Bundling against Learning*

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Abstract

A monopolist sells multiple goods to an uninformed buyer. The buyer chooses to learn any one-dimensional linear signal of their values for the goods, anticipating the seller's mechanism. The seller designs an optimal mechanism, anticipating the buyer's learning choice. In a generalized Gaussian environment, we show that every equilibrium has ***vertical learning*** where the buyer's posterior means are comonotonic, and every equilibrium is outcome-equivalent to ***nested bundling*** where the seller offers a menu of nested bundles. In equilibrium, the buyer learns more about a higher-tier good, resulting in a higher posterior variance on the log scale.

Keywords: Equilibrium learning, multidimensional screening, multidimensional learning, vertical learning, horizontal learning, nested bundling.

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1 Introduction

When a new multiproduct firm enters a market, consumers are often uncertain about their willingness to pay for the firm's various products.[1] They spend time learning about the firm's products before buying. At the same time, the new firm often conducts experimentation to optimize prices and product offerings against the demand system, which depends on what the consumers learn. What should we expect in equilibrium about the endogenous demand system and the endogenous product offerings resulting from consumer optimal learning and firm optimal pricing?

In this paper, we answer this question with a model of equilibrium learning in multiproduct pricing. We consider a simultaneous-move game between a seller and a buyer. The seller has K goods to sell. She chooses a ***selling mechanism***, a menu of bundles and prices, allowing for lotteries. The buyer is initially uninformed about his vector of values for each good $\mathbf{v} = (v_k)_k$, which is drawn from an elliptical distribution supported on $V \subset \mathbb{R}^K_+$.[2] The buyer chooses an informative signal about his values $\mathbf{v}$ at no cost but faces a dimension restriction—he can only choose to observe a one-dimensional signal $\boldsymbol{\alpha} \cdot \mathbf{v}$, where $\boldsymbol{\alpha} \in \mathbb{R}^K$ is the vector of ***learning weights***. In equilibrium, the buyer optimally chooses the learning weights to maximize his expected payoff when purchasing from the seller's menu; the seller optimally chooses a menu to maximize revenue given the endogenous demand system—the distribution of posterior means induced by the buyer's signal.

By design, the buyer faces the choice of *what* to learn: He can fully learn the value of any bundle, but must decide which one to learn. He can also learn about the differences between any two goods, or between two bundles, or more generally any linear combination of these signals.[3] However, any two non-identical

[1]For a concrete example, consider OpenAI, which launched new products such as ChatGPT (a language-generation model) and DALL·E (an image-generation model) in 2022.

[2]Elliptical distributions generalize Gaussian distributions; for foundations and economic applications of elliptical distributions, see e.g. Gupta, Varga, and Bodnar (2013), Frankel and Kartik (2019), He and Natenzon (2023), and Ball (2025).

[3]Note that we do not allow for *nonlinear* signals. If we were to allow the buyer to learn any real-valued signal, then the dimension restriction has no bite, as every random vector can be embedded into a real-valued random variable by the Borel isomorphism theorem.

informative signals in our model are *not* Blackwell ordered. Thus, the buyer's choice about what to learn depends on the equilibrium product offerings and their prices. Similarly, the seller's menu results in very different revenue depending on the endogenous distribution of posterior means, and if it fails to be revenue maximizing, the seller will re-optimize.

Our main result (Theorem 1) shows that every equilibrium features a comonotonic posterior mean distribution (***vertical learning***) and is outcome-equivalent to an equilibrium in which the seller offers a menu of nested bundles (***nested bundling***). This result holds regardless of the correlation in the underlying value distribution. In particular, even if the values $\mathbf{v}$ are negatively correlated, the buyer's equilibrium types (posterior means) must be vertically ordered. Moreover, we show that every equilibrium outcome, which consists of a learning strategy and a nested menu, has a simple structure: In equilibrium, the buyer learns more about the higher-tier goods (the upgrades), resulting in a higher posterior variance on the log scale (Proposition 1).

To illustrate the basic intuition behind our main result, consider the following example:

Illustrative Example. There are two goods whose values are drawn from a Gaussian distribution with $\mathbb{E}[v_1] = \mathbb{E}[v_2] = 2$, $\text{Var}(v_1) = \text{Var}(v_2) = 4$, and $\text{Corr}(v_1, v_2) = 0$. The support of the distribution is truncated so as to lie in the positive quadrant, as depicted in Figure 1.

Suppose for contradiction that we are in an equilibrium where the buyer chooses to learn about the difference between the two goods $v_1 - v_2$, which is a ***horizontal learning*** strategy. If he learns $v_1 - v_2 = s$, then he knows that his values lie on the corresponding 45-degree line segment (dashed red lines in the left panel of Figure 1). The goods are ex ante symmetric, so his posterior expected values for goods 1 and 2 are $2 + 0.5s$ and $2 - 0.5s$, respectively. Thus, the buyer's realized type always lies on the -45-degree line segment going through the prior mean (full red line in the middle panel). The seller correctly anticipates the buyer's chosen signal, but does not observe the signal realization. From the seller's point of view, the buyer's type then follows a (truncated) Gaussian distribution supported on the -45-degree line segment. Against this type distribu-

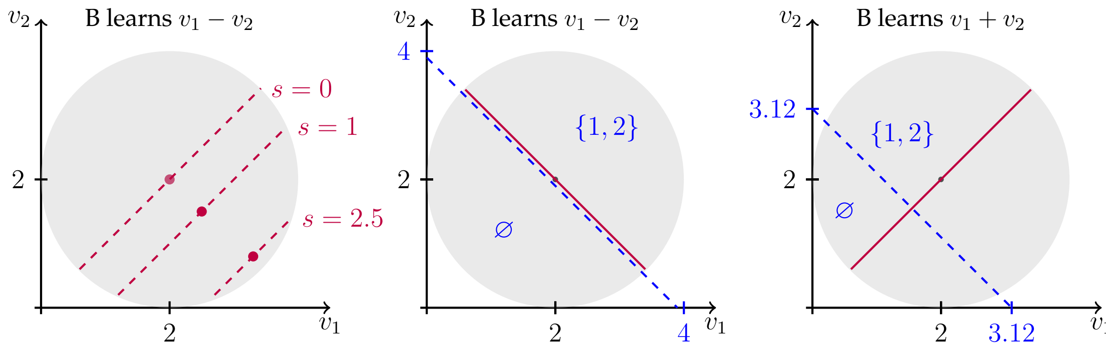

Figure 1: The shaded gray area is the set of possible values V. The left panel illustrates how the buyer updates his belief about goods' expected values upon learning $v_1 - v_2 = s$, for several realizations of s. The middle panel illustrates that the seller would offer only the bundle when the buyer chooses signal $v_1 - v_2$, leading to a contradiction. The right panel illustrates an equilibrium with pure bundling. The full red line corresponds to the support of the buyer's type distribution. The dashed blue lines partition the type space based on optimal allocations under the seller's menu: types above the dashed line purchase $\{1, 2\}$ while types below it purchase nothing $\varnothing$.

tion, to be revenue maximizing, the seller must offer the *bundle* at price 4, which can extract the full surplus since the sum of the posterior means for two goods is always $(2+0.5s)+(2-0.5s) = 4$. However, the buyer then prefers to learn about the value of the bundle $v_1 + v_2$, which is a vertical learning strategy—hence, a contradiction.

Now, suppose that the buyer indeed chooses the vertical learning strategy that reveals the bundle value $v_1 + v_2$. Upon learning $v_1 + v_2 = s$, the buyer's posterior mean for each good is simply $0.5s$. The buyer's realized type now always lies on the 45-degree line segment going through the prior mean (full red line in the right panel). Against this type distribution, it is an optimal strategy for the seller to offer a menu that consists only of the bundle $\{1, 2\}$ at price 3.12. Against this menu, it is indeed optimal for the buyer to learn his value for the bundle $v_1 + v_2$. Thus, we have found an equilibrium. □

As the illustrative example shows, the key intuition behind our result builds on the long-standing insight from multiproduct pricing (Adams and Yellen 1976): Bundling is profitable when consumers have negatively correlated preferences, since it averages out the variation in the willingness to pay for different goods. We take this insight to its logical conclusion when the consumers need to learn about their values: In equilibrium, the consumers cannot spend too much effort learning about their relative values across different goods—because if so, the seller would re-optimize to offer the bundle but then the horizontal information would be useless. Instead, in equilibrium, the consumers spend more time acquiring information that updates their beliefs about different goods in the same direction—such as the firm's reputation, the products' shared functionality, or general aspects of the new technology. Such vertical learning leads to positively correlated preferences in equilibrium. To screen such consumers, the seller then offers a menu of nested bundles with larger bundles targeting consumers with higher posterior expected values of all the goods.

The intuition behind why nested bundling is profitable against positively correlated preferences differs from the classic intuition that bundling averages out different values for different goods. Indeed, in the illustrative example, in the vertical-learning pure-bundling equilibrium, the bundle is not used to average out the willingness to pay but rather to screen the vertical information. In general, the seller offers more than the grand bundle, creating different tiers, and these tiers are ordered in such a way as to facilitate screening, as characterized in Proposition 1. For instance, in the previous illustrative example, if $\mathbb{E}[v_2]$ is perturbed to $\mathbb{E}[v_2] = 2.05$, then pure bundling ceases to be an equilibrium. There exists, however, an equilibrium in which the buyer learns $v_1 + v_2$ and the seller offers a nested menu $\{\{2\}, \{1,2\}\}$ where the base bundle $\{2\}$ is priced at 1.59, and the full bundle $\{1,2\}$ is priced at 3.16. The slightly higher mean of good 2 leads the buyer's posterior means to be more concentrated on the log scale, making it strictly profitable for the seller to offer $\{2\}$ by itself in addition to $\{1,2\}$.[4] The logic behind the buyer having a higher log-scale posterior variance about an upgrade good is due to the optimization by the *seller*. If the buyer learns less about the higher-tier goods, then his posterior means are more concentrated, but then the seller strictly benefits from switching the ordering of the goods to better

[4]See Section 3.1 for various other illustrative examples.

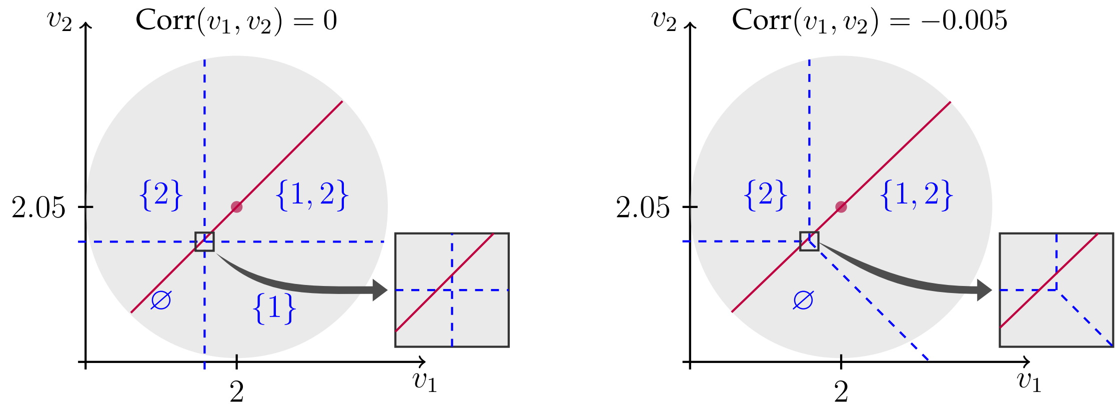

Figure 2: Two perturbations of the above illustrative example. In the left panel, $\mathbb{E}[v_2] = 2.05$ and $\mathrm{Corr}(v_1, v_2) = 0$. The figure depicts an equilibrium with separate sales. There no longer exists an equilibrium with pure bundling. In the right panel, $\mathbb{E}[v_2] = 2.05$ and $\mathrm{Corr}(v_1, v_2) = -0.005$. The figure depicts an equilibrium with nested bundling. There no longer exists an equilibrium with separate sales.

screen the buyer. Indeed, goods with lower posterior variance on the log scale have more elastic demand curves, and those must be offered in the lower tiers to facilitate screening.

Besides the screening property, nested bundling and vertical learning have a self-stabilizing aspect. To illustrate, note that in the above perturbed example, another optimal mechanism against vertical learning is to sell goods 1 and 2 separately, at price 1.57 and 1.59 respectively. When $\mathrm{Corr}(v_1, v_2) = 0$, this still constitutes an equilibrium, which is depicted in Figure 2 (left panel). This equilibrium is outcome equivalent to the nested bundling equilibrium: the buyer's learning is unchanged, and the added option in the seller's menu is never purchased. However, if we perturb the example once again and set $\mathrm{Corr}(v_1, v_2) = -0.005$, separate sales cease to be an equilibrium, but nested bundling still is (Figure 2, right panel). This is not specific to this example. Indeed, as we show, even though a separate sales mechanism can also be optimal against a vertical type distribution, it can never form an equilibrium when $\mathbf{v}$ has negative correlation because then the buyer strictly benefits from switching to a horizontal learning strategy (Proposition 3). Meanwhile, a nested bundling equilibrium always exists with symmetric distributions (Proposition 2), even when $\mathbf{v}$ has negative correlation. By decreasing the instrumental value of horizontal comparisons, bundling favors vertical learning, which is necessary for equilibrium.

To further illustrate this intuition, we also provide results on the buyer's best response to canonical selling mechanisms. For these results, we assume two goods and mostly focus on uncorrelated values. We first show that against separate sales, the buyer is indifferent between vertical and horizontal learning when $\mathrm{Corr}(v_1, v_2) = 0$ (Proposition 4). A separate sales mechanism is one in which goods are sold separately. The correlation of the posterior means is then irrelevant for the buyer, and vertical and horizontal strategies perform equally well. However, against any nested bundling mechanism, the buyer prefers vertical learning (Proposition 5). Indeed, by not allowing the sale of one of the goods by itself, nested bundling reduces the benefits from horizontal information and favors vertical information instead. Finally, if the seller only allows the buyer to buy a single good, but not both, then horizontal learning is optimal (Proposition 6). Thus, even though horizontal learning cannot be sustained in equilibrium, it can still be a best response to some mechanisms.

1.1 Related Literature

We build on a large literature on multiproduct pricing and optimal bundling (starting with Stigler 1963; Adams and Yellen 1976; McAfee, McMillan, and Whinston 1989; Armstrong 1996; Rochet and Chone 1998). Following most of this literature, we assume that the buyer has additive values (McAfee and McMillan 1988; Manelli and Vincent 2006; Pavlov 2011; Daskalakis, Deckelbaum, and Tzamos 2017; Bergemann et al. 2022).[5] There are two general insights from this literature: *(i)* Bundling is often more profitable in settings with negatively correlated values (Stigler 1963; Adams and Yellen 1976); *(ii)* some form of bundling is generically profitable, but characterizing optimal mechanisms is analytically intractable (McAfee, McMillan, and Whinston 1989; Rochet and Stole 2003).[6]

[5]For models with non-additive values, see e.g. Haghpanah and Hartline (2021), Ghili (2023), and Yang (2025).

[6]It is known that finding the optimal mechanism is also computationally intractable (Daskalakis, Deckelbaum, and Tzamos 2014). Moreover, it is known that the optimal mechanism often requires an infinite menu size (Hart and Nisan 2019), and a small perturbation of virtually any incentive-compatible mechanism can make it optimal for some type distribution

Given the difficulty in multidimensional screening, our main conceptual contribution is to take a step back and model the buyer's learning process, which disciplines what type distributions are likely to arise endogenously in markets with new firms or new products. Our results take the classic insight from the bundling literature to its logical conclusion: When consumer preferences arise endogenously from optimal learning, they are likely to be positively correlated across goods; any negative correlation invites enough bundling responses to always disincentivize horizontal learning. As we explained, our model also brings out new insights about the self-stabilizing nature of nested bundling and vertical learning. As a consequence, we provide a microfoundation for the exogenous type spaces studied in the bundling literature—in particular, Yang (2025), which *assumes* a comonotonic type distribution to characterize demand conditions under which nested bundling is optimal (allowing for non-additive values).[7]

In proving our main result, we also make a technical contribution to this literature by fully characterizing all optimal mechanisms for type distributions supported on any line segment in any dimension (see Section 4). Our analysis combines techniques from the recent works by Frick, Iijima, and Ishii (2024), who study bundling by a seller with rich consumer data, and Loertscher and Muir (2024), who study optimal auctions for selling two horizontally differentiated goods in the Hotelling sense to unit-demand bidders.

Several recent papers adopt a robustness approach to study multidimensional screening (Carroll 2017; Brooks and Du 2024; Deb and Roesler 2024; Che and Zhong 2024). In these papers, the seller evaluates the performance of a mechanism against the worst-case distribution of buyer types within some set of admissible distributions. Like in our paper, the relevant distribution is then endogenous to the mechanism. However, it is not a result of the buyer's learning incentives. Most relevant for our analysis is Deb and Roesler (2024). They consider a setting where the seller and the buyer share a common prior about the buyer's values, but the seller is agnostic as to which additional information the buyer might have. Thus, the set of admissible type distributions is the set of all

(Manelli and Vincent 2007; Lahr and Niemeyer 2024).

[7] For other sufficient conditions under which nested bundling is optimal, see also Bergemann et al. (2022) and Yang (2022).

distributions that can be induced by *some* signal, given the prior. Assuming the prior is exchangeable, they show that randomized pure bundling is worst-case optimal where the worst-case signal reveals noisy information about the grand bundle in a way that generates a truncated Pareto distribution. This implies that when the buyer moves first, such that his chosen signal is observed by the seller, it leads to pure bundling with efficient trade.[8] We complement their analysis by considering a simultaneous-move game where the buyer cannot flexibly design a signal and cannot commit to the signal—he optimally chooses what to learn, given the seller's menu, from a set of Blackwell undominated signals. As a consequence, trade is inefficient in our model, and outcomes generally involve nested bundling with the buyer learning more about higher-tier goods.[9]

Lastly, we contribute to the literature on mechanism design with information acquisition (Bergemann and Välimäki 2002; Shi 2012; Mensch 2022; Mensch and Ravid 2025). This literature has studied how the agent's learning incentives affect the principal's optimal mechanism in various settings, ranging from the design of efficient mechanisms (Bergemann and Välimäki 2002) to monopoly pricing (Mensch and Ravid 2025).[10] We depart from that literature in two main ways. To the best of our knowledge, this paper is the first to focus on learning incentives in multiproduct monopoly pricing. The complexity compared to the single-product case arises from the fact that the buyer learns about a multidimensional state.[11] Second, we consider a simultaneous-move game between the buyer and seller, while most of the existing literature gives the principal a first-mover advantage.[12] A notable exception is Ravid, Roesler, and Szentes (2022) who also consider a simultaneous-move game, but in single-good monopoly pricing.

[8] This generalizes the single-good result of Roesler and Szentes (2017).

[9] See Section 5.1 for further discussion of what happens if the buyer moves first in our model.

[10] A smaller part of the literature studies post-purchase learning and product returns (e.g. Che 1996; Matthews and Persico 2007); we abstract away from these concerns. For a discussion of bundling and product returns, see Haberman, Jagadeesan, and Yang (2025).

[11] A few papers also study multidimensional learning but in other contexts (Gleyze and Pernoud 2023; Bobkova 2024a,b; Pernoud and Gleyze 2025).

[12] See Section 5.1 for a discussion of what happens when the seller moves first; as we explain there, the timing of the moves is important for sustaining vertical learning.

Overview. The remainder of the paper proceeds as follows. Section 2 presents our model. Section 3 presents our main result and further illustrates the intuition. Section 4 sketches the proof of the main result. Section 5 discusses extensions and generalizations. Section 6 concludes. All the proofs can be found in Appendix A.

2 Model

We consider a simultaneous-move game between a seller and a buyer. The seller (she) has K indivisible goods to sell to the buyer (he). The buyer's utility is additive across goods and quasilinear in money.[13] His payoff from purchasing bundle $B \subseteq \{1, \ldots, K\}$ at price p is then

$$\sum_{k \in B} v_k - p \,,$$

where v_k denotes his value for good k. We consider the case where the buyer's values are always above the seller's costs, and normalize the cost for each good to be zero.[14]

The buyer's ***values*** $\mathbf{v} = (v_k)_k$ follow an elliptical distribution with continuous density supported on a compact set $V \subset \mathbb{R}^K_+$.[15] Let $\boldsymbol{\mu} = (\mu_k)_k$ and Σ denote the mean vector and covariance matrix of $\mathbf{v}$. Goods can differ in their prior mean and variance, but we assume that they share the same correlation $\text{Corr}(v_i, v_j) = \rho \in (-1, 1)$ for all pairs of goods i and j. We say that the values are ***positively correlated*** if $\rho > 0$, ***negatively correlated*** if $\rho < 0$, and ***uncorrelated*** otherwise.

The buyer does not observe $\mathbf{v}$ but has access to a dimension-restricted ***learning technology***: he can choose any one-dimensional linear signal of the vector of values $\mathbf{v}$. That is, a learning strategy for the buyer consists of choosing ***learning***

[13] We relax this assumption and discuss robustness to nonadditive values in Section 5.2.

[14] We relax this assumption and discuss the case of high production costs in Section 5.3.

[15] Formally, a random vector $\mathbf{v} \in \mathbb{R}^K$ has an ***elliptical distribution*** if its characteristic function ϕ satisfies $\phi_{\mathbf{v}-\boldsymbol{\mu}}(t) = \psi(t'\Sigma t)$ for any column vector $t \in \mathbb{R}^K$, where $\boldsymbol{\mu}$ is the location parameter, Σ is a positive definite matrix, and ψ is a scalar function.

weights $\boldsymbol{\alpha} \in \mathbb{R}^K$, and the buyer gets to observe the realization of $\boldsymbol{\alpha} \cdot \mathbf{v}$.[16]

Without observing the buyer's choice of signal, the seller chooses a ***selling mechanism*** $\mathcal{M} = (M, x, p)$, which consists of

$$\text{message space } M\text{, allocation rule } x : M \to \Delta(2^K)\text{, payment rule } p : M \to \mathbb{R}\,.$$

Equivalently, a mechanism can be represented as a ***menu*** $\{(x, p)\}$ of lotteries of bundles and associated prices.

The buyer maximizes his expected payoff and the seller maximizes her expected profits.

Buyer's strategy and induced type distribution. We now explain in more detail how the buyer's learning strategy maps into a type distribution. The buyer is risk-neutral, so his purchasing decision depends only on the posterior expected value for each good. Given weights $\boldsymbol{\alpha}$ and signal realization s, the buyer's ***type*** $\boldsymbol{\theta} = (\theta_k)_k$ consists of the expected value for each good k:

$$\theta_k(s; \boldsymbol{\alpha}) := \mathbb{E}[v_k \mid s] = \mu_k + \frac{\text{Cov}(v_k, \boldsymbol{\alpha} \cdot \mathbf{v})}{\text{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})}[s - \boldsymbol{\alpha} \cdot \boldsymbol{\mu}]\,.$$

Let $G_{\boldsymbol{\alpha}}$ denote the distribution of types induced by weights $\boldsymbol{\alpha}$. Note that conditional expectations, or types, are linear in the signal realization s.[17] This is an important property that we leverage in our analysis. It implies that the support of $G_{\boldsymbol{\alpha}}$ is a line segment in $\mathbb{R}^K$. It also implies that the buyer's type follows an elliptical distribution, since linear combinations of elliptical random variables are also elliptical. Without loss of generality, we assume that $\text{Cov}(v_k, \boldsymbol{\alpha} \cdot \mathbf{v}) > 0$ for at least one good k. Indeed, if $\text{Cov}(v_k, \boldsymbol{\alpha} \cdot \mathbf{v}) \leq 0$ for all goods k, then the alternative signal $\boldsymbol{\alpha}' = -\boldsymbol{\alpha}$ induces the same type distribution, but satisfies $\text{Cov}(v_k, \boldsymbol{\alpha}' \cdot \mathbf{v}) \geq 0$ for all goods k. Moreover, the uninformative signal $\boldsymbol{\alpha} = \mathbf{0}$ is dominated by any other signal and never chosen in equilibrium.

[16]This is equivalent to assuming that the buyer can choose any one-dimensional signal s that is jointly elliptically distributed with $\mathbf{v}$.

[17]The family of elliptical distributions is the most general class of distributions with this property (see Gupta, Varga, and Bodnar 2013).

A buyer with type $\boldsymbol{\theta}(s;\boldsymbol{\alpha})$ who faces a mechanism $\mathcal{M}$ solves

$$\sup_{m\in M} \sum_k \theta_k(s;\boldsymbol{\alpha})x_k(m) - p(m)\,.$$

Without loss of generality, we assume that there exists some $m_o \in M$ such that $x(m_o) = \varnothing$ and $p(m_o) = 0$ (i.e., the buyer can always walk away to obtain his outside option, which is normalized to have value 0). As is standard in the literature, we assume that when indifferent between two messages, the buyer breaks the indifference in favor of the seller. This guarantees the existence of an optimal mechanism for the seller.

Solution concept. Our solution concept is pure-strategy Nash equilibrium.[18] A strategy profile $(\boldsymbol{\alpha}, \mathcal{M})$ forms an ***equilibrium*** if the buyer's learning strategy $\boldsymbol{\alpha}$ is optimal against mechanism $\mathcal{M}$, and mechanism $\mathcal{M}$ is profit-maximizing against the type distribution induced by learning strategy $\boldsymbol{\alpha}$. Two equilibria are ***outcome-equivalent*** if they induce the same allocation and transfer.

Remark on modeling choices. Our main goal is to study the direction of learning (*what* the buyer learns about) and not the extent of learning (*how much* he learns). We thus model learning as being free but constrained in its dimensionality: the buyer can only learn along one direction, but can learn as much as possible along that direction. We could augment the model to also allow the buyer to control the precision of the signal at some cost. That is, after having chosen a direction, the buyer also chooses a level of noise, trading off higher precision for higher costs. As long as the cost is the same in all directions, then all of our results go through. The restriction to one-dimensional signals not only constrains the buyer's learning but also provides much tractability. We discuss multidimensional signals in Section 5.5.

[18]Our results hold even with a weaker solution concept as shown in Section 5.4.

3 Main Results

We say that a learning strategy $\boldsymbol{\alpha}$ is ***vertical*** if $\text{Cov}(v_k, \boldsymbol{\alpha} \cdot \mathbf{v}) \geq 0$ for every good k, and ***horizontal*** otherwise. Note that learning strategy $\boldsymbol{\alpha}$ is vertical if and only if its induced posterior mean distribution $G_{\boldsymbol{\alpha}}$ is ***comonotonic***, i.e., $\theta_i \leq \theta'_i \implies \theta_j \leq \theta'_j$ for all goods i, j, and all types $\theta, \theta' \in \operatorname{supp} G_{\boldsymbol{\alpha}}$.

We say that an equilibrium has ***nested bundling*** if the seller offers a menu of deterministic bundles that can be totally ordered by set-inclusion.

THEOREM 1. *Every equilibrium has vertical learning, and is outcome-equivalent to a nested bundling equilibrium.*

The proof is in the appendix. We provide the intuition in Section 3.1 and Section 3.2. We sketch the proof in Section 4. The first part of Theorem 1 asserts that only vertical learning can be sustained in equilibrium, and hence the type distribution endogenously features vertically differentiated types. This is true regardless of whether the underlying values $\mathbf{v}$ are positively or negatively correlated. Moreover, note that this is true even though the space of vertical learning strategies is vanishingly small as K increases—indeed, with uncorrelated values, vertical learning requires $\boldsymbol{\alpha} \in \mathbb{R}^K$ to have weights being all positive or all negative, which are only two possibilities out of 2^K possible sign combinations.

The second part of Theorem 1 asserts that for every equilibrium, either the seller is using a nested bundling strategy, or there exists a nested bundling equilibrium in which both the seller and the buyer get exactly the same outcomes.

Our next result further characterizes the buyer's learning strategy in equilibrium. For a given nested menu $\{(B_1, p_1), \ldots, (B_m, p_m)\}$, where $B_1 \subseteq \cdots \subseteq B_m$, we define the ***tier*** of an item i as the index of the smallest bundle that includes item i.

PROPOSITION 1. *Consider any nested bundling equilibrium. For any items i, j where* $\text{tier}(i) \leq \text{tier}(j)$, *we have*

$$\text{Cov}(v_i/\mu_i, \boldsymbol{\alpha} \cdot \mathbf{v}) \leq \text{Cov}(v_j/\mu_j, \boldsymbol{\alpha} \cdot \mathbf{v}) \;\; \textit{and} \;\; \text{Var}(\log(\theta_i)) \leq \text{Var}(\log(\theta_j))\,.$$

If the values are uncorrelated, then the buyer's adjusted learning weights are ordered:

$$0 \le \frac{\sigma_i^2}{\mu_i}\alpha_i \le \frac{\sigma_j^2}{\mu_j}\alpha_j \,.$$

The proof is in the appendix. We provide the intuition in Section 3.1. Proposition 1 says that, in any nested bundling equilibrium, the buyer's signal covaries more with the higher-tier good when normalized by the mean, resulting in a higher posterior variance for the higher-tier good on the log scale. Under uncorrelated values, Proposition 1 shows that this is only possible if the adjusted learning weights are ordered, where the adjustment takes into account that the same learning weight may resolve more uncertainty for one good than the other due to the difference in the prior distribution.

Since Theorem 1 shows that all equilibrium outcomes are characterized by vertical learning and nested bundling, Proposition 1 and Theorem 1 together then give a qualitative prediction of all equilibrium outcomes.

Equilibrium Existence. Since we focus on pure-strategy equilibria, an equilibrium may not always exist, but the following result gives simple sufficient conditions for the existence of an equilibrium:

PROPOSITION 2. *An equilibrium exists if* $\mathbf{v}$ *is exchangeable. Moreover, holding everything else fixed, there exists* $\underline{\rho} < 1$ *such that for all* $\rho \ge \underline{\rho}$*, an equilibrium exists.*

The proof is in the appendix. When $\mathbf{v}$ is exchangeable, by the argument given in the introduction, a pure bundling equilibrium exists. Otherwise, when ρ is sufficiently high, the buyer's problem becomes quasi-concave (while the seller's problem is linear), and hence existence of equilibrium is guaranteed by standard fixed-point arguments. We also consider a weaker solution concept in Section 5.4 that only requires the buyer to choose a Blackwell undominated signal given the seller's menu. As we show in Section 5.4, our main results continue to hold and equilibrium existence is guaranteed.

Instability of Separate Sales. Unlike nested bundling, the next result shows that it is impossible to sustain separate sales in equilibrium if the values are negatively correlated:

PROPOSITION 3. *When $\rho < 0$, there exists* no *separate sales equilibrium.*

The proof is in the appendix. We provide the intuition in Section 3.2.

3.1 Illustrative Examples

We now go through several numerical examples to illustrate and provide intuition for Theorem 1. We only consider examples with two goods for simplicity.

First, let the buyer's values be drawn from a Gaussian distribution with mean $\boldsymbol{\mu} = (1, 2)$, standard deviation $\sigma_1 = \sigma_2 = 1$, and correlation $\rho = 0.5$, whose support is truncated to lie in the positive quadrant as depicted in Figure 3. There exists an equilibrium in which the buyer chooses signal $\boldsymbol{\alpha} = (.74, .26)$, and the seller offers a menu consisting of good $\{2\}$ at price 1.51 and the bundle $\{1, 2\}$ at 2.37. This equilibrium is illustrated in the left panel of Figure 3, where the red segment is the support of the type distribution induced by $\boldsymbol{\alpha}$. The red segment is increasing, which means that the buyer is using a vertical learning strategy as required by Theorem 1. The seller uses a nested bundling mechanism, where the low tier consists of good 2 and the high tier bundles good 1 with good 2. Proposition 1 states that the buyer's posterior value for good 1 must be more dispersed on the log scale than his posterior value for good 2. This is indeed the case here since $\text{Var}(\log(\theta_1)) = 0.39$ while $\text{Var}(\log(\theta_2)) = 0.03$. To understand why the variation is measured on the log scale, note that the logic behind this comparison actually comes from the *seller*'s optimization. Indeed, if the log-scale dispersion of θ_1 is strictly lower than that of θ_2, then the endogenous demand curve of good 1 must be more elastic than that of good 2—the seller would then have an incentive to deviate by swapping the base good and the upgrade good to increase her revenue.

The equilibrium described above features both vertical learning and nested bundling, but this is not the only equilibrium. Against the same type distribution, another optimal mechanism is a separate sales mechanism that offers good 2 at price 1.51, good 1 at 0.86, and the grand bundle at $1.51 + 0.86 = 2.37$ (right panel). In this example, it happens that signal $\boldsymbol{\alpha} = (.74, .26)$ remains optimal

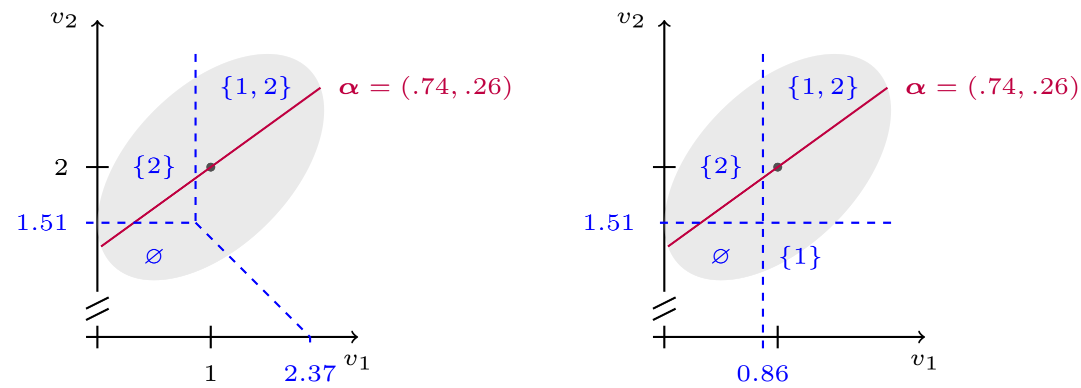

Figure 3: The shaded gray area is the set of possible values V. The left panel illustrates an equilibrium with nested bundling. The buyer chooses signal $\boldsymbol{\alpha} = (.74, .26)$, leading to a type distribution supported on the full red line. The seller offers $\{2\}$ at price 1.51 and $\{1, 2\}$ at 2.37. The right panel illustrates an equilibrium with separate sales, which is outcome-equivalent to the nested bundling equilibrium.

under separate sales, and so this also constitutes an equilibrium. This equilibrium does *not* feature nested bundling since the equilibrium mechanism offers both $\{1\}$ and $\{2\}$, but it is outcome-equivalent to a nested bundling equilibrium. Indeed, even though the buyer has the opportunity to buy each good by itself, he never does so in equilibrium, and makes the same purchasing decisions as in the nested bundling equilibrium. The players' payoffs and the equilibrium outcomes remain unchanged.

