# Terence Tao

Teaches Mathematical Thinking

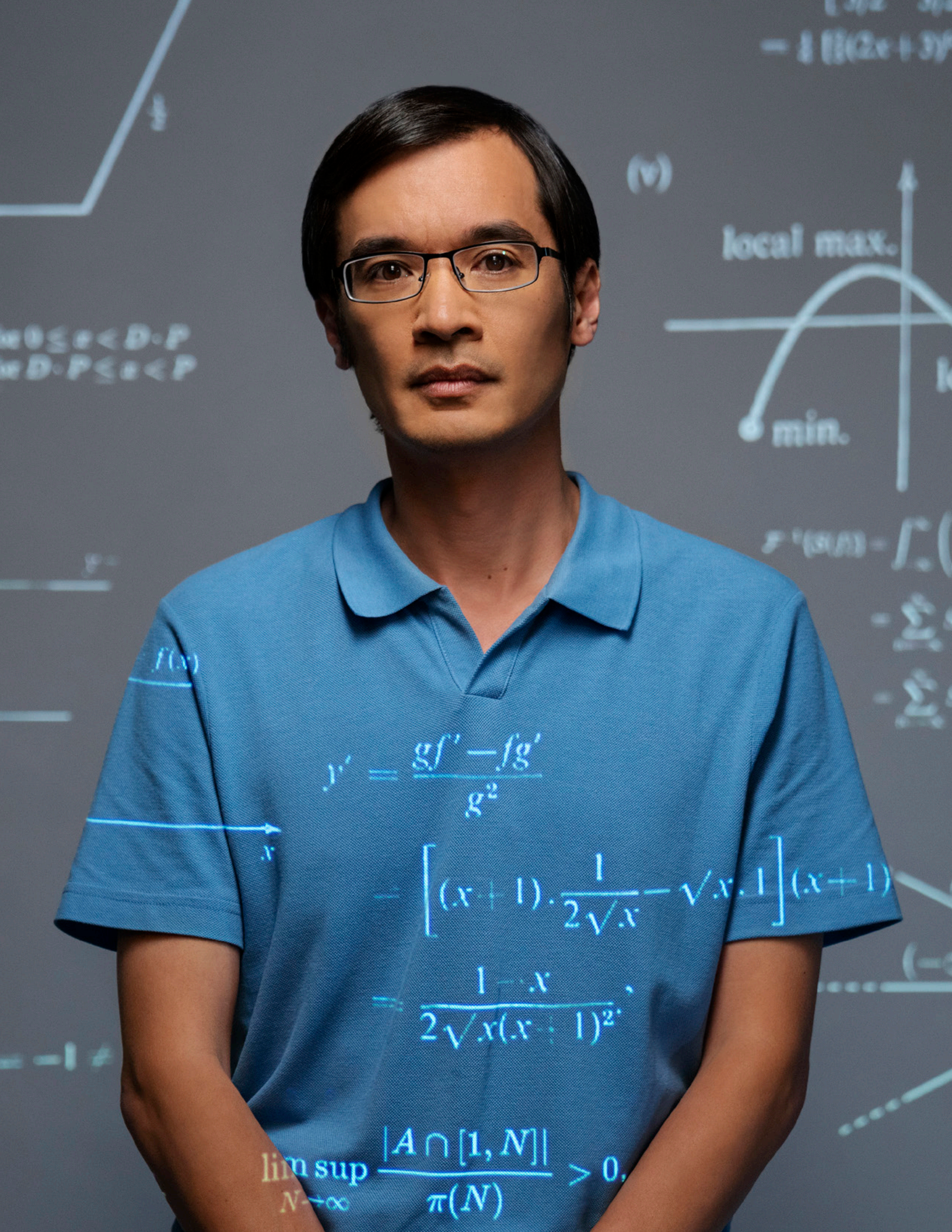
MasterClass

# Table of Contents



'The failure is often more instructive than the success.'

—TERENCE



(v)
local max.
min.
$y' = \frac{gf' - fg'}{g^2}$
$= \left[(x+1)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}\cdot 1\right](x+1)$
$= \frac{1-x}{2\sqrt{x}(x+1)^2}$.
$\limsup_{N\to\infty} \frac{|A \cap [1,N]|}{\pi(N)} > 0$.

# Terence Tao

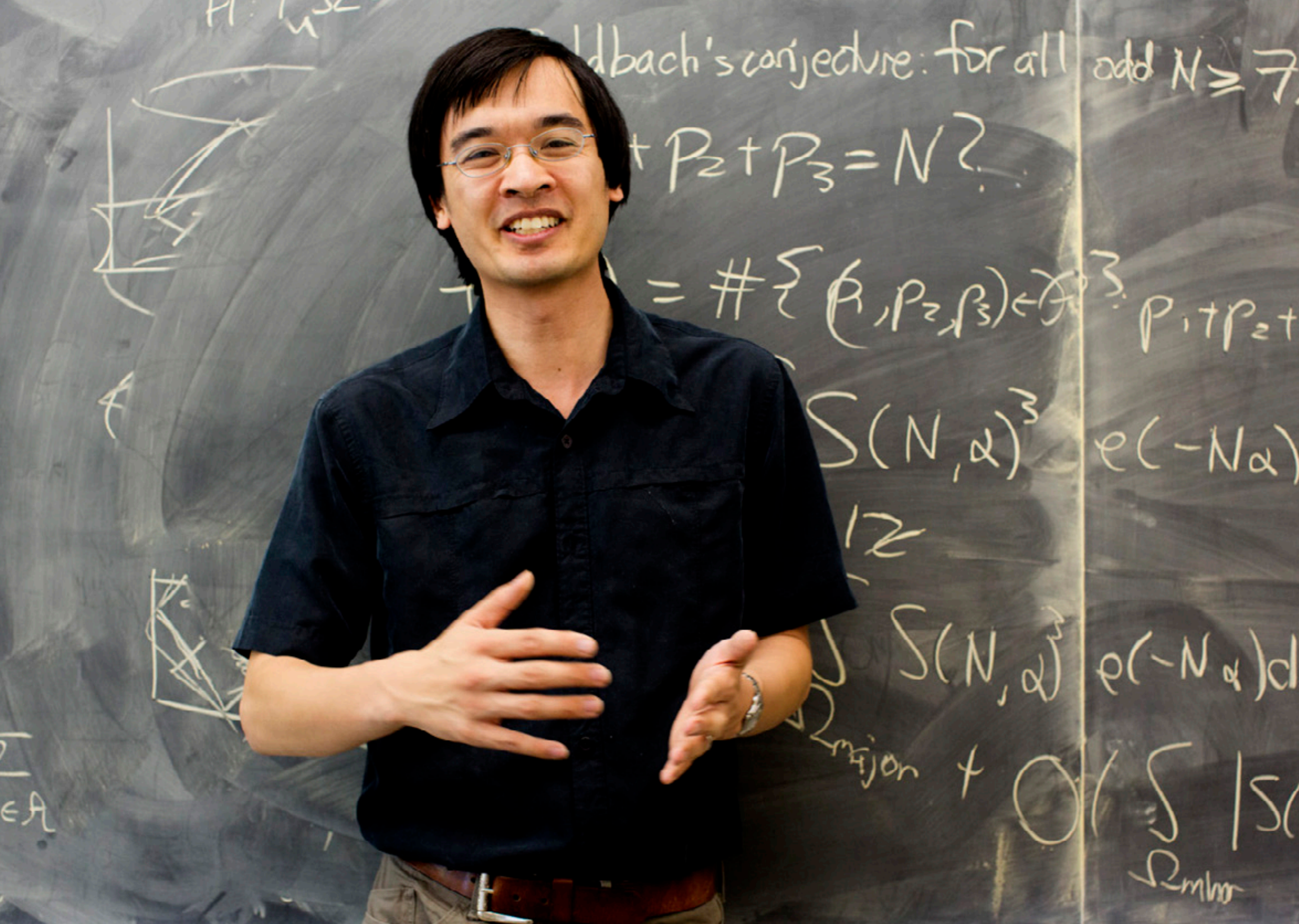
For the ground-breaking mathematician, logical thinking goes beyond theorems and solutions – it's the key to changing minds and, in some cases, saving lives

Terence Tao loves a tough problem – the kind that makes you sweat. He's taken on some of the most baffling equations known to humankind, and he's won.

In the process, he's been part of major advances in mathematics and sparked lifesaving innovation in the field of medicine. His work in number theory [1] and quantum mechanics [2] has solidified his reputation as one of the most powerful minds on the planet, all while opening countless eyes to the beauty, artistry and universal relevance of mathematical thinking.

Born and raised in Adelaide, Australia, Terence found an early mentor in his mother, a secondary school maths teacher. 'My parents tell me [that] when I was two, they found me trying to teach some older kids how to count,' he says. By the age of seven, Terence was learning calculus[3]; by nine, he was taking it at the university level. Within a few years, he was competing in international mathematics competitions.

Terence went on to earn his Ph.D. from Princeton University in New Jersey and join the faculty at the University of California, Los Angeles (UCLA). At the age of twenty-four he was promoted to full professorship, a role he continues to hold. He has received the Fields Medal, the highest prize that mathematics has to offer; a MacArthur Fellowship, also known as a 'genius grant'; and a Royal Medal from the U.K.'s Royal Society, the oldest continuously running scientific academy in the world. He's also authored or co-authored more than three hundred research papers that have been cited more than eighty-



three thousand times, and he's published more than a dozen books.

Among elite mathematicians, Terence is particularly renowned for his work on prime numbers (see 'Prime Time' on page 6). But his contributions have spilled over into other areas, too, with profoundly practical implications. In the mid- to late-2000s, he collaborated with French statistician Emmanuel Candès to advance concepts that allow the creation of high-quality images with less data, slashing the time an MRI[4] machine needs to collect information. Terence's work in this field heralded advances in other realms that rely on sensing technology, including seismology and astronomy (the study of earthquakes and space, respectively).

While few will ever grapple with radical and complex mathematical theories like these, Terence believes that the foundational elements of problem-solving are more accessible than you might expect. Now, he's here to help you reimagine how you approach problems in your own life. Knowing how to think mathematically, Terence says, makes you better prepared to attack myriad challenges. He hopes this class will make the process of problem-solving more manageable and interesting – and even fun.

## Notes

**1. Number theory** is a branch of mathematics that handles the properties and relationships of numbers larger than zero. It is an example of 'pure mathematics', or maths that's typically performed without explicit or immediate consideration of its practical application. Mathematicians often pursue it because of the intellectual challenges it poses and the aesthetic beauty it reveals.

**2. Quantum mechanics** is a mathematical field concerning the behaviour of molecules, atoms and subatomic particles like photons and electrons. Practitioners use it to understand why stars shine and atoms decay, how computer chips operate and why the floor beneath your feet doesn't spontaneously collapse.

**3. Calculus** is a field concerned with derivatives, integrals and functions. It has been described more simply as the study of continuous change.

**4. MRI**, or magnetic resonance imaging, is a technology whereby the inner workings of living beings are rendered visible using magnetic fields and radio waves.

## Prime Time

For Terence, prime numbers are endlessly fascinating. These are whole numbers larger than one that can be divided evenly by only themselves and one. (Think: 2, 3, 5, 7 and 11.) Primes are considered number theory's fundamental building blocks, but they are enigmatic: mathematicians don't exactly understand why they show up where they do. Sometimes pairs of primes, or so-called 'twin primes', appear just two numbers apart, like 5 and 7. In collaboration with the British mathematician Ben Green of the University of Oxford in England, Terence found that, no matter how long of a string of primes (like 3, 7 and 11) with a constant gap (in this case, a distance of four) between them, there would always be a string that's longer. Their proof stands among the most significant discoveries since the Greek mathematician Euclid proved – more than two millennia ago – that the number of prime numbers is infinite.

