

# Exploring the limitations of the Atelier B automated prover

Agata Borkowska, UID: 1690550, *MSc in Computer Science, University of Warwick*

**Abstract**—AtelierB is a tool for formal software development through refinement, using the B-method. It incorporates an automated prover, which has been recognized as the most thorough prover for B set theory, and has been used as a basis for many others. Nevertheless it has multiple shortcomings. Various approaches have been suggested and taken to improve its performance, including extensions to the proof rule base, plug-ins and third-party software. In this work we strive to establish the limitations of the prover without such additions, and discover at which point they become necessary. The secondary goal is to improve the robustness of the software without straying from pure B method, and taking into account the ease of use. As a metric of our success, we use the benchmarks proposed by Conchon and Iguernlala [1].

**Index Terms**—B method, formal verification, abstract machine, proof

## I. INTRODUCTION

THE aim of formal specification and verification is to ensure the correctness of software. While overall less popular than quality assurance through testing, it is used in

## II. LITERATURE REVIEW

## III. PROJECT OVERVIEW

### A. The aims of the project

The key aim of this project is to discover the limitations of the Atelier B software. In the previous section, we have mentioned multiple plugins and extensions to the Atelier B software, which were created in order to improve the functionality. They highlight what other users have found lacking in Atelier B and what they have considered in need of improvement. However none of them demonstrate what can be achieved with the original software alone. Hence we strive to assess at what point Atelier B alone becomes insufficient, and the extensions are necessary to successfully verify a project.

An extension to this aim is answering the question: at what point is it necessary to add user-created rules in order to facilitate the proofs in the automated prover?

It needs to be stressed that adding user-created rules is not advised unless it is absolutely necessary. Such rules then have to be thoroughly verified by means other than Atelier B software, to ensure that they are in fact correct. An error in an added rule would invalidate the entire proof process. Hence we will strive to circumvent obstacles by all means other than adding proof rules, before resorting to it. We hope to gather such advice and guidelines and make it accessible to others, so that they may be dissuaded from unnecessarily adding proof rules in their work. It is possible that adding proof rules to the Atelier B's rule base may be avoided entirely in the scenarios we have chosen, as they are by no means exhaustive; otherwise we will present the rules and their proofs for the use of others who wish to study the B-method.

We intend to explore various ways of expressing a specification in the B language, paying attention to how seemingly equivalent expression result in different proof obligations being generated by the automated prover, and follow it up by explaining the differences with the information contained in the source texts. This approach is often taken by students and academics beginning their work with the B-method and the Atelier B software. They are unlikely to have a thorough knowledge of the intricacies of the B-method as implemented in the Atelier B software, and would attempt to write a machine first, then search the manuals to explain

We hope to provide them with an explanation for the most commonly encountered proof obligations which are not automatically discharged, and persistent patterns among them. Both understanding the information given by the automated prover and avoiding undischarged proof obligations in the first place is of interest.

The observations arising from our work can be separated into two groups. Firstly there will be points highlighting the intricacies of the B method, which can be reasoned about and explained easily using source texts, primarily Abrial's *The B-book* [5] and Schneider's *The B-Method* [4].

Secondly, there will be disparities between the B method and its implementation in the Atelier B software. Not all of them can be classified as bugs, and some are

clearly conscious choices which diverge from the pure theory of the B method. The manuals provided with the software will aid us with comparing the implementation to the theory.

### B. The scope of the project

This work focused on abstract machines - they are the first step towards a formally verified implementation of a specification, thus making them the foundation of any project developed using the B method. An implication of this choice is focusing on set-related structures, including relations which are understood as sets of maplets, and operations, such as set comprehension, union, and intersection. These abstract constructs are not allowed in the later stages, where all data structures have to be concrete, and operations deterministic. We will also not discuss proof obligations related to loops, since those arise in the implementation stage.

Another reason for this choice is the distinct lack of literature focusing on this stage of development, in comparison to the refinement and implementation stages.[CITE] The latter stages undeniably generate more proof obligations, since on top of the proof obligations which can appear in the abstract machines, we need to consider those related to connection between a refinement machine and the machine being refined.

