Vielbein of the Schwarzchild Metric

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The Schwarzchild Metrc:

$$\mathrm{d}s^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$$

I will use natural units ($G = 1$ and $c = 1$):

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$$

This gives the following metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Using this, I can calculate the Vielbein e^a_μ.

Sanity check: The Vielbein is simply a set of basis vectors for each point of the manifold that express our metric. It will satisfy the following equation:

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \qquad a, b = 0, 1, 2, 3 \quad \mu, \nu = t, r, \theta, \phi$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ and is the Minkowski metric. Since the metric is diagonal, the equation can be simplified by saying $a = b$. I will also constrain the Vielbein to being a diagonal matrix, so $e^a_\mu = \mathrm{diag}(e^0_t, e^1_r, e^2_\theta, e^3_\phi)$, which allows me to simplify further by saying $\mu = \nu$. This also gives a correspondance between local and Schwarzchild coordinates, where $a = 0, 1, 2, 3$ correspond to $\mu = t, r, \theta, \phi$ respectively in the Vielbein. Any other combination equals 0.

So, I am left with the equation:

$$g_{\mu\mu} = \eta_{aa}(e^a_\mu)^2$$

Solving this explicitly:

$$
\begin{aligned}
g_{tt} &= -\left(1 - \frac{2M}{r}\right) = -(e_t^0)^2 \\
e_t^0 &= \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
g_{rr} &= \left(1 - \frac{2M}{r}\right)^{-1} = -(e_r^1)^2 \\
e_r^1 &= \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
g_{\theta\theta} &= r^2 = -(e_\theta^2)^2 \\
e_\theta^2 &= r
\end{aligned}
$$

$$
\begin{aligned}
g_{\phi\phi} &= r^2 \sin^2\theta = -(e_\phi^3)^2 \\
e_\phi^3 &= r \sin\theta
\end{aligned}
$$

So, the Vielbein:

$$
e_\mu^a = \begin{pmatrix}
\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} & 0 & 0 & 0 \\
0 & \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} & 0 & 0 \\
0 & 0 & r & 0 \\
0 & 0 & 0 & r\sin\theta
\end{pmatrix}
$$