# Terence Through the Years

At age 2

he taught himself how to read and do basic arithmetic

At age 8

he scored 760 out of 800 on the maths portion of the standardised test known as the SAT, which is usually administered to secondary school students

At age 43

he was named a 'Great Immigrant' by the Carnegie Corporation, a premier philanthropic fund that each year honours an elite number of naturalised Americans

At age 20

he earned a Ph.D. from New Jersey's Princeton University

At age **15**

he was **commissioned to write a training book** for secondary school maths teachers on algebra, Euclidean geometry and analytic geometry

At age **31**

he was made a MacArthur Fellow

At age **46**

he was enlisted by U.S. President Joe Biden to serve on the **President's Council of Advisors** on Science and Technology

At age **10**

he became the **youngest person in history** to win a medal in the International Mathematical Olympiad

At age **28**

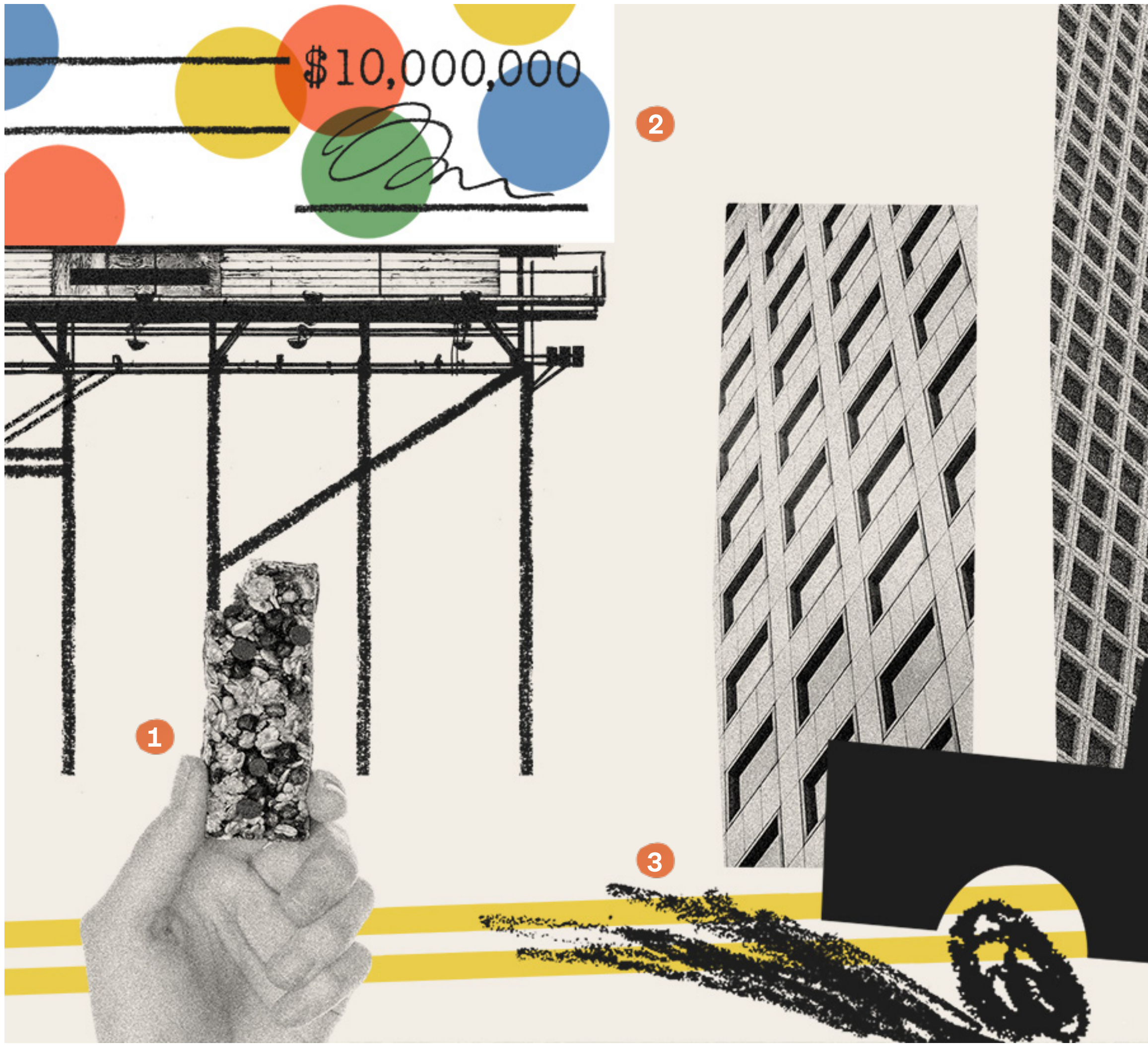
he co-authored the **Green-Tao theorem**, a masterstroke in the field of number theory

At age **24**

he became one of the **youngest tenured professors** at UCLA

# Equations Are Everywhere!

▶ Your days are powered by mathematical concepts




### 1
## Algebra

Farmers use maths every day, from measuring their fields and estimating crop yields to predicting profit over time. (Far more complicated mathematical algorithms are now being used by drones to forecast weather, monitor soil health and detect infectious diseases from above.) The oats in your breakfast bar may have been produced by Mother Nature, but she got a lot of help from mathematical concepts that allow us to work with unknown quantities.

### 2
## Permutations, Combinations, Probabilities

If you were to take a mathematical approach to the lottery, you might compute the number of possible combinations on your ticket using the combination formula (if you don't care about the order) or the permutation formula (if you do). Or you might use a probability formula to analyse the general likelihood of a win. Unfortunately, you're much more likely to be killed by an asteroid (1 in 700,000) than to win the American Powerball jackpot (1 in 292 million).

### 3
## Newton's First Law of Motion

Seventeenth-century English mathematician Isaac Newton's eponymous laws, which define the relationship between the motion of an object and the forces acting on it, quickly become apparent in life-or-death situations. His first law states that an object in motion will stay in motion, in a straight line, until an external force acts upon it – hence seat belts, without which drivers and passengers would fly towards their windscreens in the event of a sudden stop.


## 4
### Navier-Stokes Equations

These equations, which describe the movement of fluids, have been used to model everything from ocean currents to the flow of blood inside your body. (Terence has made major contributions to this field.) Formulated by French physicist Claude Louis-Navier and Irish polymath George Gabriel Stokes in the nineteenth century, they are now employed by car designers looking to ensure their roofs are in peak aerodynamic form.

## 5
### Newton's Universal Law of Gravitation

More than four thousand satellites are currently orbiting Earth, some of them no larger than a loaf of bread. These spacecraft transmit everything from television shows to high-resolution photos of receding snowcaps. But how did they get out there in the first place? More Newton: he's widely acknowledged as the first scientist to prove that a projectile launched with sufficient speed would orbit the planet.

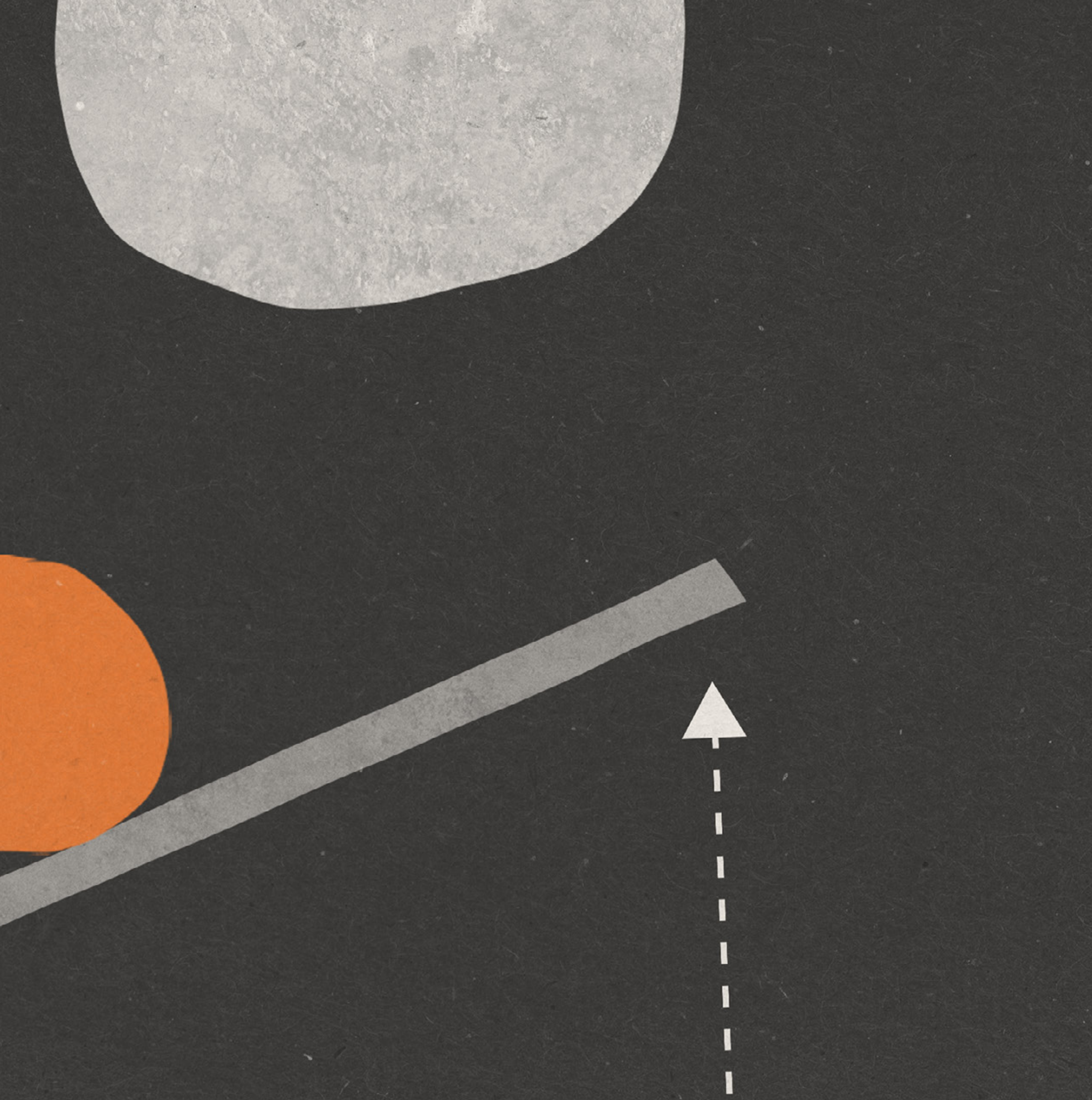
## 6
### Schrödinger Equation

In 1926, the Austrian physicist Erwin Schrödinger found that electrons were more like waves than dots. Swiss scientist Felix Bloch and others soon applied this theory to the complicated systems of atoms found in crystals and metals; they discovered that, if the crystal structure contained even a single out-of-place atom, electron waves would travel a different path through it. This led to the development of semiconductors, now a critical component in smartphones, refrigerators and more.

## 7
### Entropy

First identified by German mathematician Rudolf Clausius in 1865, the concept of entropy posits that energy disperses and systems inevitably dissolve into chaos. That may sound philosophical, but it also has profound practical implications. It can, for instance, help people measure the way heat spreads. Thanks to entropy, New York City's main energy provider, Consolidated Edison, is able to estimate how much steam power it can provide to roughly two thousand buildings throughout the city by boiling water.

# Maths Misconceptions



Chances are, you've encountered a few equations that left you feeling stumped. You might have blamed yourself – or dismissed maths entirely. But problem-solving needn't be so punishing. To get a healthier view, start by vanquishing a few common assumptions

THE MYTH

## Some people are inherently bad at maths.

THE REALITY

Nearly everyone has *some* innate mathematical talent. After all, the average child can understand that sharing a pizza among four people gets them a smaller slice than sharing it among three and that whatever number they can think of, there's always one that's larger. Perhaps people would have more confidence in their mathematical abilities, Terence posits, if they learned maths in a way that harnessed their natural intuitions and curiosities.

THE MYTH

## Maths is all about memorisation.

THE REALITY

To tackle cutting-edge problems, Terence sometimes needs to invent new formulas. In this arena, creativity is key; standard equations don't always apply. When it comes to lower-level education, though, maths is often presented as misleadingly cut-and-dried. Imagine if you weren't allowed to write creatively in a literature class and were instead confined to lessons on diagramming sentences: you might come away from these lessons thinking, *I don't understand prepositions, so I guess I can never be a writer.* Focusing only on the rules leaves out the humanity and dazzling complexity that numbers can possess.