Similarly, as can be seen in the literature review, there has been significant amount of work done on verifying user-created proof rules, despite commonly seen advice to use this option as a last resort to prove correctness. We have found little discussion on how to avoid writing user's own proof rules.

We shall approach this project from an academic rather than industrial point of view, focusing on smaller yet more illustrative examples. The main reason for it is to limit the number of factors affecting the number of generated proof obligations and be able to control them more precisely. An industrial-scale project with hundreds of proof obligations would be rather unwieldy for our purposes. Secondly a person new to formal development and the B-method is more likely to be able to follow clear, exemplar scenarios.

Nevertheless we hope that not only students and researchers will find our work helpful. Being able to minimise the number of proof obligations to be manually discharged has the potential to reduce the time required to complete any project. The constructs and expressions we discuss are the same ones as those used in industry. In fact, we found that the more complex structures in the B language, such as sequences, are rarely used in large-scale projects, as exemplified by [CITE].

## IV. METHODOLOGY

### V. PROJECT MANAGEMENT

#### A. Choice of scenarios

This work has focused on two generic scenarios and explored various ways of fulfilling a specification for each of them. The specifications were kept deliberately imprecise for two reasons. Firstly, it allowed for an in-depth exploration of the theme. More precise specification would narrow down the options significantly, limiting our findings. Secondly in an industry setting it is not unheard of to have specification documents which leave details up to interpretation or are open-ended.

After exhausting the ways each expressing the specification in the B-method, we created a few more machines for each scenario, which illustrate other variations on the theme, although they diverge further from the original intentions. They served to analyse constructs which are more particular or less suited to the chosen scenarios, but nevertheless not unheard of.

It is important to observe that in large-scale industry projects which apply the B method, the constructs are kept simple and straightforward to avoid obfuscation. Thus in the sections below, dedicated to the chosen scenarios, the examples created were ordered by complexity. The later ones, for example in the case of the Bag Machine those involving sequences or relations, were analysed to compare the number of proof obligations generated next to their simpler variants.

### VI. THE BAG MACHINE - VARIOUS APPROACHES TO DESCRIBING SETS AND COLLECTIONS

#### A. Specification

Given a set of items, we want to describe a bag containing a subset of them. Initially, the bag is empty. We can perform the following operations on the bag:

- add an item to the bag
- remove an item from the bag
- find out the number of items in the bag
- find out which items are in the bag
- query whether a given item is in the bag

#### B. Discussion of the specification

An image of a bag of items was chosen due to its simplicity, although we recognize that it is potentially an unhelpful deviation from the most general description of this scenario, which can be achieved solely in set-theoretical terminology. We argue that this scenario is applicable in many circumstances. For example, one may be asked to develop a system that controls the barriers to a private car park. Then the system would maintain a

set of registration numbers of the vehicles permitted to park there, which is a subset of all possible registration numbers. Another example is a library system, where `ITEMS` is the set of books in the library, and `content` are the books a person has on loan. We may want to impose a limit on the total number of items in the subset - bound by the number of spaces in the car park or the maximum number of books permitted to have on loan at the same time. This variation, although not explicitly required by the specification, was analysed in depth among others listed below.

The aim of this scenario is to explore different ways of expressing sets and operations on them. The two sets we are working on are `ITEMS`, consisting of all possible items that can be put in the bag, and `content`, the items contained in the bag at a given point. There are various ways of describing each of these sets.

We take the relation between them firstly to be that of subset, namely  $\text{content} \subseteq \text{ITEMS}$  or equivalently  $\text{content} \in \mathbb{P}(\text{ITEMS})$ . This already shows two different ways of expressing a simple relation like this. Furthermore we may want to impose the limitation that the content of the bag is a finite set, thus arriving at  $\text{content} \in \text{FIN}(\text{ITEMS})$  in B notation, where  $\text{FIN}(\text{ITEMS})$  denotes all finite subsets of the set of `ITEMS`.

There are other ways of describing the relation between the `ITEMS` and the `content` of the bag. For example latter can be a mapping from a subset of `ITEMS` to the number of times a given item appears in the bag.

