

Variations on the Action

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I'll start by considering the total action $S_{tot} = S_{EC} + S_m + S_{NY}$ where

$$S_{EC} = -\frac{M_{Pl}^2}{4} \int \epsilon_{abcd} V^a \wedge V^b \wedge R^{cd} \quad (1)$$

$$S_m = \frac{1}{2} \int d\phi \wedge \star d\phi \quad (2)$$

$$S_{NY} = -nf \int d\phi \wedge T^a \wedge V_a \quad (3)$$

S_{EC} is the Einstein-Cartan gravitational action, S_m is the scalar field , and S_{NY} is the coupling to a Nieh-Yan Form. I'll also define:

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \omega_c^a \wedge \omega^{cb} \\ T^a &= dV^a + \omega_b^a \wedge V^b \end{aligned}$$

The variation of the total action is

$$\delta S_{tot} = \delta S_{EC} + \delta S_m + \delta S_{NY} \quad (4)$$

1 Vielbein

1.1 Einstein-Cartan Action

$$\begin{aligned} \delta_V S_{EC} &= -\frac{M_{Pl}^2}{4} \int \epsilon_{abcd} (\delta V^a \wedge V^b + V^a \wedge \delta V^b) \wedge R^{cd} \\ &= -\frac{M_{Pl}^2}{4} \int 2\epsilon_{abcd} \delta V^a \wedge V^b \wedge R^{cd} \\ &\text{(using the antisymmetry of the Levi-Civita Symbol)} \\ &= \frac{M_{Pl}^2}{2} \int R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a \\ &= \boxed{\int \frac{M_{Pl}^2}{2} R^{cd} \wedge V^b \epsilon_{abcd} \wedge \delta V^a} \end{aligned} \quad (5)$$

1.2 Scalar Field

$$\begin{aligned}
\delta_V S_m &= \frac{1}{2} \int \delta d\phi \wedge \star d\phi + d\phi \wedge \delta \star d\phi \\
&= \frac{1}{2} \int d\phi \wedge \delta \star d\phi \quad \text{since } \delta_V d\phi = 0
\end{aligned} \tag{6}$$

Expanding:

$$\begin{aligned}
\star d\phi &= \star(\partial_\mu \phi dx^\mu) = \star(\partial_\mu \phi V_a^\mu V^a) \\
&= \frac{1}{3!} V_a^\mu \partial_\mu \phi \epsilon_{bcd}^a V^b \wedge V^c \wedge V^d
\end{aligned}$$

So,

$$\delta_V \star d\phi = \frac{1}{3!} (\delta V_a^\mu) \partial_\mu \phi \epsilon_{bcd}^a V^b \wedge V^c \wedge V^d + \frac{1}{3!} V_a^\mu \partial_\mu \phi \epsilon_{bcd}^a \delta(V^b \wedge V^c \wedge V^d) \tag{7}$$

Using the identity

$$\begin{aligned}
(\delta V_a^\mu) V_\mu^j &= -(\delta V_\mu^j) V_a^\mu \\
(\delta V_a^\mu) &= -(\delta V_\nu^j) V_a^\nu V_j^\mu
\end{aligned}$$

and plugging that into eq. (7) and using the antisymmetry of the Levi-Civita Symbol in the second term:

$$\begin{aligned}
&\delta_V \star d\phi \\
&= -\frac{1}{3!} (\delta V_\nu^j) V_a^\nu V_j^\mu \partial_\mu \phi \epsilon_{bcd}^a V^b \wedge V^c \wedge V^d + \frac{1}{3!} V_a^\mu \partial_\mu \phi (3) \epsilon_{bcd}^a \delta V^b \wedge V^c \wedge V^d \\
&= -\frac{1}{3!} (\delta V_\nu^j) V_a^\nu \partial_j \phi \epsilon_{bcd}^a V^b \wedge V^c \wedge V^d + \frac{1}{2} \partial_a \phi \epsilon_{bcd}^a \delta V^b \wedge V^c \wedge V^d
\end{aligned}$$