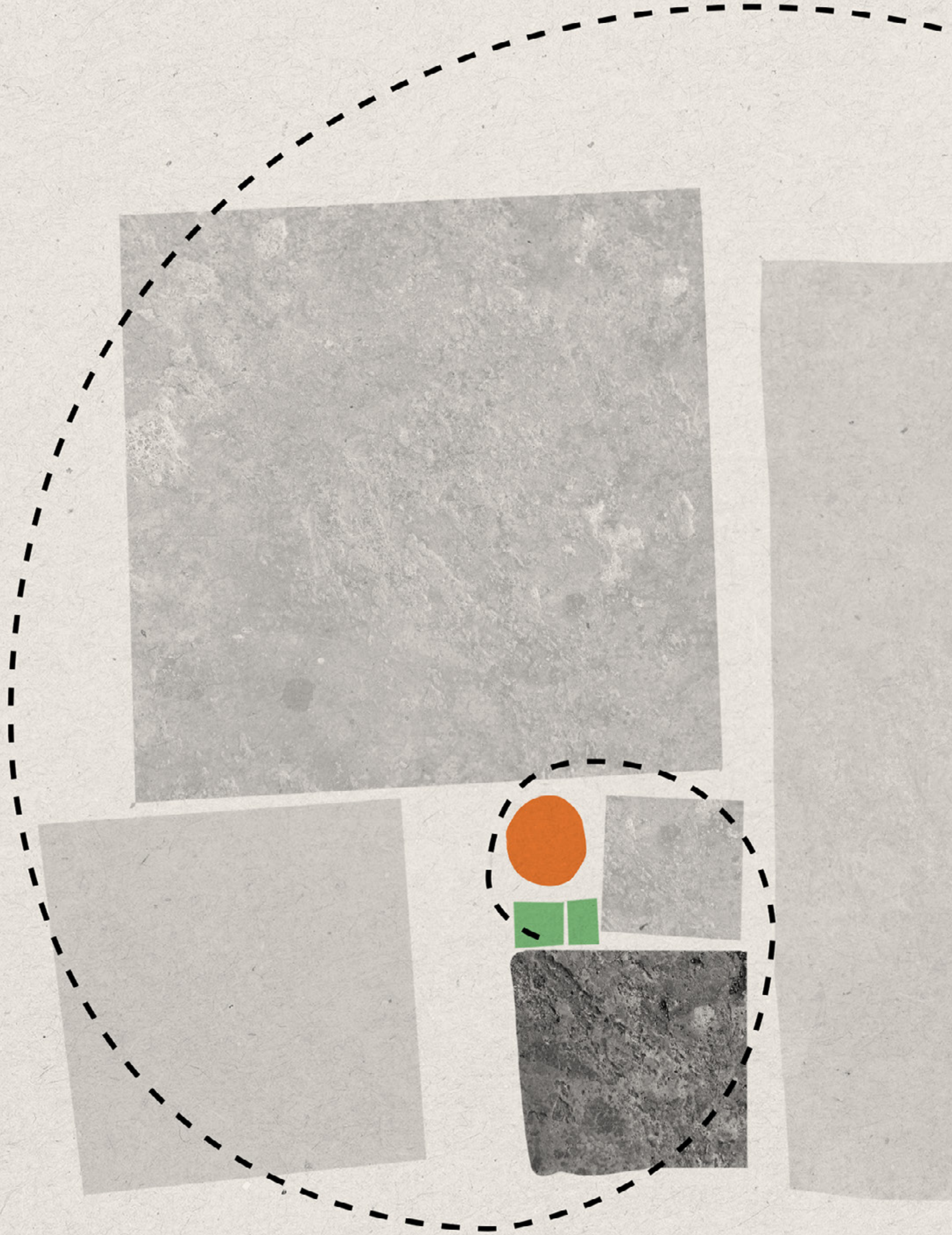
THE MYTH

## Maths is only about numbers.

THE REALITY

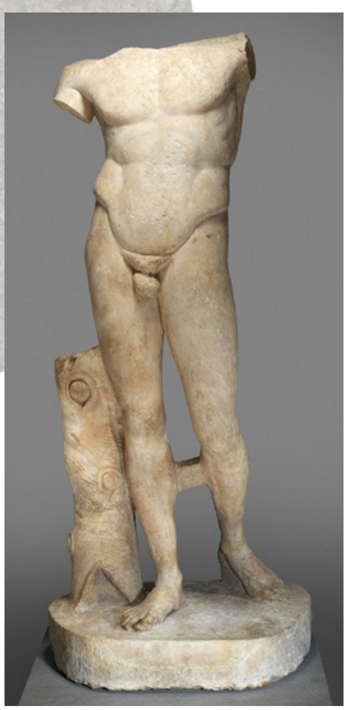
Mathematics touches nearly every aspect of modern life, from computer software design to questions about humanity's existence in the universe. Even classic children's tales like *Alice's Adventures in Wonderland* can be full of mathematical allegories. The point is that maths isn't just moving numbers around; it's a way of viewing the world. 'I think that people often have a fear of mathematics – that it is just too alien, too different from their everyday experience for it to be useful in their lives,' says Terence. '[But] knowing how to think mathematically can be a tool that you can use to solve problems in a systematic way, which can give you reassurance and can make a complex world a little bit more manageable.'

**Even classic children's tales like *Alice's Adventures in Wonderland* can be full of mathematical allegories. The point is that maths isn't just moving numbers around; it's a way of viewing the world.**



# Maths ~~and~~ Is Art

▶ Six of the many ways mathematics has played a role in major creative works



**Above:**
*Diadoumenos*
(c. 430 BC),
by Polykleitos

**Above right:**
*Twenty-three Deity Nairatma Mandala*
(14th century;
Sakya-affiliated
monastery)

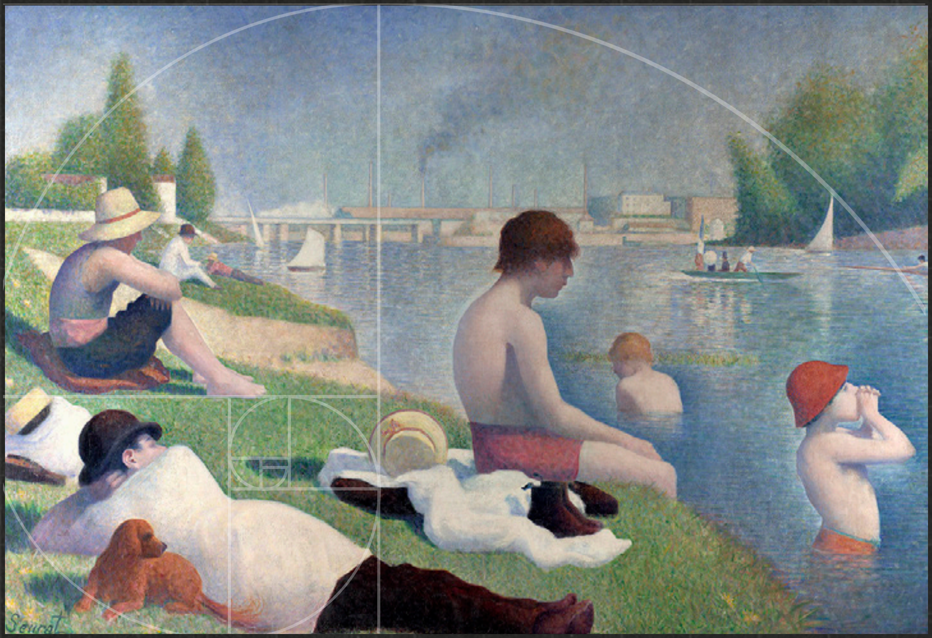
## Proportion

Want to see an example of how art and maths have evolved in tandem? Look no further than *Diadoumenos*, a renowned sculpture from Greek antiquity. The piece is credited to the artist Polykleitos, whose treatise on how to depict the ideal proportions of the human figure advises readers to see the body in four parts, all of which relate to one another in a complementary fashion. With *Diadoumenos*, Polykleitos sought to achieve an ideal balance between muscular tensions, with the figure resting his weight on his right leg while freeing his left to bend. The mathematically determined result is suffused with life.

## Symmetry

Artists have known for millennia that symmetry – a correspondence in size, form or arrangement in different sections of the image – can evoke a sense of balance, serenity and purpose. From geometric Moroccan architecture to ancient Chinese tapestries, symmetrical layouts present the universe as a highly ordered whole. Radial symmetry, achieved by placing images around a central point or axis, commonly appear in sacred images. A fourteenth-century Tibetan piece known as *Twenty-three Deity Nairatma Mandala* shows several figures radiating outward from the Buddhist deity Vajrayogini.

**Above:** ***Primavera*** **(c. 1480), by Sandro Botticelli**




**Above:** ***Bathers at Asnières*** **(1884), by Georges Seurat**

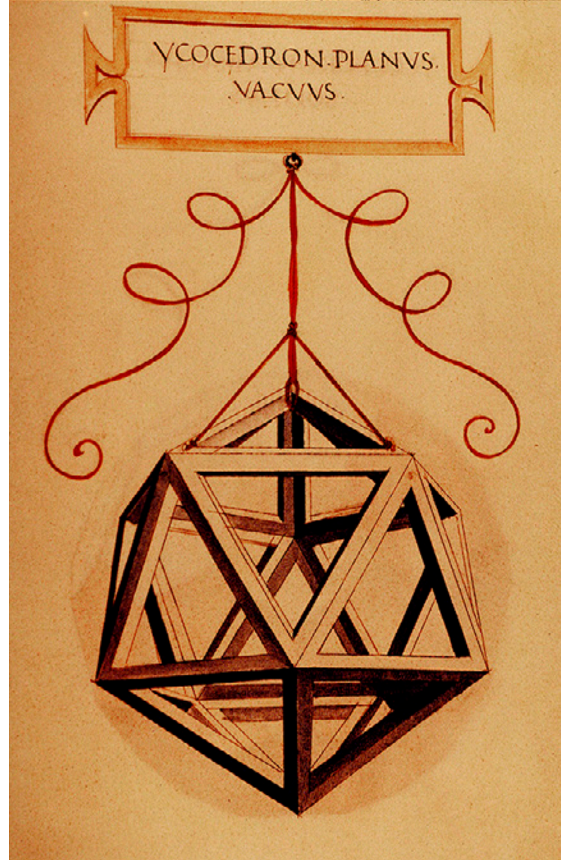
## Golden Ratio

This principle undergirds some of the greatest artworks in history. Also known as the 'divine proportion' or the 'golden section', the ratio refers to phi (Φ), an irrational number, like pi, that can go on forever. First described by the Greek mathematician Euclid around 300 BCE, the ratio is found by dividing a line into two unequal parts; the longer part, when divided by the shorter part, should then be equal to the overall length divided by the longer part. When these proportions are expressed as a spiral, it resembles geometries found in nature. These shapes are widespread in art and architecture, too: Italian Renaissance artists Sandro Botticelli and Michelangelo – and later artists like nineteenth-century French painter Georges Seurat – seem to have used the ratio in their work.


**Above:** ***The Last Judgment*** **(late 16th century), by Giorgio Vasari and Federico Zuccari**

## Linear Perspective

During most of the mediaeval era, the size of a person or object within an artwork corresponded to their thematic or religious importance, often at the expense of realism. But around 1415, the Italian Renaissance architect Filippo Brunelleschi began employing a forgotten method to give painted surfaces a more lifelike appearance. (Brunelleschi designed the dome reproduced above.) While some Chinese artists had used techniques like oblique perspective to give a more convincing view of their subjects, linear perspective drew on ancient Greek and Roman principles to create the illusion of depth. Painters used parallel lines, vanishing points and a horizon line to arrange a work's composition, similar to how the human eye sees the world. *Christ Consigning the Keys to St. Peter* (see page 21), a fifteenth-century painting by the Italian artist Pietro Perugino, is a prime example of this technique.