At the level of abstract machines, it is permitted to use set comprehension and other operations and properties of sets, according to the Zermelo-Fraenkel set theory. The B language offers abbreviations of some more common expressions, such as domain or range restrictions. Another thing for us to explore is how using these shorthand expressions rather than writing them out fully affects the proof process.

### C. Variants of the Bag Machine

There are various ways of expressing the requirements given in the specification as an abstract machine. The following files are contained in the *Test\_scenarios* archive for reference of the reader. We begin by having `ITEMS` as a deferred set - a set which will be defined at some later point of the development process, and `content` as simply a subset of `ITEMS`, then inspect different ways of including the set of `ITEMS` in the machine, then we focus on the relation between the two sets as expressed in the Invariant clause. We then move onto different ways of expressing the set of `ITEMS`, for example as

an enumerated set or one of a basic type. We finally explore various ways of describing the content of the bag, such as using sequences or relations. The reason for this order of tasks is to begin with the most intuitive implementation of the specification, before discussing less obvious changes to it.

1) *Bagmch*: is the reference machine which we take to be the core of this scenario. It implements exactly the specification without imposing any non-required conditions, such as the limit on the number of items in the bag. At the same time it includes one condition not explicitly mentioned in the specification, namely:

```
INVARIANT
  content : FIN(ITEMS)
```

The specification requires only that  $\text{content} \subseteq \text{ITEMS}$ , however in any implementation it is infeasible to have truly infinite sets, thus the software considers them to be erroneous. In the *B Language Reference Manual* [7] we find the following definition for the set of natural numbers:  $\text{NAT} = 0..\text{MAXINT}$ , where `MAXINT` can be set by the user for a given project, although it is usually understood to be  $2^{31} - 1$ . This definition is not supported by the *B-book*, thus demonstrating a small disparity between the theory of the B-method and its implementation in Atelier B. Nevertheless it is very reasonable for practical purposes.

This machine generates only four proof obligations and all of them are discharged automatically. The first three check that the `INVARIANT` is preserved in the initialisation and by the operations to add and remove items - the only three actions affecting the state of the bag. The last one is as follows:

```
"Well definedness"
=>
  content : FIN(content)
```

It is concerned with the well-defineness of the operation `howmany`, which returns the number of items in the bag. The well-defineness proof obligations arise when an expression is used which requires certain conditions to be met in order to be well-defined. In this case, `card(content)` is well-defined only if `content` is a finite set. The well-definess conditions for all such expressions can be found in *B Language Reference Manual* [7]. The following machine illustrates the proof obligations generated when the well-definess conditions are not met.

This machine is shown in Appendix A for ease of access.

2) *Bagmch\_unbounded*: illustrates the necessity to impose finiteness on the set of items contained in the

bag. The sole difference between this and the reference Bag Machine is the statement:

```
INVARIANT
  content <: ITEMS
```

This machine generates the same four proof obligations as the reference machine, however the last one remains undischarged by the automated prover and correctly points the user to a problem in the abstract machine.

3) *Bagmch\_params*: differs from the reference machine in the way the set of `ITEMS` is included in it. *Bagmch* is given the set of `ITEMS` as a deferred set, which is required to be explicitly defined at a later stage of the development. Here, it is given as a set-valued parameter, which must be instantiated any time the machine is used. [4] However at the stage of abstract machine which is not used by any other machine, it does not make a difference in the prover.

Similar behaviour can be observed later in the machines *Bagmch\_limited* and *Bagmch\_limited\_params*, with a scalar-valued parameter.

4) *Bagmch\_long\_inv*: demonstrates that a provably equivalent way of writing an expression may lead to a different behaviour of the prover.

