Equations of Motion from the Action

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From my previous notes, I had calculated the Variations on the total action:

$$
\begin{aligned}
\delta_V S_{tot} = \frac{M_{Pl}^2}{2} \int R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a \\
+ \frac{1}{2} \int (\partial^a \vartheta \mathrm{d}\vartheta \wedge {}^\star(V_a \wedge V_b) \wedge \delta V^b + \partial_b \vartheta^\star \mathrm{d}\vartheta \wedge \delta V^b) \\
- 2nf \int \mathrm{d}\vartheta \wedge \mathrm{d}V_a \wedge \delta V^a + \mathrm{d}\vartheta \wedge \omega_{bc} \wedge V^b \wedge \delta V^c
\end{aligned} \tag{1}
$$

$$
\delta_\omega S_{tot} = \frac{M_{Pl}^2}{2} \int \epsilon_{abcd} V^a \wedge V^b \wedge \omega^{fd} \wedge \delta\omega^c_f + \frac{nf}{2} \int \mathrm{d}\vartheta \wedge V_a \wedge V_c \wedge \delta\omega^{ac} \tag{2}
$$

$$
\delta_\vartheta S_{tot} = - \int \delta\vartheta \, \mathrm{d}^\star \mathrm{d}\vartheta + nf \int \mathrm{d}(T^a \wedge V_a) \wedge \delta\vartheta \tag{3}
$$

1 Vielbein

$$
\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a + \frac{1}{2}(\partial^a \vartheta \mathrm{d}\vartheta \wedge {}^\star(V_a \wedge V_b) \wedge \delta V^b + \partial_b \vartheta^\star \mathrm{d}\vartheta \wedge \delta V^b)
$$
$$
-2nf(\mathrm{d}\vartheta \wedge \mathrm{d}V_a \wedge \delta V^a + \mathrm{d}\vartheta \wedge \omega_{bc} \wedge V^b \wedge \delta V^c) = 0
$$

Relabeling indices:

$$
\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a + \frac{1}{2}(\partial^b \vartheta \mathrm{d}\vartheta \wedge {}^\star(V_b \wedge V_a) \wedge \delta V^a + \partial_a \vartheta^\star \mathrm{d}\vartheta \wedge \delta V^a)
$$
$$
-2nf(\mathrm{d}\vartheta \wedge \mathrm{d}V_a \wedge \delta V^a + \mathrm{d}\vartheta \wedge \omega_{ba} \wedge V^b \wedge \delta V^a) = 0
$$
$$
\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a + \frac{1}{2}(\partial^b \vartheta \mathrm{d}\vartheta \wedge {}^\star(V_b \wedge V_a) + \partial_a \vartheta^\star \mathrm{d}\vartheta) \wedge \delta V^a
$$
$$
-2nf(\mathrm{d}\vartheta \wedge \mathrm{d}V_a + \mathrm{d}\vartheta \wedge \omega_{ba} \wedge V^b) \wedge \delta V^a = 0
$$
$$
\int (\frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} + \frac{1}{2}(\partial^b \vartheta \mathrm{d}\vartheta \wedge {}^\star(V_b \wedge V_a) + \partial_a \vartheta^\star \mathrm{d}\vartheta)
$$
$$
-2nf(\mathrm{d}\vartheta \wedge \mathrm{d}V_a + \mathrm{d}\vartheta \wedge \omega_{ba} \wedge V^b)) \wedge \delta V^a = 0
$$

Since δV^a is arbitrary:

$$\frac{M_{Pl}^2}{2}R^{cd}\wedge V^b\epsilon_{abcd}+\frac{1}{2}(\partial^b\vartheta\mathrm{d}\vartheta\wedge{}^\star(V_b\wedge V_a)+\partial_a\vartheta^\star\mathrm{d}\vartheta)$$
$$-2nf(\mathrm{d}\vartheta\wedge\mathrm{d}V_a+\mathrm{d}\vartheta\wedge\omega_{ba}\wedge V^b)=0 \tag{4}$$

so, the Equation of Motion:

$$\frac{M_{Pl}^2}{2}R^{cd}\wedge V^b\epsilon_{abcd}-2nf(\mathrm{d}\vartheta\wedge\mathrm{d}V_a+\mathrm{d}\vartheta\wedge\omega_{ba}\wedge V^b)$$
$$=-\frac{1}{2}(\partial^b\vartheta\mathrm{d}\vartheta\wedge{}^\star(V_b\wedge V_a)+\partial_a\vartheta^\star\mathrm{d}\vartheta) \tag{5}$$

