Appendix A
'A subset of a finite set is finite' - Proof

Let A and B be sets, with $A \subseteq B$ and B finite. Let us define $[n]$ to be the set of all elements of $\mathbb{N}$ less than n, i.e. $[n] = \{0, 1, ..., n-1\}$.

Since B is finite, by the definition of finiteness there is $n \in \mathbb{N}$ such that there exists a bijection between B and $[n]$. Hence it suffices to prove that any subset of $[n]$ for $n \in \mathbb{N}$ is finite. We proceed by induction.

When $n = 0$, $[n] = \varnothing$, and trivially all subsets of the empty set are finite.

Let $n > 0$ and assume that all subsets of $[n-1]$ are finite.

Note that $[n] = \{0, 1, ..., n-1\} = \{0, 1, ..., n-2\} \cup \{n-1\} = [n-1] \cup \{n-1\}$. Let $x \subseteq [n]$. Then either $n-1 \notin x$ or $n-1 \in x$. In the first case, $x \subseteq [n-1]$, and thus it is finite.

Otherwise, $x \backslash \{n-1\} \subseteq [n-1]$ and is finite. Therefore there exists a $k \in \mathbb{N}$ such that there is a bijection $f : x \backslash \{n-1\} \rightarrow [k]$. Then $f' = f \cup \{(n-1, k)\}$ is a bijection $f' : x \rightarrow [k+1]$ and by the inductive property of the natural numbers, $k+1 \in \mathbb{N}$.

Hence, any $x \subseteq n$ is finite.

$\square$