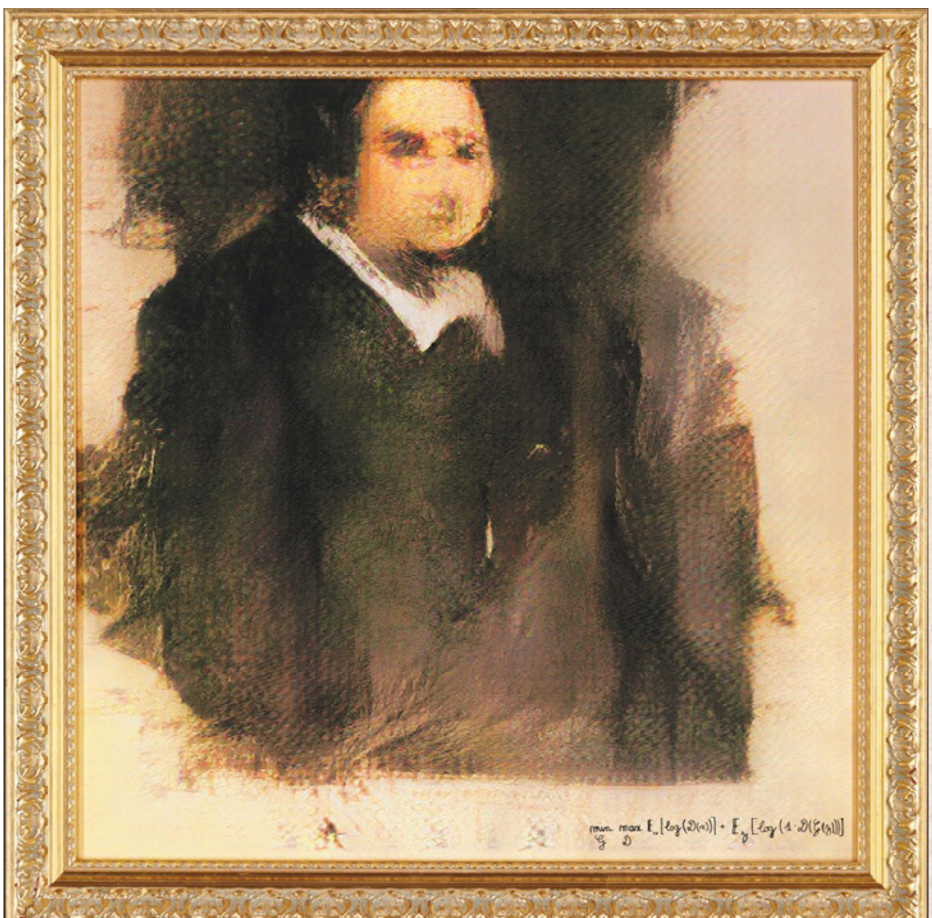



## Fractals and A.I.

By the 1980s, computer processing speeds had progressed to such an extent that algorithms (sets of rules used in computer programming) could generate art. Fractals, a form of abstract art, are patterns that can be unimaginably complex and are often generated by a computer from a single formula. Additionally, artists have been 'teaching' artificial intelligence systems to produce original visual works. A 2018 printed piece called *Edmond de Belamy, from La Famille de Belamy*, which later sold at auction for more than $400,000, was generated by an algorithm that had ingested thousands of examples of portraiture.

**Above:** ***Icosahedron*** **(c. 1509), by Leonardo Da Vinci**

**Below:** ***Edmond de Belamy, from La Famille de Belamy*** **(2018), published by Obvious Art**



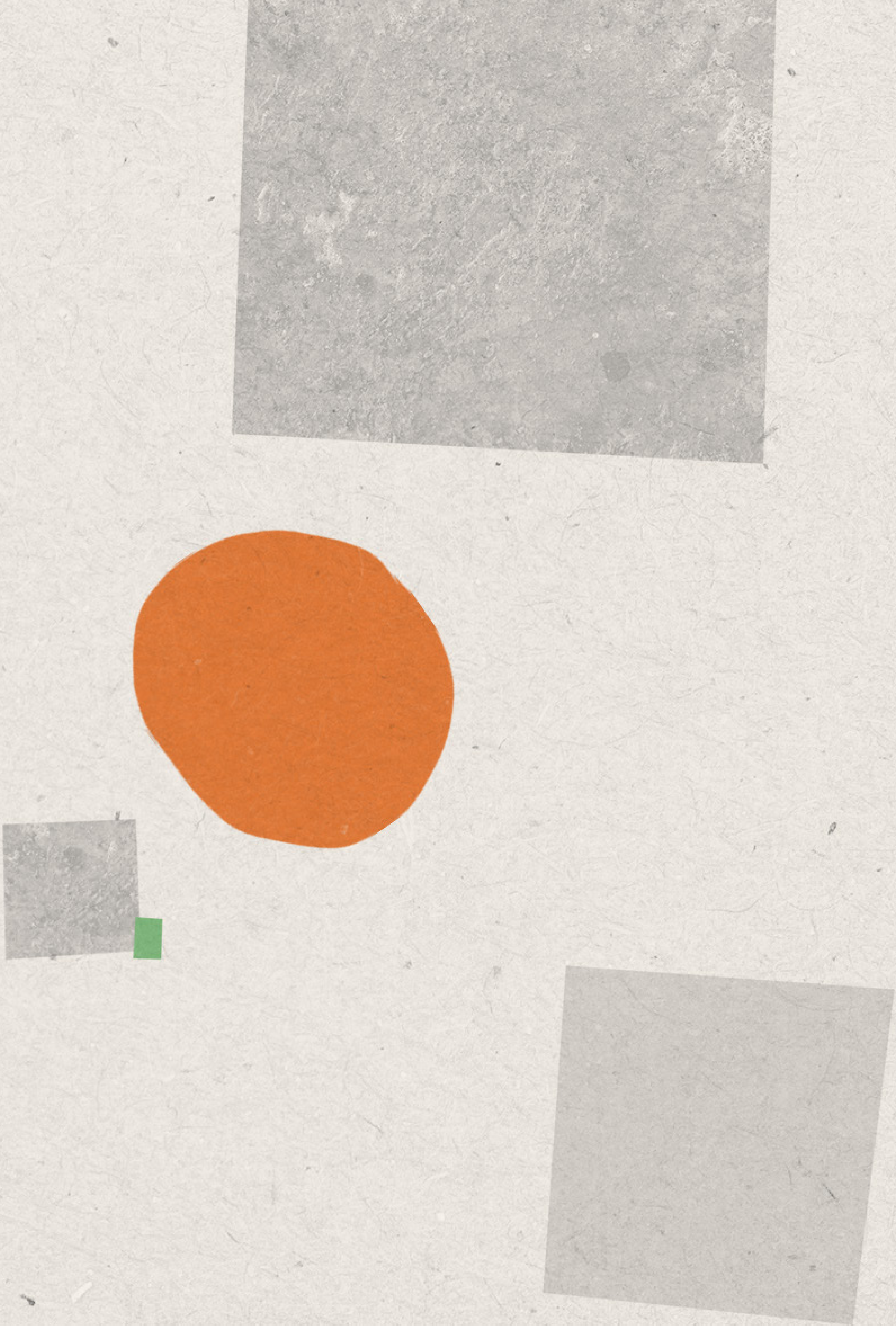
## Polyhedrons

Since the pre-Renaissance era, artists have played with polyhedrons, or shapes with a multitude of planes. (The dice in your board game sets? Polyhedrons.) These geometric figures show up everywhere in the art world, from the marble mosaics at the San Marco Basilica in Venice, Italy, to illustrations by Leonardo Da Vinci. In a more recent exploration, the twentieth-century Dutch graphic artist MC Escher garnered acclaim from critics (and the admiration of scholars) for the mathematical depth of his work – despite the artist's professed inability to fully understand the subject.

 *Christ Consigning the Keys to St. Peter* (1481), by Pietro Perugino


SIXTE OPIBVS
DISPAR
RELIGIONE
PRIOR

# What's Your Problem?



▶ 'There is a certain way in which mathematicians approach problems,' says Terence. 'We abstract them. We break them up into pieces. We make analogies. We try to find connections with other problems.' You can begin to harness this creative approach to problem-solving using a few simple strategies

## Choose Wisely

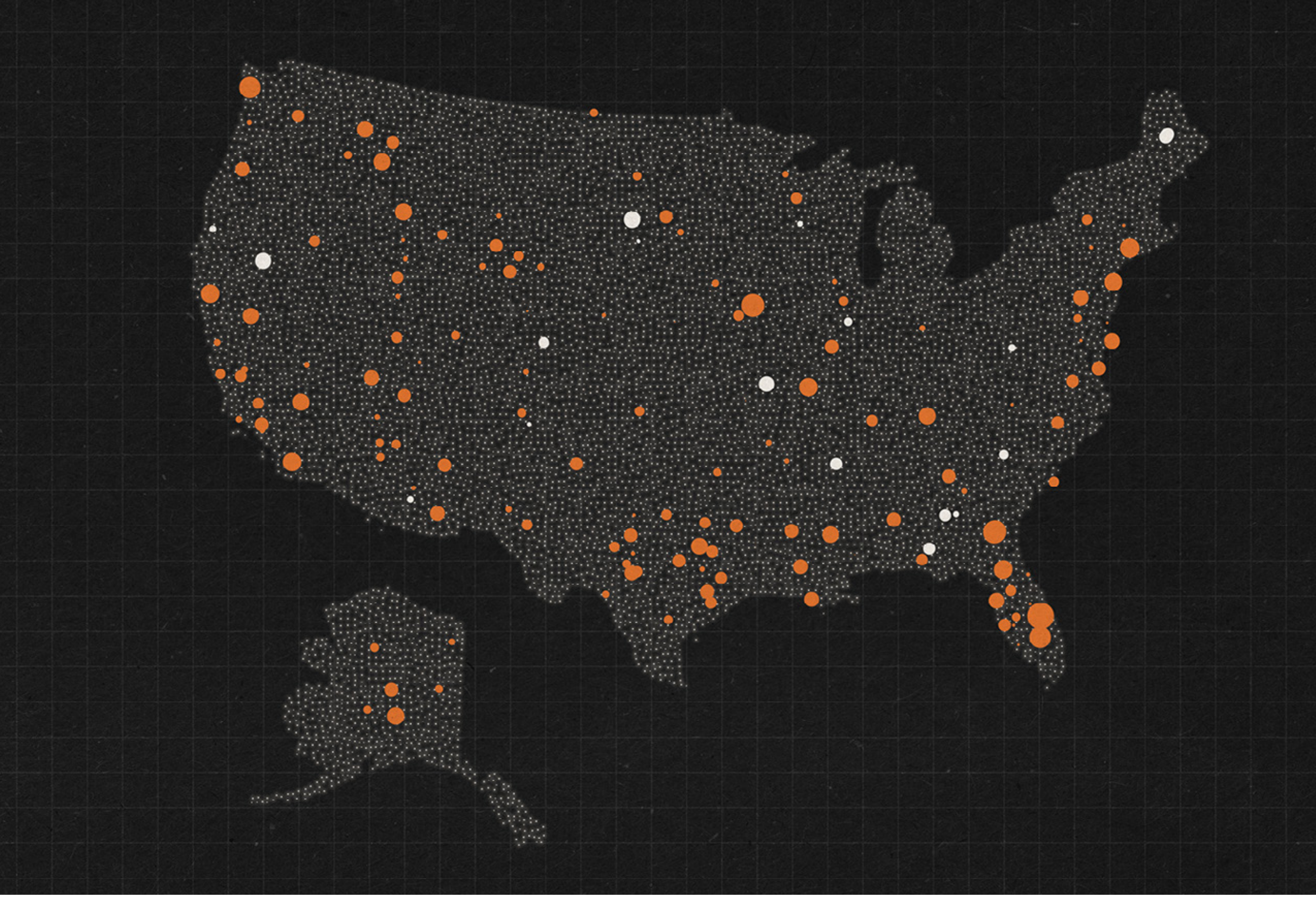
Mathematicians can often select the problems on which they work. They're also free to change the parameters of a problem to make it easier, harder, more generalised or more specific. You can essentially play around with the dials to modify all sorts of variables. What should you look for in a potential problem? Like any creative field, maths is most exciting and rewarding when you're working at the cutting edge of feasibility, Terence says. You want problems that are just barely outside the range of all your tools.

