\mathrm{d}\vartheta \wedge \mathrm{d}V_a + \mathrm{d}\vartheta \wedge \omega_{ba} \wedge V^b) \\ = -\frac{1}{2}(\partial^b \vartheta \mathrm{d}\vartheta \wedge {}^\star(V_b \wedge V_a) + \partial_a \vartheta^\star \mathrm{d}\vartheta)\end{aligned} \tag{1}$$

Set $n = 0$ to remove all torsion. At the end, this should yield the standard Schwarzchild metric:

$$\frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} = -\frac{1}{2}(\partial^b \vartheta \mathrm{d}\vartheta \wedge {}^\star(V_b \wedge V_a) + \partial_a \vartheta^\star \mathrm{d}\vartheta) \tag{2}$$

This has simplified to a standard Einstein equation. For the vielbeins, we must choose a spherically symmetrical general metric:

$$\mathrm{d}s^2 = -e^{\nu(t,r)}\,\mathrm{d}t^2 + e^{\lambda(t,r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\vartheta^2\,. \tag{3}$$

Since $V^a = V^a_\mu \mathrm{d}x^\mu$, we can use the Vielbein of the general metric, calculated in previous notes (Vielbein of the General Schwarzchild Metric):

$$V^a_\mu = \begin{pmatrix} e^{\frac{\nu(t,r)}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\lambda(t,r)}{2}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix} \tag{4}$$

The values of V^a are then:

$$\begin{aligned} V^0 &= e^{\frac{\nu(t,r)}{2}}\,\mathrm{d}t \\ V^1 &= e^{\frac{\lambda(t,r)}{2}}\,\mathrm{d}r \\ V^2 &= r\,\mathrm{d}\theta \\ V^3 &= r\sin\theta\,\mathrm{d}\vartheta \end{aligned}$$

Next, we impose the torsion-free condition to calculate the spin connection:

$$\mathrm{d}V^a + \omega^a_b \wedge V^b = 0 \tag{5}$$