All the equilibria considered so far feature vertical learning, and Theorem 1 states that only such equilibria can exist. This is true even when vertical learning seems to resolve little uncertainty for the buyer. For instance, take the extreme case of very negatively correlated values, as depicted in Figure 4. There is much more dispersion in the distribution of $\mathbf{v}$ along decreasing lines than along increasing lines. Thus, horizontal learning strategies resolve much more uncertainty than vertical ones, and lead to type distributions with higher dispersion. However, Theorem 1 asserts that horizontal learning cannot be sustained in equilibrium. To understand this, note that even though the "magnitude" of information seems to be large for horizontal learning, any two non-identical

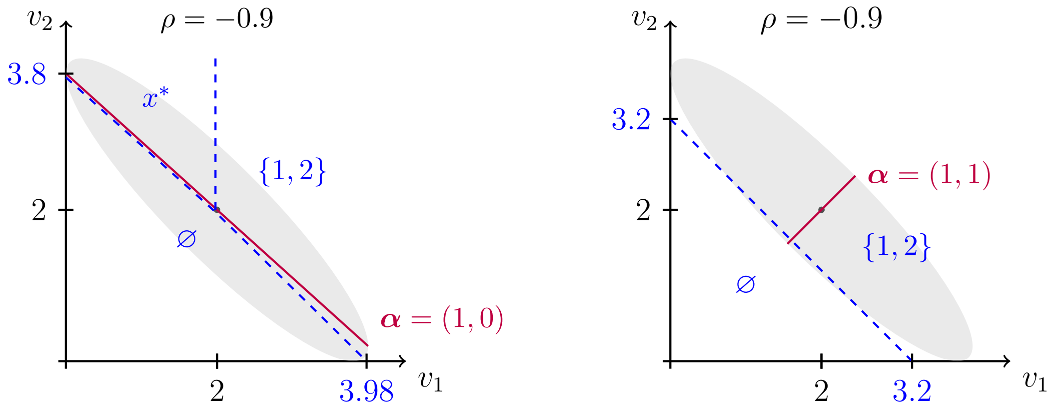

Figure 4: The shaded gray area is the set of possible values V. The left panel considers what happens when the buyer chooses horizontal learning strategy $\boldsymbol{\alpha} = (1,0)$. The type distribution is supported on the full red line. The associated optimal mechanism offers $\{1,2\}$ at price 3.98 and the lottery $x^* = 0.9 \times \{1,2\} + 0.1 \times \{2\}$ at price 3.8. This leaves little variation in the buyer's payoffs, which implies that the buyer's learning strategy must be quite uninformative about the optimal purchase decision given this menu. The right panel illustrates a pure bundling equilibrium. Even though the buyer's equilibrium signal $\boldsymbol{\alpha} = (1,1)$ leads to a less dispersed type distribution than $\boldsymbol{\alpha} = (1,0)$, it is perfectly informative about the buyer's optimal purchase decision given the menu.

signals in our model are *not* Blackwell ordered. It turns out that the seller's optimal mechanism against horizontal learning would always lead to a decision problem for which the original learning strategy is in the wrong "direction" and hence suboptimal.

To further understand the intuition, note that, as we explained in the introduction, horizontal learning leads to negatively correlated preferences, for which the seller wants to design bundling mechanisms that limit the variation in the buyer's payoff. This implies that the buyer's learning strategy necessarily leads to limited variation in payoffs when facing the seller's menu, and hence resolves "wrong" uncertainty that is irrelevant for his purchase decisions.

To illustrate, consider again the example depicted in Figure 4, where the distribution of $\mathbf{v}$ is exchangeable with $\rho = -0.9$. Signal $\boldsymbol{\alpha} = (1,0)$ fully reveals the buyer's value for good 1, but also provides information about good 2 since the values are correlated. The type distribution is then supported on a decreasing line segment, with slope $\text{Cov}(v_2, v_1)/\text{Cov}(v_1, v_1) = -0.9$. Note that all types of the buyer have the same value for a particular randomized bundle x^* that offers a lottery of getting $\{1, 2\}$ with probability 0.9 and getting $\{2\}$ with probability 0.1. The seller's optimal mechanism, also depicted in Figure 4, leverages this randomized bundle to limit the variation in the buyer's payoffs. Thus, even though the horizontal signal $\boldsymbol{\alpha} = (1, 0)$ seems to resolve a substantial amount of uncertainty about $\mathbf{v}$, it is actually very *uninformative* about what to purchase when facing the seller's menu—the seller's bundling mechanism by design limits the informational value of the original learning strategy.

By contrast, vertical learning leads to positively correlated preferences, for which the seller's optimal mechanism cannot "average out" the variation and hence takes the form of a screening mechanism with nested bundles. In this case, the buyer's learning strategy can lead to substantial variations in payoffs, and hence can be quite informative about what to purchase under the seller's menu. In the example depicted in Figure 4, the distribution of values is exchangeable, and hence there exists a pure bundling equilibrium in which the seller offers $\{1, 2\}$ at price 3.2 and the buyer chooses $\boldsymbol{\alpha} = (1, 1)$ (right panel). Even though the buyer's equilibrium signal $\boldsymbol{\alpha} = (1, 1)$ leads to a much less dispersed type distribution than $\boldsymbol{\alpha} = (1, 0)$, it resolves exactly the relevant uncertainty given the menu.

3.2 Intuition for Optimal Learning

In this section, to provide intuition, we further derive properties of the buyer's best response against deterministic mechanisms when there are two goods.

We say that the buyer is ***indifferent between vertical and horizontal learning*** if, for every horizontal learning strategy, there exists a *strictly* vertical learning strategy that gives the buyer the same expected payoff, and vice versa.[19] We say that the buyer ***prefers horizontal to vertical learning*** if, for every strictly vertical learning strategy, there exists a horizontal learning strategy that gives the buyer a weakly higher expected payoff (and the other direction is defined

[19] A strictly vertical learning strategy has $\text{Cov}(v_i, \boldsymbol{\alpha} \cdot \mathbf{v}) > 0$ for both goods, such that the type distribution is supported on an increasing line that has neither zero nor infinite slope.

analogously).[20]

Proposition 4. *With two goods, against any separate sales mechanism:*

(i) The buyer is indifferent between vertical and horizontal learning if $\rho = 0$;

(ii) The buyer prefers vertical to horizontal learning if $\rho > 0$;

(iii) The buyer prefers horizontal to vertical learning if $\rho < 0$.

Against a separate sales mechanism, the buyer's purchasing decision is separable across goods—he can separately decide whether to buy each good i at price p_i. Any correlation in the buyer's posterior values for goods (θ_1, θ_2) is then irrelevant; only the marginals of the type distribution matter. When values are uncorrelated, any vertical learning strategy can be matched to a "flipped" horizontal learning strategy that induces the same marginal type distributions, even though the joint distribution differs. The buyer is then indifferent between vertical and horizontal learning. This is illustrated in Figure 5. Any positive correlation in the distribution of values tips the scales in favor of vertical learning, and vice versa.

Proposition 3 can be understood by combining Proposition 4 and Theorem 1: If $\rho < 0$, then any separate sales mechanism induces the buyer to learn horizontally, which cannot happen in equilibrium.

By contrast, nested bundling favors vertical learning:

Proposition 5. *With two goods and uncorrelated values, against any nested bundling mechanism, the buyer prefers vertical learning to horizontal learning.*

To see the intuition, it is useful to note that a nested bundling mechanism can be constructed by removing one (or several) of the standalone goods from a separate sales mechanism. For instance, the nested bundling menu that offers $\{2\}$ at price p_2 and $\{1, 2\}$ at price p_{12} can be constructed by removing option $\{1\}$ from the separate sales mechanism that offers $\{2\}$ at price p_2, $\{1\}$ at price $p_{12} - p_2$, and $\{1, 2\}$ at price p_{12}. Removing such an option reduces the buyer's benefit from learning which good he prefers—and thus from horizontal learning—since

[20]The definition can be strengthened to strict comparisons for our results as long as the seller's mechanism is not dominated in an appropriate sense.

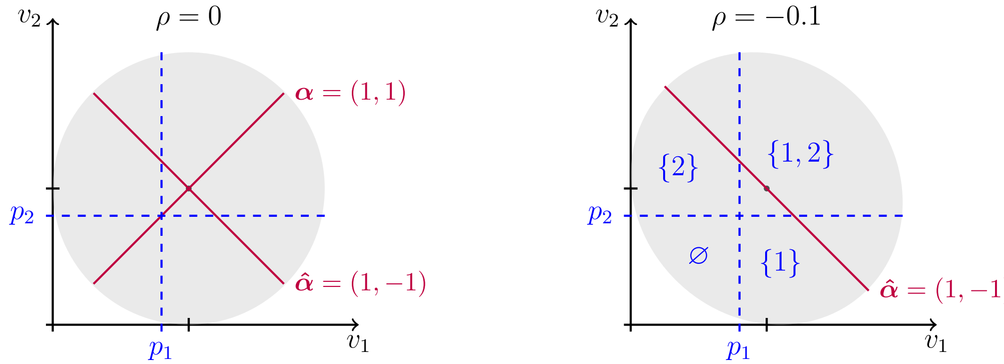

Figure 5: The shaded gray area is the set of possible values V. The left panel considers the case of $\rho = 0$. The distribution of $\mathbf{v}$ is then symmetric along the vertical and horizontal axes going through (μ_1, μ_2). Any vertical learning strategy (here $\boldsymbol{\alpha} = (1,1)$) can be flipped along these axes to construct a horizontal learning strategy (here $\hat{\boldsymbol{\alpha}} = (1,-1)$) that induces the same marginals over θ_1 and θ_2. When $\rho < 0$ (right panel), the flipped horizontal learning strategy leads to marginals that are strictly more dispersed.

one good is no longer available by itself. This breaks the buyer's indifference between vertical and horizontal learning in favor of the former. This effect is only made stronger if values are positively correlated.

Finally, we say that a mechanism only allows for the purchase of one good if it does not offer the bundle $\{1,2\}$.

Proposition 6. *With two goods and uncorrelated values, against any mechanism that only allows the purchase of one good, the buyer prefers horizontal learning to vertical learning.*

As before, such a mechanism can be constructed by removing one option—this time, the grand bundle—from a separate sales mechanism. Removing the grand bundle reduces the relative benefits from vertical learning, thus making the buyer prefer horizontal learning. This effect is only made stronger if values are negatively correlated. Proposition 6 shows that there exist mechanisms that favor horizontal learning even with uncorrelated values. Indeed, Theorem 1 is not driven by vertical learning being better under *any* mechanism—instead, only mechanisms that favor vertical learning can be sustained in equilibrium.

4 Proof Sketch of Theorem 1

In this section, we sketch the proof of Theorem 1 (see Appendix A.1 for the details). Theorem 1 consists of two parts: *(i)* every equilibrium has vertical learning, and *(ii)* every equilibrium is outcome-equivalent to a nested bundling equilibrium. Once we show part *(i)*, part *(ii)* follows relatively straightforwardly since the equilibrium type space must be comonotonic, and we can leverage known results from the literature.

Thus, the main difficulty is to prove vertical learning. The proof proceeds as follows:

Step 1. We characterize optimal mechanisms against any distribution supported on a line segment in $\mathbb{R}_+^K$.

Step 2. We characterize properties of the buyer's optimal learning strategies against any candidate optimal mechanism using **Step 1**.

Step 3. We show these together lead to a contradiction if the buyer uses a horizontal learning strategy.

Section 4.1 sketches **Step 1**; Section 4.2 sketches **Step 2**; Section 4.3 sketches **Step 3** which then completes the proof for vertical learning and then shows how the nested bundling claim follows from there.

4.1 Optimal Mechanisms

As we have discussed, a key property of our elliptical setup is that any learning weights $\boldsymbol{\alpha}$ must lead to a distribution of posterior means $\boldsymbol{\theta} \in \mathbb{R}_+^K$ supported on a line segment in $\mathbb{R}_+^K$. This line segment must pass through the prior $\boldsymbol{\mu}$ but can point in any direction. In equilibrium, the seller must use a mechanism that is optimal against such a distribution. Unlike in the standard mechanism design problem, the key issue here is that the types with a binding IR constraint (the ***worst-off types***) are endogenous to the mechanism. Moreover, unlike standard mechanism design, we need to characterize the properties that hold for ***all*** optimal mechanisms since the seller's indifference may be instrumental for sustaining an equilibrium.

We give a full characterization of every optimal mechanism in the appendix. Assuming the buyer adopts a horizontal learning strategy, we first describe the structure of the optimal mechanisms—which will be used heavily to derive a buyer deviation—and then sketch how we solve the mechanism design problem.

It turns out the structure of any optimal mechanism can be described as follows:

(Claim 1) There is a set of ***"negative" goods*** that would be allocated to every type with full probability.

(Claim 2) There is a set of ***"positive balancing" goods*** that would be allocated to every type with full probability except the types with 0 payoff.

(Claim 3) There is a set of ***"positive non-balancing" goods*** that would be allocated in a standard fashion to types with their value above a threshold.

These three sets of goods are mutually exclusive and collectively exhaustive. Moreover, they are independent of the optimal mechanism. Every optimal mechanism must satisfy **(Claim 1)** to **(Claim 3)** for the same sets of goods.

We now explain the construction of these sets and their names. Since the posterior mean $\boldsymbol{\theta} \in \mathbb{R}_+^K$ forms a line segment, we can parameterize the types on the line by $t \in [0, 1]$, and write

$$\theta_i(t) := a_i t + b_i \,.$$

Now, importantly, we make the following ***sign convention***:

$$\sum_i a_i \geq 0 \,.$$

Note that this is without loss of generality because if it fails, then we can simply redefine types $\tilde{t} = 1 - t \in [0, 1]$, and write

$$a_i t + b_i = \underbrace{-a_i}_{\tilde{a}_i} \tilde{t} + \underbrace{a_i + b_i}_{\tilde{b}_i} \,,$$

which flips the sign for each good i. Intuitively, this sign convention normalizes the direction of types so that a higher type has a higher value for the grand bundle. Under this sign convention, we define

$$I^+ := \Big\{i : a_i > 0\Big\} \quad \text{and} \quad I^- := \Big\{i : a_i \leq 0\Big\},$$

and call the goods in I^+ the ***(strictly) positive goods***, and the goods in I^- the ***negative goods***. Because of our sign normalization, the positive goods are exactly the goods whose values are positively correlated with the grand bundle value, and the negative goods are exactly the goods whose values are negatively correlated with the grand bundle value.

To define "positive balancing" and "positive non-balancing" goods, consider the following auxiliary problem:

$$\begin{aligned} \max_{\mathbf{x} \in [0,1]^K} & \sum_i b_i x_i \\ \text{subject to} & \sum_i a_i x_i = 0\,. \end{aligned} \tag{Auxiliary Problem}$$

The auxiliary problem can be viewed as a fractional knapsack problem and hence admits a greedy solution. In fact, every solution to the auxiliary problem can be characterized as follows. There exists $\kappa \geq 0$ such that for any optimal solution x^* to (Auxiliary Problem), we have:

- For any negative good i, $x_i^* = 1\,$;
- For any strictly positive good i, $x_i^* = 1$ if $b_i/a_i > \kappa$ and $x_i^* = 0$ if $b_i/a_i < \kappa\,$.

For any strictly positive good i where $b_i/a_i = \kappa$, an optimal solution has the freedom to ration it as long as the feasibility constraint is satisfied. Let $\lambda = -\kappa$. It is not hard to see that λ is exactly an optimal ***dual multiplier*** on the equality constraint in (Auxiliary Problem). Let X^* be the set of all optimal solutions to (Auxiliary Problem). Now, define $I^* \subseteq I^+$ as follows:

$$I^* := \bigcup_{x^* \in X^*} \Big\{i \in I^+ : x_i^* > 0\Big\}\,.$$

We call I^* the ***positive balancing goods***, and $I^+ \backslash I^*$ the ***positive non-balancing goods***. Intuitively, the positive balancing goods balance the negative goods so that $\sum_i a_i x_i^* = 0$ in the feasibility constraint of the auxiliary problem.

Now, we sketch the proofs of **(Claim 1)** to **(Claim 3)**. Our proof approach combines techniques from Frick, Iijima, and Ishii (2024) and Loertscher and Muir (2024). In particular, the auxiliary problem above is a special case of the program in Frick, Iijima, and Ishii (2024) that finds the lottery to use for the types with binding IR in mechanism design problems with one-dimensional linear types (which also allow for non-additive values).

To prove **(Claim 1)** to **(Claim 3)**, we adopt a saddle-point approach building on Loertscher and Muir (2024). For the mechanism to be revenue-maximizing, the IR constraint must be binding for (at least) one type. Moreover, for any type t_0 such that $U(t_0) = 0$, the standard characterization of [IC] applies in this setting: [IC] holds if and only if

$$\sum_i a_i x_i(t) \text{ is nondecreasing in } t \text{ and } U(t) = \int_{t_0}^{t} \sum_i a_i x_i(\nu) d\nu \text{ for all } t.$$

In particular, let

$$\text{MON} := \left\{ x : [0,1] \to [0,1]^K \text{ such that } \sum_i a_i x_i(t) \text{ is nondecreasing in } t \right\}.$$

For any t_0, define

$$\Phi(t; t_0) := \left(t + \frac{F(t)}{f(t)}\right) \mathbb{1}\{t \leq t_0\} + \left(t - \frac{1 - F(t)}{f(t)}\right) \mathbb{1}\{t > t_0\}.$$

By a similar argument as in Loertscher and Wasser (2019) and Loertscher and Muir (2024), given any mechanism, t_0 is a worst-off type if and only if

$$t_0 \in \arg\min_{\hat{t}} \mathbb{E}\left[\sum_i \left(a_i x_i(t) \Phi(t; \hat{t}) + b_i x_i(t) \right) \right].$$

In particular, we have that every optimal mechanism must have an allocation rule $x(\,\cdot\,)$ in the following set

$$\arg\max_{x\in\text{MON}} \min_{t_0\in[0,1]} \mathbb{E}\left[\sum_i \Big(a_i x_i(t)\Phi(t;t_0) + b_i x_i(t)\Big)\right].$$

Moreover, we have the following ***saddle point property***:

$$\max_{x\in\text{MON}} \min_{t_0\in[0,1]} \mathbb{E}\left[\sum_i \Big(a_i x_i(t)\Phi(t;t_0) + b_i x_i(t)\Big)\right] = \min_{t_0\in[0,1]} \max_{x\in\text{MON}} \mathbb{E}\left[\sum_i \Big(a_i x_i(t)\Phi(t;t_0) + b_i x_i(t)\Big)\right].$$

In fact, we will explicitly construct a saddle point. Our key technical insight is that a saddle point can always be constructed using an optimal multiplier from (Auxiliary Problem). Toward this end, for any fixed t_0, let $\overline{\Phi}(t;t_0)$ denote the ironed version of $\Phi(t;t_0)$ exactly as in Myerson (1981). Now let

$$g(t_0) := \overline{\Phi}(t_0;t_0)$$

be the value of the ironed part including t_0. Note that $g(t_0)$ is continuous in t_0 (see Lemma 3 in the appendix), and satisfies $g(1) > 0$. Moreover, by the property of the elliptical distributions (see Lemma 2 in the appendix), it can be shown that

$$g(0) = \overline{\Phi}(0;0) < -\max_{i\in I^+}\left\{\frac{b_i}{a_i}\right\}.$$

Note that under horizontal learning, at least one $a_i < 0$, and hence by our previous observation, for any $x^* \in X^*$, there exists some $i \in I^+$ such that $x_i^* > 0$. As a consequence, it must be that $\kappa \le \max_{i\in I^+}\{\frac{b_i}{a_i}\}$ and hence

$$0 \ge \lambda = -\kappa \ge -\max_{i\in I^+}\left\{\frac{b_i}{a_i}\right\}.$$

Therefore, by the intermediate value theorem, there exists some t_0^* such that

$$\overline{\Phi}(t_0^*;t_0^*) = \lambda\,.$$

We claim that the ironing interval including t_0^* must also include 0. Indeed, if not, then we have both that $\overline{\Phi}(t_0^*; t_0^*) \leq 0$ and that $\overline{\Phi}(0^+; t_0^*) > 0$ (since that value would become the ironed virtual cost), contradicting the monotonicity of $\overline{\Phi}(\,\cdot\,; t_0^*)$. As a consequence, there must exist an ironing interval $\mathcal{I} \supset [0, t_0^*]$.

Now, we claim that t_0^* is part of a saddle point. Indeed, fix t_0^* as the conjectured worst-off type and consider the pointwise maximization problem after ironing:

$$\max_{x:[0,1]\to[0,1]^K} \mathbb{E}\left[\Big(\sum_i a_i x_i(t)\Big)\overline{\Phi}(t; t_0^*) + \sum_i b_i x_i(t)\right] \tag{1}$$

First, consider the interval $t \in \mathcal{I}$, note that on that interval the pointwise maximization problem, by construction, is equivalent to

$$\max_{x\in[0,1]^K} \sum_i a_i \overline{\Phi}(t; t_0^*) x_i + \sum_i b_i x_i = \max_{x\in[0,1]^K} \sum_i b_i x_i + \lambda \sum_i a_i x_i\,, \tag{2}$$

which is the Lagrangian of the (Auxiliary Problem). By construction of λ, there must exist a solution $x^\dagger \in X^*$ to this pointwise maximization problem. Note that $\sum_i a_i x_i^\dagger = 0$, and $x_i^\dagger = 1$ for all $i \in I^-$.

Now we consider any $t \notin \mathcal{I}$. For any $i \in I^-$, we have for every type t,

$$a_i \overline{\Phi}(t; t_0^*) + b_i \geq a_i \overline{\Phi}(1; t_0^*) + b_i \geq a_i + b_i \geq 0\,,$$

where the last inequality is due to $\theta_i(t) \geq 0$ for all t, and in particular $t = 1$. Moreover, note that either the first inequality or the second inequality must be strict (which one would be a strict inequality depending on whether t and 1 are in the same ironing interval). For any $i \in I^*$, we have that for every type $t \notin \mathcal{I}$,

$$a_i \overline{\Phi}(t; t_0^*) + b_i > a_i \overline{\Phi}(t_0^*; t_0^*) + b_i = a_i \lambda + b_i \geq 0\,.$$

For any $i \in I^+ \backslash I^*$, note that since

$$a_i \overline{\Phi}(t; t_0^*) + b_i$$

is a monotone function that starts at a strictly negative value, there exists some threshold $t_i^* \notin \mathcal{I}$ such that $x_i(t) = 1\{t \geq t_i^*\}$ is pointwise optimal. In fact, because of Lemma 2 in the appendix, $a_i\overline{\Phi}(t;t_0^*) + b_i$ must be strictly single-crossing.

Now, simply define the allocation rule $x^*(\,\cdot\,)$ as: for all $t \in \mathcal{I}$, $x^*(t) = x^\dagger \in X^*$, and for all $t \notin \mathcal{I}$, $x_i^*(t) = 1$ for all $i \in I^* \cup I^-$ and $x_i^*(t) = 1\{t \geq t_i^*\}$ for all $i \in I^+ \backslash I^*$. By the above argument, $x^*(\,\cdot\,)$ must pointwise maximize the ironed objective. Note that $\sum_i a_i x_i^*(t)$ is nondecreasing since we keep adding strictly positive goods as we move from $t = 0$ to $t = 1$. Moreover, it is a ***consistent*** solution with respect to ironing intervals. Together, these imply that the constructed solution solves

$$\max_{x \in \text{MON}} \mathbb{E}\left[\sum_i \Big(a_i x_i(t)\Phi(t;t_0^*) + b_i x_i(t)\Big)\right].$$

Now, we verify that t_0^* must be a worst-off type given the constructed mechanism, which then implies that it solves

$$\min_{t_0 \in [0,1]} \mathbb{E}\left[\sum_i \Big(a_i x_i^*(t)\Phi(t;t_0) + b_i x_i^*(t)\Big)\right].$$

But that is clear by construction: Indeed, $t_0^* \in \mathcal{I}$, and hence $x^*(t_0^*) = x^\dagger$ which leaves 0 payoff to type t_0^* by construction (indeed, the payment implied by the Envelope theorem would be $\sum_i b_i x_i^\dagger$).

Therefore, we have found a saddle point (t_0^*, x^*). As a consequence, x^* is optimal. Clearly, the solution x^* as described satisfies **Claim (1)** to **Claim (3)**. Moreover, as another consequence, every other optimal x' must also form a saddle point with t_0^*, and hence they must solve the pointwise maximization problem (1) in a way such that t_0^* is a worst-off type—in particular, it implies that for *every* optimal x', we must have

$$\sum_i a_i x'(t) = 0$$

for all $t \in \mathcal{I} \supset [0, t_0^*]$. This is the rectangular property of saddle points. As a consequence, for all $t \in \mathcal{I}$, every $x'(t)$ must be maximizing (2) in a way such that $\sum_i a_i x'(t) = 0$, which happens, by construction, if and only if $x'(t) = \hat{x}$ for some $\hat{x} \in X^*$ given that λ is the optimal dual multiplier of (Auxiliary Problem). Therefore, any optimal x' must satisfy **Claim (2)**.

Now, for the types $t \notin \mathcal{I}$, note that the pointwise maximization in fact has a unique solution almost everywhere by inspecting our previous inequalities. Thus, any optimal mechanism must satisfy **Claim (1)** to **Claim (3)**.

4.2 Optimal Learning

Now, continue assuming that the buyer is using a horizontal learning strategy. From **Step 1**, we know that **Claim (1)** to **Claim (3)** must hold for the seller's mechanism in this conjectured equilibrium. We now derive properties of the buyer's best response against such a mechanism, which will eventually lead to a contradiction.

Conceptually, the buyer solves a constrained learning problem. Given the mechanism $\mathcal{M}$, there exists a menu of choices:

$$O = \bigcup_{m \in M} \Big\{ \big(x(m), p(m)\big) \Big\}$$

which is the set of potential outcomes the mechanism can induce. The choices always lead to an indirect utility function for the buyer as a function of the posterior means $\boldsymbol{\theta}$, since the buyer's payoff is linear in these choices. Therefore, the buyer solves the following problem:

$$\max_{\boldsymbol{\alpha}} \mathbb{E}\Big[U\Big(\theta_1(s;\boldsymbol{\alpha}), \ldots, \theta_K(s;\boldsymbol{\alpha})\Big)\Big]$$

for a convex function U induced by the seller's menu. By the property of elliptical distributions, choosing the learning weights $\boldsymbol{\alpha}$ here turns out to be equivalent to choosing the ***posterior mean line*** $\Theta(s)$ in $\mathbb{R}_+^K$ that passes through the prior $\boldsymbol{\mu}$. The distribution $\boldsymbol{\theta}$ supported on that line will be a mean-preserving contraction of the prior distribution **v** and pinned down by the elliptical updating rule explained in Section 2.

There are two key properties that we show must hold for every optimal $\boldsymbol{\alpha}^*$:[21]

(Claim 4) If $U(\boldsymbol{\theta})$ does not depend on θ_i, then $\alpha_i^* = 0$.

[21] These properties must hold for any convex U that is *not* affine, so we must rule out the case where information is not strictly valuable in equilibrium. We prove this separately in Lemma 8.

(Claim 5) If $U(\boldsymbol{\theta})$ depends on θ_i and θ_j symmetrically via $\theta_i + \theta_j$, then $\alpha_i^* = \alpha_j^*$.

These two properties are relatively easy to see in the uncorrelated case where $\rho = 0$, since then the learning weight put on good i has no impact on how much is learned about good $j \neq i$. However, with correlated values, the learning weights may be chosen to balance learning across different goods. We show that even though the optimal strategy $\boldsymbol{\alpha}^*$ will in fact take into account the correlation structure, **(Claim 4)** and **(Claim 5)** must hold regardless of the correlation structure (see Lemma 6 and Lemma 7 in the appendix).

The proofs exploit the following ***orthogonal decomposition*** property of elliptical distributions: for any $\boldsymbol{\alpha}$, and any $N < K$, we can write

$$\boldsymbol{\alpha} \cdot \mathbf{v} = \sum_{i \leq N} \alpha_i v_i + \sum_{j > N} \alpha_j v_j = \sum_{i \leq N} \tilde{\alpha}_i v_i + \varepsilon,$$

for some $\tilde{\boldsymbol{\alpha}}$, where $\mathrm{Cov}(v_i, \varepsilon) = 0$ for all $i \leq N$, and ε is a non-degenerate elliptical random variable. Indeed, by the linear-projection property, we can write for each $j > N$

$$v_j = \sum_{i \leq N} \beta_i^j v_i + \varepsilon_j,$$

for some $\boldsymbol{\beta}^j$ with $\mathrm{Cov}(v_i, \varepsilon_j) = 0$ for all $i \leq N$, where ε_j is constructed as

$$\varepsilon_j := v_j - \mathbb{E}\left[v_j \mid v_1, \ldots, v_N\right].$$

Now, to see **(Claim 4)**, suppose for contradiction that $\boldsymbol{\alpha}$ is optimal and yet $\alpha_i > 0$. Construct an alternative signal $\boldsymbol{\alpha}^*$ as follows. Apply the orthogonal decomposition to write $\boldsymbol{\alpha} \cdot \mathbf{v}$ as

$$\boldsymbol{\alpha} \cdot \mathbf{v} = \sum_{j \neq i} \tilde{\alpha}_j v_j + \varepsilon$$

and set $\alpha_j^* := \tilde{\alpha}_j$ for all $j \neq i$ and $\alpha_i^* := 0$. Note that in this new coordinate, the original signal is $(\tilde{\boldsymbol{\alpha}}, 1)$ while the new signal is $(\tilde{\boldsymbol{\alpha}}, 0)$ that replaces the weight 1 on the uncorrelated term ε with 0. One can then verify that such a change must lead to a new posterior mean line $\Theta^*(s) \in \mathbb{R}_+^K$ whose projection into $\mathbb{R}_+^{K-1}$ is exactly the same as the projection of the original posterior mean line generated by $\boldsymbol{\alpha}$. Moreover, it generates a strictly higher posterior variance for each good $j \neq i$—the distribution of $\boldsymbol{\theta}^*_{-i}$ is strictly higher than the distribution of $\boldsymbol{\theta}_{-i}$ in the convex order. That is, even though the signals in our model are never Blackwell ranked, this construction does lead to an improvement in the Blackwell order for the information about the goods $j \neq i$. Since U does not depend on good i, this must be a strict improvement, and hence a contradiction.

Now, for **(Claim 5)**, note that it follows as a consequence of **(Claim 4)** by writing down an augmented coordinate: Consider the elliptical random vector

$$\Big((v_k)_{k\neq i, k\neq j}, v_i, v_j, w\Big), \quad \text{where } w \ = \ v_i + v_j \, .$$

That is, we augment the original space by another random variable $w = v_i + v_j$. Since U depends on θ_i, θ_j only via $\theta_i + \theta_j$, we can write down an equivalent $\tilde{U}$ in this augmented space that depends on the posterior mean of w but not θ_i and θ_j. Applying **(Claim 4)** shows that any optimal strategy must put 0 weight on both v_i and v_j in this augmented space which implies equal weights on $v_i + v_j$ in the original space.

4.3 Completion of the Proof

Now, we are ready to complete the proof of Theorem 1.

Vertical Learning. Suppose for contradiction that there exists a horizontal-learning equilibrium $(\boldsymbol{\alpha}, \mathcal{M})$. Let $O := \cup_{m\in M}\{(x(m), p(m))\}$ be the set of all possible outcomes that can be achieved under mechanism $\mathcal{M}$. Now let

$$O^* := \bigcup_{t\in[0,1]} \Big\{\big(x(t), p(t)\big)\Big\}$$

be the set of outcomes that are chosen by some type t under the seller's mechanism in equilibrium. Note that the buyer's (ex ante) payoff is exactly the same, by construction, when facing O^* or when facing $O \supseteq O^*$. This implies that there cannot be a profitable deviation by the buyer against menu O^* since that would imply a profitable deviation against the original menu O.

As noted before, by construction, there must exist both *(i)* negative goods I^- and *(ii)* positive balancing goods I^*. By **(Claim 1)** and **(Claim 2)** together, every option in O^* that yields a strictly positive payoff for some equilibrium type t must include all goods in $I^* \cup I^-$ with full probability. However, by **(Claim 5)**, this implies that the buyer's learning strategy $\boldsymbol{\alpha}$ must put equal weights on all $i \in I^* \cup I^-$. Indeed, if not, then even ignoring some of the options in O^*, the buyer has a strictly profitable deviation by using a learning strategy that puts equal weights on all goods in $I^* \cup I^-$ by **(Claim 5)**.

Moreover, by **(Claim 1)**, every option in O^* must include all goods in I^- with full probability. However, this implies that the buyer's decision problem, when facing menu O^*, does not depend on the values of the negative goods $\left(v_i\right)_{i \in I^-}$. Therefore, by **(Claim 4)**, the buyer must put 0 weights on all the goods in I^-.

Together, these two observations imply that the buyer must put weight 0 on every good $i \in I^* \cup I^-$. But since the correlation ρ is the same across all pairs of goods, this implies that $(a_i)_{i \in I^* \cup I^-}$ must be either *(i)* all weakly positive or *(ii)* all weakly negative. Indeed, for any $i \in I^* \cup I^-$,

$$\text{sign}(a_i) = \text{sign}(\text{Cov}(\boldsymbol{\alpha} \cdot \mathbf{v}, v_i)) = \text{sign}\left(\rho \sum_{j \in I^+ \setminus I^*} \alpha_j \sigma_j\right),$$

which does not depend on i. However, by construction, $a_i > 0$ for all $i \in I^*$ and $a_i \leq 0$ for all $i \in I^-$. Moreover, since $(\boldsymbol{\alpha}, \mathcal{M})$ is a horizontal learning equilibrium, there exists some $i \in I^-$ such that $a_i < 0$. A contradiction.

Nested Bundling. As we have shown, every equilibrium must have vertical learning, and hence a comonotonic type distribution. Thus, the posterior mean distribution can be written as: for each i,

$$\theta_i = a_i t + b_i$$

where $a_i \geq 0$, $b_i \geq 0$, and $t \in [0, 1]$. We claim that, against such a posterior mean distribution, there exists a unique optimal direct-revelation mechanism (up to measure zero) that is deterministic and can be represented by nested menus.

This is a direct consequence of the optimal mechanism we give in Section 4.1 that holds against any distribution supported on a line segment in $\mathbb{R}_+^K$. Indeed, note that now there exists no good with $a_i < 0$, and hence every optimal solution x to (Auxiliary Problem) must have $x_i = 0$ for all the strictly positive goods i. Thus, there are no positive balancing goods, i.e., $I^* = \varnothing$. By **(Claim 1)**, all the goods with $a_i = 0$ must be allocated to all types with full probability. By **(Claim 3)**, all the goods with $a_i > 0$ must then be allocated in a monotone, deterministic fashion according to the threshold rules $\mathbb{1}\{t \geq t_i^*\}$.

Now, consider the induced outcomes in the equilibrium chosen by various types t:

$$O^* := \bigcup_{t\in[0,1]} \left\{ \big(x(t), p(t)\big) \right\}.$$

By the above argument, it must be that the allocations offered in O^* are deterministic and totally ordered by set inclusion. Moreover, as argued before, there cannot be a profitable deviation by the buyer against menu O^* since that would imply a profitable deviation against the original menu $O \supseteq O^*$. It follows immediately that having the seller offering the menu O^* and the buyer using the same strategy $\boldsymbol{\alpha}$ must constitute an equilibrium, completing the proof.

5 Discussions

5.1 Commitment

Seller Moves First. Our main model considers a simultaneous-move game between the seller and the buyer. This best captures markets where the seller frequently readjusts prices and may not be able to commit not to do so. We now discuss what happens if instead the seller has a first-mover advantage. That is, the seller first commits to a menu, which is observed by the buyer, who then learns and makes a purchasing decision. By choosing an appropriate menu, the seller can shape the buyer's learning incentives, and the optimal menu needs to account for this effect. A full analysis of such a model is beyond the scope of this paper, but we go through an example to illustrate the role of commitment.

Suppose that there are $K = 2$ goods. The buyer's values are drawn from a

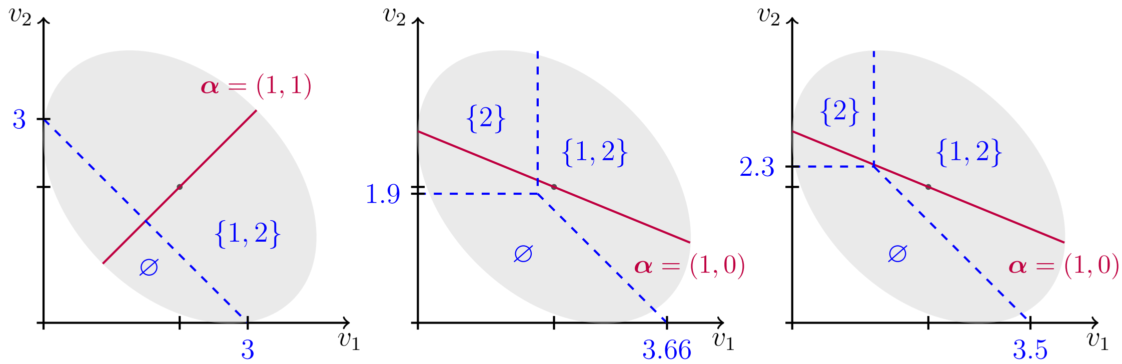

Figure 6: The shaded gray area is the set of possible values V. The left figure illustrates an equilibrium of the simultaneous-move game. The buyer chooses signal $\boldsymbol{\alpha} = (1, 1)$ and the solid red line is the support of the associated type distribution. The dashed blue line represents optimal allocations given the seller's menu: types below the line buy nothing while types above the line buy the bundle. The middle figure illustrates an optimal (deterministic) menu when the seller has commitment and the buyer's associated learning strategy. The right figure illustrates an optimal menu against learning strategy $\boldsymbol{\alpha} = (1, 0)$.