## Trim the Fat

When he's working on a problem, Terence will often use mathematical tools of abstraction to simplify things. This could involve taking key components of a problem and representing them with mathematical 'objects' like numbers or shapes. In order to do this in a way that makes sense, you first need to figure out which elements of the problem are most important. 'We're really stripping the problem down to its bare essentials,' Terence says. 'By moving the inessential components of a problem, you can focus on what's really going on.'

## Use Analogies

Analogies can help you understand the mechanics of a problem or even its philosophical underpinnings. For example, the process of polling – gathering opinions from individuals

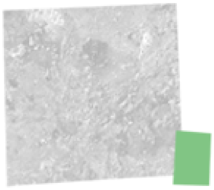


**Above:** polling depends on well-chosen sample sizes

to assess public viewpoints – often involves statistical estimations. Terence compares this to testing ocean waters for salt content: to understand the overall salinity, you need to collect results from multiple locations within the ocean. Similarly, you need to collect opinions from across a citizenry to measure a proposal's political viability. (Mathematical theories, it's worth noting, can also help correct for problems in polling.)

## Get Attached

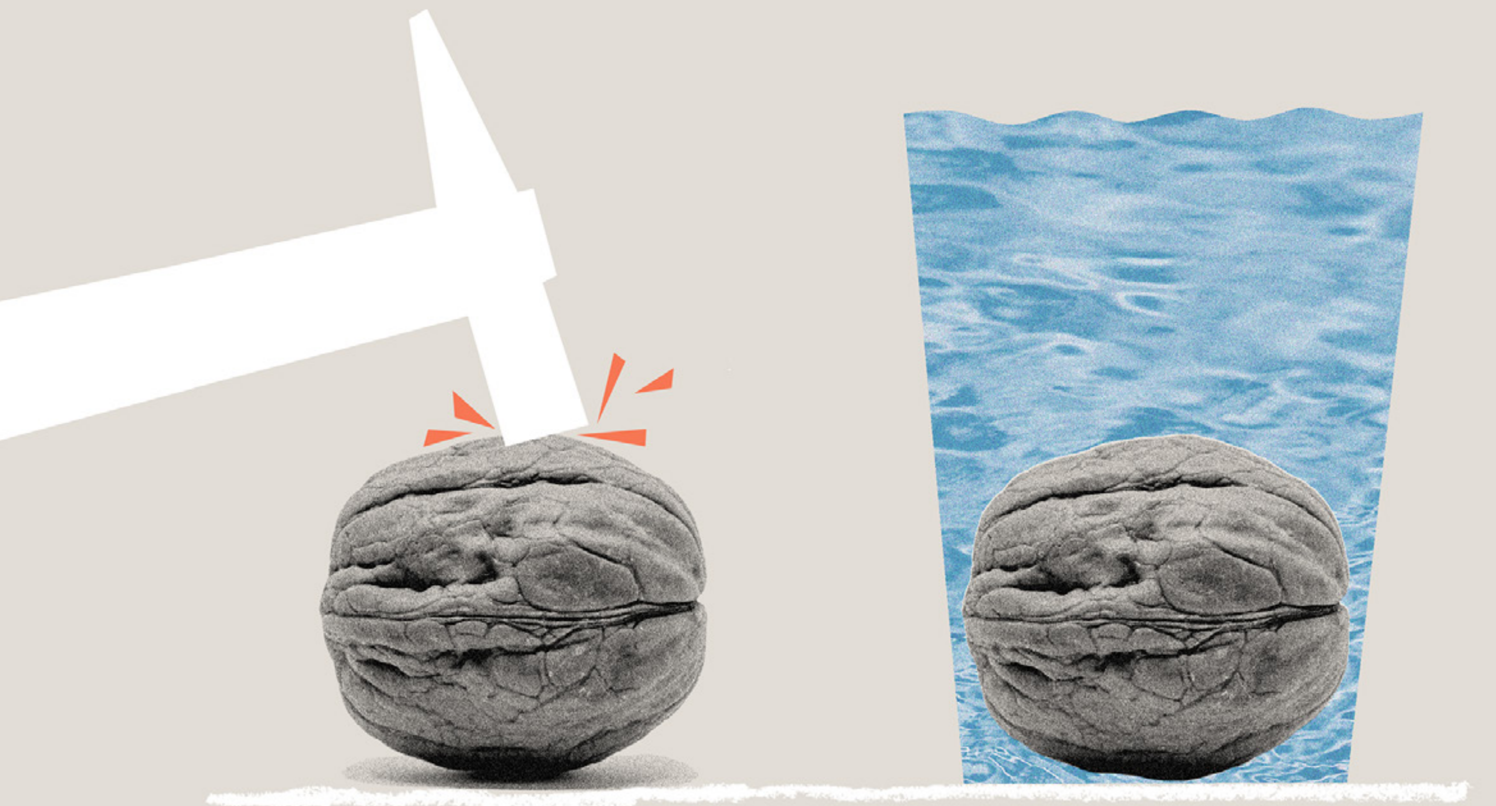
If you are straining to solve a problem, chances are you're not truly invested in the process. Instead of solving for *X* or *Y*, Terence recommends thinking of the endeavour as a search for clues or a battle against a wily enemy.



Because solving a problem can be labour-intensive and often involves failing again and again, finding some semblance of meaning in the process is crucial.

## Question Everything

As a young student, Terence was innately sceptical of the established methods for solving problems. He remembers trying to pass an exam without using the quadratic formula as recommended by his teacher; Terence tried his way first, only reverting to his teacher's recommendation after spending an hour failing to solve the equation. He urges others to challenge the status quo, even if they turn out to be wrong.



## Be Patient

Citing the late German mathematician Alexander Grothendieck, Terence likens solving a tough problem to opening a walnut: instead of hitting the nut with a sledgehammer, try soaking it in water. ‘Eventually the shell becomes so soft you can just peel it apart with your hands,’ he says. In a similar way, a problem will become far easier to solve the more you (quietly, calmly) work on it.

## Maintain Perspective

Terence has solved a few big problems and he's come close to solving a few more – but he insists that progress, not perfection, is a noble aspiration. Even with the best tools and training, sometimes a rock climber can't make it all the way to the top of the mountain. What matters is getting as far as you can. ‘That's how mathematics proceeds,’ he says. ‘You have to learn how to be part of a bigger process that's been going on for thousands of years.’

## Rethink ‘Eureka!’

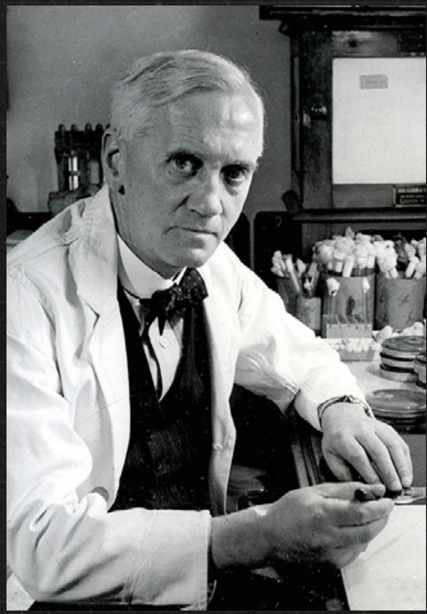
While Terence is working on a problem, moments of easily measurable progress are few and far between. More often, the progress he's making is beneath the surface. ‘You are setting the stage for putting the problem in exactly the right perspective,’ he says. When the last puzzle piece is finally in place, everything will suddenly make sense. Often, he says, your reaction will be less *Yippee!* and more *Oh, that was it?*

# Who'd Have Thought?

▶ The old adage, often attributed to American researcher and sci-fi writer Isaac Asimov, goes like this: 'The most exciting phrase to hear in science, the one that heralds new discoveries, is not "eureka" but "that's funny...." ' Here are a few laboratory accidents that were ground-breaking epiphanies in disguise

## Safety Glass

French scientist and aesthete Édouard Bénédictus cemented his legacy during a lab session in 1903, when he accidentally knocked a glass flask off his shelf. As he was grabbing a broom to collect the shards, he realised the clean-up was unnecessary: the glass had remained intact thanks to the concoction of cellulose nitrate plastic that had dried inside of it, creating an adhesive film coating. Bénédictus filed for a patent in 1910; his accidental invention – and subsequent iterations thereof – has since been used to fortify everything from eyeglasses to skyscrapers.

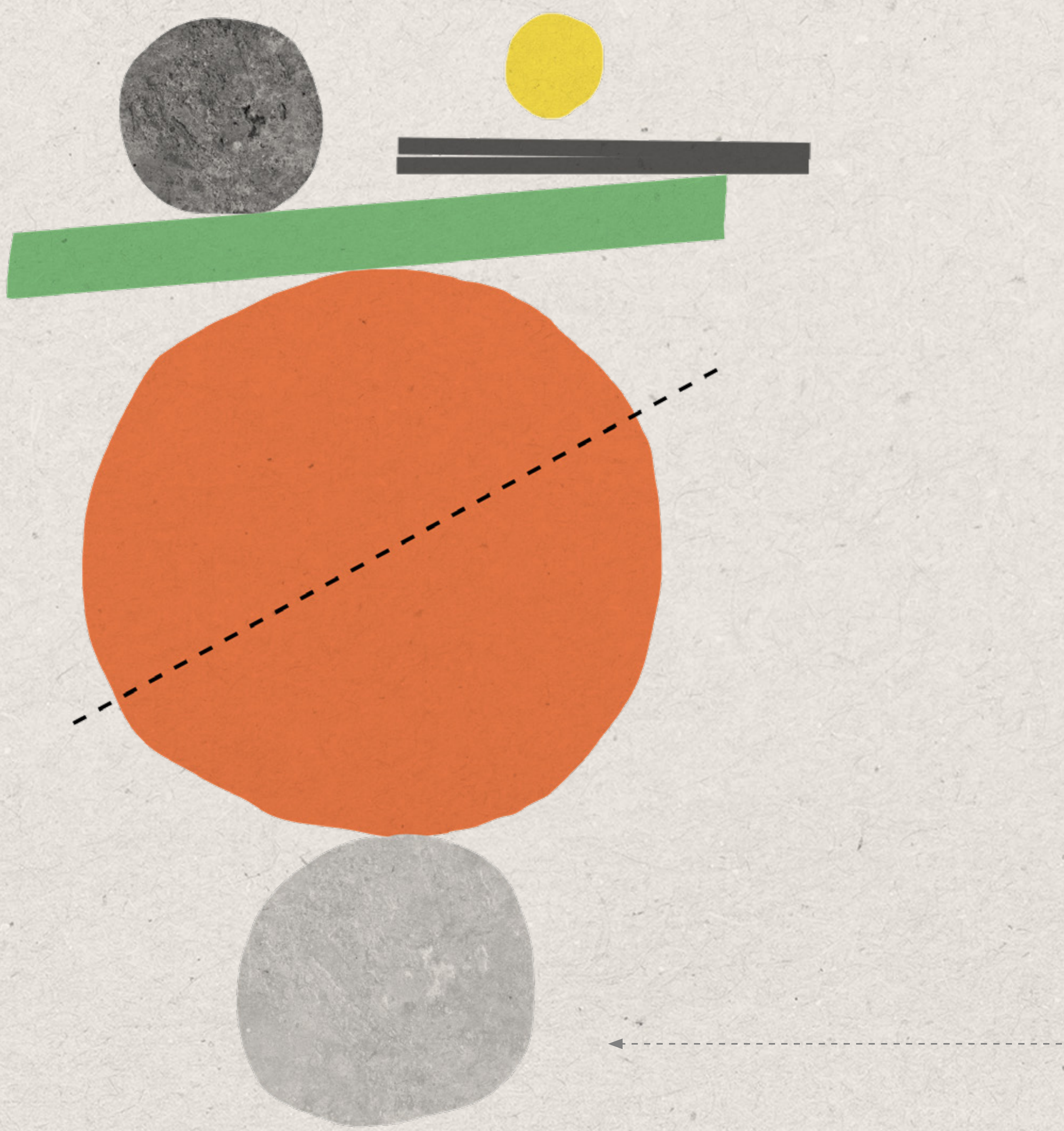



## Penicillin

In 1928, fresh off a holiday to his native Scotland, bacteriologist Alexander Fleming returned to his London laboratory only to find that a mould called *Penicillium notatum* had contaminated his petri dishes. Placing the dishes under a microscope, he noticed that this spore had inhibited growth of *Staphylococcus aureus*, which causes serious bacterial infections in patients with weakened immune systems. His discovery led to the mass-produced antibiotics that, according to the British Broadcasting Company (BBC), have saved more than two hundred million lives.

