Equations of Motion from the Action

8/11/2025

From my previous notes, I had calculated the Variations on the total action:

$$
\begin{aligned}
\delta_V S_{tot} = \frac{M_{Pl}^2}{2} \int & R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a \\
& + \frac{1}{2} \int (\partial^a \phi \mathrm{d}\phi \wedge {}^\star(V_a \wedge V_b) \wedge \delta V^b + \partial_b \phi^\star \mathrm{d}\phi \wedge \delta V^b) \\
& - 2nf \int \mathrm{d}\phi \wedge \mathrm{d}V_a \wedge \delta V^a + \mathrm{d}\phi \wedge \omega_{bc} \wedge V^b \wedge \delta V^c
\end{aligned} \tag{1}
$$

$$
\delta_\omega S_{tot} = \frac{M_{Pl}^2}{2} \int \epsilon_{abcd} V^a \wedge V^b \wedge \omega^{fd} \wedge \delta\omega_f^c + \frac{nf}{2} \int \mathrm{d}\phi \wedge V_a \wedge V_c \wedge \delta\omega^{ac} \tag{2}
$$

$$
\delta_\phi S_{tot} = -\int \delta\phi \, \mathrm{d}^\star \mathrm{d}\phi + nf \int \mathrm{d}(T^a \wedge V_a) \wedge \delta\phi \tag{3}
$$

1 Vielbein

$$
\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a + \frac{1}{2}(\partial^a \phi \mathrm{d}\phi \wedge {}^\star(V_a \wedge V_b) \wedge \delta V^b + \partial_b \phi^\star \mathrm{d}\phi \wedge \delta V^b)
$$
$$
-2nf(\mathrm{d}\phi \wedge \mathrm{d}V_a \wedge \delta V^a + \mathrm{d}\phi \wedge \omega_{bc} \wedge V^b \wedge \delta V^c) = 0
$$

Relabeling indices:

$$
\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a + \frac{1}{2}(\partial^b \phi \mathrm{d}\phi \wedge {}^\star(V_b \wedge V_a) \wedge \delta V^a + \partial_a \phi^\star \mathrm{d}\phi \wedge \delta V^a)
$$
$$
-2nf(\mathrm{d}\phi \wedge \mathrm{d}V_a \wedge \delta V^a + \mathrm{d}\phi \wedge \omega_{ba} \wedge V^b \wedge \delta V^a) = 0
$$

$$
\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a + \frac{1}{2}(\partial^b \phi \mathrm{d}\phi \wedge {}^\star(V_b \wedge V_a) + \partial_a \phi^\star \mathrm{d}\phi) \wedge \delta V^a
$$
$$
-2nf(\mathrm{d}\phi \wedge \mathrm{d}V_a + \mathrm{d}\phi \wedge \omega_{ba} \wedge V^b) \wedge \delta V^a = 0
$$

$$
\int (\frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} + \frac{1}{2}(\partial^b \phi \mathrm{d}\phi \wedge {}^\star(V_b \wedge V_a) + \partial_a \phi^\star \mathrm{d}\phi)
$$
$$
-2nf(\mathrm{d}\phi \wedge \mathrm{d}V_a + \mathrm{d}\phi \wedge \omega_{ba} \wedge V^b)) \wedge \delta V^a = 0
$$

Since δV^a is arbitrary:

$$\frac{M_{Pl}^2}{2}R^{cd}\wedge V^b\epsilon_{abcd}+\frac{1}{2}(\partial^b\phi\mathrm{d}\phi\wedge{}^\star(V_b\wedge V_a)+\partial_a\phi^\star\mathrm{d}\phi)$$
$$-2nf(\mathrm{d}\phi\wedge\mathrm{d}V_a+\mathrm{d}\phi\wedge\omega_{ba}\wedge V^b)=0 \tag{4}$$

so, the Equation of Motion:

$$\frac{M_{Pl}^2}{2}R^{cd}\wedge V^b\epsilon_{abcd}-2nf(\mathrm{d}\phi\wedge\mathrm{d}V_a+\mathrm{d}\phi\wedge\omega_{ba}\wedge V^b)$$
$$=-\frac{1}{2}(\partial^b\phi\mathrm{d}\phi\wedge{}^\star(V_b\wedge V_a)+\partial_a\phi^\star\mathrm{d}\phi) \tag{5}$$

2 Spin Connection

$$\int\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega^{fd}\wedge\delta\omega_f^c+\int\frac{nf}{2}\,\mathrm{d}\phi\wedge V_a\wedge V_c\wedge\delta\omega^{ac}=0$$
$$\int\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega_f^d\wedge\delta\omega^{fc}+\int\frac{nf}{2}\,\mathrm{d}\phi\wedge V_a\wedge V_c\wedge\delta\omega^{ac}=0$$

Relabeling indices:

$$\int\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega_f^d\wedge\delta\omega^{fc}+\int\frac{nf}{2}\,\mathrm{d}\phi\wedge V_f\wedge V_c\wedge\delta\omega^{fc}=0$$
$$\int(\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega_f^d+\frac{nf}{2}\,\mathrm{d}\phi\wedge V_f\wedge V_c)\wedge\delta\omega^{fc}=0$$

So,

$$\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega_f^d+\frac{nf}{2}\,\mathrm{d}\phi\wedge V_f\wedge V_c=0$$

3 Scalar Field

$$-\int\delta\phi\,\mathrm{d}^\star\mathrm{d}\phi+nf\int\mathrm{d}(T^a\wedge V_a)\wedge\delta\phi=0$$