Gaussian distribution with mean $\mu_1 = \mu_2 = 2$, standard deviation $\sigma_1 = \sigma_2 = 4$, and correlation $\rho = -0.4$, whose support is truncated to lie in the positive quadrant. The distribution of values is exchangeable, so the simultaneous-move game admits an equilibrium in which the seller only offers the grand bundle $\{1, 2\}$ and the buyer only learns about his value for the grand bundle $\boldsymbol{\alpha} = (1, 1)$. In this equilibrium, the grand bundle is priced at 3 and the expected revenue equals 2.35 (Figure 6, left panel).

Now, suppose that the seller has a first-mover advantage. We solve for an optimal deterministic menu numerically. An optimal menu is to offer good 2 at price 1.9 and the grand bundle at price 3.66. Against this menu, the buyer finds it optimal to only learn about his value for good 1. Because values are negatively correlated, this induces horizontal differentiation across types, as depicted in Figure 6 (middle panel). The expected revenue equals 2.92.

Two features of the example are worth noting. First, commitment is valuable—the seller achieves a strictly greater expected revenue when it has a first-mover advantage. Second, horizontal learning can be sustained in equilibrium when the seller has commitment. This is not driven by the restriction to deterministic menus: Against the above horizontal learning strategy, the optimal deterministic mechanism sells good 2 at price 2.3 and the bundle at price 3.5 (Figure 6, right panel). Thus, absent commitment, the seller would have an incentive to increase the price of the base good and slightly decrease the price of the bundle. However, if she were to do that, then the buyer would want to *deviate* to learning about the bundle, and this would not form an equilibrium. By committing to a lower price for the base good (good 2), the seller can incentivize the buyer to only learn about the upgrade good (good 1), leading to a higher expected revenue.

Buyer Moves First. We now discuss what happens when the buyer has a first-mover advantage. This can be viewed as a benchmark to understand which information structure benefits the buyer by shaping the seller's mechanism. This is also the timing considered by Deb and Roesler (2024).

A simplistic intuition behind Theorem 1 is that mechanisms that are optimal against horizontal learning leave little information rent to the buyer. However, this intuition is incomplete. Indeed, as we show next, in the model where the buyer can move first, there can exist horizontal learning strategies that secure a *higher* information rent for the buyer than any vertical learning strategies.

Consider again $K = 2$ goods. Suppose that the buyer's values are drawn from a Gaussian distribution with mean $\boldsymbol{\mu} = (0.2, 0.1)$, standard deviations $\sigma_1 = 3$, $\sigma_2 = 1$, and correlation $\rho = -0.98$, whose support is truncated to lie in the positive quadrant. For any learning strategy $\boldsymbol{\alpha}$, as part of our main analysis, we have characterized the seller's best response and the induced payoff to the buyer (see Lemma 4). We find the learning strategy that maximizes the buyer's expected payoff numerically, which is $\boldsymbol{\alpha} = (1, 2.8)$. As depicted in Figure 7, this is a horizontal learning strategy. The seller's best response against $\boldsymbol{\alpha} = (1, 2.8)$ is to offer a menu composed of the full bundle $\{1, 2\}$ at price 0.25 and a rationing option $x^* = (0.14, 1)$ at price 0.13 (a lottery of getting good 1 with probability 0.14 and good 2 for sure). The rationing option exploits the negative correlation in the buyer's posterior means distribution so as to leave no rent to any type who purchases it. However, there is sufficient rent for the types who purchase

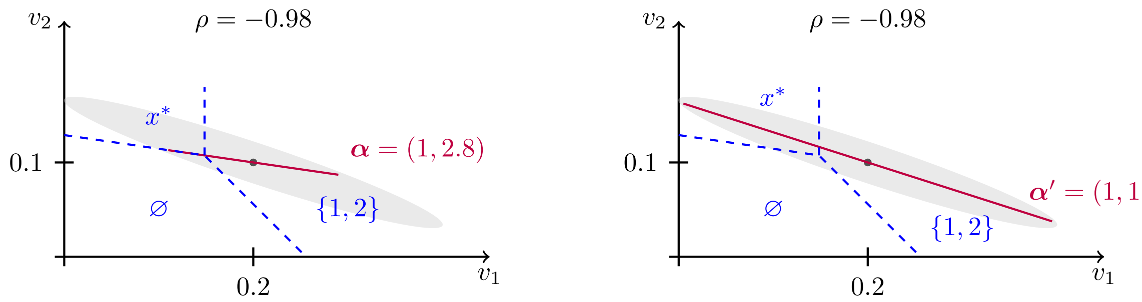

Figure 7: The left figure illustrates the optimal learning strategy $\boldsymbol{\alpha}$ when the buyer has commitment, and the seller's associated mechanism. Notably, the optimal learning strategy is horizontal. The right figure illustrates that $\boldsymbol{\alpha}$ cannot be sustained in equilibrium—against the seller's mechanism, the buyer strictly prefers $\boldsymbol{\alpha}'$ to $\boldsymbol{\alpha}$.

the full bundle to make the strategy $\boldsymbol{\alpha}$ optimal for the buyer with commitment.

We know from Theorem 1 that $\boldsymbol{\alpha}$ cannot be part of an equilibrium. Indeed, given the menu $\{(x^*, 0.13), (\{1,2\}, 0.26)\}$, the buyer strictly benefits from deviating to only learning about the full bundle (Figure 7, right panel).[22] The key intuition behind the existence of such a deviation, as discussed in Section 3.1, is that the seller's optimal mechanism, against horizontal learning, is designed to limit the *variation* in payoffs along the posterior mean line direction. This implies that the buyer's original learning strategy cannot resolve too much uncertainty relevant for his purchase decision, and hence there exists another learning direction (which may or may not be vertical learning) that resolves more relevant uncertainty given the seller's menu—hence, horizontal learning is unstable and cannot be sustained in equilibrium.

This example also provides an interesting contrast with Deb and Roesler (2024) who show that the *buyer-optimal signal* induces vertical types, efficient trade, and pure bundling when the buyer moves first. The difference can be understood as follows. Deb and Roesler (2024) allow the buyer to commit to arbitrary signals and assume that $\mathbf{v}$ has an exchangeable distribution. In particular, the buyer can commit to learning a noisy signal about the grand bundle

[22] Note that because the distribution $\mathbf{v}$ is not exchangeable, learning the bundle value here actually leads to negatively correlated posterior means—i.e., a horizontal learning strategy.

such that *(i)* the seller best responds by offering only the grand bundle and *(ii)* any other signal leads to a weakly higher revenue for the seller. The noisy signal is constructed to induce a truncated Pareto posterior mean distribution such that the seller finds it optimal not to exclude any buyer types—hence, the signal must be buyer-optimal. In the above example, *(i)* the buyer can only commit to a "direction of learning" which excludes the noisy Pareto signals and leads to inefficient trade, and *(ii)* the seller would not best respond with pure bundling even if the buyer commits to learning only about the grand bundle because of the asymmetry in the distribution of $\mathbf{v}$.[23]

5.2 Non-Additive Values

Our analysis so far assumes additive values. Certain extreme forms of non-additivity can overturn our results. For instance, if the goods are perfect substitutes, then the buyer has unit demand over the goods, which makes bundling less effective and increases the instrumental value of horizontal comparisons—hence, it would generally be difficult to sustain a vertical learning equilibrium.

However, it turns out that, when the goods are not perfect substitutes, our main result is robust to some form of complementarity and substitutability. To illustrate, suppose that we have two goods and the value for the bundle $\{1, 2\}$ is given by

$$\gamma\big(v_1 + v_2\big)\,,$$

where $\gamma > 1$ when the two goods are complements, $\gamma < 1$ when the two goods are substitutes, and $\gamma = 1$ when the two goods are additive. We assume that a larger bundle gives a weakly higher value: for all $\boldsymbol{v} \in V$,

$$\gamma\big(v_1 + v_2\big) \geq \max\{v_1, v_2\}\,.$$

As in the main model, we allow the buyer to learn any $\alpha_1 v_1 + \alpha_2 v_2$, which in this case is equivalent to learning any linear combination of various bundle values. We say that an equilibrium has ***vertical learning*** if the posterior means

[23]Indeed, the distribution of $\mathbf{v}$ in this example does not satisfy the condition in Proposition 10 which one can show is also necessary for the seller best responding with pure bundling against the buyer learning about the grand bundle.

for all bundle values are comonotonic, and ***horizontal learning*** otherwise.

Perhaps surprisingly, our main result continues to hold in this setting regardless of whether the goods are complements or substitutes:

PROPOSITION 7. *With two goods and uncorrelated values, for any γ, every equilibrium has vertical learning and is outcome-equivalent to a nested bundling equilibrium.*

The proof follows the same logic as the proof of Theorem 1. For the vertical learning part, it shows that against any horizontal learning strategy, the seller's optimal mechanism turns out to have the same structure as identified in Section 4 regardless of complementarity and substitutability, which then implies a profitable deviation by the buyer. For the nested bundling part, given that equilibrium types must be comonotonic, the proof leverages the nesting condition in Yang (2025), which does not require additive values.

5.3 Production Costs

Our main model assumes that the buyer's value for each good k is always weakly higher than the seller's cost. It is then always efficient to allocate all the goods and the only reason the seller might refrain from doing so is to extract more surplus. However, when buyer types are horizontally differentiated, the seller only needs to distort the allocation of *some* goods to maximize revenue. The remaining goods are allocated to all buyer types, who then have no incentive to learn how much they value them. This is a key step in the proof of Theorem 1, but it does rely on the seller's production costs being lower than any buyer's realized value. We now investigate the robustness of Theorem 1 when this assumption is relaxed.

Suppose that the seller has a constant marginal cost c_k of producing good k. Suppose that $c_k < \mu_k$ for all k. Thus, the buyer may have a value for good k below its production cost under some signal realization, but the expected value for good k is still above its cost.

PROPOSITION 8. *Suppose that there are two goods with uncorrelated and log-concave value distribution and that $c_k < \mu_k$ for both goods k. Then, every equilibrium has vertical learning, and is outcome-equivalent to a nested bundling equilibrium.*

The proof follows the same logic as the proof of Theorem 1. In particular, the characterization of optimal mechanisms against distributions supported on any line segment in Section 4 can be generalized to incorporate constant marginal costs. If the marginal costs are not too high, then the optimal mechanism against any horizontal learning strategy continues to involve enough bundling responses by the seller (in the form of mixed bundling) such that the buyer finds it optimal to deviate to a vertical learning strategy.

5.4 A Weakening of Nash Equilibrium

This section introduces a solution concept weaker than Nash equilibrium under which our main results in Section 3 continue to hold, and equilibrium existence is always guaranteed. The solution concept weakens the requirement that the buyer chooses a fully optimal learning strategy, and only requires an appropriate sense of Blackwell undominance.

Given some mechanism $\mathcal{M}$, we say a learning strategy $\boldsymbol{\alpha}$ is ***$\mathcal{M}$-Blackwell dominated*** by another strategy $\boldsymbol{\alpha}'$ if there exists $M^* \subseteq M$ such that *(i)* for any signal realization s in the support of $\boldsymbol{\alpha} \cdot \mathbf{v}$,

$$M^* \cap \arg\max_{m \in M} \Big\{ \boldsymbol{\theta}(s; \boldsymbol{\alpha}) \cdot \boldsymbol{x}(m) - p(m) \Big\} \neq \varnothing$$

and *(ii)* $\boldsymbol{\alpha}'$ is strictly Blackwell more informative than $\boldsymbol{\alpha}$ about the relevant normalized payoffs, in the sense that for some $m_0 \in M^*$, we have

$$\Big(\boldsymbol{\theta}(s; \boldsymbol{\alpha}) \cdot \big(\boldsymbol{x}(m) - \boldsymbol{x}(m_0) \big) \Big)_{m \in M^* \setminus \{m_0\}} \preceq_{\text{cx}} \Big(\boldsymbol{\theta}(s; \boldsymbol{\alpha}') \cdot \big(\boldsymbol{x}(m) - \boldsymbol{x}(m_0) \big) \Big)_{m \in M^* \setminus \{m_0\}},$$

where $\preceq_{\text{cx}}$ is the usual convex order and these two random vectors are not the same. In words, the set M^* includes an optimal option for each type in the support of the posterior mean distribution under signal $\boldsymbol{\alpha}$, and signal $\boldsymbol{\alpha}'$ is strictly more informative about the options in M^* (since the buyer's decision problem is mean-measurable, by Blackwell's theorem, it suffices to consider the convex order over the posterior mean distributions).[24]

[24]Moreover, what matters for decision-making is *relative* payoffs across options in the menu, hence the normalization. For example, if the buyer either buys good 1 or buys good 2 (but

We say that a learning strategy $\boldsymbol{\alpha}$ is $\mathcal{M}$-***Blackwell undominated*** given some mechanism $\mathcal{M}$ if it is not $\mathcal{M}$-Blackwell dominated by any other strategy. A strategy profile $(\boldsymbol{\alpha}, \mathcal{M})$ forms a ***weak equilibrium*** if the buyer's learning strategy $\boldsymbol{\alpha}$ is $\mathcal{M}$-Blackwell undominated, and mechanism $\mathcal{M}$ is profit-maximizing against $\boldsymbol{\alpha}$. Unlike Nash equilibrium, this solution concept does not require that the buyer fully best-responds to the seller's mechanism, but only that he chooses a signal that is not Blackwell dominated anticipating the options offered by the seller. Note that the weak equilibria form a superset of the Nash equilibria.[25]

PROPOSITION 9. *There exists a weak equilibrium. Moreover:*

(i) Every weak equilibrium has vertical learning, and is outcome-equivalent to a nested bundling weak equilibrium.

(ii) In any nested bundling weak equilibrium, the buyer's log-scale posterior variance $\mathrm{Var}(\log(\theta_i))$ *of different items are ordered according to their tiers.*

Proposition 9 shows that our main results in Section 3 generalize to this concept of weak Nash equilibrium, and such a weak equilibrium is guaranteed to exist. In particular, Theorem 1 does not rely on the buyer being able to fully optimize in response to the seller's mechanism. Fairly weak rationality requirements are sufficient to rule out horizontal learning equilibria. At the same time, even under this weak rationality requirement by the buyer, the prediction about the buyer's learning strategy as in Proposition 1 continues to hold—indeed, as we explained in Section 3, the ordering of the posterior variance is mostly due to the optimization by the *seller*, and hence continues to hold even if the buyer does not fully optimize.

Our notion of Blackwell dominance does not require the buyer to leverage any structure in the purchasing problem he faces. In particular, he does not need to recognize any separability in his purchasing problem. If we tighten

always buys something), then what matters for his decision is not the whole vector (v_1, v_2) but only $v_1 - v_2$.

[25]Indeed, for any Nash equilibrium, if the buyer's strategy were to be Blackwell dominated by another strategy $\boldsymbol{\alpha}'$, then there would exist some M^* and some m_0 such that the normalized posterior mean distributions supported on a line segment in $\mathbb{R}^{|M^*|-1}$ must coincide, which then implies that the two signals must be equivalent (see the proof of Lemma 6), a contradiction.

the rationality requirement by allowing the buyer to recognize additive structures, then we could also rule out any separate sales equilibrium under negative correlation, thus generalizing Proposition 3. Indeed, the proof of Proposition 3 shows that horizontal learning leads to distributions of posterior means whose *marginals* are strictly higher in the convex order. This is sufficient for Blackwell dominance against vertical learning in each good separately, which is all that matters under separability. But any weak equilibrium must have vertical learning, thus precluding the existence of any separate sales equilibrium.

5.5 Multidimensional Signals

We have assumed throughout that the buyer can only acquire a one-dimensional linear signal about his vector of values $\mathbf{v}$. This is a key assumption for two reasons. First, it puts constraints on how much the buyer can learn and ensures that he faces non-trivial tradeoffs when deciding what to learn about. Second, it ensures the buyer's endogenous type distribution is supported on a line segment, facilitating the characterization of the seller's best response problem, especially against horizontal learning strategies, where the buyer's types are *not* ordered.

In this section, we discuss what happens when the buyer can learn an N-dimensional signal, where $N \leq K$.

Free Signals. Note that, if $N = K$, there always exists an equilibrium in which the buyer fully learns his values. There may exist other equilibria as well. Proposition 10 gives a sufficient condition for the existence of an equilibrium in which the seller only offers the grand bundle and the buyer only learns his value for the grand bundle.

PROPOSITION 10. *If for all $i \neq j$,*

$$\mu_i = \frac{\sigma_i^2 + \rho\sigma_i \sum_{k \neq i} \sigma_k}{\sigma_j^2 + \rho\sigma_j \sum_{k \neq j} \sigma_k} \mu_j \,,$$

then vertical learning and pure bundling form an equilibrium.

The above condition guarantees that, when the buyer chooses to only learn about the grand bundle, i.e., $\boldsymbol{\alpha} = \mathbf{1}$, the seller's optimal menu is indeed to sell only the grand bundle.[26] Note that in such an equilibrium, the buyer only acquires one signal even though he could acquire up to N. Thus, pure bundling equilibria of our baseline model persist when the buyer can acquire additional signals, and are robust in that sense.

Costly Signals. In any other equilibrium of our baseline model, the buyer has a strict incentive to acquire additional signals. To assess the strength of this incentive, suppose that each additional signal costs $c > 0$ and the buyer chooses ex ante how many signals to acquire. We want to understand when the buyer does not want to acquire additional signals, such that the equilibria we analyze in the baseline model persist.

PROPOSITION 11. *Let $K = 2$ and $\rho = 0$. Fix any nested bundling equilibrium from our baseline model, where the buyer can acquire only one signal. Suppose that, without loss of generality, the equilibrium nested menu sells good 1 as the base good at price p_1. This continues to be an equilibrium when the buyer can acquire additional signals at any signal cost*

$$c \geq \mathbb{E}\big[\max\{v_1 - p_1, 0\}\big] - (\mu_1 - p_1)\,.$$

Intuitively, the value of an additional signal is bounded above by the value of fully learning about the base good. If the base good is inexpensive (low p_1), then the buyer has little incentive to acquire an additional signal. Even a relatively low cost c is enough for the one-signal equilibria of our main model to persist.

6 Conclusion

We study an equilibrium model of multiproduct pricing with consumer learning. The buyer chooses to learn any one-dimensional linear signal of their values for the goods, anticipating the seller's mechanism. The seller designs an optimal mechanism, anticipating the buyer's learning choice. In a generalized Gaussian environment, we show that every equilibrium has vertical learning where the

[26] Indeed, the condition in Proposition 10 generalizes exchangeability, and it holds if and only if the direction of the posterior mean line after learning about $\sum_i v_i$ points toward the origin (which is called "stochastic comonotonicity" by Che and Zhong 2024), ensuring that pure bundling is optimal by Haghpanah and Hartline (2021) (or by the analysis in Section 4).

buyer's posterior means are comonotonic, and every equilibrium is outcome equivalent to nested bundling where the seller offers a menu of nested bundles to screen the buyer. In equilibrium, the buyer learns more about higher-tier goods, resulting in higher posterior variances on the log scale.

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A Proofs

A.1 Proof of Theorem 1

The proof is organized as follows. First, we prove that every equilibrium must have vertical learning. Second, we prove its outcome equivalence to a nested bundling equilibrium.

A.1.1 Vertical learning

Recall that a buyer's type is linear in the signal realization, i.e.,

$$\theta_i(s;\boldsymbol{\alpha}) = \frac{\mathrm{Cov}(v_i, \boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})} s + \mu_i - \frac{\mathrm{Cov}(v_i, \boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})} \boldsymbol{\alpha}\cdot\boldsymbol{\mu}\,.$$

Thus, the endogenous type distribution is supported on a segment in $\mathbb{R}^K_+$. Let $[\underline{s}, \overline{s}]$ denote the set of possible signal realizations. That is, $\underline{s} := \min_{\mathbf{v}\in V} \boldsymbol{\alpha}\cdot\mathbf{v}$ and $\overline{s} := \max_{\mathbf{v}\in V} \boldsymbol{\alpha}\cdot\mathbf{v}$. We can re-index types by some parameter t so that $t \in [0, 1]$. Indeed, set $t = (s - \underline{s})/(\overline{s} - \underline{s})$. Rewriting the above expression yields

$$\theta_i(t;\boldsymbol{\alpha}) = (\overline{s} - \underline{s}) \frac{\mathrm{Cov}(v_i, \boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})} t + \mu_i - \frac{\mathrm{Cov}(v_i, \boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})} (\boldsymbol{\alpha}\cdot\boldsymbol{\mu} - \underline{s}) =: a_i t + b_i\,.$$

Let F be the cumulative distribution over t induced by the learning strategy $\boldsymbol{\alpha}$. Given our setup, we know that the distribution of signals s is elliptical, and thus so is F.

The proof proceeds as follows:

Step 1. We characterize optimal mechanisms against any distribution supported on a line segment in $\mathbb{R}^K_+$.

Step 2. We characterize properties of the buyer's optimal learning strategies against any candidate optimal mechanism using **Step 1**.

Step 3. We show these together lead to a contradiction if the buyer uses a horizontal learning strategy.

Optimal mechanism. We solve for every optimal direct revelation mechanism $\mathcal{M}^{DR}$ under the type distribution induced by any $\boldsymbol{\alpha}$. For the seller's strategy $\mathcal{M}$ to be a best response, the mechanism she uses must be outcome-equivalent to some $\mathcal{M}^{DR}$, when restricting to the type space induced by $\boldsymbol{\alpha}$.

We make an important sign convention:

$$\sum_i a_i \geq 0\,. \tag{Sign Convention}$$

Note that this is without loss of generality because if it fails, then we can simply redefine types $\tilde{t} = 1 - t \in [0,1]$, and write

$$a_i t + b_i = \underbrace{-a_i}_{\tilde{a}_i}\tilde{t} + \underbrace{a_i + b_i}_{\tilde{b}_i}\,,$$

which flips the sign for each good i. Under this sign convention, we define

$$I^+ := \{i : a_i > 0\} \quad \text{and} \quad I^- := \{i : a_i \leq 0\}\,,$$

and call the goods in I^+ the ***strictly positive goods***, and the goods in I^- the ***negative goods***. Note that under horizontal learning, there must exist two goods i, j such that

$$\text{sign}(\text{Cov}(v_i, \boldsymbol{\alpha}\cdot\mathbf{v})) \neq \text{sign}(\text{Cov}(v_j, \boldsymbol{\alpha}\cdot\mathbf{v}))\,,$$

and hence sign$(a_i) \neq$ sign(a_j)—there must exist both strictly positive and strictly negative goods. Under vertical learning, by the sign convention, all the goods must be either strictly positive goods, or zero-sign goods (i.e., $a_i = 0$).

While either a_i or b_i can be 0, they cannot be both zero. Indeed, Bayesian plausibility requires that $\mathbb{E}[a_i t + b_i] = \mu_i > 0$ for any good i. Since $\mathbb{E}(t) = 0.5$, this implies $(a_i, b_i) \neq (0,0)$. Moreover, since $\mathbf{v} \in \mathbb{R}^K_+$, we have that, under any learning strategy, $a_i t + b_i \geq 0$ for all $t \in [0,1]$ and hence

$$b_i \geq 0, \quad a_i + b_i \geq 0\,,$$

which combined with the observation $(a_i, b_i) \neq (0,0)$ implies that

$$a_i \leq 0 \implies b_i > 0\,.$$

Before our characterization, we first consider an auxiliary program:

$$\max_{\mathbf{x}\in[0,1]^K} \sum_i b_i x_i \quad \text{subject to} \quad \sum_i a_i x_i = 0\,. \tag{3}$$

Let X^* be the set of all optimal solutions to the above program. We first characterize every solution to this auxiliary problem:

LEMMA 1. *There exists $\kappa \geq 0$ such that for any optimal solution x^* to (3), we have:*

(i) For any negative good i, $x_i^* = 1$.

(ii) For any strictly positive good i, $x_i^* = 1$ if $b_i/a_i > \kappa$ and $x_i^* = 0$ if $b_i/a_i < \kappa$.

Proof. For part *(i)*, we prove by contradiction. Suppose by contradiction that x^* is an optimal solution such that $x_i^* < 1$ for some negative good i. By the previous observation, this implies that $b_i > 0$. Now, if $a_i = 0$, then increasing x_i^* would be strictly improving the object while satisfying the constraint. Thus, it must be the case that $a_i < 0$. However, then it must be that there exists some strictly positive good j such that $x_j^* < 1$, because otherwise we cannot satisfy the equality constraint:

$$\sum_k a_k x_k^* > a_i + \sum_{k\neq i} a_k x_k^* \geq a_i + \sum_{k\in I^+} a_k + \sum_{k\in I^-, k\neq i} a_k \geq 0\,,$$

where the last inequality is by the sign convention. Then, strictly increasing x_i^* and x_j^* can strictly increase the objective while keeping the equality constraint (given that $b_i > 0$ by the previous observation again). For part *(ii)*, note that given all negative goods must have $x_i^* = 1$, the problem of assigning the positive goods reduces to a fractional knapsack problem. Thus, by an exchange argument, every optimal solution must satisfy the greedy property of assigning in the order of b_i/a_i with possible randomization for the goods with the same b_i/a_i. The claim follows immediately. □

Now, define $I^* \subseteq I^+$ as follows:

$$I^* := \bigcup_{x^* \in X^*} \left\{ i \in I^+ : x_i^* > 0 \right\}.$$

We call these the ***balancing goods***. Note that under any horizontal learning, some balancing good always exists: Indeed, under horizontal learning, there must exist a strictly positive good and a strictly negative good. Thus, for every $x^* \in X^*$, by Lemma 1, if $x_i^* = 0$ for all $i \in I^+$, then

$$\sum_i a_i x_i^* = \sum_{i \in I^-} a_i x_i^* = \sum_{i \in I^-} a_i < 0\,,$$

which would violate the equality constraint of (3). Therefore, for every $x^* \in X^*$, $\left\{ i \in I^+ : x_i^* > 0 \right\} \neq \varnothing$, and hence $I^* \neq \varnothing$.

We also note the following observation about the distribution F:

Lemma 2. *Let $\Phi(t) := t - \frac{1-F(t)}{f(t)}$. The function Φ is strictly increasing for $t \leq 0.5$ and strictly positive for $t > 0.5$. Moreover, $\lim_{t \to 0} \Phi(t) = -\infty$. Thus, for any $a_i > 0, b_i \geq 0$, we have $a_i \Phi(t) + b_i$ is strictly single-crossing with a crossing point $t_i^* > 0$.*

Proof. We first prove that Φ is strictly increasing for $t \leq 0.5$. Recall that F and f are the CDF and PDF of an elliptical distribution supported on $[0, 1]$ and thus centered at 0.5. Elliptical distributions are unimodal, such that f is nondecreasing on $[0, 0.5]$ and nonincreasing on $[0.5, 1]$. Thus, for any $t \leq 0.5$,

$$\Phi'(t) = 1 + \frac{f(t)^2 + f'(t)[1 - F(t)]}{f(t)^2} = 2 + \frac{f'(t)[1 - F(t)]}{f(t)^2} > 0\,.$$

We now prove that Φ is strictly positive for $t \geq 0.5$. Using the fact that f is nonincreasing on $[0.5, 1]$, we get that for all $t > 0.5$:

$$1 - F(t) = \int_t^1 f(s) ds \leq \int_t^1 f(t) ds = f(t)(1 - t).$$

Thus,

$$\Phi(t) = t - \frac{1 - F(t)}{f(t)} \geq t - (1 - t) = 2t - 1 > 0\,.$$

Finally, we show that $f(0) = 0$, which implies $\lim_{t \to 0} \Phi(t) = -\infty$ since f is continuous. Recall that the buyer has type $t = 0$ if and only if he receives the lowest possible signal realization $\underline{s} = \min_{\mathbf{v} \in V} \boldsymbol{\alpha} \cdot \mathbf{v}$. The overall set of values V is an ellipsoid, so the subset of values at which the buyer receives $\underline{s}$ is a single point $|\{\mathbf{v}' \in V \mid \boldsymbol{\alpha} \cdot \mathbf{v}' = \underline{s}\}| = 1$, which has $(K-1)$-dimension Lebesgue measure 0, while $\{\mathbf{v}' \in V \mid \boldsymbol{\alpha} \cdot \mathbf{v}' = s\}$ has strictly positive $(K-1)$-dimension Lebesgue measure for all $s \in (\underline{s}, \overline{s})$. Thus, by Fubini's theorem, $f(0) = 0$.

Now, fix any i with $a_i > 0$ and $b_i \geq 0$. Suppose that $a_i \Phi(t) + b_i \geq 0$. We claim that for all $\hat{t} > t$, $a_i \Phi(\hat{t}) + b_i > 0$. Indeed, if not, then

$$a_i \Phi(\hat{t}) + b_i \leq 0$$

and hence

$$\Phi(\hat{t}) \leq -\frac{b_i}{a_i} \leq 0\,,$$

but then it implies that $\hat{t} \leq 0.5$, and hence it must be that

$$\Phi(t) < \Phi(\hat{t})\,.$$

But then it follows that

$$a_i \Phi(t) + b_i < 0\,,$$

a contradiction. Finally, we claim that the crossing point

$$t_i^* := \inf \big\{ t : a_i \Phi(t) + b_i > 0 \big\} > 0\,.$$

Indeed, if not, then for any $t > 0$, we have

$$a_i \Phi(t) + b_i > 0$$

but then

$$\lim_{t \to 0} \Phi(t) \geq -\frac{b_i}{a_i}\,,$$

a contradiction. $\square$

LEMMA 3. *For any t_0, let*

$$\Phi(t;t_0) := \Big(t + \frac{F(t)}{f(t)}\Big)\mathbb{1}\{t \le t_0\} + \Big(t - \frac{1-F(t)}{f(t)}\Big)\mathbb{1}\{t > t_0\}\,.$$

Let $\overline{\Phi}(t;t_0)$ be the ironed $\Phi(\,\cdot\,;t_0)$. Then $g(t_0) := \overline{\Phi}(t_0;t_0)$ is continuous in t_0.

Proof. By definition, we can write

$$H(q;t_0) := \int_0^q \Phi(F^{-1}(u);t_0)du\,,$$

and

$$\overline{\Phi}(t;t_0) = \partial_+\text{conv}[H(\,\cdot\,;t_0)](F(t))\,.$$

Note first that

$$\sup_{q\in[0,1]} \big|H(q;t_0') - H(q;t_0)\big| \to 0$$

as $t_0' \to t_0$, since

$$\big|H(q;t_0') - H(q;t_0)\big| \le |F(t_0') - F(t_0)| \sup_{t\in[\min\{t_0',t_0\},\max\{t_0',t_0\}]} \frac{1}{f(t)}\,,$$

which converges to 0 as $t_0' \to t_0$. Therefore, $H(\,\cdot\,;t_0)$ is continuous in t_0 in the sup norm. We claim that

$$||\text{conv}[H_1] - \text{conv}[H_2]||_{\sup} \le ||H_1 - H_2||_{\sup}\,.$$

Indeed, note that if $H_2(z) - \delta \le H_1(z) \le H_2(z) + \delta$, then we have

$$\text{conv}[H_2 - \delta] \le \text{conv}[H_1] \le \text{conv}[H_2 + \delta]$$

and hence

$$\text{conv}[H_2] - \delta \le \text{conv}[H_1] \le \text{conv}[H_2] + \delta\,.$$

Now, we claim that for any interval $\mathcal{I}$ on which a sequence of convex H_n converge to H uniformly, and $t_n \to t \in \mathcal{I}$, where H is differentiable at t, we have

$$\partial_+ H_n(t_n) \to \partial H(t)\,.$$

Indeed, the uniform convergence implies epi-convergence, which by Attouch's theorem, implies that the subdifferentials must converge in the sense of graphical convergence (Rockafellar and Wets 1998). Then, the claim follows immediately given that H is differentiable at x.

Now let

$$H_n(\,\cdot\,) := \mathrm{conv} H(\,\cdot\,; t_0^n)\,.$$

Combining earlier observations, we have that $H_n(\,\cdot\,)$ converges uniformly to

$$H^\star(\,\cdot\,) := \mathrm{conv} H(\,\cdot\,; t_0)$$

as $t_0^n \to t_0$. By construction $H^\star(\,\cdot\,)$ is differentiable at $F(t_0)$. It follows by the previous claim that

$$\partial_+ H_n(F(t_0^n)) \to \partial H^\star(F(t_0))\,,$$

as $t_0^n \to t_0$. Therefore, we have

$$g(t_0^n) = \overline{\Phi}(t_0^n; t_0^n) = \partial_+ H_n(F(t_0^n)) \to \partial H^\star(F(t_0)) = \overline{\Phi}(t_0; t_0) = g(t_0)$$

as $t_0^n \to t_0$, proving the result. $\square$

Our next result characterizes the structure of every optimal mechanism against any linear projection of elliptical distribution in $\mathbb{R}_+^K$. For any given mechanism, let $U(t)$ denote the indirect utility of type-t buyer.

LEMMA 4. *There exists a unique $\bar{t}_0 > 0$, and unique $t_i^* > \bar{t}_0$ for all $i \in I^+ \backslash I^*$ such that for any optimal mechanism $\mathcal{M}^{DR}$:*

(i) $x(t) \in X^*$ *and* $U(t) = 0$ *for all* $t \in [0, \bar{t}_0]$*;*

(ii) *For all* $i \in I^*$, $x_i(t) = 1$ *for all* $t \in (\bar{t}_0, 1]$*;*

(iii) *For all* $i \in I^-$, $x_i(t) = 1$ *for all* $t \in [0, 1]$*;*

(iv) *For all* $i \in I^+ \backslash I^*$, $x_i(t) = 1_{t \geq t_i^*}$ *for all* $t \in [0, 1]$*.*

Proof. Any optimal direct-revelation mechanism solves the following problem:

$$\max_{x,p} \quad \mathbb{E}\big[p(t)\big] \quad \text{s.t.}$$

$$U(t) = t\sum_i x_i(t)a_i + \sum_i b_i x_i(t) - p(t) \geq 0 \quad \forall t \qquad \text{[IR]}$$
$$U(t) \geq t\sum_i x_i(t')a_i + \sum_i b_i x_i(t') - p(t') \quad \forall t, t' \qquad \text{[IC]}\,.$$

For the mechanism to be revenue-maximizing, the IR constraint must be binding for (at least) one type. Moreover, for any t_0 such that $U(t_0) = 0$, the standard characterization of [IC] applies in this setting: [IC] holds if and only if

$$\sum_i a_i x_i(t) \text{ is nondecreasing in } t \text{ and } U(t) = \int_{t_0}^{t} \sum_i a_i x_i(\nu) d\nu \text{ for all } t.$$

In particular, let

$$\text{MON} := \left\{ x : [0,1] \to [0,1]^n \text{ such that } \sum_i a_i x_i(t) \text{ is nondecreasing in } t \right\}.$$

For any t_0, define

$$\Phi(t; t_0) := \left(t + \frac{F(t)}{f(t)}\right)\mathbb{1}\{t \leq t_0\} + \left(t - \frac{1-F(t)}{f(t)}\right)\mathbb{1}\{t > t_0\}\,.$$

By a similar argument as in Loertscher and Wasser (2019) and Loertscher and Muir (2024), given any mechanism, t_0 is a worst-off type (i.e., $U(t_0) = 0$) if and only if

$$t_0 \in \arg\min_{\hat{t}} \mathbb{E}\left[\sum_i \left(a_i x_i(t)\Phi(t;\hat{t}) + b_i x_i(t)\right)\right].$$

In particular, we have that every optimal mechanism must have an allocation rule $x(\,\cdot\,)$ in the following set

$$\arg\max_{x \in \text{MON}} \min_{t_0 \in [0,1]} \mathbb{E}\left[\sum_i \left(a_i x_i(t)\Phi(t;t_0) + b_i x_i(t)\right)\right].$$

Moreover, we have the following saddle point property:

$$\max_{x \in \text{MON}} \min_{t_0 \in [0,1]} \mathbb{E}\left[\sum_i \left(a_i x_i(t)\Phi(t;t_0) + b_i x_i(t)\right)\right] = \min_{t_0 \in [0,1]} \max_{x \in \text{MON}} \mathbb{E}\left[\sum_i \left(a_i x_i(t)\Phi(t;t_0) + b_i x_i(t)\right)\right].$$

In fact, we will explicitly construct a saddle point (x^*, t_0^*) shortly that satisfies the above equality.