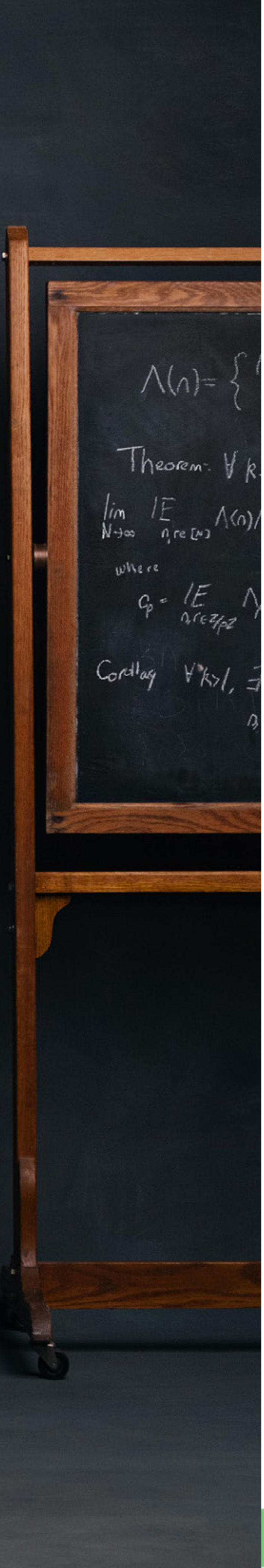
## New Blue

In 2009, a team led by Mas Subramanian, a professor of materials science at Oregon State University, was studying the electronic properties of manganese oxide, a chemical compound, by heating it to a temperature of 1,095°C. One day, a sample developed a brilliant, ultramarine hue. Subramanian named the sample 'YInMn Blue' because of its chemical components (yttrium, indium and manganese) and filed for a patent. Not only is Subramanian's invention popular with artists; it's also nontoxic and easier to produce than many other colours.



# What to Do When You're Stumped

▶ Even the most accomplished mathematicians spend great swaths of their careers pushed to the limits of their abilities. But being stumped isn’t cause for despair. Quite the opposite: it’s usually a sign that you’re on the right track. And continuing to fight with problems you don’t think you can solve is an essential aspect of learning




## Jump Right In

Stop worrying and take a stab at it. Sure, there's a chance that you won't land on an answer at first. But putting in the effort could give you ideas that lead to success down the line – or prime your brain for solving future problems.

## Simplify, Simplify, Simplify

In mathematics, this might entail using smaller numbers, removing or adding restrictions and combining like terms by addition or subtraction. In life, it could involve breaking large problems into small chunks, listing what you don't understand and trying to take a bird's-eye view of your tasks.

## Build Yourself Up

Recall the problems you *do* know how to solve and see if you can apply the knowledge you gained to the dilemma in front of you. You've done a lot thus far in your life, no doubt. You've probably even solved problems similar to the one you're currently facing.

## Don't Lament Time Spent

The Danish physicist Niels Bohr supposedly said that an expert is someone 'who has found out by his own painful experience all the mistakes that one can make in a very narrow field.' This concept helps Terence remind himself that there's no such thing as wasted time, so long as you're learning: the goal isn't just to solve a problem quickly or efficiently, but to enjoy yourself and draw lessons from the process.

## Cheat (Ethically)

In this context, *cheating* means approaching a novel problem with tools you already possess. Like a video game's 'cheat code', which allows you to skip over some tasks or stages, maths cheats can get you further into a problem faster. Terence, for example, will sometimes take a curve and make it a straight line to simplify a problem – a mathematical tool called linearisation.

## Try a New Approach

When Terence finds himself in a rut, sometimes it's because he's been looking at a problem and thinking, *I must solve it this specific way.* To get past this mental dead end, he tries to let go of his preconceived beliefs. He says that ideas formed in desperation can be your friend in these moments: 'Sometimes they can jar you out of one thought process and into a better one.'

## Collaborate!

The more people working on a single problem, the better. Maybe one member of the team is an expert in an obscure mathematical field and another has read a piece of arcane-yet-relevant literature. Good! Terence also loves solving problems with collaborators because it heightens the joy of finding a solution. 'When you and your collaborator are both standing at the blackboard... and everything checks out...that's a great experience [and] more fun than if you're working on your own.'

# Puzzle Time With Terence

▶ The group testing and counterfeit coin puzzles, explained by the man himself

## The Group Testing Puzzle

During his class, Terence presents some formidable logic puzzles. First, he asks you to identify a hypothetical infected person out of a population of a hundred. Later, you must weigh twelve seemingly identical coins to find the one that's counterfeit. In both cases, Terence gets to the answer in remarkably few steps. Here, in his own words, he lays out all the logic – and explains how the maths behind brain twisters like these can change the world.

### The Problem

In my class, I showed you how to use group testing to efficiently locate an infected patient from a pool of 100 by mixing blood samples from a subgroup of this pool and testing that mixture. The first step is to test one-half of the patients – say, patients 1–50. If the blood test returns positive, then the infected patient must be in the range 1–50. If the test is negative, the infected patient must instead be in the range 51–100. Either way, you cut down the number of possibilities by a factor of two.

But what if you knew that there were *two* infected patients? If you test patients 1–50 and the test is negative, then you can conclude that both infected patients are in the group of 51–100. But if this test comes back positive, it could be that both infected patients are in the range 1–50, but it could also be that one patient is in this range and the other is in the complementary range 51–100. This is a more complicated conclusion, and it is not as clear what the next step should be.

### The Solution

It becomes easier to see how to proceed by transforming the problem a little – by focusing on possible scenarios rather than individual patients. One scenario might be that patients 24 and 58 are infected; you can represent this mathematically by a pair (24,58), where you order the pair in increasing order. Every pair of numbers from 1 to 100, with the second number larger than the first, is a possible scenario that you have to either select or eliminate with your testing (see figure 1). As it turns out, the total number of scenarios is 4,950.

### Figure 1

```
(1,2), (1,3), (1,4), …,(1,100),
(2,3), (2,4),…,(2,100),
(3,4),…,(3,100),
.
.
.
(98, 99), (98, 100)
(99, 100)
```

This is an intimidatingly large number, but the procedure is actually much the same as it was in the case of one infected patient: you should always aim to design a test that, regardless of the outcome, is going to

Terence uses one hundred ping-pong balls to explain group testing

cut down the number of possible scenarios by a factor of about two. If, for instance, you test the group 1–50 as suggested above, then on a negative outcome you would only be left with the scenarios below.

### Figure 2

(51,52), (51,53), (51,54),…,(51,100),
(52,53), (52,54),…, (52,100),
(53,54),…, (53,100),
.
.
.
(98, 99), (98, 100)
(99, 100)

These turn out to be 1,225 in number – much less than half of the original count. But if the outcome of the group test was positive, you would be left with 4,950 – 1,225 = 3,725, which is much larger than half of the original count. So this was not an optimal choice for the first test, as it did not divide the scenarios into two roughly equal halves.

If you start off by testing the group 1–29, this will leave you with 2,485 scenarios when the test is negative and 2,465 scenarios if the test is positive, so either way you are left with about half of the original scenarios. By continuing in this fashion, you can isolate both infected patients using no more than fourteen blood tests, though the mathematics of verifying this are a little complicated and will not be detailed here.

## Real-Life Applications

What if you don't know the precise number of infected patients? Now a lot more scenarios are in play. You'd need to use more advanced mathematics from the fields of *probability theory* and *information theory* – in which one assigns a probability to each potential scenario – and design tests that eliminate about half of the scenarios (as measured by probability) regardless of the outcome. Nevertheless, the basic ideas are the same as in the simple case of having just one infected patient. It is therefore important to work on these simpler 'toy' problems first, to gather the intuition and experience needed to tackle the more complicated and realistic problems.

## Further Reading

For an in-depth article on group testing, I recommend the article 'Pools of Blood' by Keith Ball, published by *Plus* magazine in 2004.

# The 12 Coins Puzzle

## The Problem

You are given twelve identical-looking coins; one is counterfeit and the rest are genuine. All the genuine coins have the same weight, but the counterfeit coin is either heavier or lighter than the rest. Your task is to determine which coin is counterfeit and whether it is heavier or lighter. The only tool you are given for this is a balance scale, which you can use three times. How can you use these three weighings to guarantee that you will locate the counterfeit coin *and* determine whether it is heavier or lighter?

## The Solution

At first glance it appears that you have far too few weighings to solve the problem. If, for instance, you tried to test each coin separately, you would expect to need up to eleven weighings instead of three. So you have to somehow use the weighings much more efficiently.

There are many ways to solve this puzzle. One of them involves a mathematical concept known as a *matrix:* a rectangular array of numbers. To make the puzzle more numerical, you will assign numbers to the twelve coins, from coin number 1 to coin number 12; you will also label the right side of the balance scale by +1 and the left side by −1. To describe how you will weigh the coins, you will use a specific matrix, with three rows and twelve columns (see next page).

Let's call this the *measurement matrix*. The twelve columns of the measurement matrix correspond to the twelve coins, and the three rows correspond to the three weighings. The procedure for using the balance scale is as follows: for each weighing, look at the row corresponding to that weighing. All the coins whose entry in this row is +1 are placed on the right of the scale (which you labelled +1); all the coins whose entry in this row is −1 are placed on the left of the scale (which you labelled −1); and the coins whose entry is 0 are set aside.

MEASUREMENT MATRIX

$$\begin{bmatrix} +1 & 0 & 0 & 0 & 0 & -1 & -1 & +1 & -1 & -1 & +1 & +1 \\ 0 & +1 & 0 & +1 & -1 & 0 & 0 & -1 & -1 & +1 & -1 & +1 \\ 0 & 0 & +1 & -1 & -1 & +1 & -1 & 0 & 0 & +1 & +1 & -1 \end{bmatrix}$$

Thus, in the second weighing one places coins 2, 4, 10 and 12 on the right scale, and coins 5, 8, 9 and 11 on the left scale, as directed by the second row of the measurement matrix. Note that each row has the same number of +1s as –1s, so each weighing involves the same number of coins on both sides.

You record the outcome of the three weighings by a column of three numbers called the *outcome vector.* If a weighing shows that the coins on one side of the scale are heavier, you record the label of that scale (+1 if it is the right side, –1 if it is the left side) on the row of the vector corresponding to that weighing; if the scales are balanced instead, you place a 0. For instance, if in the first weighing the left scale is heavier, in the second weighing the right scale is heavier and in the third weighing the scales are balanced, the outcome vector would be the one you see to the left.

VECTOR 1

$$\begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix}$$

Now you can study what happens to this outcome vector in various scenarios. Suppose that it is coin 4 that is counterfeit, and it is heavier than the others. In the first weighing, this coin is not used at all, because the fourth entry on the first row of the measurement matrix is 0; so the first row of the outcome vector is 0. In the second weighing, the coin is placed on the right scale that is labelled +1, which makes that scale heavier; so the second row of the outcome vector is also +1. Similarly, in the third weighing, the coin is placed on the left scale that is labelled –1, making that scale heavier; so the third and final row of the outcome vector is –1, giving you the outcome vector you see to the right. Note that this is exactly the same as the fourth column of the measurement matrix!

VECTOR 2

$$\begin{bmatrix} 0 \\ +1 \\ -1 \end{bmatrix}$$

Now suppose instead that coin 4 is still counterfeit, but it is now *lighter* than the other coins. Then the outcome of the weighings will all be reversed: scales that were balanced in the previous scenario will remain balanced, but scales that tilted one way would now tilt the other. So the outcome vector is now the one you see to the right – the negation of the fourth column of the measurement matrix, in which all the +1s are replaced with –1s and vice versa.