Since we are in an orthonormal frame defined by Vielbeins, we can also say that $\partial_a \phi \epsilon_{bcd}^a = \partial^a \phi \epsilon_{abcd}$. Using this in the second term:

$$\begin{aligned}
\delta_V \star d\phi &= -\frac{1}{3!} (\delta V_\nu^j) V_a^\nu \partial_j \phi \epsilon_{bcd}^a V^b \wedge V^c \wedge V^d + \frac{1}{2} \partial^a \phi \epsilon_{abcd} \delta V^b \wedge V^c \wedge V^d \\
&= -(\delta V_\nu^j) V_a^\nu \partial_j \phi \left(\frac{1}{3!} \epsilon_{bcd}^a V^b \wedge V^c \wedge V^d \right) + \partial^a \phi \left(\frac{1}{2} \epsilon_{abcd} V^c \wedge V^d \right) \wedge \delta V^b
\end{aligned}$$

Finally, using the definition of a dual, we can simplify the terms in parenthesis:

$$\begin{aligned}
\delta_V^* d\phi &= -(\delta V_\nu^j) V_a^\nu \partial_j \phi (^* V^a) + \partial^a \phi^* (V_a \wedge V_b) \wedge \delta V^b \\
&= -\partial_j \phi (\delta V_a^j) (^* V^a) + \partial^a \phi \delta V^b \wedge (^* (V_a \wedge V_b)) \\
&= -\partial_j \phi^* (\delta V_a^j V^a) + \partial^a \phi \delta V^b \wedge (^* (V_a \wedge V_b)) \\
&= -\partial_j \phi^* (\delta V^j) + \partial^a \phi \delta V^b \wedge (^* (V_a \wedge V_b))
\end{aligned}$$

Relabeling indices, we are left with

$$\delta_V^* d\phi = \partial^a \phi \delta V^b \wedge (^* (V_a \wedge V_b)) - \partial_b \phi^* (\delta V^b) \quad (8)$$

Plugging this back into eq. (6):

$$\begin{aligned}
\delta_V S_m &= \frac{1}{2} \int d\phi \wedge (\partial^a \phi \delta V^b \wedge (^* (V_a \wedge V_b)) - \partial_b \phi^* (\delta V^b)) \\
&= \frac{1}{2} \int \partial^a \phi d\phi \wedge \delta V^b \wedge (^* (V_a \wedge V_b)) - \partial_b \phi d\phi \wedge (^* (\delta V^b))
\end{aligned}$$

Using the identity $A \wedge ^* B = B \wedge ^* A$ when A and B have the same dimension in the second term:

$$\begin{aligned}
\delta_V S_m &= \frac{1}{2} \int \partial^a \phi d\phi \wedge (^* (V_a \wedge V_b)) \wedge \delta V^b - \partial_b \phi (\delta V^b) \wedge ^* d\phi \\
&= \boxed{\int \frac{1}{2} (\partial^a \phi d\phi \wedge (^* (V_a \wedge V_b)) \wedge \delta V^b + \partial_b \phi^* d\phi \wedge \delta V^b)} \quad (9)
\end{aligned}$$

1.3 Nieh-Yan Form

$$\delta_V S_{NY} = -nf \int d\phi \wedge (\delta T^a \wedge V_a + T^a \wedge \delta V_a)$$

Expanding:

$$\begin{aligned}
& \delta_V S_{NY} \\
&= -nf \int d\phi \wedge ((\delta dV^a + \omega_b^a \wedge \delta V^b) \wedge V_a + (dV^a + \omega_b^a \wedge V^b) \wedge \delta V_a) \\
&= -nf \int d\phi \wedge (\delta dV^a \wedge V_a + \omega_b^a \wedge \delta V^b \wedge V_a + dV^a \wedge \delta V_a + \omega_b^a \wedge V^b \wedge \delta V_a) \\
&= -nf \int d\phi \wedge (\delta dV^a \wedge V_a + dV^a \wedge \delta V_a + \omega_b^a \wedge (\delta V^b \wedge V_a + V^b \wedge \delta V_a))
\end{aligned} \tag{10}$$