Before that, we make an observation about the maximization problem over $x \in \text{MON}$ for any fixed t_0^*. Let

$$\overline{\Phi}(t; t_0^*)$$

be the ironed version of $\Phi(t; t_0^*)$ exactly as in Myerson (1981). By Myerson (1981), we have that for any $x \in \text{MON}$,

$$\mathbb{E}\left[\sum_i \Big(a_i x_i(t)\Phi(t; t_0^*) + b_i x_i(t)\Big)\right] \leq \mathbb{E}\left[\Big(\sum_i a_i x_i(t)\Big)\overline{\Phi}(t; t_0^*) + \sum_i b_i x_i(t)\right].$$

Moreover, by an argument similar to that of Myerson (1981), we also know that there exists some $\tilde{x} \in \text{MON}$ that solves the following unconstrained problem:

$$\max_x \mathbb{E}\left[\Big(\sum_i a_i x_i(t)\Big)\overline{\Phi}(t; t_0^*) + \sum_i b_i x_i(t)\right], \tag{4}$$

with the optimal value given by

$$\text{OPT} := \mathbb{E}\left[\Big(\sum_i a_i \tilde{x}_i(t)\Big)\Phi(t; t_0^*) + \sum_i b_i \tilde{x}_i(t)\right].$$

Note that this implies that every optimal mechanism x' must also solve (4) since by construction

$$\begin{aligned}
\mathbb{E}\left[\Big(\sum_i a_i \tilde{x}_i(t)\Big)\Phi(t; t_0^*) + \sum_i b_i \tilde{x}_i(t)\right] &\leq \mathbb{E}\left[\Big(\sum_i a_i x_i'(t)\Big)\Phi(t; t_0^*) + \sum_i b_i x_i'(t)\right] \\
&\leq \mathbb{E}\left[\Big(\sum_i a_i x_i'(t)\Big)\overline{\Phi}(t; t_0^*) + \sum_i b_i x_i'(t)\right],
\end{aligned}$$

but the left-hand side is the optimal value of (4) and hence these inequalities must hold with equality.

Moreover, every optimal mechanism x' must also be ***consistent*** with respect to $\overline{\Phi}(t; t_0^*)$ in the sense that $\sum_i a_i x_i'(t)$ must be constant on any ironing interval where $\overline{\Phi}(t;t_0^*)$ is constant and differs from $\Phi(t;t_0^*)$. Indeed, if this were not to be the case, then by an argument similar to that of Myerson (1981), we have

$$\mathbb{E}\left[\sum_i \left(a_i x_i'(t)\Phi(t;t_0^*) + b_i x_i'(t)\right)\right] < \mathbb{E}\left[\left(\sum_i a_i x_i'(t)\right)\overline{\Phi}(t;t_0^*) + \sum_i b_i x_i'(t)\right],$$

contradicting to what we have just shown.

Now, let

$$g(t_0) := \overline{\Phi}(t_0;t_0)$$

be the value of the ironed part including t_0. Note that $g(t_0)$ is continuous in t_0 by Lemma 3, and satisfies $g(1) > 0$. Moreover, by Lemma 2, we have that

$$g(0) = \overline{\Phi}(0;0) < -\max_{i\in I^+}\left\{\frac{b_i}{a_i}\right\}.$$

Indeed, Lemma 2 implies that $a_i\Phi(t,0)+b_i$ is strictly single-crossing for all $i \in I^+$ with a strictly positive crossing point, which implies that the ironed version $a_i\overline{\Phi}(t,0) + b_i$ must also be strictly single-crossing with the same crossing point, and hence $a_i\overline{\Phi}(0,0) + b_i < 0$ for all $i \in I^+$.[27]

Note that under horizontal learning, at least one $a_i < 0$, and hence by our previous observation, for any $x^* \in X^*$, there exists some $i \in I^+$ such that $x_i^* > 0$. Let

$$\lambda := -\kappa$$

and note that λ must be an optimal dual multiplier for the problem (3). As a consequence, it must be that $\kappa \leq \max_{i\in I^+}\{\frac{b_i}{a_i}\}$ and hence

$$0 \geq \lambda \geq -\max_{i\in I^+}\left\{\frac{b_i}{a_i}\right\}.$$

Therefore, by the continuity of $g(\,\cdot\,)$ and the intermediate value theorem, there exists some t_0^* such that

$$\overline{\Phi}(t_0^*;t_0^*) = \lambda.$$

[27] To see this, note that the concave envelope of a strictly quasi-concave function must touch the original function at the original peak.

We claim that the ironing interval including t_0^* must also include 0. Indeed, if not, then we have both that $\overline{\Phi}(t_0^*; t_0^*) \leq 0$ and that $\overline{\Phi}(0^+; t_0^*) > 0$ (since that value would become the ironed virtual cost), contradicting the monotonicity of $\overline{\Phi}(\,\cdot\,; t_0^*)$. As a consequence, there must exist an ironing interval $\mathcal{I} \supset [0, t_0^*]$.

Now, we claim that t_0^* is part of a saddle point. Indeed, fix t_0^* as the conjectured worst-off type and consider the pointwise maximization problem after ironing:

$$\max_{x:[0,1]\to[0,1]^K} \mathbb{E}\left[\Big(\sum_i a_i x_i(t)\Big)\overline{\Phi}(t; t_0^*) + \sum_i b_i x_i(t)\right] \tag{5}$$

First, consider the interval $t \in \mathcal{I}$, note that on that interval the pointwise maximization problem, by construction, is equivalent to

$$\max_{x\in[0,1]^K} \sum_i a_i \overline{\Phi}(t; t_0^*) x_i + \sum_i b_i x_i = \max_{x\in[0,1]^K} \sum_i b_i x_i + \lambda \sum_i a_i x_i\,, \tag{6}$$

which is the Lagrangian of (3). By construction of λ, there must exist a solution $x^\dagger \in X^*$ to this pointwise maximization problem. Note that $\sum_i a_i x_i^\dagger = 0$, and $x_i^\dagger = 1$ for all $i \in I^-$.

Now we consider any $t \notin \mathcal{I}$. For any $i \in I^-$, we have for every type t,

$$a_i \overline{\Phi}(t; t_0^*) + b_i \geq a_i \overline{\Phi}(1; t_0^*) + b_i \geq a_i + b_i \geq 0\,,$$

where the last inequality is due to $\theta_i(t) \geq 0$ for all t, and in particular $t = 1$. Moreover, note that either the first inequality or the second inequality must be strict (which one would be a strict inequality depending on whether t and 1 are in the same ironing interval). For any $i \in I^*$, we have that for every type $t \notin \mathcal{I}$,

$$a_i \overline{\Phi}(t; t_0^*) + b_i > a_i \overline{\Phi}(t_0^*; t_0^*) + b_i = a_i \lambda + b_i \geq 0\,.$$

For any $i \in I^+ \backslash I^*$, note that since

$$a_i \overline{\Phi}(t; t_0^*) + b_i$$

is a monotone function that starts at a strictly negative value, there exists some threshold $t_i^* \notin \mathcal{I}$ such that $x_i(t) = 1\{t \geq t_i^*\}$ is pointwise optimal. In fact, because of Lemma 2 in the appendix, $a_i\overline{\Phi}(t;t_0^*)+b_i$ must be strictly single-crossing.

Now, simply define the allocation rule $x^*(\,\cdot\,)$ as: for all $t\in\mathcal{I}$, $x^*(t)=x^\dagger\in X^*$, and for all $t\notin\mathcal{I}$, $x_i^*(t)=1$ for all $i\in I^*\cup I^-$ and $x_i^*(t)=1\{t\geq t_i^*\}$ for all $i\in I^+\backslash I^*$. By the above argument, $x^*(\,\cdot\,)$ must pointwise maximize the ironed objective. Note that $\sum_i a_i x_i^*(t)$ is nondecreasing since we keep adding strictly positive goods as we move from $t=0$ to $t=1$. Moreover, it is a ***consistent*** solution with respect to ironing intervals. Together, these imply that the constructed solution solves

$$\max_{x\in\text{MON}}\mathbb{E}\left[\sum_i\Big(a_i x_i(t)\Phi(t;t_0^*)+b_i x_i(t)\Big)\right].$$

Now, we verify that t_0^* must be a worst-off type given the constructed mechanism, which then implies that it solves

$$\min_{t_0\in[0,1]}\mathbb{E}\left[\sum_i\Big(a_i x_i^*(t)\Phi(t;t_0)+b_i x_i^*(t)\Big)\right].$$

But that is clear by construction: Indeed, $t_0^*\in\mathcal{I}$, and hence $x^*(t_0^*)=x^\dagger$ which leaves 0 payoff to type t_0^* by construction (indeed, the payment implied by the Envelope theorem would be $\sum_i b_i x_i^\dagger$).

Therefore, we have found a saddle point (t_0^*,x^*). Now, let

$$\bar{t}_0:=\sup\,\mathcal{I}\,,$$

and let

$$t_i^*:=\inf\big\{t: a_i\overline{\Phi}(t;t_0^*)+b_i>0\big\}$$

for all $i\in I^+\backslash I^*$. Since, by construction, for such i

$$a_i\overline{\Phi}(\bar{t}_0;t_0^*)+b_i=a_i\lambda+b_i<0\,,$$

and hence $t_i^*>\bar{t}_0$. Note that the solution x^* as described satisfies parts *(i)* to parts *(iv)*. Moreover, as another consequence, every other optimal x' must also form a saddle point with t_0^* (this is the rectangular property of saddle points), and hence they must solve the pointwise maximization problem (5) in a way such that t_0^* is a worst-off type—in particular, it implies that for *every* optimal x', we must have

$$\sum_i a_i x'(t) = 0$$

for all $t \in \mathcal{I} = [0, \bar{t}_0]$. To see this, note that, as argued before, x' must also be consistent with respect to the ironing interval $\mathcal{I}$:

$$\sum_i a_i x'(t) = \sum_i a_i x'(t_0^*)$$

for all $t \in \mathcal{I}$, which, combined with that t_0^* is a worst-off type, implies that

$$\sum_i a_i x'(t) = \sum_i a_i x'(t_0^*) = 0\,,$$

for all $t \in \mathcal{I}$. As a consequence, for all $t \in \mathcal{I}$, every $x'(t)$ must be maximizing (6) in a way such that $\sum_i a_i x'(t) = 0$, which happens, by construction, if and only if $x'(t) = \hat{x}$ for some $\hat{x} \in X^*$ given that λ is the optimal dual multiplier of (3). Therefore, any optimal x' must satisfy part *(i)*.

Now, for the types $t \notin \mathcal{I}$, note that the pointwise maximization in fact has a unique solution almost everywhere by inspecting our previous inequalities. Thus, any optimal mechanism must satisfy parts *(i)* to *(iv)*. □

Optimal learning. We start with showing that, all else equal, the buyer strictly benefits from a more dispersed type distribution whenever his indirect utility from the mechanism is not affine.

LEMMA 5. *Let U be a piecewise-linear convex continuous function. Let t be a random variable and t' a strict mean-preserving spread of t in the following sense: There exists a continuous random variable ε such that $t' = t + \varepsilon$, with $\mathbb{E}(\varepsilon \mid t) = 0$ and $\mathrm{Var}(\varepsilon \mid t) > 0$ in the interior of $\mathrm{supp}(t)$ and continuous in t. If U is not affine over the support of t, then we must have*

$$\mathbb{E}[U(t')] > \mathbb{E}[U(t)]\,.$$

Proof. Using the law of iterated expectation, $\mathbb{E}[U(t')] = \mathbb{E}_t[\mathbb{E}_{t'}[U(t') \mid t]]$. The function U is convex, so by Jensen's inequality we have that, for each realization of t,

$$\mathbb{E}_{t'}[U(t') \mid t] \geq U(t).$$

Thus, we just need to show that the inequality holds strictly for a set of realizations of t that has positive mass.

Let $T \subseteq \mathbb{R}$ be the support of t. The function U is piecewise linear, so its graph consists of linear segments separated by kinks. Let $\partial U(t)$ be the set of sub-gradients of U at t. Let $\{t_k\}_{k=1,\ldots K}$ be the points in T at which U admits a kink, i.e., $\{t_k\}_k := \{t \in T \mid |\partial U(t)| > 1\}$, and label kinks in the increasing order. There must exist at least one such kink, since U is not affine over the support of t. Clearly, $\{t_k\}_k$ are in the interior of $\mathrm{supp}(t)$. Thus, there exists a compact interval $[a, b]$ in the interior of T such that $a < t_1$ and $t_K < b$. Note that $\mathrm{Var}(\varepsilon \mid t)$ is strictly positive, and continuous on $[a, b]$. Let

$$\sigma^2 := \min_{t \in [a,b]} \mathrm{Var}(\varepsilon \mid t) > 0\,.$$

For each t, pick any $g \in \partial U(t)$ and let $l_t(x) = U(t) + g(x - t)$ be the global supporting line at t. By construction, $U(x) \geq l_t(x)$ for any x, t, with equality only if x and t lie in the same segment of the graph (i.e., $x, t \in [t_k, t_{k+1}]$ for some k).

For all realizations of $t \in [a, b]$, the random variable $\varepsilon \mid t$ has variance at least σ^2 and is centered at 0. Thus, there exists $\eta > 0$ such that for all $t \in [a, b]$ we have $[-\eta, \eta] \subset \mathrm{co}(\mathrm{supp}(\varepsilon \mid t))$.

Note that for any $t \in [a, b]$ that is within $\eta/2$ of a kink, the random variable $\varepsilon \mid t$ puts a positive mass on the values t' that are on the other side of the kink where $U(t') > l_t(t')$, and hence $\mathbb{E}_{t'}[U(t') \mid t] > U(t)$ at these values of t. Moreover, there exists a positive measure of such t since $a < t_1$ and $t_K < b$. Integrating over all values of t, we thus get

$$\mathbb{E}[U(t')] > \mathbb{E}[U(t)]\,,$$

completing the proof. □

We now show two properties of optimal learning strategies.

LEMMA 6. *Let $N < K$. Consider the following optimization problem:*

$$\max_{\boldsymbol{\alpha}} \mathbb{E}\big[\, U\big(\theta_1(s; \boldsymbol{\alpha}), \ldots, \theta_N(s; \boldsymbol{\alpha})\big)\big]\,,$$

where U is a piecewise-linear convex function. Then, every optimal solution $\boldsymbol{\alpha}^*$ must put zero weight on $(v_{N+1}, \ldots, v_K)$ unless $\boldsymbol{\alpha}^* = \mathbf{0}$ is optimal.

Proof. Assume that $\mathbf{0}$ is suboptimal. Suppose for contradiction that $\boldsymbol{\alpha}$ is optimal and that it puts non-zero weights on $(v_{N+1}, \ldots, v_K)$. Since $\mathbf{0}$ is suboptimal,

$$\mathbb{E}\big[\, U\big(\theta_1(s;\boldsymbol{\alpha}), \ldots, \theta_N(s;\boldsymbol{\alpha})\big)\big] > \mathbb{E}\big[\, U\big(\mu_1, \ldots, \mu_N\big)\big]\,,$$

and thus along the posterior mean line generated by $\boldsymbol{\alpha}$, the function U is not affine.

Note that

$$\boldsymbol{\alpha} \cdot \mathbf{v} = \sum_{i \le N} \alpha_i v_i + \sum_{j > N} \alpha_j v_j = \sum_{i \le N} \tilde{\alpha}_i v_i + \varepsilon\,,$$

for some $\tilde{\boldsymbol{\alpha}}$, where $\mathrm{Cov}(v_i, \varepsilon) = 0$ for all $i \le N$, and ε is a non-degenerate elliptical random variable. Indeed, by the linear-projection property, we can write for each $j > N$

$$v_j = \sum_{i \le N} \beta_i v_i + \varepsilon_j\,,$$

with $\mathrm{Cov}(v_i, \varepsilon_j) = 0$ for all $i \le N$. Note that $\tilde{\boldsymbol{\alpha}} \neq \mathbf{0}$ by the previous observation.

Consider the signal structure

$$\boldsymbol{\alpha}^* = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N, 0, \ldots, 0)\,.$$

Under the original $\boldsymbol{\alpha}$, each posterior mean is

$$\theta_i(s;\boldsymbol{\alpha}) = a_i s + b_i, \quad a_i = \frac{\mathrm{Cov}\big(v_i, \sum_{k \le N} \tilde{\alpha}_k v_k\big)}{\mathrm{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})} = \frac{\sum_{k \le N} \tilde{\alpha}_k \,\mathrm{Cov}(v_i, v_k)}{\mathrm{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})}\,,$$

whereas under $\boldsymbol{\alpha}^*$,

$$\theta_i^*(s^*, \boldsymbol{\alpha}^*) = a_i^* s^* + b_i^*, \quad a_i^* = \frac{\mathrm{Cov}\big(v_i, \sum_{k \le N} \tilde{\alpha}_k v_k\big)}{\mathrm{Var}(\boldsymbol{\alpha}^* \cdot \mathbf{v})} = \frac{\sum_{k \le N} \tilde{\alpha}_k \,\mathrm{Cov}(v_i, v_k)}{\mathrm{Var}(\boldsymbol{\alpha}^* \cdot \mathbf{v})} = a_i \frac{\mathrm{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}^* \cdot \mathbf{v})}\,.$$

Clearly the posterior lines

$$\big(\theta_1(s;\boldsymbol{\alpha}), \ldots, \theta_N(s;\boldsymbol{\alpha})\big) \quad \text{and} \quad \big(\theta_1(s^*;\boldsymbol{\alpha}^*), \ldots, \theta_N(s^*;\boldsymbol{\alpha}^*)\big)$$

in $\mathbb{R}^N$ both pass through the posterior mean $(\mu_1, \ldots, \mu_N)$, and their directional derivatives are

$$\partial_s \theta(s) = (a_1, \ldots, a_N), \qquad \partial_{s^*} \theta^*(s^*) = \frac{\text{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})}{\text{Var}(\boldsymbol{\alpha}^* \cdot \mathbf{v})}(a_1, \ldots, a_N)\,,$$

so the two lines are collinear.

Leveraging this collinearity, we can write for each $i \leq N$:

$$\theta_i^*(s^*, \boldsymbol{\alpha}^*) = a_i \frac{\text{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})}{\text{Var}(\boldsymbol{\alpha^*} \cdot \mathbf{v})} s^* + b_i^* = a_i \hat{s}^* + b_i\,,$$

where

$$\hat{s}^* := \frac{\text{Var}(s)}{\text{Var}(s^*)} s^* + \left(\boldsymbol{\alpha} \cdot \boldsymbol{\mu} - \frac{\text{Var}(s)}{\text{Var}(s^*)} \boldsymbol{\alpha}^* \cdot \boldsymbol{\mu}\right) = \frac{\text{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})}{\text{Var}(\boldsymbol{\alpha^*} \cdot \mathbf{v})} s^* + \frac{b_i^* - b_i}{a_i}\,,$$

where we have used that $b_i = \mu_i - a_i \boldsymbol{\alpha} \cdot \boldsymbol{\mu}$, $b_i^* = \mu_i - a_i^* \boldsymbol{\alpha}^* \cdot \boldsymbol{\mu}$, and $a_i^*/a_i = \text{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})/\text{Var}(\boldsymbol{\alpha}^* \cdot \mathbf{v})$. Then,

$$\mathbb{E}\big[\,U\big(\theta_1(s^*; \boldsymbol{\alpha}^*), \ldots, \theta_N(s^*; \boldsymbol{\alpha}^*)\big)\big] = \mathbb{E}\big[\,U\big(\theta_1(\hat{s}^*; \boldsymbol{\alpha}), \ldots, \theta_N(\hat{s}^*; \boldsymbol{\alpha})\big)\big]\,.$$

To reach a contradiction, we have left to show that

$$\mathbb{E}\big[\,U\big(\theta_1(\hat{s}^*; \boldsymbol{\alpha}), \ldots, \theta_N(\hat{s}^*; \boldsymbol{\alpha})\big)\big] > \mathbb{E}\big[\,U\big(\theta_1(s; \boldsymbol{\alpha}), \ldots, \theta_N(s; \boldsymbol{\alpha})\big)\big]\,.$$

Recall that

$$s = \boldsymbol{\alpha} \cdot v = \sum_{i \leq N} \tilde{\alpha}_i v_i + \varepsilon = s^* + \varepsilon\,.$$

Thus,

$$\hat{s}^* = \frac{\text{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})}{\text{Var}(\boldsymbol{\alpha^*} \cdot \mathbf{v})}(s - \varepsilon) + \underbrace{\left(\boldsymbol{\alpha} \cdot \boldsymbol{\mu} - \frac{\text{Var}(s)}{\text{Var}(s^*)} \boldsymbol{\alpha}^* \cdot \boldsymbol{\mu}\right)}_{=:Z}\,.$$

Note that

$$\mathbb{E}[\hat{s}^* \mid s] = \mathbb{E}\left[\frac{\text{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})}{\text{Var}(\boldsymbol{\alpha^*} \cdot \mathbf{v})} s^* + \frac{b_i^* - b_i}{a_i} \mid s\right] = \frac{\text{Var}(\boldsymbol{\alpha} \cdot \mathbf{v})}{\text{Var}(\boldsymbol{\alpha^*} \cdot \mathbf{v})} \mathbb{E}[s^* \mid s] + Z$$

$$
\begin{aligned}
&= \frac{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}^*\cdot\mathbf{v})}\left(\boldsymbol{\alpha}^*\cdot\boldsymbol{\mu} + \frac{\mathrm{Cov}(\boldsymbol{\alpha}^*\cdot\mathbf{v},\boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}(s-\boldsymbol{\alpha}\cdot\boldsymbol{\mu})\right) + Z \\
&= \frac{\mathrm{Cov}(\boldsymbol{\alpha}^*\cdot\mathbf{v},\boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}^*\cdot\mathbf{v})}s + \frac{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}^*\cdot\mathbf{v})}\boldsymbol{\alpha}^*\cdot\boldsymbol{\mu} - \frac{\mathrm{Cov}(\boldsymbol{\alpha}^*\cdot\mathbf{v},\boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}^*\cdot\mathbf{v})}\boldsymbol{\alpha}\cdot\boldsymbol{\mu} + Z \\
&= s + \frac{\mathrm{Var}(s)}{\mathrm{Var}(s^*)}\boldsymbol{\alpha}^*\cdot\boldsymbol{\mu} - \boldsymbol{\alpha}\cdot\boldsymbol{\mu} + Z \\
&= s\,,
\end{aligned}
$$

since $\mathrm{Cov}(\boldsymbol{\alpha}^*\cdot\mathbf{v},\boldsymbol{\alpha}\cdot\mathbf{v}) = \mathrm{Cov}(\boldsymbol{\alpha}^*\cdot\mathbf{v},\boldsymbol{\alpha}^*\cdot\mathbf{v}+\varepsilon) = \mathrm{Var}(\boldsymbol{\alpha}^*\cdot\mathbf{v})$. Moreover,

$$
\begin{aligned}
\mathrm{Var}(\hat{s}^* \mid s) = \left(\frac{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}^*\cdot\mathbf{v})}\right)^2 \mathrm{Var}(\varepsilon\mid s) &= \left(\frac{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}^*\cdot\mathbf{v})}\right)^2 \kappa(s)\left[\mathrm{Var}(\varepsilon) - \frac{\mathrm{Cov}(\varepsilon,\boldsymbol{\alpha}\cdot\mathbf{v})^2}{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}\right] \\
&= \left(\frac{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}{\mathrm{Var}(\boldsymbol{\alpha}^*\cdot\mathbf{v})}\right)^2 \kappa(s)\mathrm{Var}(\varepsilon)\left(1 - \frac{\mathrm{Var}(\varepsilon)}{\mathrm{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}\right) > 0\,,
\end{aligned}
$$

where the scaling factor $\kappa(s) > 0$ for all s in the interior of the support and is continuous in s (Gupta, Varga, and Bodnar 2013).

Thus, $\hat{s}^*$ is a strict mean-preserving spread of s—it can be written as $\hat{s}^* = s+\hat{\varepsilon}$ where $\mathbb{E}[\hat{\varepsilon}\mid s] = 0$ and $\mathrm{Var}(\hat{\varepsilon}\mid s) > 0$ for all interior s. By Lemma 5, this implies

$$
\mathbb{E}\big[\,U\big(\theta_1(\hat{s}^*;\boldsymbol{\alpha}),\ldots,\theta_N(\hat{s}^*;\boldsymbol{\alpha})\big)\big] > \mathbb{E}\big[\,U\big(\theta_1(s;\boldsymbol{\alpha}),\ldots,\theta_N(s;\boldsymbol{\alpha})\big)\big]\,,
$$

contradicting the optimality of $\boldsymbol{\alpha}$. Hence, every optimal $\boldsymbol{\alpha}^*$ must place zero weight on $(v_{N+1},\ldots,v_K)$. □

LEMMA 7. *Let $N < K$. Consider the following optimization problem*

$$
\max_{\boldsymbol{\alpha}} \mathbb{E}\Big[\,U\Big(\theta_1(s;\boldsymbol{\alpha}),\ldots,\theta_{N-1}(s;\boldsymbol{\alpha}),\sum_{j\geq N}\theta_j(s;\boldsymbol{\alpha})\Big)\Big]\,,
$$

where U is a piecewise-linear convex function. Then, every optimal solution $\boldsymbol{\alpha}^*$ assigns equal weights to each of $v_N,\ldots,v_K$ unless $\boldsymbol{\alpha}^* = \mathbf{0}$ is optimal.

Proof. Consider the elliptical random vector

$$
\big(v_1,\ldots,v_{N-1},\,w_N,\,v_N,\ldots,v_K\big), \qquad w_N \;=\; \sum_{j\geq N} v_j\,.
$$

Clearly any linear signal in the original space is equivalent to one in this $(K+1)$-dimensional space, and vice versa. By Lemma 6 (applied with dimensions $N < K+1$) to this augmented vector, any optimal signal must be equivalent to some signal that puts zero weights on each of $v_N, \ldots, v_K$, unless $\mathbf{0}$ is optimal (in particular, note that the constructed ε in Lemma 6 is again non-degenerate here). Therefore, any optimal signal must be equivalent to one that puts non-zero weights only on $v_1, \ldots, v_{N-1}, w_N$, which implies that in the original coordinates it assigns the same weights to each of $v_N, \ldots, v_K$, unless $\mathbf{0}$ is optimal. This completes the proof. $\square$

We next show that in every horizontal learning equilibrium, information must be strictly valuable:

LEMMA 8. *For any horizontal learning strategy $\boldsymbol{\alpha}$ and any mechanism $\mathcal{M}$ that is optimal against the associated type distribution, information is strictly valuable under $\mathcal{M}$, i.e., $\tilde{\boldsymbol{\alpha}} = \mathbf{0}$ is strictly suboptimal against $\mathcal{M}$.*

Proof. Fix any such $(\boldsymbol{\alpha}, \mathcal{M})$. We denote the posterior mean distribution under $\boldsymbol{\alpha}$ by $(a_i t + b_i)_i \in \mathbb{R}^K_+$ following the previous notation. Suppose for contradiction that $\tilde{\boldsymbol{\alpha}} = \mathbf{0}$ is optimal against $\mathcal{M}$.

We consider two cases. Case *(i)*: the optimal mechanism has an allocation rule with $\bar{t}_0 < 1$ where $\bar{t}_0$ is identified in Lemma 4. By Lemma 4, we know that the indirect utility function $U(t)$ of the types t must be a convex (piecewise linear) function, with $U(t) = 0$ for $t \le \bar{t}_0$ and $U(t) > 0$ for $t > \bar{t}_0$. Thus, U is not affine on $[0,1]$. But then given that type t has an elliptical distribution with full-support on $[0,1]$, by Lemma 5, we have

$$\int U(t)dF(t) > U\left(\int tdF(t)\right) = U\left(\frac{1}{2}\right).$$

Note that, by construction, type $t = \frac{1}{2}$ has posterior mean given by $\boldsymbol{\theta}(t) = \boldsymbol{\mu}$. Therefore, by definition,

$$U\left(\frac{1}{2}\right)$$

is the buyer's (ex ante) payoff under strategy $\tilde{\boldsymbol{\alpha}} = \mathbf{0}$. Since $\int U(t)dF(t)$ is the buyer's (ex ante) payoff under strategy $\boldsymbol{\alpha}$, it follows immediately that $\tilde{\boldsymbol{\alpha}} = \mathbf{0}$

cannot be optimal.

Now, consider case *(ii)*: the optimal mechanism has an allocation rule with $\bar{t}_0 = 1$ where $\bar{t}_0$ is identified in Lemma 4. Then, by Lemma 4, it must be that for all types t, we have

$$U(t) = 0\,.$$

In particular, this holds for $t = \frac{1}{2}$ and hence the buyer's (ex ante) payoff under strategy $\tilde{\boldsymbol{\alpha}} = \mathbf{0}$ must be 0. Note that the seller cannot offer $x(t) = 0$ to all types t since that would imply 0 revenue, while the seller can clearly secure a strictly positive revenue (by even selling one good). Then, by Lemma 4, we know that there must exist some type t such that $x(t) = x^*$ for some non-zero $x^* \in X^*$ and $p(t) = \sum_i b_i x_i^*$. Since $\sum_i a_i x_i^* = 0$, we know that $p(t) = \sum_i b_i x_i^* = \sum_i \mu_i x_i^*$. Now consider the strategy $\hat{\boldsymbol{\alpha}} = x^*$. Note that the buyer's (ex ante) payoff from this strategy must be bounded from below by

$$\int \max\Big\{0, s - p^*\Big\} dG(s) > 0\,,$$

where G is the distribution of $\boldsymbol{v} \cdot x^*$ which is non-degenerate, and $p^* = \sum_i \mu_i x_i^* = \mathbb{E}[\boldsymbol{v} \cdot x^*]$. But then $\tilde{\boldsymbol{\alpha}} = \mathbf{0}$ cannot be optimal, since it gives 0 payoff. $\square$

Now, we exploit the characterization of the optimal mechanism in Lemma 4 to further pin down the buyer's learning in equilibrium:

LEMMA 9. *Let $(\boldsymbol{\alpha}, \mathcal{M})$ be an equilibrium that exhibits horizontal learning. Then $\boldsymbol{\alpha}$ must put zero weights on all goods in $I^* \cup I^-$.*

Proof. Fix any horizontal learning equilibrium $(\boldsymbol{\alpha}, \mathcal{M})$. We follow the same notation as before. As noted before, under any horizontal learning, we must have the set $I^* \neq \varnothing$. Let

$$O := \Big\{\big(x(t), p(t)\big)\Big\}_{t \in [0,1]}$$

denote the minimal menu that implements the seller's optimal mechanism (which would give the same ex ante payoff to the buyer under strategy $\boldsymbol{\alpha}$). Note that as long as we can construct a profitable deviation for the buyer against this menu, then it must be a profitable deviation considering the other possible options offered by the seller.

Now, note that by Lemma 4, for any $(x, p) \in O$ such that $x_i < 1$ for some $i \in I^*$, we must have that

$$x \in X^*,$$

and hence $(x, p) \in O$ yields zero payoff to every realized type t under the equilibrium strategy. Thus, we may treat them as the outside option 0 (in the deviation strategy we construct these options can only bring non-negative payoffs, and we bound them from below by 0). Moreover, for any other $(x, p) \in O$, we must have

$$x_i = 1 \text{ for all } i \in I^* \cup I^-.$$

By Lemma 8, information must be strictly valuable for the buyer against mechanism $\mathcal{M}$. Therefore, by Lemma 7, it must be the case that $\boldsymbol{\alpha}$ puts equal weights on all goods in $I^* \cup I^-$, since otherwise the buyer has a profitable deviation of assigning equal weights on all goods in $I^* \cup I^-$. Indeed, by Lemma 7, there exists one such strategy that results in a mean-preserving spread on

$$\Big((\theta_i)_{i \in I^+ \setminus I^*}, \sum_{j \in I^* \cup I^-} \theta_j\Big)$$

that strictly improves the expected payoff. Thus, the buyer's strategy $\boldsymbol{\alpha}$ must put equal weights, say α_c, on all goods in $I^* \cup I^-$.

Note that if $\alpha_c = 0$, then we are done. Otherwise, consider another signal

$$\boldsymbol{\alpha}' = \Big((\tilde{\boldsymbol{\alpha}}_i)_{i \in I^+}, 0, \ldots, 0\Big)$$

where $\boldsymbol{\alpha}'$ modifies $\boldsymbol{\alpha}$ by changing the weights to 0 for all the goods in I^- and keeping the posterior-mean line of $(\theta_i)_{i \in I^+}$ in the space $\mathbb{R}^{|I^+|}$ collinear—such a signal exists by the proof of Lemma 6 and leads to a strict mean-preserving spread along the posterior-mean line of $(\theta_i)_{i \in I^+}$. Since every element $x^* \in X^*$ also has $x_i^* = 1$ for all $i \in I^-$, every option (x, p) in O satisfies that $x_i = 1$ for all $i \in I^-$. Therefore, for the buyer's decision problem from menu O, the negative goods are irrelevant. By the proof of Lemma 6, this implies that $\boldsymbol{\alpha}'$ must be a strict improvement unless $\varepsilon = 0$ in Lemma 6, but that could only happen if

$$\alpha_c \sum_{j\in I^-} \left(v_j - \sum_{i\in I^+} \beta_i v_i \right) = 0 \, .$$

However, that is impossible since the random vector

$$(\varepsilon_j)_{j\in I^-}, \text{ where } \varepsilon_j = v_j - \sum_{i\in I^+} \beta_i v_i \, ,$$

is a full-dimension elliptical distribution by the assumption that $\rho \in (-1, 1)$. This concludes the proof. $\square$

Completion of the proof. Suppose for contradiction that there exists a horizontal-learning equilibrium $(\boldsymbol{\alpha}, \mathcal{M})$. As noted before, we must then have $I^* \neq \varnothing$ and $I^- \neq \varnothing$. By Lemma 9, it must be the case that for all $i \in I^* \cup I^-$, we have

$$\alpha_i = 0 \, .$$

Since the correlation ρ is the same across all pairs of goods, this implies that $(a_i)_{i\in I^*\cup I^-}$ must be either *(i)* all weakly positive or *(ii)* all weakly negative. Indeed, for any $i \in I^* \cup I^-$,

$$\text{sign}(a_i) = \text{sign}(\text{Cov}(\boldsymbol{\alpha} \cdot \mathbf{v}, v_i))) = \text{sign} \left(\rho \sum_{j\in I^+\setminus I^*} \alpha_j \sigma_j \right) \, ,$$

which does not depend on i. However, by construction, $a_i > 0$ for all $i \in I^*$ and $a_i \leq 0$ for all $i \in I^-$. Moreover, since $(\boldsymbol{\alpha}, \mathcal{M})$ is a horizontal learning equilibrium, there exists some $i \in I^-$ such that $a_i < 0$. A contradiction.