There are multiple ways to denote subsets and finite subsets in these two machines. For unbounded subsets  $\text{content} \subseteq \text{ITEMS}$  is equivalent to  $\text{content} \in \text{POW}(\text{ITEMS})$ . We include the latter form in the Invariant of this machine

As we have discovered earlier, we need to limit the content of the bag to only finite subsets of `ITEMS` in the Invariant clause. A phrasing has been suggested by the well-defineness proof obligation generated by the reference machine. Thus, another way of expressing it to the one seen in the *Bagmch* is:

```
INVARIANT
  content : POW(ITEMS) &
  content : FIN(content)
```

This time there are six proof obligations generated, all discharged automatically, and they are concerned only with the Invariant being preserved. There is a proof obligation for each clause of the Invariant, for each one of the initialisation and the two operation affecting the content of the bag.

Firstly, comparing the proof obligation related to the first clause of the Invariant, we see that it is not as simple as the one in *Bagmch\_unbounded*. For example in the initialisation we see:

```
"Invariant is preserved" &
"Check invariant ((content):(POW(ITEMS)))"
=>
  {} <: ITEMS
```

Where we previously had:

```
"Invariant is preserved" => {} <: ITEMS
```

It strongly suggests that the notation  $a \subseteq b$  is preferable to  $a \in \mathbb{P}(b)$ , for any two sets  $a$  and  $b$ . While the result is still the same, the former seems to require less work from the prover, and thus especially for large projects may save some computation time.

Next we compare the proof obligations regarding finiteness of `content`. The one generated now are of the form:

```
"Invariant is preserved" &
"Check invariant ((content):(POW(ITEMS)))"
=>
  {} : FIN({})
```

We can compare it to the finiteness POs in the reference *Bagmch*:

```
"Invariant is preserved" => {}: FIN(ITEMS)
```

This change in expression does not appear to make a difference for the proof obligations regarding the Invariant being preserved. However the key difference between this machine and the reference is that now there is now no proof obligation concerning the well-defineness of the expression `card(content)`. In this case, by adding the goal of a proof obligation regarding well-defineness, as it was seen before, to the Invariant, we have avoided the proof obligation altogether. A similar behaviour will be observed in later examples.

We conclude that these two ways of expressing the relation 'content is a finite subset of `ITEMS`', while equivalent in theory, may affect the performance and the time taken to complete computations by the prover in different ways. We recognize that it is something that may be noticeable at a very large-scale projects, and therefore we have no means of exploring it further in this work, but we highlight it as something users should be aware of.

It is regrettable that the prover does not allow us to inspect the proof rules applied when proof obligations are automatically discharged, leaving us with speculations about its inner workings.

5) *Bagmch\_finite\_items*: is an attempt to impose finiteness earlier on, on the set of `ITEMS`. We do it by adding the clause ' $\text{ITEMS} \in \text{FIN}(\text{ITEMS})$ ' to the Properties clause. Otherwise the machine is identical to *Bagmch\_unbounded*. We rely on the fact that any subset of a finite set is necessarily finite. Appendix B contains a proof of this claim.

We see the same four proof obligations as in *Bagmch\_unbounded*, and again the one regarding well-defineness is not discharged automatically. We can of

course add the clause ' $\text{content} \in \text{FIN}(\text{content})$ ' or otherwise explicitly impose the finiteness property on the contents of the bag, but it should not be necessary. The machine is correct, and as the aforementioned proof suggests, it can be verified manually.

Thus we move on to the next stage in our process, as described in the Methodology section. The interactive prover allows users to search rules by the goal, as well as browse the file containing the integrated proof rule base. However none of the rules contains the goal required. No other manipulation in the interactive prover leads to discharging the proof obligation either.

The claim 'a subset of a finite set is finite' can be expressed as a B rule as follows:

```
THEORY userInFINXY IS
  band(binhyp(b: FIN(b)),
    binhyp(a <: b))
=>
  a: FIN(a)
END
```

This rule can be added to the *.pmm* file for this component, so that it is not visible during other proofs, or to the Patch Prover, which contains user-added rules available to all components in a project. Either way it has to be then called explicitly in the interactive prover, with the command to apply rule ' $\text{ar}(\langle \text{theory name} \rangle)$ ', to discharge the obligations with this goal. The project archive contains the *Bagmch\_finite\_items.pmm* file with this rule to illustrate how it works.