VECTOR 3

$$\begin{bmatrix} 0 \\ -1 \\ +1 \end{bmatrix}$$

By pursuing this line of thought, you can come up with the solution to the twelve coins puzzle. To recap:

Perform the three weighings according to the measurement matrix given above and record the outcome vector of these weighings.

Compare the outcome vector against the twelve columns of the matrix. If there is a match, the column will correspond to the counterfeit coin, which will be heavier than the other coins.

If the outcome vector matches the negation of one of the twelve columns, then that column will corre-

VECTOR 4

$$\begin{bmatrix} -1 \\ +1 \\ -1 \end{bmatrix}$$

'One can determine the location of an object using only a small number of measurements.'

—TERENCE

spond to the counterfeit coin, which will be lighter than the other coins.

For instance, if the outcome vector is the one you see to the left, this does not match any of the twelve columns of the matrix – but it matches the negation of column 11, so you can conclude that it is coin 11 that is counterfeit and that it is lighter than the rest. Note that all twelve columns of the matrix are different from each other, as well as from the flipped versions of themselves, so the procedure is completely unambiguous. This solves the puzzle!

The mathematical study of matrices and vectors is known as *linear algebra.*

## Challenge

Can you devise a similar procedure to identify one counterfeit coin amongst three coins rather than twelve – using only two weighings?

## Takeaways

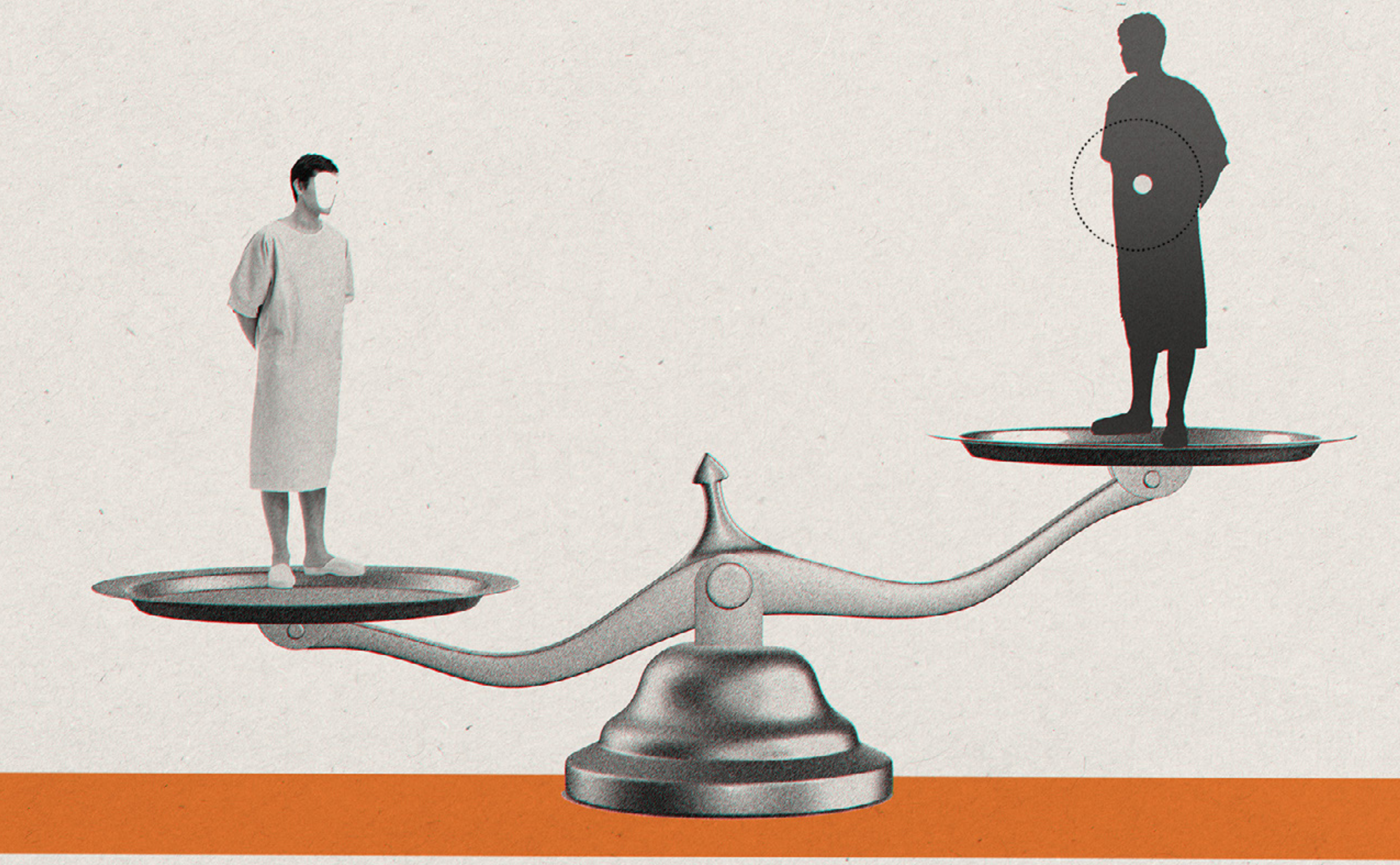
This puzzle shows that by using a bit of maths, one can determine the location of an object using only a small number of measurements. This is somewhat surprising at first, since there are twelve possible locations for this coin and one might have imagined that twelve weighings would have been needed to test each location separately. By using a carefully chosen measurement matrix, one can significantly cut down the number of weighings required.

## Real-Life Applications

Magnetic resonance imaging (MRI) is a medical imaging technique used to scan patients for abnormal features such as tumours or lesions. An MRI scanner immerses the patient in a magnetic field and then sends a pulse of radio waves through the patient's body, measuring the resulting changes in the magnetic field. By executing a sequence of such pulses, the scanner obtains a sequence of measurements of the internal structure of the patient. It used to be that in order to get a high-resolution image of the patient, many such pulses were required and a patient often had to stay immobile in the scanner for several minutes – with certain scans taking more than an hour. In some cases, particularly when scanning children, the patient had to be anaesthetised to get a good scan.

In 2004, I met with Emmanuel Candès and Justin Romberg, two mathematicians who were experimenting with a new mathematical technique to reconstruct a usable MRI image using a much shorter scan. To their surprise, when applying this technique to standard test images, the reconstruction method worked extraordinarily well, yielding almost perfect images from a much smaller set of measurements than what was traditionally used. It was as if they had discovered a way to cut down the number of weighings needed to locate a counterfeit coin from twelve to three.

When they first told me about this discovery, I was extremely sceptical. In fact, after they showed me their results, I spent an entire evening trying to prove that what they were claiming was mathematically impossible. But after several hours, I realised their method could work. An MRI scan can be described mathematically using a measurement matrix that is much larger and more complicated than the one used to solve the twelve coins puzzle, but it is still a matrix that can be studied by the mathematics of linear algebra. The col-



umns of this matrix turn out to be very different from each other and this is the key to solving the MRI scanning problem, much as it is in the twelve coins puzzle (although the precise meaning of what 'very different' means is much more technical).

After showing my preliminary calculations to Emmanuel and Justin, we worked together to refine the results and eventually published one of the pioneering papers in the field now known as *compressed sensing*. Using these methods, modern MRI machines significantly cut down the time needed to obtain a medical image. A scan that may have taken three or four minutes might now only take twenty seconds, and patients who may have required risky anaesthesia before a scan no longer need it. Most of my work has been purely theoretical, so it's been deeply satisfying to know that I helped contribute to a field that has real-world impact on everyday people.

## Further Reading

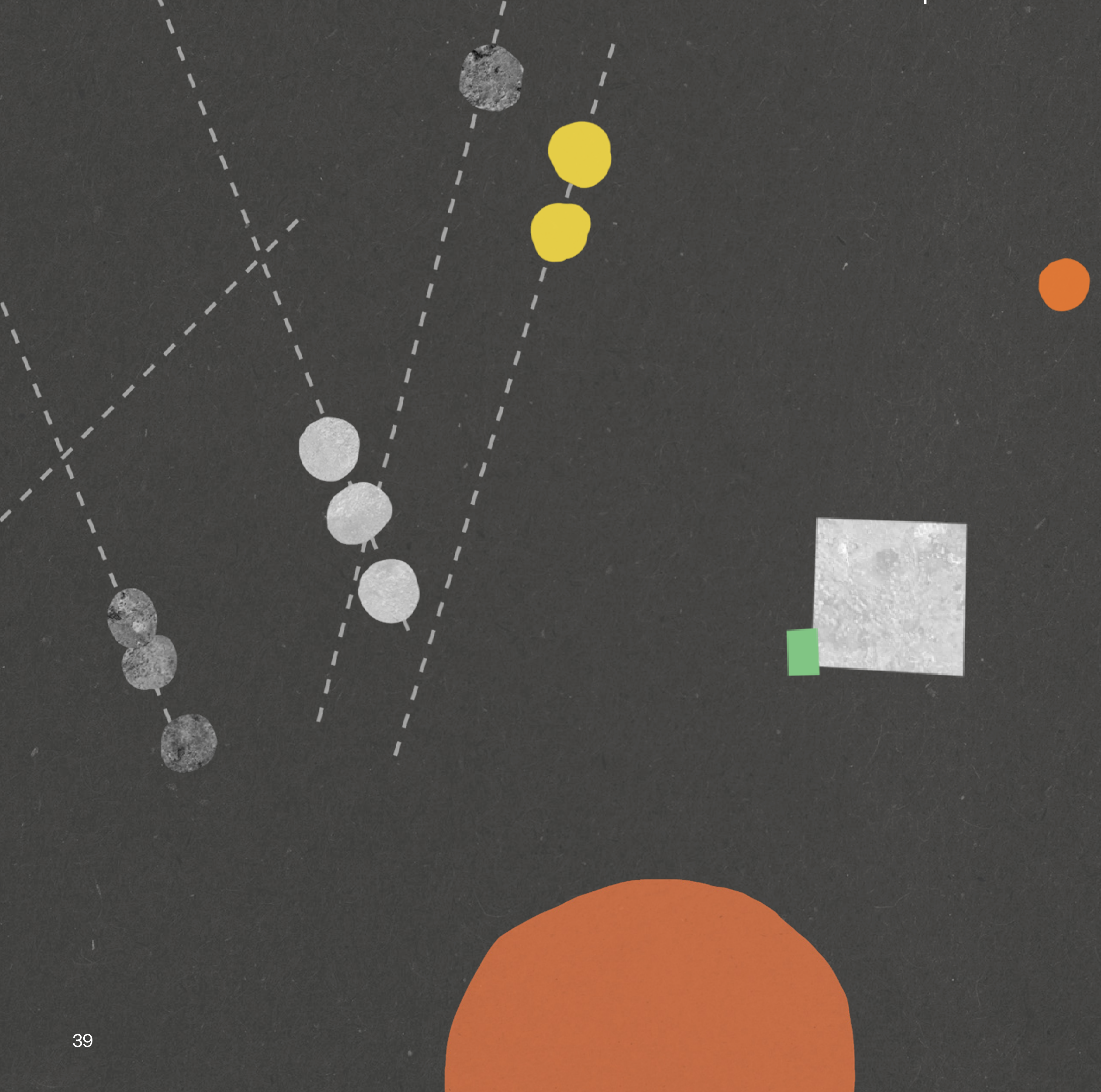
For some recollections of my role in the story of compressed sensing, see post 225 in the mathoverflow.net thread titled, 'Examples of unexpected mathematical images'. My paper with Candès and Romberg ('Robust Uncertainty Principles: Exact Signal Reconstruction from Highly Incomplete Frequency Information') can be found on various scholarly websites.