A.1.2 Nested Bundling

By the previous results, we know that in every equilibrium, the buyer uses a vertical learning strategy. Thus, the posterior mean distribution can be written as: for each i,

$$\theta_i = a_i t + b_i$$

where $a_i \geq 0$, $b_i \geq 0$, and $t \in [0,1]$. Moreover, $(a_i, b_i) \neq (0,0)$.[28]

We claim that, against such a posterior mean distribution, there exists a unique optimal direct-revelation mechanism (up to measure zero) that is deterministic and can be represented by a nested menu. To prove that, we show that there exists a unique solution to the relaxed problem. In particular, we maximize the virtual value function pointwise and show that the unique solution is implementable in the original problem. Indeed, consider

$$\max_{x:[0,1]\to[0,1]^K} \mathbb{E}\left[\sum_i (a_i \Phi(t) + b_i) x_i(t)\right].$$

This is a relaxed problem by the proof of Lemma 4. This problem is decomposable across items: for each i, consider

$$\max_{x:[0,1]\to[0,1]} \mathbb{E}\left[(a_i \Phi(t) + b_i) x_i(t)\right].$$

By Lemma 2, for any i such that $a_i > 0$, $a_i \Phi(t) + b_i$ is strictly single-crossing, and hence there exists a unique pointwise solution (up to a measure-zero set):

$$x_i^*(t) = \mathbb{1}\{t \geq t_i^*\}$$

where t_i^* is the crossing point of $a_i \Phi(t) + b_i$ identified in Lemma 2. For any i such that $a_i = 0$, we have that $b_i > 0$, and hence there also exists a unique solution:

$$x_i^*(t) = \mathbb{1}\{t \geq 0\}.$$

It follows immediately that *(i)* for all $t < t'$, we have

$$x^*(t) \leq x^*(t')$$

and *(ii)* $x^*(t) \in \{0,1\}$ for all t. Moreover, note that $\sum_i a_i x_i^*(t)$ is nondecreasing

[28] If $\boldsymbol{\alpha} = 0$, then the seller would extract full surplus, but that cannot form an equilibrium.

since $a_i \geq 0$ for all i. Thus, x^* is implementable and hence optimal (combined with the payment implied by the Envelope theorem).

Let p^* be the transfer rule that implements x^*, implied by the Envelope theorem. To see that (x^*, p^*) must be the unique optimal mechanism, note that for any other optimal mechanism (x', p'), by the Envelope theorem, we must have that the expected revenue is given by

$$\mathbb{E}\left[\sum_i (a_i \Phi(t) + b_i) x_i'(t)\right].$$

By construction of x^*, we have that for all t,

$$\sum_i (a_i \Phi(t) + b_i) x_i^*(t) \geq \sum_i (a_i \Phi(t) + b_i) x_i'(t),$$

and if x^* and x' differ on a strictly positive measure of types, then we also have

$$\mathbb{E}\left[\sum_i (a_i \Phi(t) + b_i) x_i^*(t)\right] > \mathbb{E}\left[\sum_i (a_i \Phi(t) + b_i) x_i'(t)\right],$$

contradicting that (x', p') is optimal.

Thus, up to a measure-zero set of types, in the equilibrium, it must be that the type space can be partitioned into a finite number of intervals: $[0,1] = [0, t_0) \cup [t_0, t_1) \cup \cdots \cup [t_{L-1}, t_L]$ such that all types $t \in [t_l, t_{l+1})$ get allocated a bundle $B_l \subseteq \{1, \ldots, K\}$ at price p_l with $B_l \subset B_{l+1}$ for all l.

Now, consider another strategy profile $(\boldsymbol{\alpha}, \mathcal{M}^{NB})$ where $\mathcal{M}^{NB}$ is a nested bundling mechanism with message space $M^{NB} = \{0, \ldots, L\}$, allocation rule $x_i^{NB}(l) = \mathbb{1}\{i \in B_l\}$ and payment rule $p^{NB}(l) = p_l$. Fixing $\boldsymbol{\alpha}$, the mechanism $\mathcal{M}^{NB}$ induces the exact same revenue as $\mathcal{M}$, and so is optimal. It remains to argue that learning strategy $\boldsymbol{\alpha}$ is a best response to $\mathcal{M}^{NB}$. What matters for the buyer is not the label of the message sent to the seller, but the induced allocation and payment. Let $O^{NB} = \bigcup_{m \in M^{NB}} (x^{NB}(m), p^{NB}(m))$ and $O = \bigcup_{m \in M} (x(m), p(m))$ be the set of outcomes that can be induced under mechanisms $\mathcal{M}^{NB}$ and $\mathcal{M}$, respectively. By contradiction, suppose that $\boldsymbol{\alpha}$ is not a best response. This means that there exists $\boldsymbol{\alpha}'$ such that

$$\mathbb{E}_{\boldsymbol{\alpha}'}\left[\max_{(x,p)\in O^{NB}}\sum_i \theta_i(s';\boldsymbol{\alpha}')x_i - p\right] > \mathbb{E}_{\boldsymbol{\alpha}}\left[\max_{(x,p)\in O^{NB}}\sum_i \theta_i(s;\boldsymbol{\alpha})x_i - p\right].$$

However,

$$\mathbb{E}_{\boldsymbol{\alpha}}\left[\max_{(x,p)\in O^{NB}}\sum_i \theta_i(s;\boldsymbol{\alpha})x_i - p\right] = \mathbb{E}_{\boldsymbol{\alpha}}\left[\max_{(x,p)\in O}\sum_i \theta_i(s;\boldsymbol{\alpha})x_i - p\right]$$

by construction. Furthermore,

$$\mathbb{E}_{\boldsymbol{\alpha}'}\left[\max_{(x,p)\in O}\sum_i \theta_i(s';\boldsymbol{\alpha}')x_i - p\right] \geq \mathbb{E}_{\boldsymbol{\alpha}'}\left[\max_{(x,p)\in O^{NB}}\sum_i \theta_i(s';\boldsymbol{\alpha}')x_i - p\right],$$

since $O^{NB} \subseteq O$. Therefore, we have

$$\mathbb{E}_{\boldsymbol{\alpha}'}\left[\max_{(x,p)\in O}\sum_i \theta_i(s';\boldsymbol{\alpha}')x_i - p\right] > \mathbb{E}_{\boldsymbol{\alpha}}\left[\max_{(x,p)\in O}\sum_i \theta_i(s;\boldsymbol{\alpha})x_i - p\right],$$

and hence $\boldsymbol{\alpha}'$ is also a profitable deviation under $\mathcal{M}$, which contradicts the initial assumption that $(\boldsymbol{\alpha}, \mathcal{M})$ is an equilibrium.

Thus, $(\boldsymbol{\alpha}, \mathcal{M}^{NB})$ is also an equilibrium. Therefore, any equilibrium is outcome-equivalent to a nested bundling equilibrium.

A.2 Proof of Proposition 1

Let $(\boldsymbol{\alpha}, \mathcal{M})$ be a nested-bundling equilibrium. Suppose for contradiction that there exist some goods i, j where tier$(i) \leq$ tier(j) and

$$\text{Var}(\log(\theta_i)) > \text{Var}(\log(\theta_j)).$$

By Theorem 1, the equilibrium learning strategy is vertical. By the linear projection property of elliptical distribution, it must be the case that there exists a type parameterization $t \in \mathbb{R}$ such that for all goods k,

$$\theta_k = a_k t + b_k \,,$$

with $a_k \geq 0$. Clearly, we have a contradiction if $a_i = 0$. Moreover, if $a_j = 0$, then it implies that $a_i = 0$ by the proof of Theorem 1. Hence, assume $a_i, a_j > 0$. Now, write

$$\text{Var}(\log(\theta_k)) = \text{Var}\Big[\log\Big(t + \frac{b_k}{a_k}\Big) + \log a_k\Big] = \text{Var}\Big[\log\Big(t + \frac{b_k}{a_k}\Big)\Big] \,.$$

By the proof of Theorem 1, since item j has a higher tier than item i, there exist some $t_1 < t_2$ and some nondecreasing function Φ^* such that

$$a_i\Phi^*(t_2) + b_i \geq 0 \,, \text{ and } a_j\Phi^*(t_2) + b_j \geq 0 \,;$$

$$a_i\Phi^*(t_1) + b_i \geq 0 \,, \text{ and } a_j\Phi^*(t_1) + b_j \leq 0 \,,$$

which implies that

$$\frac{b_i}{a_i} \geq -\Phi^*(t_1) \geq \frac{b_j}{a_j} \,.$$

Therefore, we can write the random variable

$$t + \frac{b_i}{a_i} = t + \frac{b_j}{a_j} + \underbrace{\Big(\frac{b_i}{a_i} - \frac{b_j}{a_j}\Big)}_{\geq 0} \,,$$

and hence

$$\log\Big(t + \frac{b_i}{a_i}\Big) \preceq_{\text{disp}} \log\Big(t + \frac{b_j}{a_j}\Big)$$

where $\preceq_{\text{disp}}$ is the dispersive order.[29] Since the variance operator respects the dispersive order, we immediately have that

$$\text{Var}\Big[\log\Big(t + \frac{b_i}{a_i}\Big)\Big] \leq \text{Var}\Big[\log\Big(t + \frac{b_j}{a_j}\Big)\Big] \,,$$

which is a contradiction.

[29] For this claim, see e.g. Lemma 2 of Yang, Dworczak, and Akbarpour (2023).

Now, note that for any good k, we have the following

$$\theta_k(s;\boldsymbol{\alpha}) = \mu_k + \frac{\text{Cov}(v_k, \boldsymbol{\alpha}\cdot\mathbf{v})}{\text{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}[s - \boldsymbol{\alpha}\cdot\boldsymbol{\mu}]\,.$$

Therefore,

$$\theta_k(s;\boldsymbol{\alpha}) = \frac{\text{Cov}(v_k, \boldsymbol{\alpha}\cdot\mathbf{v})}{\text{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}s + \left(\mu_k - \frac{\text{Cov}(v_k, \boldsymbol{\alpha}\cdot\mathbf{v})}{\text{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}\boldsymbol{\alpha}\cdot\boldsymbol{\mu}\right).$$

Write

$$\theta_k(s;\boldsymbol{\alpha}) = a_k s + b_k$$

where

$$a_k = \frac{\text{Cov}(v_k, \boldsymbol{\alpha}\cdot\mathbf{v})}{\text{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}, \qquad b_k = \mu_k - \frac{\text{Cov}(v_k, \boldsymbol{\alpha}\cdot\mathbf{v})}{\text{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}\boldsymbol{\alpha}\cdot\boldsymbol{\mu}\,.$$

As argued before, we can assume $a_i, a_j > 0$. Thus, for $k = i, j$, we can write

$$\frac{b_k}{a_k} = \frac{\text{Var}(\boldsymbol{\alpha}\cdot\mathbf{v})}{\text{Cov}(v_k, \boldsymbol{\alpha}\cdot\mathbf{v})}\mu_k - \boldsymbol{\alpha}\cdot\boldsymbol{\mu}\,.$$

Thus, by the previous argument, it follows that

$$\text{Var}(\log(\theta_i)) \leq \text{Var}(\log(\theta_j)) \iff \frac{b_i}{a_i} \geq \frac{b_j}{a_j} \iff \text{Cov}(v_i/\mu_i, \boldsymbol{\alpha}\cdot\mathbf{v}) \leq \text{Cov}(v_j/\mu_j, \boldsymbol{\alpha}\cdot\mathbf{v})\,.$$

Moreover, note that

$$\text{Cov}(v_i, \boldsymbol{\alpha}\cdot\mathbf{v}) = \alpha_i\sigma_i^2 + \alpha_j\rho\sigma_i\sigma_j + \text{Cov}\Big(v_i, \sum_{k\neq i,j}\alpha_k v_k\Big)\,.$$

$$\text{Cov}(v_j, \boldsymbol{\alpha}\cdot\mathbf{v}) = \alpha_j\sigma_j^2 + \alpha_i\rho\sigma_i\sigma_j + \text{Cov}\Big(v_j, \sum_{k\neq i,j}\alpha_k v_k\Big)\,.$$

If $\rho = 0$, then these together imply that

$$\left(\alpha_i\sigma_i^2/\mu_i - \alpha_j\sigma_j^2/\mu_j\right) \leq 0\,,$$

and hence

$$0 \leq \alpha_i\sigma_i^2/\mu_i \leq \alpha_j\sigma_j^2/\mu_j\,,$$

where the first inequality is due to $a_i \geq 0$.

A.3 Proof of Proposition 2

If $\mathbf{v}$ is exchangeable, then the condition in Proposition 10 holds and hence there exists a pure bundling equilibrium.

Now, we prove equilibrium existence for ρ high enough. We first state three lemmas and then prove equilibrium existence given the lemmas, and finally prove the lemmas.

Throughout, we normalize learning weights to have unit length. Let $\boldsymbol{\alpha}(\boldsymbol{\varphi})$ denote the $\boldsymbol{\alpha} \in \mathbb{R}^K$ vector on the unit sphere $\mathbb{S}^{K-1}$ with spherical coordinates $\boldsymbol{\varphi}$. Let $\underline{\theta}_i =: \lim\limits_{\rho \to 1} \min\limits_{\mathbf{v} \in V} v_i$.

The first lemma shows that it is without loss of optimality for the buyer to only consider learning weights with $\boldsymbol{\varphi} \in [0, \frac{\pi}{2}]^{K-1}$, whenever prices fall in $\boldsymbol{p} \in \prod_i [\underline{\theta}_i + \varepsilon', \mu_i]$. The second lemma shows that, once the correlation is sufficiently high, it is without loss of optimality for the seller to choose prices in $\boldsymbol{p} \in \prod_i [\underline{\theta}_i + \varepsilon', \mu_i]$ when the buyer's learning strategy is $\boldsymbol{\varphi} \in [0, \frac{\pi}{2}]^{K-1}$. Finally, the third lemma shows that the buyer's optimization problem becomes quasi-concave over $\boldsymbol{\varphi} \in [0, \frac{\pi}{2}]^{K-1}$ against any undominated separate sales mechanism, once the correlation is sufficiently high.

Lemma 10. *If $\rho \geq 0$, then for any $\boldsymbol{\varphi}$, and any $\boldsymbol{p}$, there exists some $\boldsymbol{\varphi}' \in [0, \frac{\pi}{2}]^{K-1}$ such that $U(\boldsymbol{\alpha}(\boldsymbol{\varphi}')) \geq U(\boldsymbol{\alpha}(\boldsymbol{\varphi}))$, where*

$$U(\boldsymbol{\alpha}(\boldsymbol{\varphi})) := \sum_i \mathbb{E}\Big[\max\big\{\theta_i(s; \boldsymbol{\alpha}(\boldsymbol{\varphi})) - p_i, 0\big\}\Big]$$

Lemma 11. *There exist some $\varepsilon' > 0$ and $\underline{\rho}' < 1$ such that for all $\rho \in [\underline{\rho}', 1)$, the following holds: for any separate sales prices $\boldsymbol{p}$, and any $\boldsymbol{\varphi} \in [0, \frac{\pi}{2}]^{K-1}$, there exists some $\boldsymbol{p}' \in \prod_i [\underline{\theta}_i + \varepsilon', \mu_i]$ such that $\boldsymbol{p}'$ leads to weakly higher expected revenue than $\boldsymbol{p}$.*

Lemma 12. *For any $\varepsilon > 0$, there exists some $\underline{\rho} < 1$ such that for all $\rho \in [\underline{\rho}, 1)$, for all $\boldsymbol{p} \in \prod_i [\underline{\theta}_i + \varepsilon, \mu_i]$, we have that U has a unique maximizer on $[0, \frac{\pi}{2}]^{K-1}$.*

Proof of Equilibrium Existence. Fix any $\rho > \max\{\rho_1, \rho_2, 0\}$, where ρ_1 is given by Lemma 12, ρ_2 is given by Lemma 11. Let ε also be that given by Lemma 11.

Define the buyer's strategy space as $[0, \frac{\pi}{2}]^{K-1}$, with payoff function given by $U(\boldsymbol{\alpha}(\boldsymbol{\varphi}))$. Define the seller's strategy space as $\prod_i [\underline{\theta}_i + \varepsilon, \mu_i]$ with payoff function given by $\sum_i \mathbb{E}\left[p_i \mathbb{1}\{\theta_i(s; \boldsymbol{\alpha}(\boldsymbol{\varphi})) \geq p_i\}\right]$. It is easy to verify that both the buyer's payoff and the seller's payoff are continuous in their joint action. By Lemma 12, the buyer's best-reply correspondence $\text{BR}_B(\boldsymbol{p})$ is single-valued. By Berge's maximum theorem, $\text{BR}_B(\boldsymbol{p})$ is also continuous. By Lemma 2, the seller has a unique best reply for any action of the buyer, and hence $\text{BR}_S(\boldsymbol{\varphi})$ is nonempty and single-valued. $\text{BR}_S(\boldsymbol{\varphi})$ is also continuous by Berge's theorem. Now, define the map: $\mathbf{B}(\boldsymbol{\varphi}, \boldsymbol{p}) := (\text{BR}_B(\boldsymbol{p}), \text{BR}_S(\boldsymbol{\varphi}))$. By Kakutani's fixed point theorem, $\mathbf{B}$ has a fixed point $(\boldsymbol{\varphi}^*, \boldsymbol{p}^*)$ such that $\boldsymbol{\varphi}^* \in \text{BR}_B(\boldsymbol{p}^*)$ and $\boldsymbol{p}^* \in \text{BR}_S(\boldsymbol{\varphi}^*)$. Thus, the defined game has an equilibrium.

We now show that $(\boldsymbol{\varphi}^*, \boldsymbol{p}^*)$ forms a Nash equilibrium in the original game. Note that $\boldsymbol{\varphi}^*$ defines a vertical learning strategy, given that $\rho > 0$. By the proof of Theorem 1, note that, by Lemma 11, $\boldsymbol{p}^*$ as a separate sales mechanism is actually optimal against $\boldsymbol{\varphi}^*$ even if the seller can choose any mechanism.[30] Therefore, the seller has no profitable deviation in the original game. Now, by Lemma 10, the buyer also has no profitable deviation since $\boldsymbol{\varphi}^*$ must yield an optimal payoff for the buyer against $\boldsymbol{p}^*$ even if the buyer can choose any $\boldsymbol{\varphi}$. Thus, we have found an equilibrium.

A.3.1 Proof of Lemma 10

Rewriting of the buyer's payoff. We start by rewriting the buyer's expected payoff under any separate sales mechanism from any learning strategy $\boldsymbol{\alpha}$, in a way that highlights the geometry of the buyer's problem.

Under separate sales, the buyer's payoff is separable across goods:

$$U(\boldsymbol{\alpha}) = \mathbb{E}\left[\sum_i (\theta_i(s; \boldsymbol{\alpha}) - p_i)_+\right]$$

with

$$\boldsymbol{\theta}(s; \boldsymbol{\alpha}) = \boldsymbol{\mu} + \Sigma\boldsymbol{\alpha}\left(\boldsymbol{\alpha}^\top \Sigma \boldsymbol{\alpha}\right)^{-1}\left(s - \boldsymbol{\alpha}^\top \boldsymbol{\mu}\right) \quad \text{and} \quad s = \boldsymbol{\alpha}^\top \mathbf{v}\,.$$

[30] In particular, by the proof of the nested bundling part of Theorem 1, under vertical learning, one can also implement the optimal direct-revelation mechanism as a separate sales mechanism.

Instead of optimizing over $\boldsymbol{\alpha}$, it is equivalent to optimize over $\boldsymbol{a} := \Sigma\boldsymbol{\alpha}$. Note that we can write

$$\begin{aligned}\boldsymbol{\mu} + \Sigma\boldsymbol{\alpha}\left(\boldsymbol{\alpha}^\top\Sigma\boldsymbol{\alpha}\right)^{-1}\left(s - \boldsymbol{\alpha}^\top\boldsymbol{\mu}\right) &= \boldsymbol{\mu} + \boldsymbol{a}\left(\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a}\right)^{-1}\left(\boldsymbol{a}^\top\Sigma^{-1}\mathbf{v} - \boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{\mu}\right)\\ &= \boldsymbol{\mu} + \boldsymbol{a}\left(\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a}\right)^{-1/2}\overline{s}\,,\end{aligned}$$

where

$$\overline{s} := \frac{\boldsymbol{a}^\top\Sigma^{-1}\mathbf{v} - \boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{\mu}}{\sqrt{\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a}}}$$

is a standardized signal. Note that $\overline{s}$ is elliptical, scalar-valued, has mean 0 and variance 1. Importantly, the distribution of $\overline{s}$ is then independent of $\boldsymbol{\alpha}$.

Therefore, for any choice $\boldsymbol{a}$, the buyer's expected payoff is:

$$\overline{U}(\boldsymbol{a}) := \sum_i \mathbb{E}\left[\left(\boldsymbol{\theta}(s;\Sigma^{-1}\boldsymbol{a}) - p_i\right)_+\right] = \sum_i \mathbb{E}\left[\left(\mu_i - p_i + \frac{a_i}{\sqrt{\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a}}}\overline{s}\right)_+\right].$$

The following facts will prove important. First, consider the function

$$g_i(l) := \mathbb{E}\left[(\mu_i - p_i + l\overline{s})_+\right].$$

Note that

$$g_i'(l) = \mathbb{E}\big[\overline{s}\mathbb{1}\{\mu_i - p_i + l\overline{s} \geq 0\}\big] = \mathbb{E}\big[\overline{s}\mathbb{1}\{\overline{s} \geq -(\mu_i - p_i)/l\}\big] \geq \mathbb{E}\big[\overline{s}\big] = 0\,,$$

where the last inequality comes from $l > 0$. Second, the distribution of $\overline{s}$ is symmetric around zero, and so the function g_i is even: $g_i(l) = g_i(-l)$ for all l, i.

Second, in the above expression for $\overline{U}$, the correlation ρ only enters through $\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a}$. We can simplify this term further to make the dependence on ρ explicit. Because $\mathrm{Corr}(v_i, v_j)$ is constant and equal to ρ for all i, j, we can decompose the variance-covariance matrix into

$$\Sigma = \mathrm{diag}(\sigma_i)\,R\,\mathrm{diag}(\sigma_i) \quad \text{where} \quad R = (1-\rho)I + \rho\mathbf{1}\mathbf{1}^\top.$$

Letting $\boldsymbol{\omega} := (a_i/\sigma_i)_i$, we get

$$\begin{aligned}\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a} = \boldsymbol{a}^\top \mathrm{diag}(1/\sigma_i)\, R^{-1}\, \mathrm{diag}(1/\sigma_i) \boldsymbol{a} &= \boldsymbol{\omega}^\top\, R^{-1}\, \boldsymbol{\omega} \\ &= \boldsymbol{\omega}^\top \frac{1}{1-\rho} \left(I - \frac{\rho}{1+\rho(K-1)} \mathbf{1}\mathbf{1}^\top \right) \boldsymbol{\omega} \\ &= \frac{1}{1-\rho} \left(||\boldsymbol{\omega}||^2 - \frac{\rho}{1+\rho(K-1)} (\mathbf{1}^\top \boldsymbol{\omega})^2 \right) .\end{aligned}$$

Note that by construction we also have $1 + \rho(K-1) > 0$ (for Σ to be positive definite).

Completion of the proof. We first show that there exists an optimal solution $\boldsymbol{a} \in \mathbb{R}_+^K$, and then show that in fact, we can focus on

$$\boldsymbol{a} \in \mathcal{A} := \left\{ \frac{\boldsymbol{a}'}{||\boldsymbol{a}'||} : \boldsymbol{a}' = \sum_i c_i \frac{\Sigma e_i}{||\Sigma e_i||} \text{ for some } \boldsymbol{c} \geq 0 \right\} .$$

Fix any $\boldsymbol{a}$. Let $\hat{\boldsymbol{a}} := (|a_i|)_i$. Using the above notation, let $\boldsymbol{\omega} = (a_j/\sigma_j)_j$ and $\hat{\boldsymbol{\omega}} = (\hat{a}_j/\sigma_j)_j$. By construction:

$$|\hat{a}_j| = |a_j| \quad \forall j \quad \text{and} \quad ||\hat{\boldsymbol{\omega}}|| = ||\boldsymbol{\omega}||.$$

Moreover, by construction:

$$|\mathbf{1}^\top \hat{\boldsymbol{\omega}}| = \sum_j \frac{\hat{a}_j}{\sigma_j} = \sum_j \left|\frac{a_j}{\sigma_j}\right| \geq \left| \sum_j \frac{a_j}{\sigma_j} \right| = \left| \mathbf{1}^\top \boldsymbol{\omega} \right| .$$

Since $\rho \geq 0$, this implies:

$$\hat{\boldsymbol{a}}^\top \Sigma^{-1} \hat{\boldsymbol{a}} \leq \boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a} \,,$$

and hence for all j,

$$\frac{|\hat{a}_j|}{\sqrt{\hat{\boldsymbol{a}}^\top \Sigma^{-1} \hat{\boldsymbol{a}}}} \geq \frac{|a_j|}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}} \,.$$

The function g_j is even and increasing over the positive range. Therefore:

$$\mathbb{E}\left[\left(\mu_j - p_j + \frac{\hat{a}_j}{\sqrt{\hat{\boldsymbol{a}}^\top \Sigma^{-1} \hat{\boldsymbol{a}}}} \overline{s} \right)_+ \right] \geq \mathbb{E}\left[\left(\mu_j - p_j + \frac{a_j}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}} \overline{s} \right)_+ \right] \quad \forall j \,.$$

It follows immediately that

$$\overline{U}(\hat{\boldsymbol{a}}) \geq \overline{U}(\boldsymbol{a})\,,$$

and hence it is without loss of optimality to focus on $\boldsymbol{a} \in \mathbb{R}_+^K$.

Now, for the second claim, note that if we define

$$\beta_i := \frac{a_i}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}}$$

then

$$\boldsymbol{\beta}^\top \Sigma^{-1} \boldsymbol{\beta} = 1\,.$$

Conversely, for any $\boldsymbol{\beta}^\top \Sigma^{-1} \boldsymbol{\beta} = 1$, we can define $\boldsymbol{a} := \boldsymbol{\beta}/||\boldsymbol{\beta}||$. Then

$$\frac{a_i}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}} = \frac{\boldsymbol{\beta}}{||\boldsymbol{\beta}||/||\boldsymbol{\beta}||} = \boldsymbol{\beta}\,.$$

Therefore, the buyer's problem is also equivalent to the following:

$$\max_{\boldsymbol{\beta}^\top \Sigma^{-1} \boldsymbol{\beta} = 1} \sum_i g_i(\beta_i)\,.$$

By the first claim, we know that there exists an optimal solution $\boldsymbol{\beta}^* \in \mathbb{R}_+^K$. Then, at this optimal point, there must exist multiplier $\lambda \in \mathbb{R}$ such that for all i,

$$g_i'(\beta_i^*) - 2\lambda(\Sigma^{-1}\boldsymbol{\beta}^*)_i = 0\,.$$

Let

$$\delta_i := g_i'(\beta_i^*)\,.$$

Note that $\boldsymbol{\delta} \in \mathbb{R}_+^K$ by our previous observation. Moreover,

$$2\lambda = 2\lambda(\boldsymbol{\beta}^*)^\top \Sigma^{-1} \boldsymbol{\beta}^* = (\boldsymbol{\beta}^*)^\top \boldsymbol{\delta} \geq 0\,.$$

First, consider the case $\lambda > 0$. Then, we must have

$$\boldsymbol{\beta}^* = \frac{1}{2\lambda} \Sigma \boldsymbol{\delta}\,.$$

Thus, there must exists an optimal solution $\boldsymbol{a}^*$ such that

$$\boldsymbol{a}^* = \boldsymbol{\beta}^*/||\boldsymbol{\beta}^*|| \in \mathcal{A}\,,$$

as desired.

Now, consider the case $\lambda = 0$. Then it must be that $g'(\beta_i^*) = 0$ for all i (since otherwise there exists some i such that $\beta_i^* g'(\beta_i^*) > 0$). Thus, by our previous observation, it must be that $\Pr(\mu_i - p_i + \beta_i^* \overline{s} \geq 0) = 1$ for all i, and hence $\overline{U}(\boldsymbol{\beta}^*/||\boldsymbol{\beta}^*||) = \sum_i (\mu_i - p_i) = \sum_i g_i(0)$. But then any feasible $\hat{\boldsymbol{\beta}}$ would be optimal, and hence the claim trivially follows.

A.3.2 Proof of Lemma 11

For any $\boldsymbol{\varphi} \in [0, \frac{\pi}{2}]^{K-1}$, note that we have

$$\boldsymbol{a}(\boldsymbol{\varphi}) \in \mathcal{A}(\rho) := \left\{ \frac{\boldsymbol{a}'}{||\boldsymbol{a}'||} : \boldsymbol{a}' = \sum_i c_i \frac{\Sigma e_i}{||\Sigma e_i||} \text{ for some } \boldsymbol{c} \geq 0 \right\}.$$

We first show that

$$\sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} ||\boldsymbol{a} - \hat{\boldsymbol{\sigma}}|| \leq h(1-\rho)\,,$$

where $\hat{\boldsymbol{\sigma}} := \boldsymbol{\sigma}/||\boldsymbol{\sigma}||$ and h is a continuous function with $h(0) = 0$. Indeed, note that

$$u_i(\rho) := \Sigma e_i = (1-\rho)\sigma_i^2 e_i + \rho \sigma_i \boldsymbol{\sigma}\,,$$

and hence

$$||u_i(\rho)|| = \sigma_i \sqrt{\rho^2 ||\boldsymbol{\sigma}||^2 + (1-\rho^2)\sigma_i^2}\,,$$

and hence

$$\frac{u_i(\rho)}{||u_i(\rho)||} \cdot \hat{\boldsymbol{\sigma}} = \frac{(1-\rho)\sigma_i^2/||\boldsymbol{\sigma}|| + \rho||\boldsymbol{\sigma}||}{\sqrt{\rho^2||\boldsymbol{\sigma}||^2 + (1-\rho^2)\sigma_i^2}}\,,$$

which converges to 1 as $\rho \to 1$. Therefore,

$$\sup_i \left|\left| \frac{u_i(\rho)}{||u_i(\rho)||} - \hat{\boldsymbol{\sigma}} \right|\right|^2 = \sup_i 2\Big(1 - \frac{u_i(\rho)}{||u_i(\rho)||} \cdot \hat{\boldsymbol{\sigma}}\Big) \leq m(\rho)\,,$$

for some continuous m such that $m(1) = 0$. Thus, for any $a \in \mathcal{A}(\rho)$ we have

$$||\boldsymbol{a} - \hat{\boldsymbol{\sigma}}||^2 = 2\big(1 - \boldsymbol{a} \cdot \hat{\boldsymbol{\sigma}}\big) = 2\Big(1 - \frac{\boldsymbol{a}' \cdot \hat{\boldsymbol{\sigma}}}{||\boldsymbol{a}'||}\Big) \leq 2\Big(1 - \frac{(1 - m(\rho)/2)\sum_i c_i}{\sum_i c_i}\Big) = m(\rho)\,.$$

Thus,

$$\sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} ||\boldsymbol{a} - \hat{\boldsymbol{\sigma}}|| \leq \sqrt{m(\rho)}\,,$$

and hence the claim follows.

Now, for any i, let

$$\beta_i(\boldsymbol{a}; \rho) := \frac{a_i}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}}\,.$$

Now we claim that

$$\sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} ||\boldsymbol{\beta}(\boldsymbol{a}; \rho) - \boldsymbol{\sigma}|| \to 0$$

as $\rho \to 1$. Indeed, first, note that for any $(\boldsymbol{a}, \rho) \to (\hat{\boldsymbol{\sigma}}, 1)$, we have

$$\boldsymbol{\beta}(\boldsymbol{a}; \rho) \to \boldsymbol{\sigma}$$

by direct calculation. Now, suppose for contradiction that the claimed uniform convergence does not hold. Then there exist some $\delta > 0$ and a sequence $\rho_n \to 1$ such that

$$\sup_{\boldsymbol{a} \in \mathcal{A}(\rho_n)} ||\boldsymbol{\beta}(\boldsymbol{a}_n; \rho_n) - \boldsymbol{\sigma}|| \geq \delta$$

for all n. Note that the above $\sup$ can always be attained by compactness and continuity. Therefore, there exists a sequence $\boldsymbol{a}_n \in \mathcal{A}(\rho_n)$ such that

$$\lim_{n \to \infty} ||\boldsymbol{\beta}(\boldsymbol{a}_n; \rho_n) - \boldsymbol{\sigma}|| \geq \delta\,.$$

Note that $(\boldsymbol{a}_n, \rho_n) \to (\hat{\boldsymbol{\sigma}}, 1)$ since

$$||\boldsymbol{a}_n - \hat{\boldsymbol{\sigma}}|| \leq \sup_{\boldsymbol{a} \in \mathcal{A}(\rho_n)} ||\boldsymbol{a} - \hat{\boldsymbol{\sigma}}|| \to 0$$

as $n \to \infty$. But then we must have that

$$\lim_{n \to \infty} ||\boldsymbol{\beta}(\boldsymbol{a}_n; \rho_n) - \boldsymbol{\sigma}|| = 0\,,$$

a contradiction.

Now, note that by Lemma 2, the set of monopoly prices p_i^* against the distribution of θ_i generated by strategy $\boldsymbol{\alpha} = \Sigma^{-1}\boldsymbol{a}$ is single-valued, and moreover, it is only a function of $\boldsymbol{\beta}(\boldsymbol{a};\rho)$ (see the proof of Lemma 10). By the uniform convergence of $\boldsymbol{\beta}$ above, for any $\varepsilon > 0$, there exists some ρ_ε such that for all $\rho > \rho_\varepsilon$, we have $\boldsymbol{\beta} \in \boldsymbol{\sigma} + \mathrm{b}_\varepsilon$, where b_ε is the ε-radius ball in $\mathbb{R}^K$. Let $\boldsymbol{p}^*(\boldsymbol{\beta})$ be the monopoly prices across the goods, given $\boldsymbol{\beta}$. Fix some ε such that $\boldsymbol{\sigma} + \mathrm{b}_\varepsilon \geq \boldsymbol{\sigma}/2$. By Berge's theorem, $\boldsymbol{p}^*$ is continuous in $\boldsymbol{\beta}$. By the Heine–Cantor theorem, it follows that $\boldsymbol{p}^*$ is uniformly continuous on the ball $\boldsymbol{\sigma} + \mathrm{b}_\varepsilon$. In particular, for any $\varepsilon_2 > 0$, there exists some $\delta > 0$ such that for any $\boldsymbol{\beta}, \boldsymbol{\beta}' \in \boldsymbol{\sigma} + \mathrm{b}_\varepsilon$ where $||\boldsymbol{\beta} - \boldsymbol{\beta}'|| < \delta$, we have $||\boldsymbol{p}^*(\boldsymbol{\beta}) - \boldsymbol{p}^*(\boldsymbol{\beta}')|| < \varepsilon_2$. Let $\boldsymbol{\beta}' = \boldsymbol{\sigma}$. By Lemma 2, $p_i^*(\boldsymbol{\sigma}) > \underline{\theta}_i$ for all i. Let

$$\varepsilon_2 := \min_i \{p_i^*(\boldsymbol{\sigma}) - \underline{\theta}_i\}/2 > 0\,.$$

Let $\delta > 0$ be the one given by the uniform continuity of $\boldsymbol{p}^*$. Then, consider any $\rho > \rho_\delta$. It follows that for any $\boldsymbol{\varphi} \in [0, \frac{\pi}{2}]^{K-1}$, the induced $\boldsymbol{\beta}(\boldsymbol{\varphi};\rho) \in \boldsymbol{\sigma} + \mathrm{b}_\delta$ by construction. Therefore, $||\boldsymbol{\beta}(\boldsymbol{\varphi};\rho) - \boldsymbol{\sigma}|| < \delta$ by construction, and hence

$$||\boldsymbol{p}^*(\boldsymbol{\beta}(\boldsymbol{\varphi};\rho)) - \boldsymbol{p}^*(\boldsymbol{\sigma})|| < \varepsilon_2\,.$$

This implies that for all i

$$|p_i^*(\boldsymbol{\beta}(\boldsymbol{\varphi};\rho)) - p_i^*(\boldsymbol{\sigma})| < \varepsilon_2 \leq (p_i^*(\boldsymbol{\sigma}) - \underline{\theta}_i)/2\,,$$

and hence

$$p_i^*(\boldsymbol{\beta}(\boldsymbol{\varphi};\rho)) > p_i^*(\boldsymbol{\sigma}) - (p_i^*(\boldsymbol{\sigma}) - \underline{\theta}_i)/2 = \underline{\theta}_i + (p_i^*(\boldsymbol{\sigma}) - \underline{\theta}_i)/2 \geq \underline{\theta}_i + \varepsilon_2\,.$$

The result follows immediately by letting $\varepsilon' := \varepsilon_2$ and $\rho' := \rho_\delta$.