An alternative to including this rule in the PatchProver or the *.pmm* file is to add the goal of the proof obligation to the Invariant, as can be seen in *Bagmch\_long\_inv*. This is a safer alternative, because it will certainly not permit incorrect statements to be accepted by the prover, although it may restrict the number of legal states of the machine. At the same time it leads to more proof obligations and it can be considered the user performing part of the proof manually.

6) *Bagmch\_constant\_set*: is very similar to the previous machines. This time the superset of *content* is a set described in the Constants clause, with the following properties:

```
CONSTANTS
  items
PROPERTIES
  items <: ITEMS &
  items : FIN(items)
```

The behaviour of this constant is identical to that of the deferred set of *ITEMS*, as seen in *Bagmch\_finite\_items*.

7) *Bagmch\_limited*: imposes a limit on the number of items that can be included in the bag at once. We define

this limit as *max\_elem* in the *CONSTANTS* clause of the static part of the machine and give its type in the *PROPERTIES* clause:

```
CONSTANTS
  max_elem
PROPERTIES
  max_elem : NAT
```

Here, we explicitly restrict *max\_elem* to be non-negative. Changing this expression to ' $\text{max\_elem} \in \mathbb{Z}$ ' results in an unprovable proof obligation in the Initialisation, the goal of which is  $\text{card}(\{\}) \leq \text{max\_elem}$  for all possible values of *max\_elem*. It clearly does not hold for negative values of the constant, hence it is limited to natural numbers.

Furthermore, there is no difference between defining just the type of *max\_elem* or its value, forgoing the type declaration, i.e. having only the clause ' $\text{max\_elem} = x$ ' for some chosen  $x \in \mathbb{N}$  in the Properties clause. The type of it is defined implicitly and it does not affect the proof process.

This restriction impacts the *additem* operation, as we now need to check that adding an item to the bag will not exceed the limit. It can be done in the preconditions section of the operation, as is shown in the example included in the archive, or in an if-else statement. We found these ways to be equivalent for our purposes, as they do not generate different proof obligations. It is worth noting that they will impact the refinement stages of the development. We have elected to stick to the former, since it is less restricting for potential refinement.

There are nine proof obligations generated. The initialisation and each of the operations to add and remove items results in two: one being a type check, the other making sure that the cardinality of *content* does not exceed *max\_elem*. The last three proof obligations are concerned with the well-definedness of the expression  $\text{card}(\text{content})$  in the invariant and the operation *additem* and *howmany*, that is anywhere where the expression appears.

If the expression  $\text{content} \in \text{FIN}(\text{ITEMS})$  is replaced with  $\text{content} \subseteq \text{ITEMS} \wedge \text{content} \in \text{FIN}(\text{content})$  there are again nine proof obligations. However this time there are three proof obligations for each of the Initialisation and the adding and removing operations - one proof obligation for each clause of the Invariant. At the same time the well-definedness proof obligations are not generated. This is in line with the observations regarding *Bagmch\_long\_inv*.

8) *Bagmch\_limited\_params*: differs from the previous version by passing *max\_elem* as a scalar-valued parameter instead. The software requires defining at

least the type of `max_elem` in the Constraints clause, while the Constants and Properties clauses can now be removed. As in the case of *Bagmch\_params* with a set-valued parameter, it does not affect the proof process.

9) *Bagmch\_wrong\_order*: is of particular interest, as it highlights a discrepancy between the theory of the B-method and its implementation in Atelier B. The Invariant clause contains a series of conjuncts which by the rules of logic are commutative. However they are not such in the software.

Firstly, recall that in the *Bagmch\_long\_inv*, the Invariant was as follows:

```
INVARIANT
  content <: ITEMS &
  content : FIN(content)
```

Switching these two statements around results in an error message as static analysis is taking place, demanding that the type of `content` is defined before applying the built-in operator `FIN` to it. The software does not allow the user to proceed with the proof as long as such errors exist.