2 Spin Connection

$$\int\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega^{fd}\wedge\delta\omega_f^c+\int\frac{nf}{2}\,\mathrm{d}\vartheta\wedge V_a\wedge V_c\wedge\delta\omega^{ac}=0$$
$$\int\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega_f^d\wedge\delta\omega^{fc}+\int\frac{nf}{2}\,\mathrm{d}\vartheta\wedge V_a\wedge V_c\wedge\delta\omega^{ac}=0$$

Relabeling indices:

$$\int\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega_f^d\wedge\delta\omega^{fc}+\int\frac{nf}{2}\,\mathrm{d}\vartheta\wedge V_f\wedge V_c\wedge\delta\omega^{fc}=0$$
$$\int(\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega_f^d+\frac{nf}{2}\,\mathrm{d}\vartheta\wedge V_f\wedge V_c)\wedge\delta\omega^{fc}=0$$

So,

$$\frac{M_{Pl}^2}{2}\epsilon_{abcd}V^a\wedge V^b\wedge\omega_f^d+\frac{nf}{2}\,\mathrm{d}\vartheta\wedge V_f\wedge V_c=0$$

3 Scalar Field

$$-\int\delta\vartheta\,\mathrm{d}^\star\mathrm{d}\vartheta+nf\int\mathrm{d}(T^a\wedge V_a)\wedge\delta\vartheta=0$$

Simplifying:

$$\int \delta\vartheta \left(-\,\mathrm{d}^{\star}\mathrm{d}\vartheta + nf\,\mathrm{d}(T^{a}\wedge V_{a})\right) = 0$$

This implies:

$$\begin{aligned} -\,\mathrm{d}^{\star}\mathrm{d}\vartheta + nf\,\mathrm{d}(T^{a}\wedge V_{a}) &= 0 \\ nf\,\mathrm{d}(T^{a}\wedge V_{a}) &= \mathrm{d}^{\star}\mathrm{d}\vartheta \\ nf\,\mathrm{d}(V^{a}\wedge T_{a}) &= \mathrm{d}^{\star}\mathrm{d}\vartheta \end{aligned}$$

From section 2 of Pollari (in literature):

$$\frac{nf}{2}\partial_{\mu}\left(V^{a}_{\nu}T_{\rho\sigma a}\right)\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\rho}\wedge\mathrm{d}x^{\sigma} = \mathrm{d}^{\star}\mathrm{d}\vartheta$$

Writing this in terms of the D'Alembert operator, which can be written as $\Box\vartheta = {}^{\star}\mathrm{d}^{\star}\mathrm{d}\vartheta$ since ϑ is a 0-form. Therefore: $\mathrm{d}^{\star}\mathrm{d}\vartheta = \Box\vartheta^{\star}1$. The equation above is written as:

$$\begin{aligned} \Box\vartheta &= \frac{nf}{2}\partial_{\mu}\left(V^{a}_{\nu}T_{\rho\sigma a}\right){}^{\star}(\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\rho}\wedge\mathrm{d}x^{\sigma}) \\ \Box\vartheta &= \frac{nf}{2}\partial_{\mu}\left(V^{a}_{\nu}T_{\rho\sigma a}\right)\frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma} \end{aligned}$$

Where ε is the Levi-Civita Symbol. Writing the D'Alembertian in coordinate form:

$$\begin{aligned} \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\vartheta\right) &= \frac{nf}{2}\partial_{\mu}\left(V^{a}_{\nu}T_{\rho\sigma a}\right)\frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma} \\ \partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\vartheta\right) &= \frac{nf}{2}\partial_{\mu}\left(T^{a}_{\rho\sigma}V_{a\nu}\right)\varepsilon^{\mu\nu\rho\sigma} \\ \partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\vartheta\right) &= \frac{nf}{2}\partial_{\mu}\left(T^{a}_{\nu\rho}V_{a\sigma}\right)\varepsilon^{\mu\nu\rho\sigma} \end{aligned}$$