A.3.3 Proof of Lemma 12

Using the same notation as in the proof of Lemma 10, we can rewrite the buyer's payoff from any learning weights $\boldsymbol{\alpha}(\boldsymbol{\varphi})$ as

$$U(\boldsymbol{\alpha}(\boldsymbol{\varphi})) = \overline{U}(\boldsymbol{a}(\boldsymbol{\varphi})) = \sum_i \mathbb{E}\left[\left(\mu_i - p_i + \frac{a_i(\boldsymbol{\varphi})}{\sqrt{\boldsymbol{a}(\boldsymbol{\varphi})^\top \Sigma^{-1} \boldsymbol{a}(\boldsymbol{\varphi})}} \overline{s}\right)_+\right],$$

where $\boldsymbol{a}(\boldsymbol{\varphi}) = \Sigma \boldsymbol{\alpha}(\boldsymbol{\varphi})$ and $\overline{s}$ is an elliptical random variable whose law does not depend on $\boldsymbol{\varphi}$.

We want to show $\overline{U}(\boldsymbol{a}(\,\cdot\,))$ has a unique maximizer over $\boldsymbol{\varphi} \in [0, \frac{\pi}{2}]^{K-1}$ for ρ high enough. As argued before, it is equivalent to show that there exists some $\underline{\rho} < 1$ such that for all $\rho > \underline{\rho}$, and all $\boldsymbol{p} \in \prod_i [\underline{\theta}_i + \varepsilon, \mu_i]$, we have that $\overline{U}(\boldsymbol{a}; \rho, \boldsymbol{p})$ has a unique maximizer in the following set

$$\mathcal{A}(\rho) := \left\{ \frac{\boldsymbol{a}'}{||\boldsymbol{a}'||} : \boldsymbol{a}' = \sum_i c_i \frac{\Sigma e_i}{||\Sigma e_i||} \text{ for some } \boldsymbol{c} \geq 0 \right\}.$$

The remainder of the proof proceeds in several steps. First, we derive bounds on elements of $\overline{U}$ and their derivative as ρ goes to one. Second, we show that along any short geodesic $\boldsymbol{a}(t) \subset \mathcal{A}$, the function $t \to \overline{U}(\boldsymbol{a}(t))$ is strictly unimodal. Finally, we show that $\overline{U}$ attains a unique maximizer $\boldsymbol{a}^* \in \mathcal{A}$.

Let $Q(\boldsymbol{a}; \rho) := \frac{1}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}}$.

Limits of $\boldsymbol{a}$ and Q. As shown in the proof of Lemma 11,

$$\sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} ||\boldsymbol{a} - \hat{\boldsymbol{\sigma}}|| \to 0$$

as $\rho \to 1$, where $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/||\boldsymbol{\sigma}|| > 0$. Moreover, note that $\hat{\boldsymbol{\sigma}}$ is the principal eigenvector of $\Sigma(\rho)$ when $\rho = 1$. Also, as shown in the proof of Lemma 11, we have

$$\sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} ||\boldsymbol{\beta}(\boldsymbol{a}; \rho) - \boldsymbol{\sigma}|| \to 0$$

as $\rho \to 1$, where

$$\boldsymbol{\beta}(\boldsymbol{a}; \rho) = \boldsymbol{a} Q(\boldsymbol{a}; \rho).$$

By the same argument as in the proof of Lemma 11, it also follows that

$$\sup_{\boldsymbol{a}\in\mathcal{A}(\rho)} \Big| Q(\boldsymbol{a};\rho) - ||\boldsymbol{\sigma}|| \Big| \to 0 \,.$$

Moreover, note that $||\boldsymbol{\sigma}||^2$ is the only non-zero eigenvalue of $\Sigma(1)$.

Limits of radial derivatives of Q. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K > 0$ be the eigenvalues of $\Sigma(\rho)$, which depend on ρ and are strictly positive for $\rho < 1$. Let $\{\boldsymbol{\nu}_i\}_i$ be the associated eigenvectors. Recall that Σ^{-1} shares the same eigenvectors as Σ but with eigenvalues $\{1/\lambda_i\}_i$. The product $\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a}$ is the Rayleigh coefficient of Σ^{-1} and $\boldsymbol{a}$. Given that $||\boldsymbol{a}|| = 1$, it can be written as

$$\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a} = \sum_i \frac{(\boldsymbol{\nu}_i^\top\,\boldsymbol{a})^2}{\lambda_i}.$$

The eigenvectors of Σ are orthonormal and can be used as a base. We can thus write $\boldsymbol{a} = \boldsymbol{\nu}_1\cos(\psi) + \sin(\psi)\sum_{i>1} c_i\boldsymbol{\nu}_i$ with $\psi := \angle(\boldsymbol{\nu}_1, \boldsymbol{a})$ and $\sum_{i>1} c_i^2 = 1$. Then,

$$\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a} = \frac{\cos^2(\psi(\boldsymbol{a}))}{\lambda_1} + \sin^2(\psi(\boldsymbol{a}))\sum_{i>1}\frac{c_i^2}{\lambda_i}.$$

Furthermore, for any $\boldsymbol{a} \in \mathcal{A}(\rho)$, we have

$$|1 - \cos\psi(\boldsymbol{a})| \leq \max_i \Big| 1 - \left(\frac{\Sigma e_i}{||\Sigma e_i||}\right)^\top \boldsymbol{\nu}_1 \Big| \,,$$

where, for each i:

$$\Big|1-\left(\frac{\Sigma e_i}{||\Sigma e_i||}\right)^\top\boldsymbol{\nu}_1\Big| = \frac{1}{2}||\frac{\Sigma e_i}{||\Sigma e_i||}-\boldsymbol{\nu}_1||^2 \leq \frac{1}{2}\big(||\boldsymbol{\nu}_1-\hat{\boldsymbol{\sigma}}||+||\hat{\boldsymbol{\sigma}}-\frac{\Sigma e_i}{||\Sigma e_i||}||\big)^2 = O((1-\rho)^2)\,,$$

where the rate estimate in the last equality follows from the bound given in the Lemma 11. Therefore, there exists some constant $C_1 > 0$ such that for all $\boldsymbol{a} \in \mathcal{A}(\rho)$, we have

$$\cos^2\psi(\boldsymbol{a}) \geq 1 - C_1\cdot(1-\rho)^2 \quad \text{and} \quad \sin^2\psi(\boldsymbol{a}) = 1 - \cos^2\psi(\boldsymbol{a}) \leq C_1\cdot(1-\rho)^2.$$

Furthermore, by standard arguments, the eigenvalues of Σ (other than the largest one) all converge to zero at rate $\lambda_i = \Theta(1-\rho)$ for all $i > 1$.

Fixing $\{c_i\}_{i>1}$ and differentiating $Q(\boldsymbol{a})$ along the curve $\psi \to \boldsymbol{\nu}_1 \cos(\psi) + \sin(\psi) \sum_{i>1} c_i \boldsymbol{\nu}_i$ gives

$$
\begin{aligned}
\frac{\partial Q}{\partial \psi} = -\frac{1}{2(\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a})^{\frac{3}{2}}} \frac{\partial \boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}{\partial \psi} &= -\frac{1}{2(\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a})^{\frac{3}{2}}} \frac{\partial}{\partial \psi} \left(\frac{\cos^2(\psi)}{\lambda_1} + \sin^2(\psi) \sum_{i>1} \frac{c_i^2}{\lambda_i} \right) \\
&= -\frac{1}{2(\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a})^{\frac{3}{2}}} \left(\frac{-\sin(2\psi)}{\lambda_1} + \sin(2\psi) \sum_{i>1} \frac{c_i^2}{\lambda_i} \right) \\
&= \frac{\sin(2\psi)}{2(\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a})^{\frac{3}{2}}} \left(\frac{1}{\lambda_1} - \sum_{i>1} \frac{c_i^2}{\lambda_i} \right).
\end{aligned}
$$

Moreover,

$$
\frac{1}{\lambda_2} \le \sum_{i>1} \frac{c_i^2}{\lambda_i} \le \frac{1}{\lambda_K}.
$$

Therefore, there exists some constant C_2 such that

$$
\sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} \left| \frac{\partial Q}{\partial \psi}(\boldsymbol{a}; \rho) \right| \le \sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} \left| \frac{1}{2} Q^3(\boldsymbol{a}; \rho) \right| \cdot \sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} \left| \sin(2\psi(\boldsymbol{a})) \right| \cdot \left| \frac{1}{\lambda_1} - \frac{1}{\lambda_K} \right| \to C_2
$$

as $\rho \to 1$, since the first term converges to a constant, the second term is bounded above by $C'(1-\rho)$, and the third term is bounded above by $C'' \frac{1}{1-\rho}$ for some constants C', C''.

Similarly, we also look at the second derivative: Fixing $\{c_i\}_{i>1}$, differentiating $Q(\boldsymbol{a})$ along ψ twice gives:

$$
\frac{\partial^2 Q}{\partial \psi^2} = \cos(2\psi) Q^3(\boldsymbol{a}; \rho) \left(\frac{1}{\lambda_1} - \sum_{i>1} \frac{c_i^2}{\lambda_i} \right) + \frac{3}{4} Q^5(\boldsymbol{a}; \rho) \sin^2(2\psi) \left(\frac{1}{\lambda_1} - \sum_{i>1} \frac{c_i^2}{\lambda_i} \right)^2.
$$

By the same argument, the absolute value of the second term is bounded from above by

$$
\sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} \frac{3}{4} Q^5(\boldsymbol{a}; \rho) \sin^2(2\psi(\boldsymbol{a})) \left| \frac{1}{\lambda_1} - \frac{1}{\lambda_K} \right|^2 \to C_3
$$

as $\rho \to 1$, where C_3 is a constant. The first term, however, satisfies

$$\cos(2\psi(\boldsymbol{a}))Q^3(\boldsymbol{a};\rho)\left(\frac{1}{\lambda_1} - \sum_{i>1}\frac{c_i^2}{\lambda_i}\right) \le \cos(2\psi(\boldsymbol{a}))Q^3(\boldsymbol{a};\rho)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right).$$

Moreover,

$$\sup_{\boldsymbol{a}\in\mathcal{A}(\rho)}\left\{\cos(2\psi(\boldsymbol{a}))Q^3(\boldsymbol{a};\rho)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\right\} \le \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\underbrace{\inf_{\boldsymbol{a}\in\mathcal{A}(\rho)}\left\{\cos(2\psi(\boldsymbol{a}))Q^3(\boldsymbol{a};\rho)\right\}}_{=:w(\rho)}.$$

Note that, as $\rho \to 1$, $\frac{1}{\lambda_1}$ converges to $1/||\boldsymbol{\sigma}||$, $\frac{1}{\lambda_2}$ diverges to ∞ at a rate $\Theta(\frac{1}{1-\rho})$, and $w(\rho)$ converges to a constant. Therefore,

$$\sup_{\boldsymbol{a}\in\mathcal{A}(\rho)}\left\{\cos(2\psi(\boldsymbol{a}))Q^3(\boldsymbol{a};\rho)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\right\} \to -\infty\,,$$

as $\rho \to 1$. It follows immediately that

$$\sup_{\boldsymbol{a}\in\mathcal{A}(\rho)}\frac{\partial^2 Q}{\partial\psi^2}(\boldsymbol{a};\rho) \to -\infty$$

as $\rho \to 1$.

Limits of derivatives of $\overline{U}$. For any i, let

$$\beta_i(\boldsymbol{a};\rho) := \frac{a_i}{\sqrt{\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a}}}.$$

As shown in the proof of Lemma 10,

$$\frac{\partial\overline{U}}{\partial\beta_i} = \mathbb{E}\Big[\overline{s}\mathbb{1}\big\{\mu_i - p_i + \beta_i\overline{s} \ge 0\big\}\Big] \ge \mathbb{E}\Big[\overline{s}\mathbb{1}\big\{\mu_i - (\underline{\theta}_i + \varepsilon) + \beta_i\overline{s} \ge 0\big\}\Big],$$

where the inequality is due to $\underline{\theta}_i + \varepsilon \le p_i \le \mu_i$ and $\beta_i \ge 0$. Therefore,

$$\inf_{\boldsymbol{a}\in\mathcal{A}(\rho)}\frac{\partial\overline{U}}{\partial\beta_i}(\boldsymbol{a};\rho) \ge \inf_{\boldsymbol{a}\in\mathcal{A}(\rho)}\mathbb{E}\Big[\overline{s}\mathbb{1}\big\{\mu_i - (\underline{\theta}_i + \varepsilon) + \beta_i(\boldsymbol{a};\rho)\overline{s} \ge 0\big\}\Big]$$

$$= \mathbb{E}\Big[\overline{s}\mathbb{1}\big\{\mu_i - (\underline{\theta}_i + \varepsilon) + \overline{s} \cdot \inf_{\boldsymbol{a}\in\mathcal{A}(\rho)} \beta_i(\boldsymbol{a};\rho) \geq 0\big\}\Big]\,,$$

where the last equality uses the convexity of $\overline{U}$ in β_i. Let $\mathcal{P} := \prod_i [\underline{\theta}_i + \varepsilon, \mu_i]$.

By the proof of Lemma 11, we know that $\inf_{\boldsymbol{a}\in\mathcal{A}(\rho)} \beta_i(\boldsymbol{a};\rho) \to \sigma_i$, as $\rho \to 1$, and hence

$$\lim_{\rho\to 1} \inf_{\boldsymbol{a}\in\mathcal{A}(\rho),\, \boldsymbol{p}\in\mathcal{P}} \frac{\partial \overline{U}}{\partial \beta_i}(\boldsymbol{a};\rho) \geq \mathbb{E}\Big[\overline{s}\mathbb{1}\big\{\mu_i - (\underline{\theta}_i + \varepsilon) + \sigma_i \overline{s} \geq 0\big\}\Big] > 0\,,$$

where the strict inequality is due to

$$\Pr\left(\mu_i + \sigma_i \overline{s} < \underline{\theta}_i + \varepsilon\right) > 0\,,$$

which holds by construction given $\varepsilon > 0$ and $\underline{\theta}_i := \lim_{\rho\to 1} \inf_{v\in V} v_i$. By a similar argument, we also have

$$\lim_{\rho\to 1} \sup_{\boldsymbol{a}\in\mathcal{A}(\rho),\, \boldsymbol{p}\in\mathcal{P}} \frac{\partial \overline{U}}{\partial \beta_i}(\boldsymbol{a};\rho) \leq \mathbb{E}\Big[\overline{s}\mathbb{1}\big\{\overline{s} \geq 0\big\}\Big] < \infty\,.$$

Also, we have that

$$\sup_{\boldsymbol{a}\in\mathcal{A}(\rho),\, \boldsymbol{p}\in\mathcal{P}} \left|\frac{\partial^2 \overline{U}}{\partial \beta_i^2}(\boldsymbol{a};\rho)\right| \leq \sup_{\boldsymbol{a}\in\mathcal{A}(\rho)} \frac{\mu_i^2}{\beta_i^3(\boldsymbol{a};\rho)} \cdot f(0) \to \frac{\mu_i^2}{\sigma_i^3} f(0)\,,$$

as $\rho \to 1$.

Concavity of $\overline{U}$ on a particular geodesic. Fixing $\{c_i\}_{i>1}$ and differentiating $\overline{U}(\boldsymbol{a})$ along the curve $\boldsymbol{a} : \psi \to \boldsymbol{\nu}_1 \cos(\psi) + \sin(\psi) \sum_{i>1} c_i \boldsymbol{\nu}_i$ with respect to ψ gives:

$$\begin{aligned}
\frac{\partial \overline{U}}{\partial \psi} &= \sum_i \frac{\partial \overline{U}}{\partial \beta_i}\Big(a_i(\psi) Q(\boldsymbol{a}(\psi))\Big) \times \left[a_i'(\psi) \times Q(\boldsymbol{a}(\psi)) + a_i(\psi) \times \frac{\partial Q}{\partial \psi}(\boldsymbol{a}(\psi))\right] \\
&= Q(\boldsymbol{a}(\psi)) \sum_i a_i'(\psi) \frac{\partial \overline{U}}{\partial \beta_i}(a_i(\psi) Q(\boldsymbol{a}(\psi))) \\
&\qquad\qquad + \frac{\partial Q}{\partial \psi}(\boldsymbol{a}(\psi)) \sum_i a_i(\psi) \frac{\partial \overline{U}}{\partial \beta_i}(a_i(\psi) Q(\boldsymbol{a}(\psi))).
\end{aligned}$$

Let $h_i(\beta_i) := \frac{\partial \overline{U}}{\partial \beta_i}$. Also abuse the notation to write $Q' := \frac{\partial Q}{\partial \psi}$, $a_i' := \frac{\partial a_i}{\partial \psi}$, and $\beta_i' := \frac{\partial \beta_i}{\partial \psi}$. The above can be written as

$$\frac{\partial \overline{U}}{\partial \psi} = \sum_i h_i(\beta_i)\beta_i' = Q \sum_i a_i' h_i(\beta_i) + Q' \sum_i a_i h_i(\beta_i) \,.$$

Therefore, we also have

$$\frac{\partial^2 \overline{U}}{\partial \psi^2} = \sum_i h_i'(\beta_i)(\beta_i')^2 + \sum_i h_i(\beta_i)\beta_i'' \,,$$

where

$$\beta_i' = a_i' Q + a_i Q' \quad \text{and} \quad \beta_i'' = a_i'' Q + 2a_i' Q' + a_i Q'' \,.$$

Equivalently, we can write

$$\frac{\partial^2 \overline{U}}{\partial \psi^2} = \underbrace{Q'' \sum_i a_i h_i(\beta_i)}_{\mathrm{I}} + \underbrace{2Q' \sum_i a_i' h_i(\beta_i)}_{\mathrm{II}} + \underbrace{Q \sum_i a_i'' h_i(\beta_i)}_{\mathrm{III}} + \underbrace{\sum_i h_i'(\beta_i)(a_i' Q + a_i Q')^2}_{\mathrm{IV}} \,.$$

Note that

$$\boldsymbol{a}'(\psi) = -\boldsymbol{\nu}_1 \sin(\psi) + \cos(\psi) \sum_{i>1} c_i \boldsymbol{\nu}_i \text{ and } \boldsymbol{a}''(\psi) = -\boldsymbol{\nu}_1 \cos(\psi) - \sin(\psi) \sum_{i>1} c_i \boldsymbol{\nu}_i \,,$$

and hence

$$||\boldsymbol{a}'(\psi)|| = ||\boldsymbol{a}''(\psi)|| = 1 \,.$$

Thus, $|a_i'|, |a_i''| \leq 1$ uniformly. Now, we show that the parts II, III, IV in the above expression are uniformly bounded. First, note that

$$\lim_{\rho \to 1} \sup_{\boldsymbol{a} \in \mathcal{A}(\rho),\, \boldsymbol{p} \in \mathcal{P}} |\mathrm{II}(\boldsymbol{a}; \boldsymbol{p}, \rho)| \leq 2 \lim_{\rho \to 1} \Big(\sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} |Q'| \Big) \cdot \Big(\sum_i \sup_{\boldsymbol{a} \in \mathcal{A}(\rho),\, \boldsymbol{p} \in \mathcal{P}} |h_i(\beta_i)| \Big) \leq C_{\mathrm{II}} \,,$$

for some constant C_{II}. Second, note that

$$\lim_{\rho \to 1} \sup_{\boldsymbol{a} \in \mathcal{A}(\rho),\, \boldsymbol{p} \in \mathcal{P}} |\mathrm{III}(\boldsymbol{a}; \boldsymbol{p}, \rho)| \leq \lim_{\rho \to 1} \Big(\sup_{\boldsymbol{a} \in \mathcal{A}(\rho)} |Q| \Big) \cdot \Big(\sum_i \sup_{\boldsymbol{a} \in \mathcal{A}(\rho),\, \boldsymbol{p} \in \mathcal{P}} |h_i(\beta_i)| \Big) \leq C_{\mathrm{III}} \,,$$

for some constant C_{III}. Third, note that

$$\lim_{\rho\to 1}\sup_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}|\mathrm{IV}(\boldsymbol{a};\boldsymbol{p},\rho)| \leq \lim_{\rho\to 1}\sum_i \sup_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}|h_i'(\beta_i)|\Big(\sup_{\boldsymbol{a}\in\mathcal{A}(\rho)}|Q|+\sup_{\boldsymbol{a}\in\mathcal{A}(\rho)}|Q'|\Big)^2 \leq C_{\mathrm{IV}}\,,$$

for some constant C_{IV}. Now, we show that part I in the previous expression diverges to $-\infty$ uniformly:

$$\lim_{\rho\to 1}\sup_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}\mathrm{I}(\boldsymbol{a};\boldsymbol{p},\rho) \leq \lim_{\rho\to 1}\Big(\sup_{\boldsymbol{a}\in\mathcal{A}(\rho)}Q''\Big)\cdot\lim_{\rho\to 1}\sum_i \inf_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}\Big(a_i h_i(\beta_i)\Big)\,,$$

Moreover, as we have shown,

$$\lim_{\rho\to 1}\Big(\sup_{\boldsymbol{a}\in\mathcal{A}(\rho)}Q''\Big) = -\infty\,,$$

and

$$\begin{aligned}\lim_{\rho\to 1}\sum_i \inf_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}\Big(a_i h_i(\beta_i)\Big) &\geq \lim_{\rho\to 1}\sum_i \inf_{\boldsymbol{a}\in\mathcal{A}(\rho)}\Big(a_i\Big)\inf_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}\Big(h_i(\beta_i)\Big)\\ &= \sum_i \hat{\sigma}_i \lim_{\rho\to 1}\inf_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}\Big(h_i(\beta_i)\Big) > 0\,,\end{aligned}$$

where the strict inequality is due to $\hat{\sigma}_i > 0$ and $\inf_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}} h_i(\beta_i) > 0$ for each i, as we have shown. Together, these imply that

$$\lim_{\rho\to 1}\sup_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}\mathrm{I}(\boldsymbol{a};\boldsymbol{p},\rho) = -\infty\,.$$

Fix some $\delta > 0$. By the above arguments, there exists some ρ_0 such that for all $\rho > \rho_0$, we have

$$\sup_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}|\mathrm{II}(\boldsymbol{a};\boldsymbol{p},\rho)|+|\mathrm{III}(\boldsymbol{a};\boldsymbol{p},\rho)|+|\mathrm{IV}(\boldsymbol{a};\boldsymbol{p},\rho)| \leq C_{\mathrm{II}}+C_{\mathrm{III}}+C_{\mathrm{IV}}+\delta\,.$$

Moreover, there also exists some ρ_1 such that for all $\rho > \rho_1$, we have

$$\sup_{\boldsymbol{a}\in\mathcal{A}(\rho),\,\boldsymbol{p}\in\mathcal{P}}\mathrm{I}(\boldsymbol{a};\boldsymbol{p},\rho) < -2\cdot\Big(C_{\mathrm{II}}+C_{\mathrm{III}}+C_{\mathrm{IV}}+\delta\Big)\,.$$

Let $\underline{\rho} := \max\{\rho_0, \rho_1\}$. It follows immediately that for all $\rho > \underline{\rho}$, we have

$$\frac{\partial^2 \overline{U}}{\partial \psi^2}(\boldsymbol{a}; \boldsymbol{p}, \rho) < -\delta\,.$$

for all $\boldsymbol{a} \in \mathcal{A}(\rho)$ and all $\boldsymbol{p} \in \mathcal{P}$.

Generalize to arbitrary geodesics in $\mathcal{A}(\rho)$. We now generalize the above argument to cover any geodesics in $\mathcal{A}(\rho)$. Let $\boldsymbol{a}(\psi)$ be a geodesic in $\mathcal{A}(\rho)$ connecting any two points $\boldsymbol{a}_0, \boldsymbol{a}_1 \in \mathcal{A}(\rho)$, and let P be the 2-D plane containing $\boldsymbol{a}(\psi)$.

Define

$$u_\star \;:=\; \frac{\mathrm{Proj}_P(\boldsymbol{\nu}_1)}{\left\|\mathrm{Proj}_P(\boldsymbol{\nu}_1)\right\|}\,, \qquad w_\star \in P,\;\; \|w_\star\| = 1,\;\; w_\star \perp u_\star.$$

The geodesic can be written as

$$\boldsymbol{a}(\psi) \;=\; \cos\psi\, u_\star + \sin\psi\, w_\star\,.$$

Let $m_{ab} := e_a^\top \Sigma^{-1} e_b$ with $a, b \in \{1, 2\}$, $e_1 = u_\star$, $e_2 = w_\star$. Then

$$q(\psi) := \boldsymbol{a}(\psi)^\top \Sigma^{-1}\, \boldsymbol{a}(\psi) = m_{11}\cos^2\psi + 2m_{12}\sin\psi\cos\psi + m_{22}\sin^2\psi,$$

and hence

$$Q'(\psi) = -\tfrac{1}{2}\, q(\psi)^{-3/2}\, q'(\psi), \qquad Q''(\psi) = -\tfrac{1}{2}\, q(\psi)^{-3/2}\, q''(\psi) + \tfrac{3}{4}\, q(\psi)^{-5/2}\left(q'(\psi)\right)^2.$$

By the same argument as before, we only need to consider $|\sin(2\psi)| \le O(1-\rho)$ and $\cos(2\psi) \ge 1 - O((1-\rho)^2)$. Moreover, as noted before, $\frac{1}{\lambda_1} \to \frac{1}{||\boldsymbol{\sigma}||^2}$, while $\frac{1}{\lambda_i} = \Theta(\frac{1}{1-\rho})$ for all $i > 1$, as $\rho \to 1$. We decompose $u_\star$ and $w_\star$ in the $\{\boldsymbol{\nu}_i\}$ basis:

$$u_\star = c_1\boldsymbol{\nu}_1 + \sum_{i>1} c_i\boldsymbol{\nu}_i \qquad w_\star = \sum_{i>1} d_i\boldsymbol{\nu}_i\,,$$

where we have also used that $w_\star \perp \boldsymbol{\nu}_1$. Note that by construction

$$1 - c_1^2 = \sum_{i>1} c_i^2 = O((1-\rho)^2)\,, \qquad \sum_{i>1} d_i^2 = 1\,.$$

Since $\Sigma^{-1}\boldsymbol{\nu}_i = \frac{1}{\lambda_i}\boldsymbol{\nu}_i$, we have

$$\begin{aligned} m_{11} = u_\star^\top A u_\star = c_1^2 \frac{1}{\lambda_1} + \sum_{i>1} \frac{c_i^2}{\lambda_i} &= c_1^2 \frac{1}{\lambda_1} + O((1-\rho)^2) \cdot \Theta(\frac{1}{1-\rho}) \\ &= \frac{1 - O((1-\rho)^2)}{\lambda_1} + O(1-\rho) \to \frac{1}{||\boldsymbol{\sigma}||^2} \end{aligned}$$

as $\rho \to 1$. Similarly, note that

$$m_{22} = w_\star^\top \Sigma^{-1} w_\star = \sum_{i>1} \frac{d_i^2}{\lambda_i} \geq \frac{1}{\lambda_2} = \Theta(\frac{1}{1-\rho})\,.$$

Moreover,

$$|m_{12}| = |u_\star^\top \Sigma^{-1} w_\star| = \big|\sum_{i>1} \frac{c_i d_i}{\lambda_i}\big| \leq \sqrt{\sum_{i>1} c_i^2}\sqrt{\sum_{i>1} \frac{d_i^2}{\lambda_i^2}} = O(1-\rho) \cdot O(\frac{1}{1-\rho}) = O(1)\,,$$

and hence we can bound $|m_{12}|$ uniformly by some constant.

Now, we start bounding. First,

$$\begin{aligned} q(\psi) &= m_{11}\cos^2\psi + 2m_{12}\sin\psi\cos\psi + m_{22}\sin^2\psi \\ &= m_{11}(1 - O((1-\rho)^2)) + O(1) \cdot O(1-\rho) \cdot (1 - O((1-\rho)^2)) + O(\frac{1}{1-\rho}) \cdot O((1-\rho)^2) \\ &\to \frac{1}{||\boldsymbol{\sigma}||^2} \end{aligned}$$

as $\rho \to 1$ (uniformly over all such planes P). The same uniform convergence holds for $q^{-1/2}$, $q^{-3/2}$ and $q^{-5/2}$.

Second,

$$\begin{aligned} |q'(\psi)| &= \big|\sin(2\psi)(m_{22} - m_{11}) + 2m_{12}\cos(2\psi)\big| \\ &\leq |\sin(2\psi)|(m_{22} + m_{11}) + 2|m_{12}| \end{aligned}$$

$$\leq O(1-\rho)O(\frac{1}{1-\rho}) + O(1-\rho) + O(1) = O(1)\,,$$

as $\rho \to 1$.

Third,

$$\begin{aligned} q''(\psi) &= 2\cos(2\psi)(m_{22}-m_{11}) - 4m_{12}\sin(2\psi) \\ &\geq 2(1-O((1-\rho)^2))(\Theta(\frac{1}{1-\rho}) - O(1)) - O(1)\cdot O(1-\rho) \geq \Theta(\frac{1}{1-\rho})\,, \end{aligned}$$

as $\rho \to 1$.

Let $\Psi(\boldsymbol{a}_0, \boldsymbol{a}_1)$ be the feasible set of ψ such that it stays in $\mathcal{A}(\rho)$. Combining all the above together, we have that

$$\sup_{\boldsymbol{a}_0,\boldsymbol{a}_1\in\mathcal{A}(\rho);\psi\in\Psi(\boldsymbol{a}_0,\boldsymbol{a}_1)} |Q'(\psi;\boldsymbol{a}_0,\boldsymbol{a}_1)| \leq C_1(\rho) \to C_1$$

as $\rho \to 1$, and

$$\sup_{\boldsymbol{a}_0,\boldsymbol{a}_1\in\mathcal{A}(\rho);\psi\in\Psi(\boldsymbol{a}_0,\boldsymbol{a}_1)} Q''(\psi;\boldsymbol{a}_0,\boldsymbol{a}_1) \leq C_2(\rho) \to -\infty$$

as $\rho \to 1$. Recall the decomposition we had:

$$\frac{\partial^2 \overline{U}}{\partial \psi^2} = \underbrace{Q''\sum_i a_i h_i(\beta_i)}_{\mathrm{I}} + \underbrace{2Q'\sum_i a_i' h_i(\beta_i)}_{\mathrm{II}} + \underbrace{Q\sum_i a_i'' h_i(\beta_i)}_{\mathrm{III}} + \underbrace{\sum_i h_i'(\beta_i)(a_i'Q + a_iQ')^2}_{\mathrm{IV}}\,.$$

Note that, as before, we have

$$||\boldsymbol{a}'(\psi)|| = 1\,, \text{ and } ||\boldsymbol{a}''(\psi)|| = 1\,.$$

Moreover, the bounds we had for h_i, h_i', and Q continue to hold here. It follows immediately by our previous argument that parts II, III, and IV are uniformly bounded by a constant for sufficiently high ρ, while part I uniformly diverges to $-\infty$ for sufficiently high ρ. By the same argument as before, for any $\delta > 0$, there exists some $\underline{\rho} < 1$ such that for all $\rho > \underline{\rho}$, we have

$$\frac{\partial^2 \overline{U}}{\partial \psi^2}(\boldsymbol{a}; \boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{p}, \rho) < -\delta$$

for any $\boldsymbol{a}_0, \boldsymbol{a}_1 \in \mathcal{A}(\rho)$, any $\boldsymbol{a}$ in the geodesic that connects $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$, and any $\boldsymbol{p} \in \mathcal{P}$.

Completion of the proof. Now we complete the proof by showing that for any $\rho > \underline{\rho}$, where $\underline{\rho}$ is given in the previous step, for any $\boldsymbol{p} \in \mathcal{P}$, we have that

$$\arg\max_{\boldsymbol{a} \in \mathcal{A}(\rho)} \overline{U}(\boldsymbol{a}; \boldsymbol{p}, \rho)$$

is single-valued. Suppose for contradiction that there exist some $\rho > \underline{\rho}$ and some $\boldsymbol{p} \in \mathcal{P}$ such that

$$|\arg\max_{\boldsymbol{a} \in \mathcal{A}(\rho)} \overline{U}(\boldsymbol{a}; \boldsymbol{p}, \rho)| > 1\,.$$

Fix any two optimizers $\boldsymbol{a}, \hat{\boldsymbol{a}} \in \mathcal{A}(\rho)$. Note that there must be a geodesic connecting $\boldsymbol{a}$ and $\hat{\boldsymbol{a}}$. It follows immediately by the strict concavity proved in the previous step that there exists some $\tilde{\boldsymbol{a}}$ such that

$$\overline{U}(\tilde{\boldsymbol{a}}; \boldsymbol{p}, \rho) > \overline{U}(\boldsymbol{a}; \boldsymbol{p}, \rho)\,,$$

a contradiction.

A.4 Proof of Proposition 3

Let $\rho < 0$. Toward a contradiction, suppose that there exists an equilibrium $(\boldsymbol{\alpha}^*, \mathcal{M}^*)$ in which the seller offers a separate sales mechanism. That is, mechanism $\mathcal{M}^*$ is a menu that offers each good i at price p_i and each bundle $B \subseteq \{1, \ldots, K\}$ at price $\sum_{i \in B} p_i$. Recall that optimal prices always lie below the mean since the virtual value function crosses zero only once at some $t_i^* \leq 0.5$ (Lemma 2). So $p_i \leq \mu_i$ for all i. By Theorem 1, the buyer's learning must be vertical. However, we show that against separate sales, any vertical learning strategy is strictly dominated by a horizontal learning strategy when $\rho < 0$.

Thus, $(\boldsymbol{\alpha}^*, \mathcal{M}^*)$ cannot be an equilibrium.

Rewriting of the buyer's payoff. We follow the same reformulation as in the proof of Lemma 10. Under separate sales, the buyer's payoff is separable across goods:

$$U(\boldsymbol{\alpha}) = \mathbb{E}\left[\sum_i (\theta_i(s;\boldsymbol{\alpha}) - p_i)_+\right].$$

Instead of optimizing over $\boldsymbol{\alpha}$, it is equivalent to optimizing over $\boldsymbol{a} := \Sigma\boldsymbol{\alpha}$. As in the proof of Lemma 10, for any choice $\boldsymbol{a}$, the buyer's expected payoff equals:

$$\overline{U}(\boldsymbol{a}) := \sum_i \mathbb{E}\left[(\theta_i(s;\boldsymbol{\alpha}(\boldsymbol{a})) - p_i)_+\right] = \sum_i \mathbb{E}\left[\left(\mu_i - p_i + \frac{a_i}{\sqrt{\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a}}}\overline{s}\right)_+\right],$$

where

$$\overline{s} := \frac{\boldsymbol{a}^\top\Sigma^{-1}\mathbf{v} - \boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{\mu}}{\sqrt{\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{a}}}$$

is a standardized signal, whose law does not depend on $\boldsymbol{a}$.

Consider the function

$$g_i(l) := \mathbb{E}\left[(\mu_i - p_i + l\overline{s})_+\right].$$

As in the proof of Lemma 10, note that

$$g_i'(l) = \mathbb{E}\big[\overline{s}\mathbb{1}\{\mu_i - p_i + l\overline{s} \geq 0\}\big] = \mathbb{E}\big[\overline{s}\mathbb{1}\{\overline{s} \geq -(\mu_i - p_i)/l\}\big] \geq \mathbb{E}\big[\overline{s}\big] = 0\,,$$

where the last inequality comes from $l > 0$. The inequality is strict whenever $-(\mu_i - p_i)/l$ is strictly greater than the lowest possible realization of $\overline{s}$. Also note that the distribution of $\overline{s}$ is symmetric around zero, and hence the function g_i is even: $g_i(l) = g_i(-l)$ for all l, i.