With the addition of the limit on the number of items in the bag, this problem becomes much less obvious. For example, if the Invariant is formulated like this:

```
INVARIANT
  content <: ITEMS &
  card(content) <= max_elem &
  content : FIN(content)
```

The static analysis does not give any errors, however an additional proof obligation is generated, on top of the nine we get in the alteration to the *Bagmch\_limited* mentioned above. The proof obligation is:

```
content <: ITEMS &
"Well definedness"
=>
content: FIN(content)
```

It does not get discharged by any means available to the user within the interactive prover.

Based on these observations we reach the conclusion that the conjuncts in the Invariant are applied sequentially, in the order in which they are written. It is very much like the Assertion clause, which is defined to be checked sequentially in the B method. It is also understandable from the implementation point of view, since checking all permutations of the conjuncts would be computationally hard.

## VII. RESULTS

Equivalence of expressions  
Ordering of the Invariant

### A. Additional rules

```
THEORY userInFINXY IS
  band(binhyp(b: FIN(b)),
  binhyp(a <: b))
=>
  a: FIN(a)
END
```

## VIII. CONCLUDING REMARKS

It was an initial goal of this project, to create a collection of proof rules which simplify the verification process in the automated prover. This collection was meant to include especially the rules that were found to be needed for the proofs of various scenarios. However as the work has progressed, we have found that it was not necessary to create additional rules, and instead any obstacle could be circumvented with a deeper understanding of the inner workings of the prover.

Instead we

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## APPENDIX A

### THE REFERENCE BAG MACHINE

```

MACHINE
  Bagmch
SETS
  // possible items we can put in the bag
  ITEMS
VARIABLES
  // contents of the bag
  content
INVARIANT
  content : FIN(ITEMS) // content is a *finite* subset of ITEMS
INITIALISATION
  content := {} // we start with an empty bag
OPERATIONS
  /* Adds item aa to the bag*/
  additem(aa) =
  PRE
    aa : ITEMS
  THEN
    content := content \/ {aa}
  END;

  /* removes aa from the bag (does nothing if aa not in the bag) */
  removeitem(aa) =
  PRE
    aa : ITEMS
  THEN
    content := content - {aa}
  END;

  /* getter for the content*/
  items <-- getcontents = items := content;

  /* query how many items are in the bag */
  nn <-- howmany = nn := card(content);

  /* checks if the item aa is in the bag */
  check <-- isin(aa) =
  PRE
    aa : ITEMS
  THEN
    IF
      aa : content
    THEN
      check := TRUE
    ELSE
      check := FALSE
    END
  END
END
END

```



## APPENDIX B

## 'A SUBSET OF A FINITE SET IS FINITE' - PROOF

Let  $A$  and  $B$  be sets, with  $A \subseteq B$  and  $B$  finite. Let us define  $[n]$  to be the set of all elements of  $\mathbb{N}$  less than  $n$ , i.e.  $[n] = \{0, 1, \dots, n-1\}$ .

Since  $B$  is finite, by the definition of finiteness there is  $n \in \mathbb{N}$  such that there exists a bijection between  $B$  and  $[n]$ . Hence it suffices to prove that any subset of  $[n]$  for  $n \in \mathbb{N}$  is finite. We proceed by induction.

When  $n = 0$ ,  $[n] = \emptyset$ , and trivially all subsets of the empty set are finite.

Let  $n > 0$  and assume that all subsets of  $[n-1]$  are finite.

Note that  $[n] = \{0, 1, \dots, n-1\} = \{0, 1, \dots, n-2\} \cup \{n-1\} = [n-1] \cup \{n-1\}$ . Let  $x \subseteq [n]$ . Then either  $n-1 \notin x$  or  $n-1 \in x$ . In the first case,  $x \subseteq [n-1]$ , and thus it is finite.

Otherwise,  $x \setminus \{n-1\} \subseteq [n-1]$  and is finite. Therefore there exists a  $k \in \mathbb{N}$  such that there is a bijection  $f : x \setminus \{n-1\} \rightarrow [k]$ . Then  $f' = f \cup \{(n-1, k)\}$  is a bijection  $f' : x \rightarrow [k+1]$  and by the inductive property of the natural numbers,  $k+1 \in \mathbb{N}$ .

Hence, any  $x \subseteq n$  is finite.

□