Focusing on the term with the spin connection:

$$\begin{aligned}
& \omega_b^a \wedge (\delta V^b \wedge V_a + V^b \wedge \delta V_a) \\
&= \omega_b^a \wedge \delta V^b \wedge V_a + \omega_b^a \wedge V^b \wedge \delta V_a \\
&= \omega_b^a \wedge \delta V^b \wedge \eta_{ac} V^c + \omega_b^a \wedge V^b \wedge \eta_{ac} \delta V^c \\
&= \eta_{ac} \omega_b^a \wedge \delta V^b \wedge V^c + \eta_{ac} \omega_b^a \wedge V^b \wedge \delta V^c \\
&= \omega_{bc} \wedge \delta V^b \wedge V^c + \omega_{bc} \wedge V^b \wedge \delta V^c
\end{aligned} \tag{11}$$

Focusing on the second term:

$$\begin{aligned}
& \omega_{bc} \wedge V^b \wedge \delta V^c \\
&= \omega_{cb} \wedge V^c \wedge \delta V^b \quad (\text{swapped indices}) \\
&= -\omega_{cb} \wedge \delta V^b \wedge V^c \\
&= \omega_{bc} \wedge \delta V^b \wedge V^c \\
& \quad (\text{using the antisymmetry of the spin connection})
\end{aligned}$$

Plugging this back into eq. (11):

$$\omega_{bc} \wedge \delta V^b \wedge V^c + \omega_{bc} \wedge \delta V^b \wedge V^c = 2\omega_{bc} \wedge \delta V^b \wedge V^c$$

Plugging this back into eq.(10):

$$\delta_V S_{NY} = -nf \int d\phi \wedge (\delta dV^a \wedge V_a + dV^a \wedge \delta V_a + 2\omega_{bc} \wedge \delta V^b \wedge V^c) \tag{12}$$

Now, focusing on $\delta dV^a \wedge V_a + dV^a \wedge \delta V_a$. We can use the Leibniz Rule:

$$\begin{aligned}
d(\delta V^a \wedge V_a) &= d\delta V^a \wedge V_a + (-1)\delta V^a \wedge dV_a \\
&= d\delta V^a \wedge V_a - \delta V^a \wedge dV_a \\
&= d\delta V^a \wedge V_a - dV_a \wedge \delta V^a \\
&= d\delta V^a \wedge V_a - dV^a \wedge \delta V_a \\
d(\delta V^a \wedge V_a) + dV^a \wedge \delta V_a &= \delta dV^a \wedge V_a
\end{aligned}$$

Plugging this into eq. (12):

$$\begin{aligned}
&\delta_V S_{NY} \\
&= -nf \int d\phi \wedge (d(\delta V^a \wedge V_a) + dV^a \wedge \delta V_a + dV^a \wedge \delta V_a + 2\omega_{bc} \wedge \delta V^b \wedge V^c) \\
&= -nf \int d\phi \wedge (d(\delta V^a \wedge V_a) + 2dV^a \wedge \delta V_a + 2\omega_{bc} \wedge \delta V^b \wedge V^c)
\end{aligned}$$

The first term ends up being a boundary term due to, so we can drop the term. Therefore:

$$\begin{aligned}
\delta_V S_{NY} &= -2nf \int d\phi \wedge (dV^a \wedge \delta V_a + \omega_{bc} \wedge \delta V^b \wedge V^c) \\
&= -2nf \int d\phi \wedge dV^a \wedge \delta V_a + d\phi \wedge \omega_{bc} \wedge \delta V^b \wedge V^c \\
&= -2nf \int d\phi \wedge dV_a \wedge \delta V^a - d\phi \wedge \omega_{bc} \wedge V^c \wedge \delta V^b \\
&= -2nf \int d\phi \wedge dV_a \wedge \delta V^a - d\phi \wedge \omega_{cb} \wedge V^b \wedge \delta V^c \\
&= \boxed{\int -2nf (d\phi \wedge dV_a \wedge \delta V^a + d\phi \wedge \omega_{bc} \wedge V^b \wedge \delta V^c)}
\end{aligned}$$