Moreover, as in the proof of Lemma 10, write

$$\Sigma = \mathrm{diag}(\sigma_i)\, R\, \mathrm{diag}(\sigma_i) \quad \text{where} \quad R = (1-\rho)I + \rho\boldsymbol{\iota}\boldsymbol{\iota}^\top,$$

where ι is the constant $\mathbf{1}$ vector. Letting $\boldsymbol{\omega} := (a_i/\sigma_i)_i$, we have

$$\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a} = \frac{1}{1-\rho} \left(||\boldsymbol{\omega}||^2 - \frac{\rho}{1+\rho(K-1)} (\boldsymbol{\iota}^\top \boldsymbol{\omega})^2 \right) .$$

Note that by construction we also have $1 + \rho(K-1) > 0$.

Vertical learning is dominated. Fix any vertical learning strategy $\boldsymbol{\alpha}$. Let $\boldsymbol{a} = \Sigma\boldsymbol{\alpha}$, so we have $\overline{U}(\boldsymbol{a}) = U(\boldsymbol{\alpha})$. Note that $a_i = \text{Cov}(v_i, \boldsymbol{\alpha}^\top \mathbf{v})$ for all i, by construction. By definition of vertical learning, $\text{Cov}(v_i, \boldsymbol{\alpha}^\top \mathbf{v}) \geq 0$ for all i, and hence $a_i \geq 0$ for all i.

Furthermore, it must be that $a_i > 0$ for some i. If not, then it implies that $\boldsymbol{\alpha} = 0$. However, such a learning strategy cannot be sustained in equilibrium, as we know that information is strictly valuable in any equilibrium (Lemma 8).

Now, we consider two cases.

Case (A): Suppose that there exist at least two goods with $a_j > 0$. We construct an alternative learning strategy $\hat{\boldsymbol{\alpha}}$ that strictly improves over $\boldsymbol{\alpha}$. Pick any good i such that

$$i \in \arg\min \left\{ a_j/\sigma_j : a_j > 0 \right\} .$$

Set $\hat{a}_i = -a_i$, $\hat{\boldsymbol{a}}_{-i} = \boldsymbol{a}_{-i}$. That is, $\hat{\boldsymbol{a}}$ simply flips the sign of a_i but keeps everything else constant. Finally, let $\hat{\boldsymbol{\alpha}} = \Sigma^{-1}\hat{\boldsymbol{a}}$. Note that $U(\hat{\boldsymbol{\alpha}}) = \overline{U}(\hat{\boldsymbol{a}})$.

Using the above notation, let $\boldsymbol{\omega} = (a_j/\sigma_j)_j$ and $\hat{\boldsymbol{\omega}} = (\hat{a}_j/\sigma_j)_j$. By construction:

$$|\hat{a}_j| = |a_j| \quad \forall j \quad \text{and} \quad ||\hat{\boldsymbol{\omega}}|| = ||\boldsymbol{\omega}|| .$$

Furthermore, by construction:

$$0 \leq \boldsymbol{\iota}^\top \hat{\boldsymbol{\omega}} = \sum_j \frac{\hat{a}_j}{\sigma_j} < \sum_j \frac{a_j}{\sigma_j} = \boldsymbol{\iota}^\top \boldsymbol{\omega} .$$

Since $\rho < 0$, this implies:

$$\hat{\boldsymbol{a}}^\top \Sigma^{-1} \hat{\boldsymbol{a}} < \boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a} ,$$

and

$$\frac{|\hat{a}_j|}{\sqrt{\hat{\boldsymbol{a}}^\top \Sigma^{-1} \hat{\boldsymbol{a}}}} \geq \frac{|a_j|}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}} \quad \forall j ,$$

and strictly so for $j = i$. The function g_j is even and increasing over the positive range. Therefore:

$$\mathbb{E}\left[\left(\mu_j - p_j + \frac{\hat{a}_j}{\sqrt{\hat{\boldsymbol{a}}^\top \Sigma^{-1} \hat{\boldsymbol{a}}}}\overline{s}\right)_+\right] \geq \mathbb{E}\left[\left(\mu_j - p_j + \frac{a_j}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}}\overline{s}\right)_+\right] \quad \forall j\,.$$

Furthermore, the inequality holds strictly for $j = i$ as long as

$$\Pr\left(\frac{a_i}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}}\overline{s} < -(\mu_i - p_i)\right) > 0\,.$$

Since $a_i > 0$, $\mathrm{Var}(\theta_i(s;\boldsymbol{\alpha})) > 0$. We know from Lemma 4 that an optimal mechanism cannot allocate good i to *all* buyer-types, that is:

$$\Pr\Big(\theta_i(s;\boldsymbol{\alpha}) < p_i\Big) > 0 \implies \Pr\left(\mu_i + \frac{a_i}{\sqrt{\boldsymbol{a}^\top \Sigma^{-1} \boldsymbol{a}}}\overline{s} < p_i\right) > 0\,.$$

Thus, the inequality holds strictly for good i. Summing over all goods j, we get

$$\overline{U}(\hat{\boldsymbol{a}}) > \overline{U}(\boldsymbol{a}) \implies U(\hat{\boldsymbol{\alpha}}) > U(\boldsymbol{\alpha})\,.$$

Therefore, against the separate sales mechanism $\mathcal{M}^*$, the buyer has a strictly profitable deviation under any vertical learning strategy. But by Theorem 1, every equilibrium must have vertical learning. Thus, $(\boldsymbol{\alpha}^*, \mathcal{M}^*)$ cannot be an equilibrium.

Case (B): Suppose that there exists only one good i such that $a_i > 0$ (and hence $a_j = 0$ for all $j \neq i$). By optimality of $\mathcal{M}^*$, it must be that all equilibrium types t consume goods $j \neq i$ (with prices being μ_j). Moreover, as noted before, there must exist a positive measure of equilibrium types t that do not consume good i. Therefore,

$$U(\boldsymbol{\alpha}^*) = \mathbb{E}\left[\sum_j \left(\theta_j(s;\boldsymbol{\alpha}^*) - p_j\right)_+\right] = \mathbb{E}\left[\Big(\theta_i(s;\boldsymbol{\alpha}^*) - p_i\Big)_+\right].$$

However, consider the learning strategy $\hat{\alpha}_i = 1$ and $\hat{\alpha}_j = 0$ for all $j \neq i$. Since $\rho < 0$, note that this is a horizontal learning strategy (since $a_j < 0$ for all $j \neq i$). Therefore, $\hat{\boldsymbol{\alpha}}$ is a different signal from $\boldsymbol{\alpha}^*$ and induces a strict mean-preserving spread of θ_i. By Lemma 5, we have

$$U(\hat{\boldsymbol{\alpha}}) = \mathbb{E}\left[\sum_j \left(\theta_j(s;\hat{\boldsymbol{\alpha}}) - p_j\right)_+\right] \geq \mathbb{E}\left[\left(\theta_i(s;\hat{\boldsymbol{\alpha}}) - p_i\right)_+\right] > U(\boldsymbol{\alpha}^*)\,,$$

and hence $\hat{\boldsymbol{\alpha}}$ is a strict improvement for the buyer, contradicting $(\boldsymbol{\alpha}^*, \mathcal{M}^*)$ being an equilibrium.

A.5 Proof of Proposition 4

Fix any separate sales mechanism and let p_i be the price of good i. Under a separate sales mechanism, the objective of the buyer is separable across goods. Thus, the buyer's expected utility when using learning strategy $\boldsymbol{\alpha}$ can be written as:

$$\sum_i \mathbb{E}_{\boldsymbol{\theta}\sim G_{\boldsymbol{\alpha}}}\left[\max\{\theta_i - p_i, 0\}\right].$$

Case (A): Uncorrelated values. Take any horizontal learning strategy $\boldsymbol{\alpha}^H$ with $\text{sign}(\alpha_1^H)\cdot\text{sign}(\alpha_2^H) < 0$. We show that the buyer is indifferent between $\boldsymbol{\alpha}^H$ and the vertical learning strategy $\boldsymbol{\alpha}^V = (\alpha_1^H, -\alpha_2^H)$.

Let $s^H = \boldsymbol{\alpha}^H \cdot \mathbf{v}$ and $s^V = \boldsymbol{\alpha}^V \cdot \mathbf{v}$. Note that $\text{Var}(s^H) = \sum_i (\alpha_i^H \sigma_i)^2 = \text{Var}(s^V)$. Thus, once de-meaned, the two signals follow the same elliptical distribution: $s^H - \mathbb{E}(s^H)$ and $s^V - \mathbb{E}(s^V)$ have both mean zero and the same variance. Note that:

$$\theta_1(s^H;\boldsymbol{\alpha}^H) = \mu_1 + \frac{\alpha_1^H\sigma_1^2}{\text{Var}(s^H)}(s^H - \mathbb{E}(s^H))$$

and

$$\theta_1(s^V;\boldsymbol{\alpha}^V) = \mu_1 + \frac{\alpha_1^V\sigma_1^2}{\text{Var}(s^V)}(s^V - \mathbb{E}(s^V)) = \mu_1 + \frac{\alpha_1^H\sigma_1^2}{\text{Var}(s^H)}(s^V - \mathbb{E}(s^V))\,.$$

Thus $\theta_1(s^H;\boldsymbol{\alpha}^H)$ and $\theta_1(s^V;\boldsymbol{\alpha}^V)$ follow the same elliptical distribution and

$$\mathbb{E}_{\boldsymbol{\theta}\sim G_{\boldsymbol{\alpha}^H}}\left[\max\{\theta_1 - p_1, 0\}\right] = \mathbb{E}_{\boldsymbol{\theta}\sim G_{\boldsymbol{\alpha}^V}}\left[\max\{\theta_1 - p_1, 0\}\right].$$

Similarly,

$$\theta_2(s^H; \boldsymbol{\alpha}^H) = \mu_2 + \frac{\alpha_2^H \sigma_2^2}{\text{Var}(s^H)}(s^H - \mathbb{E}(s^H))$$

and

$$\theta_2(s^V; \boldsymbol{\alpha}^V) = \mu_2 + \frac{\alpha_2^V \sigma_2^2}{\text{Var}(s^V)}(s^V - \mathbb{E}(s^V)) = \mu_2 - \frac{\alpha_2^H \sigma_2^2}{\text{Var}(s^H)}(s^V - \mathbb{E}(s^V)) \,.$$

Since the distribution of $(s^V - \mathbb{E}(s^V))$ is symmetric around zero, $\theta_2(s^H; \boldsymbol{\alpha}^H)$ and $\theta_2(s^V; \boldsymbol{\alpha}^V)$ also follow the same elliptical distribution, and

$$\mathbb{E}_{\boldsymbol{\theta} \sim G_{\boldsymbol{\alpha}^H}}\left[\max\{\theta_2 - p_2, 0\}\right] = \mathbb{E}_{\boldsymbol{\theta} \sim G_{\boldsymbol{\alpha}^V}}\left[\max\{\theta_2 - p_2, 0\}\right] .$$

Overall, the horizontal and vertical learning strategies induce the same marginal type distributions (though not the same joint distribution). Under a separate sales mechanism, they then yield the same expected payoff to the buyer.

Case (B): Negatively correlated values. This is the case considered in the proof of Proposition 3. For every vertical learning strategy $\boldsymbol{\alpha}^V$, the proof there shows that there exists an alternative horizontal learning strategy $\boldsymbol{\alpha}^H$ such that the latter induces marginal distributions over θ_1 and θ_2 that are mean-preserving spreads of those induced by $\boldsymbol{\alpha}^V$.

Case (C): Positively correlated values. We can use a similar construction as in the proof of Proposition 3; in particular, see Lemma 10. When $\rho > 0$, the opposite result obtains: For every horizontal learning strategy $\boldsymbol{\alpha}^H$, there exists a vertical learning strategy $\boldsymbol{\alpha}^V$ such that the latter induces marginal distributions over θ_1 and θ_2 that are mean-preserving spreads of those induced by $\boldsymbol{\alpha}^H$.

A.6 Proof of Proposition 5

Assume $\rho = 0$. Take any nested bundling mechanism and, without loss, suppose that good 1 is the base good. Let p_1 be the price of good 1 when sold alone and

$p_{12} \geq p_1$ the price of the bundle $\{1, 2\}$.[31] Take any horizontal learning strategy $\boldsymbol{\alpha}^H$ with $\operatorname{sign}(\alpha_1^H) \cdot \operatorname{sign}(\alpha_2^H) < 0$. We show that the buyer is weakly better off under the vertical learning strategy $\boldsymbol{\alpha}^V = (\alpha_1^H, -\alpha_2^H)$.

Let U_{NB}^H (respectively U_{NB}^V) be the buyer's expected utility when he uses learning strategy $\boldsymbol{\alpha}^H$ (respectively $\boldsymbol{\alpha}^V$) against the above nested bundling mechanism. Similarly, let U_{SS}^H (respectively U_{SS}^V) be the buyer's expected utility when he uses learning strategy $\boldsymbol{\alpha}^H$ (respectively $\boldsymbol{\alpha}^V$) against a separate sales mechanism that sells good 1 at p_1, good 2 at $p_{12} - p_1$, and the bundle at p_{12}. We know from Proposition 4 that $U_{SS}^H = U_{SS}^V$. Thus, $U_{NB}^H \leq U_{NB}^V$ if and only if $U_{SS}^H - U_{NB}^H \geq U_{SS}^V - U_{NB}^V$. We show that the latter inequality always holds.

By construction, $\boldsymbol{\alpha}^H$ and $\boldsymbol{\alpha}^V$ lead to the same distribution over posterior values for good 1. We thus index types by θ_1, and let G denote the distribution of θ_1 under both $\boldsymbol{\alpha}^H$ and $\boldsymbol{\alpha}^V$. We denote by $\theta_2^H(\theta_1)$ and $\theta_2^V(\theta_1)$ the buyer's posterior value for good 2 when his posterior value for good 1 is θ_1, under the horizontal and vertical learning strategies respectively. That is,

$$\theta_2^H(\theta_1) = \mu_2 + \frac{\operatorname{Cov}(v_2, \boldsymbol{\alpha}^H \cdot \mathbf{v})}{\operatorname{Cov}(v_1, \boldsymbol{\alpha}^H \cdot \mathbf{v})}[\theta_1 - \mu_1] = \mu_2 + \frac{\alpha_2^H \sigma_2^2}{\alpha_1^H \sigma_1^2}[\theta_1 - \mu_1]$$

and

$$\theta_2^V(\theta_1) = \mu_2 + \frac{\operatorname{Cov}(v_2, \boldsymbol{\alpha}^V \cdot \mathbf{v})}{\operatorname{Cov}(v_1, \boldsymbol{\alpha}^V \cdot \mathbf{v})}[\theta_1 - \mu_1] = \mu_2 + \frac{\alpha_2^V \sigma_2^2}{\alpha_1^V \sigma_1^2}[\theta_1 - \mu_1].$$

Recall that $\theta_2^H(\,\cdot\,)$ is strictly decreasing since $\boldsymbol{\alpha}^H$ corresponds to horizontal learning. Furthermore, by construction of $\boldsymbol{\alpha}^V$, note that $\theta_2^V(\,\cdot\,)$ is strictly increasing and that $0.5 \times [\theta_2^H(\theta_1) + \theta_2^V(\theta_1)] = \mu_2$.

The only difference between the separate sales and the nested bundling mechanisms is that the former has one additional option: it allows the buyer to buy only good 2 at price $p_{12} - p_1$. Thus, the only buyer types that get a different payoff under these two mechanisms are those that purchase only good 2 when feasible. Let $\Theta_{\{2\}}^H := \{\theta_1 : \theta_1 < p_1, \theta_2^H(\theta_1) \geq p_{12} - p_1\}$ and $\Theta_{\{2\}}^V := \{\theta_1 : \theta_1 <$

[31] If $p_{12} < p_1$ then no buyer type ever prefers buying good 1 only. The uniquely optimal learning strategy is then to learn the value of the bundle, which is vertical.

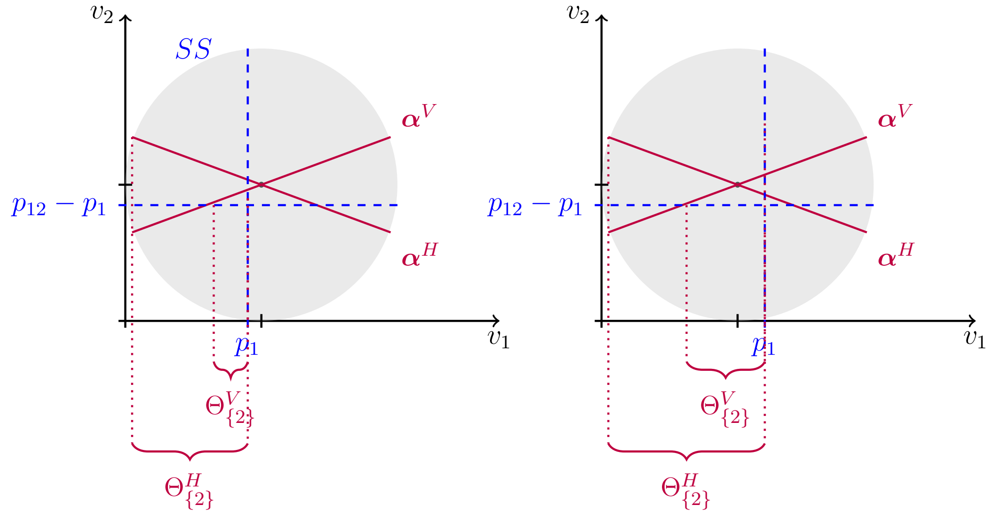

Figure 8: Illustration of the two cases. Case (A) is on the left and Case (B) on the right.

$p_1, \theta_2^V(\theta_1) \geq p_{12} - p_1\}$ denote the set of such types. Then,

$$U^H_{SS} - U^H_{NB} = \int_{\theta_1 \in \Theta^H_{\{2\}}} \underbrace{\left[\theta_2^H(\theta_1) - (p_{12} - p_1) - \max\{0, \theta_1 + \theta_2^H(\theta_1) - p_{12}\}\right]}_{:=\Delta^H(\theta_1)} dG(\theta_1)$$

$$U^V_{SS} - U^V_{NB} = \int_{\theta_1 \in \Theta^V_{\{2\}}} \underbrace{\left[\theta_2^V(\theta_1) - (p_{12} - p_1) - \max\{0, \theta_1 + \theta_2^V(\theta_1) - p_{12}\}\right]}_{:=\Delta^V(\theta_1)} dG(\theta_1)\,,$$

where $\Delta^X(\theta_1) \geq 0$ is the gain in payoff to type $(\theta_1, \theta_2^X(\theta_1))$ from being able to purchase good 2 by itself, for $X \in \{H, V\}$. For any $\theta_1 \notin \Theta^H_{\{2\}}$, there is no gain, and $\Delta^H(\theta_1) = 0$. Similarly, $\Delta^V(\theta_1) = 0$ for any $\theta_1 \notin \Theta^V_{\{2\}}$.

There are two cases: either $\max\{\theta_1 : \theta_1 \in \Theta^V_{\{2\}}\} \leq \mu_1$ or $\max\{\theta_1 : \theta_1 \in \Theta^V_{\{2\}}\} > \mu_1$. The first case is more straightforward so we start with that one.

Case (A): If $\max\{\theta_1 : \theta_1 \in \Theta^V_{\{2\}}\} \leq \mu_1$, then $\theta_2^V(\theta_1) \leq \theta_2^H(\theta_1)$ for all $\theta_1 \in \Theta^V_{\{2\}}$. Then, for any $\theta_1 \in \Theta^V_{\{2\}}$, the following holds:

$$\begin{aligned}
&\Delta^H(\theta_1) - \Delta^V(\theta_1) \\
&\qquad = \theta_2^H(\theta_1) - \theta_2^V(\theta_1) - \max\{0, \theta_1 + \theta_2^H(\theta_1) - p_{12}\} + \max\{0, \theta_1 + \theta_2^V(\theta_1) - p_{12}\}
\end{aligned}$$

$$\geq 0\,.$$

Furthermore, $\Theta^V_{\{2\}} \subseteq \Theta^H_{\{2\}}$. Thus, $U^H_{SS} - U^H_{NB} \geq U^V_{SS} - U^V_{NB}$.

Case (B): If $\max\{\theta_1 : \theta_1 \in \Theta^V_{\{2\}}\} > \mu_1$, then it is no longer the case that $\theta^V_2(\theta_1) \leq \theta^H_2(\theta_1)$ for all $\theta_1 \in \Theta^V_{\{2\}}$, and the above argument does not hold. We can however leverage the symmetry around the mean of the type distribution: type $\theta_1 > \mu_1$ has the same probability as type $2\mu_1 - \theta_1$, and, by construction, $\theta^H_2(2\mu_1 - \theta_1) = \theta^V_2(\theta_1)$. Thus, if $\theta_1 \in \Theta^V_{\{2\}}$ and $\theta_1 > \mu_1$, then $2\mu_1 - \theta_1 \in \Theta^H_{\{2\}}$. We show that $\Delta^V(\theta_1) + \Delta^V(2\mu_1 - \theta_1) \leq \Delta^H(\theta_1) + \Delta^H(2\mu_1 - \theta_1)$ for any $\theta_1 \in \Theta^V_{\{2\}}$, $\theta_1 > \mu_1$.

Take any $\theta_1 \in \Theta^V_{\{2\}}$, $\theta_1 > \mu_1$. If $\theta_1 \in \Theta^H_{\{2\}}$ then $2\mu_1 - \theta_1 \in \Theta^V_{\{2\}}$, and:

$$\begin{aligned}
&\Delta^H(\theta_1) - \Delta^V(\theta_1)\\
&\quad = \theta^H_2(\theta_1) - \theta^V_2(\theta_1) - \max\{0, \theta_1 + \theta^H_2(\theta_1) - p_{12}\} + \max\{0, \theta_1 + \theta^V_2(\theta_1) - p_{12}\}\\
&\quad = \theta^V_2(2\mu_1 - \theta_1) - \theta^H_2(2\mu_1 - \theta_1)\\
&\qquad - \max\{0, \theta_1 + \theta^V_2(2\mu_1 - \theta_1) - p_{12}\} + \max\{0, \theta_1 + \theta^H_2(2\mu_1 - \theta_1) - p_{12}\}\\
&\quad = \Delta^V(2\mu_1 - \theta_1) - \Delta^H(2\mu_1 - \theta_1)\\
&\qquad + \max\{0, 2\mu_1 - \theta_1 + \theta^V_2(2\mu_1 - \theta_1) - p_{12}\} - \max\{0, 2\mu_1 - \theta_1 + \theta^H_2(2\mu_1 - \theta_1) - p_{12}\}\\
&\qquad - \max\{0, \theta_1 + \theta^V_2(2\mu_1 - \theta_1) - p_{12}\} + \max\{0, \theta_1 + \theta^H_2(2\mu_1 - \theta_1) - p_{12}\}\\
&\quad \geq \Delta^V(2\mu_1 - \theta_1) - \Delta^H(2\mu_1 - \theta_1)\,.
\end{aligned}$$

If $\theta_1 \notin \Theta^H_{\{2\}}$, it must be that $\theta^H_2(\theta_1) < p_{12} - p_1$. But then, $\theta^V_2(2\mu_1 - \theta_1) < p_{12} - p_1$, such that $2\mu_1 - \theta_1 \notin \Theta^V_{\{2\}}$, and:

$$\begin{aligned}
\Delta^H(\theta_1) - \Delta^V(\theta_1) &= -\Delta^V(\theta_1) = -\theta^V_2(\theta_1) + p_{12} - p_1 + \max\{0, \theta_1 + \theta^V_2(\theta_1) - p_{12}\}\\
&= -\theta^H_2(2\mu_1 - \theta_1) + p_{12} - p_1 + \max\{0, \theta_1 + \theta^H_2(2\mu_1 - \theta_1) - p_{12}\}\\
&= -\Delta^H(2\mu_1 - \theta_1) - \max\{0, 2\mu_1 - \theta_1 + \theta^H_2(2\mu_1 - \theta_1) - p_{12}\}\\
&\qquad\qquad + \max\{0, \theta_1 + \theta^H_2(2\mu_1 - \theta_1) - p_{12}\}\\
&\geq -\Delta^H(2\mu_1 - \theta_1) = \Delta^V(2\mu_1 - \theta_1) - \Delta^H(2\mu_1 - \theta_1)\,.
\end{aligned}$$

Thus $\Delta^V(\theta_1) + \Delta^V(2\mu_1 - \theta_1) \leq \Delta^H(\theta_1) + \Delta^H(2\mu_1 - \theta_1)$ for any $\theta_1 \in \Theta^V_{\{2\}}$, $\theta_1 > \mu_1$. Combined with $\Delta^H(\theta_1) \geq \Delta^V(\theta_1)$ for any $\theta_1 \in \Theta^V_{\{2\}}$, $\theta_1 \leq \mu_1$, this implies

$U^H_{SS} - U^H_{NB} \geq U^V_{SS} - U^V_{NB}$.

A.7 Proof of Proposition 6

The proof is very similar to that of Proposition 5.

Assume $\rho = 0$. Take any mechanism $\mathcal{M}$ that only sells one good and let p_1 be the price of good 1 and p_2 the price of good 2. So $\mathcal{M}$ can be written as a menu $\{(\varnothing, 0), (\{1\}, p_1), (\{2\}, p_2)\}$. Take any strictly vertical learning strategy $\boldsymbol{\alpha}^V$ with $\text{sign}(\alpha^V_1) \cdot \text{sign}(\alpha^V_2) > 0$. We show that the buyer is weakly better off under the horizontal learning strategy $\boldsymbol{\alpha}^H = (\alpha^V_1, -\alpha^V_2)$.

Let U^H_M (respectively U^V_M) be the buyer's expected utility when he uses learning strategy $\boldsymbol{\alpha}^H$ (respectively $\boldsymbol{\alpha}^V$) against the mechanism $\mathcal{M}$. Similarly, let U^H_{SS} (respectively U^V_{SS}) be the buyer's expected utility when he uses learning strategy $\boldsymbol{\alpha}^H$ (respectively $\boldsymbol{\alpha}^V$) against a separate sales mechanism that sells good 1 at p_1, good 2 at p_2, and the bundle at $p_1 + p_2$. We know from Proposition 4 that $U^H_{SS} = U^V_{SS}$. Thus, $U^V_{NB} \leq U^H_{NB}$ if and only if $U^V_{SS} - U^V_M \geq U^H_{SS} - U^H_M$. We show that the latter inequality always holds.

By construction, $\boldsymbol{\alpha}^H$ and $\boldsymbol{\alpha}^V$ lead to the same distribution over posterior values for good 1. We thus index types by θ_1, and let G denote the distribution of θ_1 under both $\boldsymbol{\alpha}^H$ and $\boldsymbol{\alpha}^V$. We denote by $\theta^H_2(\theta_1)$ and $\theta^V_2(\theta_1)$ the buyer's posterior value for good 2 when his posterior value for good 1 is θ_1, under the horizontal and vertical learning strategy respectively. That is,

$$\theta^H_2(\theta_1) = \mu_2 + \frac{\alpha^H_2 \sigma^2_2}{\alpha^H_1 \sigma^2_1}[\theta_1 - \mu_1] \quad \text{and} \quad \theta^V_2(\theta_1) = \mu_2 + \frac{\alpha^V_2 \sigma^2_2}{\alpha^V_1 \sigma^2_1}[\theta_1 - \mu_1].$$

Recall that $\theta^H_2(\,\cdot\,)$ is strictly decreasing since $\boldsymbol{\alpha}^H$ corresponds to horizontal learning. Furthermore, by construction of $\boldsymbol{\alpha}^V$, note that $\theta^V_2(\,\cdot\,)$ is strictly increasing and that $0.5 \times [\theta^H_2(\theta_1) + \theta^V_2(\theta_1)] = \mu_2$.

The only difference between the separate sales mechanism and mechanism $\mathcal{M}$ is that the former has one additional option: it allows the buyer to buy the grand bundle at price $p_1 + p_2$. Thus, the only buyer types that get a different payoff under these two mechanisms are those that purchase both goods when feasible. Let $\Theta^H_{\{1,2\}} := \{\theta_1 : \theta_1 \geq p_1, \theta^H_2(\theta_1) \geq p_2\}$ and $\Theta^V_{\{1,2\}} := \{\theta_1 : \theta_1 \geq$

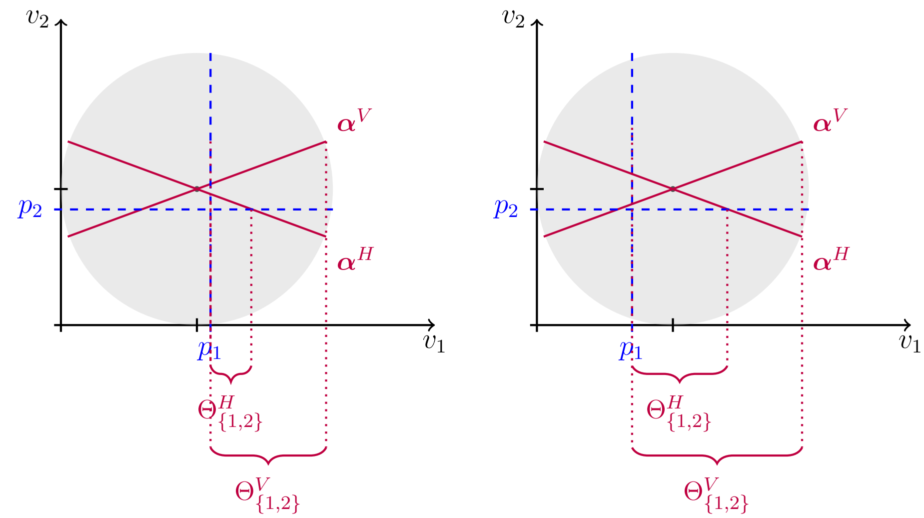

Figure 9: Illustration of the two cases. Case (A) is on the left and Case (B) on the right.

$p_1, \theta_2^V(\theta_1) \geq p_2\}$ denote the set of such types. Then,

$$U^H_{SS} - U^H_M = \int_{\theta_1 \in \Theta^H_{\{1,2\}}} \underbrace{\left[\theta_1 + \theta_2^H(\theta_1) - (p_1 + p_2) - \max\{\theta_1 - p_1, \theta_2^H(\theta_1) - p_2\}\right]}_{:=\Delta^H(\theta_1)} dG(\theta_1)$$

$$U^V_{SS} - U^V_M = \int_{\theta_1 \in \Theta^V_{\{1,2\}}} \underbrace{\left[\theta_1 + \theta_2^V(\theta_1) - (p_1 + p_2) - \max\{\theta_1 - p_1, \theta_2^V(\theta_1) - p_2\}\right]}_{:=\Delta^V(\theta_1)} dG(\theta_1)\,,$$

where $\Delta^X(\theta_1) \geq 0$ is the gain in payoff to type $(\theta_1, \theta_2^X(\theta_1))$ from being able to purchase both goods, for $X \in \{H, V\}$. For any $\theta_1 \notin \Theta^H_{\{1,2\}}$, there is no gain, and $\Delta^H(\theta_1) = 0$. Similarly, $\Delta^V(\theta_1) = 0$ for any $\theta_1 \notin \Theta^V_{\{1,2\}}$.

There are two cases: either $\min\{\theta_1 : \theta_1 \in \Theta^H_{\{1,2\}}\} \geq \mu_1$ or $\min\{\theta_1 : \theta_1 \in \Theta^H_{\{1,2\}}\} < \mu_1$. The first case is more straightforward so we start with that one.

Case (A): If $\min\{\theta_1 : \theta_1 \in \Theta^H_{\{1,2\}}\} \geq \mu_1$, then $\theta_2^V(\theta_1) \geq \theta_2^H(\theta_1)$ for all $\theta_1 \in \Theta^H_{\{1,2\}}$. Then, for any $\theta_1 \in \Theta^H_{\{1,2\}}$, the following holds:

$$\begin{aligned}&\Delta^V(\theta_1) - \Delta^H(\theta_1) \\ &\qquad = \theta_2^V(\theta_1) - \theta_2^H(\theta_1) - \max\{\theta_1 - p_1, \theta_2^V(\theta_1) - p_2\} + \max\{\theta_1 - p_1, \theta_2^H(\theta_1) - p_2\}\end{aligned}$$

$$\geq 0\,.$$

Furthermore, $\Theta^H_{\{1,2\}} \subseteq \Theta^V_{\{1,2\}}$. Thus, $U^V_{SS} - U^V_M \geq U^H_{SS} - U^H_M$.

Case (B): If $\min\{\theta_1 : \theta_1 \in \Theta^H_{\{1,2\}}\} < \mu_1$, then it is no longer the case that $\theta^V_2(\theta_1) \geq \theta^H_2(\theta_1)$ for all $\theta_1 \in \Theta^H_{\{1,2\}}$, and the above argument does not hold. We can however leverage the symmetry around the mean of the type distribution: type $\theta_1 < \mu_1$ has the same probability as type $2\mu_1 - \theta_1$, and, by construction, $\theta^H_2(2\mu_1 - \theta_1) = \theta^V_2(\theta_1)$. Thus, if $\theta_1 \in \Theta^H_{\{1,2\}}$ and $\theta_1 < \mu_1$, then $2\mu_1 - \theta_1 \in \Theta^V_{\{1,2\}}$. We show that $\Delta^H(\theta_1) + \Delta^H(2\mu_1 - \theta_1) \leq \Delta^V(\theta_1) + \Delta^V(2\mu_1 - \theta_1)$ for any $\theta_1 \in \Theta^H_{\{1,2\}}$, $\theta_1 < \mu_1$.

Take any $\theta_1 \in \Theta^H_{\{1,2\}}$, $\theta_1 < \mu_1$. If $\theta_1 \in \Theta^V_{\{1,2\}}$ then $2\mu_1 - \theta_1 \in \Theta^H_{\{1,2\}}$, and:

$$\begin{aligned}
&\Delta^V(\theta_1) - \Delta^H(\theta_1)\\
&\quad = \theta^V_2(\theta_1) - \theta^H_2(\theta_1) - \max\{\theta_1 - p_1, \theta^V_2(\theta_1) - p_2\} + \max\{\theta_1 - p_1, \theta^H_2(\theta_1) - p_2\}\\
&\quad = \theta^H_2(2\mu_1 - \theta_1) - \theta^V_2(2\mu_1 - \theta_1)\\
&\qquad - \max\{\theta_1 - p_1, \theta^H_2(2\mu_1 - \theta_1) - p_2\} + \max\{\theta_1 - p_1, \theta^V_2(2\mu_1 - \theta_1) - p_2\}\\
&\quad = \Delta^H(2\mu_1 - \theta_1) - \Delta^V(2\mu_1 - \theta_1)\\
&\qquad + \max\{2\mu_1 - \theta_1 - p_1, \theta^H_2(2\mu_1 - \theta_1) - p_2\} - \max\{2\mu_1 - \theta_1 - p_1, \theta^V_2(2\mu_1 - \theta_1) - p_2\}\\
&\qquad - \max\{\theta_1 - p_1, \theta^H_2(2\mu_1 - \theta_1) - p_2\} + \max\{\theta_1 - p_1, \theta^V_2(2\mu_1 - \theta_1) - p_2\}\\
&\quad \geq \Delta^H(2\mu_1 - \theta_1) - \Delta^V(2\mu_1 - \theta_1)\,.
\end{aligned}$$

If $\theta_1 \notin \Theta^V_{\{1,2\}}$, it must be that $\theta^V_2(\theta_1) < p_2$. But then, $\theta^H_2(2\mu_1 - \theta_1) < p_2$, such that $2\mu_1 - \theta_1 \notin \Theta^H_{\{1,2\}}$, and:

$$\begin{aligned}
&\Delta^V(\theta_1) - \Delta^H(\theta_1) = -\Delta^H(\theta_1) = -\theta_1 - \theta^H_2(\theta_1) + p_1 + p_2 + \max\{\theta_1 - p_1, \theta^H_2(\theta_1) - p_2\}\\
&\quad = -\theta_1 - \theta^V_2(2\mu_1 - \theta_1) + p_1 + p_2 + \max\{\theta_1 - p_1, \theta^V_2(2\mu_1 - \theta_1) - p_2\}\\
&\quad = -\Delta^V(2\mu_1 - \theta_1) + 2\mu_1 - 2\theta_1 - \max\{2\mu_1 - \theta_1 - p_1, \theta^V_2(2\mu_1 - \theta_1) - p_2\}\\
&\qquad\qquad + \max\{\theta_1 - p_1, \theta^V_2(2\mu_1 - \theta_1) - p_2\}\\
&\quad = -\Delta^V(2\mu_1 - \theta_1) + \min\{p_1 - \theta_1, p_2 - \theta^V_2(2\mu_1 - \theta_1) + 2\mu_1 - 2\theta_1\}\\
&\qquad\qquad - \min\{p_1 - \theta_1, p_2 - \theta^V_2(2\mu_1 - \theta_1)\}\\
&\quad \geq -\Delta^V(2\mu_1 - \theta_1) = \Delta^H(2\mu_1 - \theta_1) - \Delta^V(2\mu_1 - \theta_1)\,.
\end{aligned}$$

Thus $\Delta^H(\theta_1)+\Delta^H(2\mu_1-\theta_1) \leq \Delta^V(\theta_1)+\Delta^V(2\mu_1-\theta_1)$ for any $\theta_1 \in \Theta^H_{\{1,2\}}$, $\theta_1 < \mu_1$. Combined with $\Delta^V(\theta_1) \geq \Delta^H(\theta_1)$ for any $\theta_1 \in \Theta^H_{\{1,2\}}$, $\theta_1 \geq \mu_1$, this implies $U^V_{SS} - U^V_M \geq U^H_{SS} - U^H_M$.