**'A scan that may have taken three or four minutes might now only take twenty seconds.'**

—TERENCE

# Assignments

▶ Test your newfound mathematical prowess with three deceptively difficult word problems



Assignment 1

# The Talent Show

This year, creating the line-up for Boulderwood High School's talent show is proving to be a major headache. Seven students – Satomi, Louis, Jeff, Viraj, Teresa, Marianne and Norris – all want to sing various versions of the same pop song in seven consecutive slots and they've got some conditions that you, the beleaguered drama teacher, must abide by. Here's what you know so far:

- Jeff *must* perform immediately before Viraj, so as to intimidate him
- Jeff and Norris are coming from an audition; Jeff's audition is before Norris's, so Norris will perform after him
- Satomi and Teresa loathe one another and refuse to perform in consecutive slots; in fact, they've requested that two performances take place between them

## PROBLEM #1

If Satomi was the second to perform, who was the third performer in the ceremony?
A. Louis
B. Jeff
C. Viraj
D. Teresa
E. Marianne
F. Norris

## THE SOLUTION

First, make a table:

| SLOT | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| PERFORMER | | | | | | | |

Since there must be two spots between Satomi and Teresa, the next step looks like this:

| SLOT | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| PERFORMER | | S | | | T | | |

You know that Jeff and Viraj are performing consecutively with Norris following them. Thus, Jeff and Viraj must take up the third and fourth slots. Norris can be either in the sixth or seventh slot.

Case I:

| SLOT | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| PERFORMER | | S | J | V | T | N | |

Case II:

| SLOT | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| PERFORMER | | S | J | V | T | | N |

In either scenario, the third performer is Jeff – so the answer to the problem is B.

## PROBLEM #2

Use the blank table to work out which of the following slots Norris is *not* able to perform in.

A. Second
B. Third
C. Fourth
D. Fifth
E. Sixth

## THE SOLUTION

You already know that Norris must follow both Jeff and Viraj. The earliest Norris can perform is when Jeff is first and Viraj is second, which puts Norris at third. Thus, Norris cannot be the second performer. The answer is A.

Assignment 2

# The Bus Stop

## THE PROBLEM

David's two aunts want to see him every weekend, but they live on opposite sides of town. To appease them, David tells both aunts that he'll head to the bus stop nearest to his flat every Saturday at a random time and board the first bus that arrives. If it happens to be an eastbound bus, he'll see Aunt Sarah. If it happens to be a westbound bus, he'll see his Aunt Maura. Both aunts agree to the plan because they know both eastbound and westbound buses arrive every fifteen minutes.

But after a few months of visiting under this arrangement, Aunt Maura is crestfallen: David has only visited her around one out of four Saturdays. How could this be? (Hint: the buses are always on time.)

## THE SOLUTION

Both buses may come every fifteen minutes, but that doesn't mean the likelihood of David getting on a westbound bus is the same as the likelihood of his getting on an eastbound bus.

Suppose eastbound buses come on the hour, the 15, the 30 and the 45, but westbound buses come on the 4, 19, 34 and 49. This means that, in any given hour, there's only 16 minutes in which the westbound bus will be the next bus to arrive. If David arrives between 9:00–9:04, 9:15–9:19, 9:30–9:34 or 9:45–9:49, he'll board the westbound bus. Otherwise, he'll get on the eastbound bus. The bus with the longer wait is the bus he's more likely to catch.

Since David arrives at a random time every hour, his chance of getting on the westbound bus will always be 16 out of 60 or around 25 percent of the time. That's the answer.

Aunt Maura may be upset with this arrangement – but she's lucky her bus isn't scheduled to arrive one minute after the eastbound bus or David would only visit her a handful of times each year!

Assignment 3

# The Bicycle Wheel

## THE PROBLEM

Imagine you have a brand-new front tyre on your bicycle. You adore it, and your excitement blossoms into visions of smooth rides down long roads into the distant future. Then your fantasy is cut short. Your neighbour – nay, your *nemesis* – Winston pulls up. He has the same bicycle that you do, with the same size front wheel. He also has a new front tyre.

Winston smugly remarks that the circumference of his new tyre is 12 millimetres longer than the circumference of your new tyre. Then he speeds off before you get a chance to measure his tyre. Now you're really fuming. You're determined to figure out the difference between his tyre and your own.

Using the knowledge that the total circumference of Winston's tyre is 12 millimetres longer than yours, how much farther from the centre of the wheel does his tyre extend compared with your front tyre? (Hint: you don't need to know the precise circumference of either tyre to solve this problem.)

## THE SOLUTION

Let's say your tyre's radius – an imaginary straight line running from the centre of the wheel to the tyre's outer edge – is *R,* measured in millimetres. You already know that the length around your entire wheel is like the circumference of a circle. And so you use the standard circumference equation to find the length of your tyre:

Circumference (your tyre) = $2 \times \prod \times R$

Winston's tyre is 12 millimetres longer:

Circumference (Winston's tyre) = $2 \times \prod \times R + 12$

Let's call the distance you're looking for *D*. The radius of the circle that Winston's tyre encompasses is then *R+D*. So the length of his tyre is its circumference, or: $2 \times \prod \times (R+D)$. But you already know his tyre is $2 \times \prod \times R + 12$.

Hooray, there's an equation for this situation:

$$2 \times \prod \times (R+D) = 2 \times \prod \times R + 12$$

If you distribute the $2 \times \prod$ on the left side of the equation over both variables inside the parenthesis, it looks like this:

$$2 \times \prod \times R + 2 \times \prod \times D = 2 \times \prod \times R + 12$$

You can drop the identical $2 \times \prod \times R$ on both sides.

$$2 \times \prod \times D = 12$$

From there you can isolate *D* by transposing $2 \times \prod$—that is, by moving it to the other side of the equation and reversing its sign from multiplication to division.

$$D = 12 \div 2 \times \prod$$

Now divide 12 by 2:

$$D = 6 \div \prod$$

Since $\prod$ is about 3, *D* is about 2 millimetres! Or, if you use 3.14 for $\prod$, *D* is 1.91 millimetres.

# Maths Apps

▶ If you're looking to sharpen your maths skills, these mobile marvels can help

## Khan Academy

Khan Academy's free lesson plans start easy and get progressively more complex. The app can help you brush up on mathematical concepts, from basic geometry to multivariable calculus, while learning entirely new ways to solve problems.

## Sumaze!

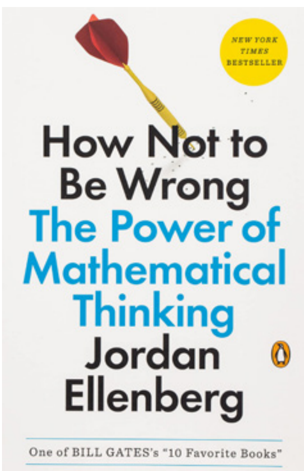
Sumaze! is an educational puzzle for children and adults that can help you solve problems like a pro. With beginner and advanced versions, the game explores arithmetic, inequalities, logarithms, powers, primes and more.

## Maths | Riddles and Puzzles

Train your mathematical brain by solving geometric and number-based puzzles in this free, minimalist game. Featuring a hundred levels, the app can get you into the spirit of problem-solving.

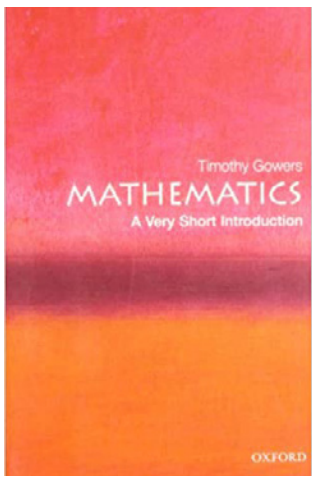
# Recommended Reading

▶ To nurture your passion for maths, logic and problem-solving, add these Terence-approved books to your shelf



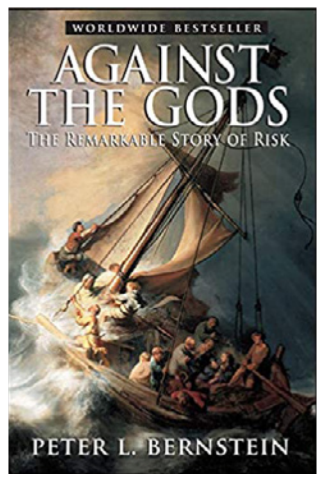

*How Not to Be Wrong: The Power of Mathematical Thinking*
BY JORDAN ELLENBERG

**Terence says:**
'A very entertaining and accessible read, giving many real-world examples of how mathematical thinking can explain a complex world and prevent one from being seriously misled by incorrect reasoning.'



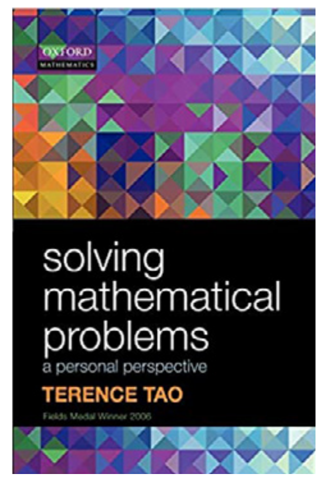

*Mathematics: A Very Short Introduction*
BY TIMOTHY GOWERS

**Terence says:**
'The author is one of the most eminent mathematicians in the world, but he made great efforts to keep this book as nontechnical as possible. It should be accessible to anyone with a secondary school education in mathematics.'




*Against the Gods: The Remarkable Story of Risk*
BY PETER L. BERNSTEIN

**Terence says:**
'This is a fascinating and well researched recounting of the ways humanity has managed uncertainty over the centuries, and how the mathematics of probability theory and statistics gave rise to modern finance and risk management.'




*Solving Mathematical Problems: A Personal Perspective*
BY TERENCE TAO

**Terence says:**
'This is my own book on problem-solving, first written when I was fifteen based on my experience in mathematics competitions. I revised it when I was thirty to add a slightly more mature perspective.'

# What Maths Can Teach You About Life

▶ Beyond the halls of academia, maths can help you find connections, gain insight and expand your sense of what's possible. Here are some life lessons Terence has learned since attending his first university-level maths course at age nine

## #1 You Can't Predict Your Legacy

Your ideas could have a major impact in your lifetime, or they could influence scholars hundreds of years from now. (We're still using concepts developed in ancient Egypt more than two thousand years ago.) But these considerations can be distracting, so don't fixate on fame or posterity. Beginning the work is what's most important.

## #2 Getting Stuck Can Be a Good Thing

When Terence is stuck on a problem, he knows he's pushing himself to the edge of his abilities. Success, to him, means ending up in a better place than where he started. After all, incremental progress is still progress.

## #3 Know When to Let Go

When solving a specific problem becomes an obsession – and other pressing problems are falling by the wayside – sometimes Terence is forced to admit defeat. In those moments, he aims for acceptance. 'We all have to learn to sometimes take a breath, step back and admit that some problems are out of one's reach,' he says.

## #4 There's Power in Numbers

Mathematics might seem like solitary work, but collaboration is key to success. The best teams, Terence says, include an optimist who can serve as an endless font of ideas and a pessimist who can shoot down the untenable ones. He also recommends divvying up your workload according to interest and skill levels, rather than aiming for perfect parity.

## #5 Share Your Successes

Solving problems in the company of others? Fun, rewarding, cathartic! Solving problems alone? Not so much. Being part of a team isn't just helpful in the nitty-gritty of coming up with solutions; it also makes the entire process more enjoyable.

# Credits

## Assorted stills

Courtesy Alamy; the American Mathematical Society (https://doi.org/10.1090/S0894-0347-07-00555-3); matematicasVisuales.com; Getty Images; Getty Images/ Vincenzo Pinto; the International Mathematical Union; Graeme Mitchell/Redux; Sputnik/Science Photo Library; Olena Shmahalo/*Quanta Magazine*; Mas Subramanian/ Oregon State University; the University of California, Los Angeles; the Master and Fellows of Trinity College, Cambridge, Add.MS.b.175

## Assorted footage

Courtesy Australian Broadcasting Corporation Library Sales; the Australian Consulate-General Los Angeles; Pond5; *The Colbert Report*/ViacomCBS; Simons Foundation; the University of California, Los Angeles

## Footage of Terence's lecture on universality for random matrix ensembles

This video is of a lecture delivered by Terence Tao at the 2017 IAS/Park City Mathematics Institute on July 3, 2017. This Institute, known as PCMI, is an outreach effort of the Institute for Advanced Study in Princeton, NJ

MasterClass