2 Spin Connection

2.1 Einstein-Cartan Action

$$\begin{aligned}
\delta_\omega S_{EC} &= -\frac{M_{Pl}^2}{4} \int \epsilon_{abcd} V^a \wedge V^b \wedge \delta R^{cd} \\
&= -\frac{M_{Pl}^2}{4} \int \epsilon_{abcd} V^a \wedge V^b \wedge (d \delta \omega^{cd} + \delta \omega_f^c \wedge \omega^{fd} + \omega_f^c \wedge \delta \omega^{fd}) \\
&= -\frac{M_{Pl}^2}{4} \int V^a \wedge V^b \wedge (\epsilon_{abcd} d \delta \omega^{cd} + \epsilon_{abcd} \delta \omega_f^c \wedge \omega^{fd} + \epsilon_{abcd} \omega_f^c \wedge \delta \omega^{fd})
\end{aligned} \tag{13}$$

Simplifying:

$$\begin{aligned}
&\epsilon_{abcd} \delta \omega_f^c \wedge \omega^{fd} + \epsilon_{abcd} \omega_f^c \wedge \delta \omega^{fd} \\
&= \epsilon_{abcd} \delta \omega_f^c \wedge \eta^{fi} \omega_i^d + \epsilon_{abcd} \omega_f^c \wedge \eta^{fi} \delta \omega_i^d \\
&= \epsilon_{abcd} \eta^{fi} \delta \omega_f^c \wedge \omega_i^d - \epsilon_{abcd} \eta^{fi} \delta \omega_i^d \wedge \omega_f^c \\
&\quad (\text{swapping indices } c \leftrightarrow d \text{ and } f \leftrightarrow i) \\
&= \epsilon_{abcd} \eta^{fi} \delta \omega_f^c \wedge \omega_i^d - \epsilon_{abdc} \eta^{if} \delta \omega_f^c \wedge \omega_i^d \\
&= \epsilon_{abcd} \eta^{fi} \delta \omega_f^c \wedge \omega_i^d + \epsilon_{abcd} \eta^{fi} \delta \omega_f^c \wedge \omega_i^d \\
&= 2\epsilon_{abcd} \eta^{fi} \delta \omega_f^c \wedge \omega_i^d
\end{aligned}$$

Pluggin this into eq. (14):

$$\delta_\omega S_{EC} = -\frac{M_{Pl}^2}{4} \int V^a \wedge V^b \wedge (\epsilon_{abcd} d \delta \omega^{cd} + 2\epsilon_{abcd} \eta^{fi} \delta \omega_f^c \wedge \omega_i^d)$$

2.2 Scalar Field

$\delta_\omega S_m$ is trivially 0.

2.3 Nieh Yan Term

$$\begin{aligned}
\delta_\omega S_{NY} &= -nf \int d\phi \wedge \delta T^a \wedge V_a \\
&= -nf \int d\phi \wedge (\delta\omega_b^a \wedge V^b) \wedge V_a \\
&= -nf \int d\phi \wedge \delta\omega_b^a \wedge \eta^{bc} V_c \wedge V_a \\
&= -nf \int d\phi \wedge \eta^{bc} \delta\omega_b^a \wedge V_c \wedge V_a \\
&= -nf \int d\phi \wedge \delta\omega^{ac} \wedge V_c \wedge V_a
\end{aligned}$$

Antisymmetrizing:

$$\delta_\omega S_{NY} = -\frac{nf}{2} \int d\phi \wedge \delta\omega^{ac} \wedge V_c \wedge V_a$$