A.8 Proof of Proposition 7

We use $B \in \big\{\{1\},\{2\},\{1,2\}\big\}$ to denote a non-empty bundle. The allocation probabilities are given by x_B, which must satisfy $\sum_B x_B \leq 1$.

Given a learning strategy $\boldsymbol{\alpha}$, the buyer's posterior means over these bundles are given by $a_B t + b_B$, where $t \in [0,1]$, and

$$a_{\{1,2\}} = \gamma\big(a_1 + a_2\big)\,,\; b_{\{1,2\}} = \gamma\big(b_1 + b_2\big)\,.$$

Since $\gamma(v_1+v_2) \geq \max\{v_1, v_2\}$, we have that for any $t \in [0,1]$,

$$a_{\{1,2\}}t + b_{\{1,2\}} \geq \max\Big\{a_1 t + b_1\,,\, a_2 t + b_2\Big\}\,.$$

Moreover, note that under any horizontal learning strategy, the above inequality must be strict for all $t \in [0,1]$ which implies, in particular, $b_{\{1,2\}} > \max\{b_1, b_2\}$.

The proof proceeds in the same way as the proof of Theorem 1. Suppose for contradiction that there exists an equilibrium where the buyer uses a horizontal learning strategy. We first derive the properties of the optimal mechanism and then construct a deviation by the buyer.

Optimal Mechanism. As before, under horizontal learning, it must be that one good has strictly positive sign and one good has strictly negative sign. We follow the same sign convention as before:

$$a_1 + a_2 \geq 0\,.$$

Note that in equilibrium, it cannot be that $a_1 + a_2 = 0$, because if so, by the logic in the introduction, the seller's mechanism would be to offer the grand bundle which extracts the full surplus of the buyer, but then the buyer would deviate to learn about $v_1 + v_2$, leading to a contradiction.

Thus, suppose that $a_1 + a_2 > 0$, which implies that

$$a_{\{1,2\}} = \gamma\big(a_1 + a_2\big) > 0\,.$$

Now, consider the following auxiliary problem:

$$\max_{\mathbf{x}\in[0,1]^3;\,\sum_B x_B \le 1} \sum_B b_B x_B \quad \text{subject to} \quad \sum_B a_B x_B = 0\,. \tag{7}$$

By strong duality, let λ be an optimal dual multiplier on the equality constraint in (7). We claim that $\lambda < 0$. Indeed, by strong duality, we know that every optimal solution to (7) must solve the following problem

$$\max_{\mathbf{x}\in[0,1]^3;\,\sum_B x_B \le 1} \sum_B \big(a_B\lambda + b_B\big)x_B\,,$$

and also satisfy the equality constraint. However, if $\lambda \geq 0$, since $b_{\{1,2\}} > \max\{b_1, b_2\}$, then for the negative good i,

$$a_{\{1,2\}}\lambda + b_{\{1,2\}} \geq b_{\{1,2\}} > b_i \geq a_i\lambda + b_i\,.$$

Thus, every optimal solution to the dualized problem must assign zero probability to the negative good. But then every optimal solution to the dualized problem must violate the equality constraint $\sum_B a_B x_B = 0$, which is impossible by strong duality.

Now, since $\lambda < 0$, we claim that every optimal solution to the dualized problem must assign zero probability to the positive good j. Clearly, this would be the case if

$$\lambda a_j + b_j < 0\,,$$

and hence suppose otherwise. Now, if $\gamma \leq 1$, then

$$a_{\{1,2\}}\lambda + b_{\{1,2\}} - \big(a_j\lambda + b_j\big) = \big(a_{\{1,2\}} - a_j\big)\lambda + \big(b_{\{1,2\}} - b_j\big) > 0\,,$$

since $b_{\{1,2\}} - b_j > 0$ and

$$a_{\{1,2\}} - a_j = \gamma a_i + \gamma a_j - a_j = \gamma a_i - (1-\gamma)a_j \leq 0\,.$$

If $\gamma > 1$, then we also have that

$$a_{\{1,2\}}\lambda + b_{\{1,2\}} - \big(a_j\lambda + b_j\big) \geq (a_1 + a_2)\lambda + (b_1 + b_2) - \big(a_j\lambda + b_j\big) = a_i\lambda + b_i > 0\,,$$

where the first inequality is due to that

$$(a_1 + a_2)\lambda + (b_1 + b_2) = \underbrace{a_j\lambda + b_j}_{\geq 0} + \underbrace{a_i\lambda + b_i}_{>0} > 0\,.$$

It follows immediately that every optimal solution to the dualized problem cannot assign positive probability to the positive good j. However, in order to satisfy the equality constraint, this implies that every optimal solution must assign a strictly positive probability on $\{1, 2\}$ and a strictly positive probability on the negative good i. It follows that we must have

$$a_{\{1,2\}}\lambda + b_{\{1,2\}} = a_i\lambda + b_i \geq 0\,,$$

for the optimal dual multiplier λ.

Following the notation in Section 4, by the proof of Theorem 1, we know that there exists some $t_0^* > 0$ such that[32]

$$0 > \lambda = \overline{\Phi}(t_0^*; t_0^*)\,.$$

Moreover, the ironing interval $\mathcal{I}$ that includes t_0^* must also include 0. We claim that there exists some x^* such that (x^*, t_0^*) forms a saddle point:

$$\max_{x \in \text{MON}} \min_{t_0 \in [0,1]} \mathbb{E}\left[\sum_B \Big(a_B x_B(t)\Phi(t;t_0) + b_B x_B(t)\Big)\right]$$
$$= \min_{t_0 \in [0,1]} \max_{x \in \text{MON}} \mathbb{E}\left[\sum_B \Big(a_B x_B(t)\Phi(t;t_0) + b_B x_B(t)\Big)\right].$$

[32]In particular, note that by the same reasoning as before, $a_{\{1,2\}}\overline{\Phi}(0;0) + b_{\{1,2\}}$ must be strictly negative, and hence $\overline{\Phi}(0;0) < \lambda$.

where

$$\text{MON} := \left\{ x : [0,1] \to [0,1]^3 \text{ s.t. } \sum_B a_B x_B(t) \text{ is nondecreasing in } t \text{ and } \sum_B x_B(t) \le 1 \right\}.$$

By the proof of Theorem 1, it suffices to show that there exists some x^* such that *(i)* it maximizes the following ironed objective pointwise:

$$\mathbb{E}\left[\sum_B \Big(a_B x_B(t) \overline{\Phi}(t; t_0^*) + b_B x_B(t) \Big) \right],$$

and *(ii)* it is consistent with ironing, and *(iii)* t_0^* is a worst-off type under the induced mechanism.

For any $t \in \mathcal{I}$, note that the pointwise maximization problem is

$$\max_{x \in [0,1]^3;\, \sum_B x_B \le 1} \sum_B a_B x_B \overline{\Phi}(t; t_0^*) + b_B x_B = \max_{x \in [0,1]^3;\, \sum_B x_B \le 1} \sum_B \big(a_B \lambda + b_B\big) x_B \,.$$

By construction, there exists some solution $x^\dagger$ such that

$$\sum_B a_B x_B^\dagger = 0\,.$$

Now, for any $t \notin \mathcal{I}$, note that the pointwise maximization problem is

$$\max_{x \in [0,1]^3;\, \sum_B x_B \le 1} \sum_B a_B x_B \tilde{\lambda}_t + b_B x_B\,,$$

where

$$1 \ge \tilde{\lambda}_t := \overline{\Phi}(t; t_0^*) > \lambda\,.$$

Note that for the negative good i, we have

$$a_{\{1,2\}} \tilde{\lambda}_t + b_{\{1,2\}} > a_{\{1,2\}} \lambda + b_{\{1,2\}} = a_i \lambda + b_i > a_i \tilde{\lambda}_t + b_i\,,$$

and hence its ironed virtual value function is everywhere strictly dominated by that of the bundle $\{1,2\}$ for all types $t \notin \mathcal{I}$. Now, for the positive good j, note that if $\gamma \leq 1$, then since

$$a_{\{1,2\}} + b_{\{1,2\}} > a_j + b_j \,,$$

which is equivalent to

$$(a_{\{1,2\}} - a_j) + b_{\{1,2\}} - b_j > 0 \,,$$

we must have

$$\tilde{\lambda}_t \underbrace{(a_{\{1,2\}} - a_j)}_{=\gamma a_i - (1-\gamma) a_j \leq 0} + b_{\{1,2\}} - b_j \geq (a_{\{1,2\}} - a_j) + b_{\{1,2\}} - b_j > 0 \,,$$

and thus

$$\tilde{\lambda}_t a_{\{1,2\}} + b_{\{1,2\}} > \tilde{\lambda}_t a_j + b_j \,.$$

Moreover, if $\gamma > 1$, we also have that

$$\tilde{\lambda}_t a_{\{1,2\}} + b_{\{1,2\}} = \gamma\big(\tilde{\lambda}_t a_i + b_i + \tilde{\lambda}_t a_j + b_j\big) > \tilde{\lambda}_t a_i + b_i + \tilde{\lambda}_t a_j + b_j \geq \tilde{\lambda}_t a_j + b_j \,,$$

where the first inequality is due to that

$$\tilde{\lambda}_t a_{\{1,2\}} + b_{\{1,2\}} > \lambda a_{\{1,2\}} + b_{\{1,2\}} \geq 0 \,,$$

and the second inequality is due to that

$$\tilde{\lambda}_t a_i + b_i \geq a_i + b_i \geq 0 \,.$$

Therefore, combining these two cases, we have that the virtual value function of the positive good j is also everywhere strictly dominated by that of the bundle $\{1,2\}$ for all types $t \notin \mathcal{I}$. It follows immediately that for all types $t \notin \mathcal{I}$, the pointwise maximization has a unique solution that puts full probability on $\{1,2\}$.

Now, we construct x^* as follows. Let $x^*(t) = x^\dagger$ for all $t \in \mathcal{I}$, and let x^* put full probability on $\{1,2\}$ for all $t \notin \mathcal{I}$. By the above arguments, clearly the constructed x^* satisfies properties *(i)* and *(ii)*. It remains to show that t_0^* is a worst-off type under x^* (and the payment rule induced by the Envelope theorem). However, this is immediate by construction. Together, these certify that (x^*, t_0^*) is a saddle point.

Now, for every optimal mechanism by the seller, it must induce some optimal allocation rule x' such that (x', t_0^*) form a saddle point (by the rectangular property of saddle points). As in the proof of Theorem 1, it follows immediately that every optimal mechanism by the seller must maximize the ironed objective pointwise in such a way that it is consistent with the ironing interval $\mathcal{I}$ and induces t_0^* as a worst-off type. These two features together imply that every optimal mechanism must assign every type $t \in \mathcal{I}$ some allocation $x(t)$ that solves the auxiliary problem. Thus, for all types $t \in \mathcal{I}$, we have that the buyer's indirect utility $U(t) = 0$. For all $t \notin \mathcal{I}$, by the previous arguments, the pointwise solution is unique, and hence every optimal mechanism must assign full probability to the bundle $\{1, 2\}$ for all $t \notin \mathcal{I}$.

Optimal Learning. By the previous part, we know that in the equilibrium, the seller must be offering $\{1, 2\}$ with full probability, and any other option in the menu consumed by some type must yield $U(t) = 0$ to all types t. By the same argument in the proof of Theorem 1, it must be that in this equilibrium, the buyer learns about $v_1 + v_2$, but that would lead to a comonotonic distribution of θ_1 and θ_2 under uncorrelated values—hence, a contradiction.

Nested Bundling. We claim that given vertical learning, there exists a unique optimal direct-revelation mechanism (up to measure zero) that is deterministic and can be represented by a nested menu.

To prove the claim, one can verify the conditions in Yang (2025). For completeness, we prove the claim directly. Under vertical learning, the posterior mean distribution can be written as: for each B,

$$\theta_B = a_B t + b_B$$

where $a_{\{1,2\}} = \gamma(a_1 + a_2)$, $b_{\{1,2\}} = \gamma(b_1 + b_2)$, and $a_1 \geq 0$, $a_2 \geq 0$, $b_1 \geq 0$, $b_2 \geq 0$. The claim is easy to see if $a_i = 0$ for some good i, since then good i must be sold to all types. Thus, suppose $a_i > 0$ for all goods i. Moreover, the claim is also easy to see if $b_1 + b_2 = 0$, since then $b_1 = b_2 = 0$, and pure bundling is optimal. Thus, suppose $b_1 + b_2 > 0$.

Without loss of generality, suppose that

$$\frac{b_2}{a_2} \geq \frac{b_1}{a_1}\,.$$

It follows that

$$\frac{b_2}{a_2} \geq \frac{b_{\{1,2\}}}{a_{\{1,2\}}} = \frac{b_1 + b_2}{a_1 + a_2} \geq \frac{b_1}{a_1}\,.$$

We make two observations. First, note that

$$\gamma(b_1 + b_2) \geq b_2$$

and hence

$$\gamma \geq \frac{b_2}{b_1 + b_2} \geq \frac{a_2}{a_1 + a_2}\,.$$

Note that if

$$\gamma = \frac{a_2}{a_1 + a_2}\,,$$

then

$$\frac{b_2}{a_2} = \frac{b_1}{a_1} = \frac{b_1 + b_2}{a_1 + a_2}\,,$$

and $a_{\{1,2\}} = a_2\,, b_{\{1,2\}} = b_2$, in which case it is easy to see that pure bundling is also optimal. Thus, suppose $\gamma > \frac{a_2}{a_1+a_2}$. Then, $(a_{\{1,2\}}t + b_{\{1,2\}}) - (a_2 t + b_2)$ is strictly increasing in t. Moreover, by Lemma 2, the virtual value function induced by $(a_{\{1,2\}}t + b_{\{1,2\}}) - (a_2 t + b_2)$ is strictly single-crossing.

Second, consider good 1 and any t such that

$$a_1 \Phi(t) + b_1 > 0\,.$$

Then

$$\begin{aligned} a_1 t + b_1 - \frac{1 - F(t)}{f(t)} a_1 &= \big(a_1 t + b_1\big)\big(1 - \frac{1 - F(t)}{f(t)} \frac{a_1}{a_1 t + b_1}\big) \\ &< \big(a_{\{1,2\}} t + b_{\{1,2\}}\big)\big(1 - \frac{1 - F(t)}{f(t)} \frac{a_{\{1,2\}}}{a_{\{1,2\}} t + b_{\{1,2\}}}\big) \end{aligned}$$

$$= a_{\{1,2\}}\Phi(t) + b_{\{1,2\}}\,.$$

As in the proof of Theorem 1, consider the following relaxed problem:

$$\max_{x:[0,1]\to[0,1]^3\,;\sum_B x_B \leq 1} \mathbb{E}\left[\sum_B (a_B\Phi(t) + b_B)x_B(t)\right],$$

where we maximize pointwise the unironed objective. By Lemma 2, we have that for all B, $a_B\Phi(t) + b_B$ is strictly single-crossing. By our second observation, every optimal solution must assign good 1 with probability 0 (almost everywhere). Moreover, the crossing point t_2^* of $a_2\Phi(t) + b_2$ is strictly less than the crossing point $t^*_{\{1,2\}}$ of $a_{\{1,2\}}\Phi(t) + b_{\{1,2\}}$, which implies that $a_{\{1,2\}}\Phi(t) + b_{\{1,2\}}$ single-crosses $a_2\Phi(t) + b_2$ from below at some point $t^\dagger > t^*_{\{1,2\}} > t^*_2$. It follows immediately that the relaxed problem has a unique solution given by assigning $\varnothing$ on $[0, t_2^*)$, assigning good 2 with full probability on $[t_2^*, t^\dagger)$, and assigning the bundle $\{1,2\}$ with full probability on $[t^\dagger, 1]$. The allocation rule is implementable given the first observation that the values for the bundle $\{1,2\}$ and for good 2 satisfy increasing differences.

The rest of the proof is identical to the proof of Theorem 1. In this equilibrium, the options in the seller's menu that are consumed by some types must be $\{2\}$ and $\{1,2\}$, but then removing the other options in the menu results in a nested menu, under which the buyer's strategy continues to be optimal, and the seller's menu continues to be optimal. Thus, we have found an outcome-equivalent nested bundling equilibrium.

A.9 Proof of Proposition 8

The proof of Proposition 8 is similar to that of Theorem 1. We first show that every equilibrium has vertical learning, and then that it is outcome equivalent to a nested bundling equilibrium. We assume throughout that $K = 2$ and $\rho = 0$.

A.9.1 Vertical learning

Toward a contradiction, suppose that there exists an equilibrium $(\boldsymbol{\alpha}, \mathcal{M})$ with horizontal learning, such that $\text{sign}(\alpha_1) \cdot \text{sign}(\alpha_2) < 0$ (recall that we assume uncorrelated values here). The proof follows the same steps as that of Theorem 1. First, we characterize optimal mechanisms against the type distribution induced by $\boldsymbol{\alpha}$. Second, we construct a profitable deviation for the buyer.

Optimal mechanisms against $\boldsymbol{\alpha}$. Using the same normalization as in the proof of Theorem 1, we can index types by some parameter $t \in [0,1]$ such that a type-t buyer has posterior expected value $\theta_i(t;\boldsymbol{\alpha}) = a_i t + b_i$ for each good i. We can furthermore normalize signs such that $\sum_i a_i \geq 0$. Since $\boldsymbol{\alpha}$ is a horizontal learning strategy, we must have $a_i > 0$ for one good i and $a_j < 0$ for the other. Let good 1 be the positive good. Note that the sign normalization is equivalent to $a_1 \geq -a_2 > 0$.

With production costs, what matters for the seller are effective types:

$$\tilde{\theta}_i(t;\boldsymbol{\alpha}) = a_i t + b_i - c_i =: \tilde{a}_i t + \tilde{b}_i.$$

To characterize optimal mechanisms, we can then use the same arguments as in the proof of Theorem 1, replacing the buyer's types by effective types. The main difference is that effective types can be negative. However, we know that $\tilde{b}_2 > 0$. Indeed, $\tilde{b}_2 > 0.5\tilde{a}_2 + \tilde{b}_2 = \mu_2 - c_2 > 0$. Furthermore, $\tilde{a}_1 + \tilde{b}_1 > 0$. Indeed, $\tilde{a}_1 + \tilde{b}_1 > 0.5\tilde{a}_1 + \tilde{b}_1 = \mu_1 - c_1 > 0$. Moreover, we have assumed in this proposition that the distribution $\mathbf{v}$ is log-concave, which implies that θ_i must be log-concave by the linear projection property of elliptical distribution and Prékopa's Theorem, and hence $a_i\Phi(t) + b_i$ is strictly increasing for $a_i > 0$ and strictly decreasing for $a_i < 0$.

We start by solving the following auxiliary problem:

$$\max_{\mathbf{x}\in[0,1]^K} \sum_i \tilde{b}_i x_i \qquad \text{(Auxiliary Problem)}$$
$$\text{subject to } \sum_i \tilde{a}_i x_i = 0\,.$$

Letting λ denote an optimal multiplier on the equality constraint, any solution

x^* to the auxiliary problem must solve

$$x_i^* \in \arg\max_{x_i \in [0,1]} (\tilde{b}_i + \lambda \tilde{a}_i) x_i \,.$$

We show that $(-a_2/a_1, 1) =: \bar{x}$ is the unique solution to this problem.

First, we argue that any solution must have $x_2^* > 0$. If not, then the only candidate that satisfies the equality constraint is $x = (0, 0)$. But this is strictly worse than $\bar{x}$ since

$$\begin{aligned}\sum_i \tilde{b}_i \bar{x}_i = -\frac{a_2}{a_1}\tilde{b}_1 + \tilde{b}_2 &= -\frac{a_2}{a_1}[b_1 - c_1] + b_2 - c_2 \\ &= -\frac{a_2}{a_1}[\mu_1 - 0.5a_1 - c_1] + \mu_2 - 0.5a_2 - c_2 \\ &= -\frac{a_2}{a_1}[\mu_1 - c_1] + \mu_2 - c_2 > 0 \,.\end{aligned}$$

Thus, any solution has $x_2^* > 0$ and the equality constraint pins down $x_1^* = -(a_2/a_1)x_2^*$. But since $-\frac{a_2}{a_1}\tilde{b}_1 + \tilde{b}_2 > 0$, any optimal solution must set $x_2^* = 1$.

The following facts are worth noting. First, since $\bar{x}$ allocates a positive amount of both goods, it must be that $\tilde{a}_i \lambda + \tilde{b}_i \geq 0$ for $i = 1, 2$. Furthermore, if $-a_2 < a_1$, then good 1 is rationed. Optimality then requires $\tilde{a}_1 \lambda + \tilde{b}_1 = 0$ and $\tilde{a}_2 \lambda + \tilde{b}_2 = (-a_2/a_1) \times \tilde{b}_1 + \tilde{b}_2 > 0$. Finally, we can set $\lambda \leq 0.5$. There are two cases. Either $-a_2 < a_1$, in which case:

$$\lambda = -\frac{\tilde{b}_1}{\tilde{a}_1} = \frac{-\mu_1 + 0.5a_1 + c_1}{a_1} = 0.5 - \frac{\mu_1 - c_1}{a_1} < 0.5.$$

If $-a_2 = a_1$, then there are many optimal multipliers, which only need to satisfy $\tilde{a}_i \lambda + \tilde{b}_i \geq 0$ for $i = 1, 2$. Setting $\lambda = 0.5$ satisfies both constraints.

Unlike in our baseline model, the multiplier on the equality constraint can be either positive or negative depending on parameter values. Thus, we distinguish between several cases in our characterization of optimal mechanisms.

LEMMA 13. *First, let $\lambda \leq 0$. Then, there exist thresholds $0 < \bar{t}_0 < \bar{t}_1 \leq 1$ such that, for any optimal mechanism,*

(i) $x(t) = \bar{x}$ and $U(t) = 0$ for all $t \in [0, \bar{t}_0]$;

(ii) $x(t) = (1,1)$ *for all* $t \in (\bar{t}_0, \bar{t}_1]$;

(iii) $x(t) = (1,0)$ *for all* $t > \bar{t}_1$.

Now, let $\lambda > 0$. *Then, either* $x_2(t) = 1$ *for all* t *under any optimal mechanism, or there exists thresholds* $0 \le \bar{t}_0 < \bar{t}_1 \le \bar{t}_2 \le 1$ *such that, for any optimal mechanism,*

(i) $x(t) = (0,1)$ *for all* $t \in [0, \bar{t}_0]$;

(ii) $x(t) = \bar{x}$ *and* $U(t) = 0$ *for all* $t \in (\bar{t}_0, \bar{t}_1]$;

(iii) $x(t) = (1,1)$ *for all* $t \in (\bar{t}_1, \bar{t}_2]$;

(iv) $x(t) = (1,0)$ *for all* $t > \bar{t}_2$.

Furthermore, $\bar{t}_1 = \bar{t}_2$ *only if* $\bar{x} = (1,1)$.

Proof. Following the same argument as in Lemma 4, we know that if (t_0^*, x^*) is a saddle point of

$$\mathbb{E}\left[\sum_i \left(\tilde{a}_i x_i(t)\overline{\Phi}(t;t_0) + \tilde{b}_i x_i(t)\right)\right],$$

then x^* is an optimal mechanism and any optimal mechanism x' must also form a saddle point with t_0^*. We first construct a saddle point, which then allows us to characterize all optimal mechanisms.

Case (A). First, consider the case of $\lambda \le 0$, which is the only case possible absent production costs. Following the same argument as in Lemma 4, we know that there exists some t_0^* such that

$$\overline{\Phi}(t_0^*; t_0^*) = \lambda\,.$$

We also know that the ironing interval including t_0^* must also include 0. That is, there exists $\bar{t}_0 \ge t_0^*$ such that $\overline{\Phi}(t; t_0^*) = \lambda$ for all $t \in [0, \bar{t}_0]$ and $\overline{\Phi}(t; t_0^*) > \lambda$ for all $t > \bar{t}_0$.

We show that t_0^* is part of a saddle point. Fixing the conjectured worst-off type t_0^*, consider the maximization problem:

$$\max_{x:[0,1]\to[0,1]^K} \mathbb{E}\left[\sum_i \left(\tilde{a}_i x_i(t)\overline{\Phi}(t;t_0^*) + \tilde{b}_i x_i(t)\right)\right].$$

We can solve this problem pointwise. By construction, for any $t \in [0, \bar{t}_0]$,

$$\tilde{a}_i \overline{\Phi}(t; t_0^*) + \tilde{b}_i = \tilde{a}_i \lambda + \tilde{b}_i \quad \forall i\,,$$

which can be maximized by setting $x(t) = \bar{x}$. Furthermore, for any $t > \bar{t}_0$, $\overline{\Phi}(t; t_0^*) > \lambda$. Thus, $\tilde{a}_1 \overline{\Phi}(t; t_0^*) + \tilde{b}_1 > 0$, and any solution must set $x_1(t) = 1$ for all $t > \bar{t}_0$. Finally, $\tilde{a}_2 \overline{\Phi}(t; t_0^*) + \tilde{b}_2 \geq 0$ at $t = \bar{t}_0$ and weakly decreases in t over $[\bar{t}_0, 1]$. Let $\bar{t}_1 := \max\{t \mid \tilde{a}_2 \overline{\Phi}(t; t_0^*) + \tilde{b}_2 \geq 0\}$. By construction $\bar{t}_1 \geq \bar{t}_0$. Any solution to the above problem must set $x_2(t) = 1$ for all $\bar{t}_0 < t < \bar{t}_1$ and $x_2(t) = 0$ for $t > \bar{t}_1$.

Consider the following allocation rule: $x^*(t) = \bar{x}$ for $t \leq \bar{t}_0$, $x^*(t) = (1, 1)$ for $t \in (\bar{t}_0, \bar{t}_1]$, and $x^*(t) = (1, 0)$ for $t > \bar{t}_1$. By the above argument, x^* pointwise maximizes the ironed objective given t_0^*. Now we verify that t_0^* is a worst-off type given x^*. This is indeed the case since type t_0^* belongs to the ironing interval with allocation $\bar{x}$, which means that it gets zero payoff under x^*.

Therefore, (t_0^*, x^*) is a saddle point, which implies x^* is an optimal mechanism. Furthermore, any other optimal mechanism x' must also form a saddle point with t_0^*. However, up to measure-zero types, x^* is the only mechanism that maximizes the ironed virtual objective given t_0^* and is consistent with t_0^*.

We have left to prove that $\bar{t}_1 > \bar{t}_0$. Since the function $\overline{\Phi}(t; t_0^*)$ is continuous in t, if $\bar{t}_1 = \bar{t}_0$, then $\tilde{a}_2 \overline{\Phi}(\bar{t}_0; t_0^*) + \tilde{b}_2 = 0$, which is equivalent to $\tilde{a}_2 \lambda + \tilde{b}_2 = 0$. Thus, if $\bar{t}_1 = \bar{t}_0$, then $\lambda = -\tilde{b}_2 / \tilde{a}_2 > 0$, a contradiction.

Case (B). Now consider the case of $\lambda > 0$. First, we show that there exists t_0^* such that $\overline{\Phi}(t_0^*; t_0^*) = \lambda$. The function $g(t_0) = \overline{\Phi}(t_0; t_0)$ is continuous in t_0 (Lemma 3) and negative at $t_0 = 0$ (Lemma 2). Thus, we only need to show that $g(1) \geq \lambda$. Recall that at $t_0 = 1$, we have

$$\Phi(t; 1) = t + \frac{F(t)}{f(t)}\,,$$

which, by log-concavity, is strictly increasing in t. Thus, $\overline{\Phi}(1; 1) = \Phi(1; 1) \geq 1$. Therefore, $g(1) \geq 1/2 \geq \lambda$, and, by the intermediate value theorem, there exists t_0^* such that $\overline{\Phi}(t_0^*; t_0^*) = \lambda$.

We construct a saddle point (t_0^*, x^*). Given the conjectured worst-off type t_0^*, consider the maximization problem:

$$\max_{x:[0,1]\to[0,1]^K} \mathbb{E}\left[\sum_i \left(\tilde{a}_i x_i(t)\overline{\Phi}(t;t_0^*) + \tilde{b}_i x_i(t)\right)\right].$$

Let $\mathcal{I} \subseteq [0,1]$ be the ironing interval that includes t_0^*. There are two subcases: either the ironing interval that includes t_0^* also includes 1 (i.e., $1 \in \mathcal{I}$) or it does not ($1 \notin \mathcal{I}$).

Case (B1). In the first case, by construction, there exists a solution $x(t) = \bar{x}$ to the above problem for all $t \in \mathcal{I}$. Furthermore, for all $t < \min_{t' \in \mathcal{I}} t'$, $\tilde{a}_2\overline{\Phi}(t;t_0^*) + \tilde{b}_2 > \tilde{a}_2\overline{\Phi}(t_0^*;t_0^*) + \tilde{b}_2 \geq \tilde{a}_2\lambda + \tilde{b}_2 \geq 0$. Thus, any solution must set $x_2(t) = 1$ for all $t < \min_{t' \in \mathcal{I}} t'$. The allocation

$$x_2^*(t) = 1 \quad \forall t \quad \text{and} \quad x_1^*(t) = \begin{cases} \bar{x}_1 & \text{if } t \in \mathcal{I} \\ \mathbb{1}\{\tilde{a}_i\overline{\Phi}(t;t_0^*) + \tilde{b}_i \geq 0\} & \text{if } t \notin \mathcal{I} \end{cases}$$

then maximizes the ironed objective pointwise given t_0^*. Furthermore, type t_0^* gets zero payoff under x^* and is indeed a worst-off type: the tuple (t_0^*, x^*) forms a saddle point. This implies, any optimal mechanism x' must also form a saddle point with t_0^*. Combined with the above argument, this requires that any optimal mechanism sets $x_2'(t) = 1$ for all t.

Case (B2). Finally, suppose $1 \notin \mathcal{I}$. Let $\bar{t}_2 = \max\{t \mid \tilde{a}_2\overline{\Phi}(t;t_0^*) + \tilde{b}_2 \geq 0\}$ and $\bar{t}_0 = \min\{t \mid \tilde{a}_1\overline{\Phi}(t;t_0^*) + \tilde{b}_1 \geq 0\}$. Since $\overline{\Phi}(t;t_0^*)$ is monotonically increasing and equals $\lambda > 0$ for all $t \in \mathcal{I}$, it must be that $\bar{t}_2 \geq \max_{t' \in \mathcal{I}} t'$ and $\bar{t}_0 \leq \min_{t' \in \mathcal{I}} t'$. The allocation

$$x^*(t) = \begin{cases} \bar{x} & \text{if } t \in \mathcal{I} \\ (\mathbb{1}_{t \geq \bar{t}_0}, \mathbb{1}_{t \leq \bar{t}_2}), & \text{if } t \notin \mathcal{I}\,, \end{cases}$$

maximizes the ironed objective pointwise. Furthermore, it forms a saddle point with t_0^* since type t_0^* falls in the rationing interval, and thus gets zero surplus under x^*. Any other optimal mechanism must also form a saddle point with t_0^*. But any such mechanism must then be identical to x^*, since the pointwise optimum is uniquely pinned down for almost all types outside of $\mathcal{I}$, and also uniquely pinned down for the types in $\mathcal{I}$ in order to be consistent with t_0^*.

Recall that if $-a_2 < a_1$, then $\tilde{a}_1\lambda + \tilde{b}_1 = 0$. This means that $\tilde{a}_1\overline{\Phi}(t; t_0^*) + \tilde{b}_1 = 0$ for all $t \in \mathcal{I}$, and $\bar{t}_0 = \min_{t' \in \mathcal{I}} t'$. Under x^*, the allocation is then $x^*(t) = (0, 1)$ for $t \leq \bar{t}_0$ and $x_1^*(t) = 1$ for $t > \max_{t' \in \mathcal{I}} t' =: \bar{t}_1$. If $-a_2 = a_1$, then $\bar{x} = (1, 1)$, and $x^*(t) = (0, 1)$ for $t \leq \bar{t}_0$ and $x_1^*(t) = 1$ for $t > \bar{t}_0$.

We have left to prove that $\bar{t}_2 = \max_{t' \in \mathcal{I}} t'$ only if $\bar{x} = (1, 1)$. Since $\overline{\Phi}(\cdot\,, t_0^*)$ is continuous, $\bar{t}_2 = \max_{t' \in \mathcal{I}} t'$ if and only if $\tilde{a}_2\lambda + \tilde{b}_2 = 0$. Thus, if $\bar{t}_2 = \max_{t' \in \mathcal{I}} t'$, we then have $\lambda = -\tilde{b}_2/\tilde{a}_2$ and $\tilde{a}_1\lambda + \tilde{b}_1 = \tilde{b}_1 - (a_1/a_2)\tilde{b}_2 > 0$ (as shown before). Thus, it must be that $\bar{x} = (1, 1)$ and $-a_2 = a_1$. $\square$

Optimal Learning. We show that, against any optimal mechanism $\mathcal{M}$, the buyer has a strictly profitable deviation under horizontal learning.

First, we establish that in any horizontal learning equilibrium $(\boldsymbol{\alpha}, \mathcal{M})$, information must be strictly valuable. That is, $\tilde{\boldsymbol{\alpha}} = \mathbf{0}$ cannot be optimal against $\mathcal{M}$. The argument is the same as in the proof of Lemma 8. There are two cases. If the optimal ironing interval constructed in Lemma 13 is a strict subset of the type space $[0, 1]$, then the indirect utility function $U(t)$ is convex and not affine. The buyer's expected payoff must then be strictly higher than if his type distribution was degenerate at the prior. If the ironing interval covers the whole type space, then all types get allocation $\bar{x}$ at price $\bar{p} = (-a_2/a_1) \times \tilde{b}_1 + \tilde{b}_2$, and get zero utility. But then deviating to $\hat{\boldsymbol{\alpha}} = \bar{x}$ guarantees a strictly positive expected payoff to the buyer and constitutes a strictly profitable deviation.

We now argue that $\boldsymbol{\alpha}$ cannot be optimal against $\mathcal{M}$. Let

$$O := \Big\{ \big(x(t), p(t)\big) \Big\}_{t \in [0,1]}$$

denote the minimal menu that implements the seller's optimal mechanism (which would give the same ex ante payoff to the buyer under strategy $\boldsymbol{\alpha}$). The characterization of optimal mechanisms (Lemma 13) distinguishes between several cases, and so does the construction of a deviation for the buyer.

Case (B1). The simplest case is when $x_2(t) = 1$ for all t under optimal mechanism $\mathcal{M}$. This corresponds to **Case (B1)** in the proof of Lemma 13. In words, this means that in the conjectured equilibrium $(\boldsymbol{\alpha}, \mathcal{M})$, the buyer always purchases good 2 irrespective of the signal realization he received: for any $(x, p) \in O$,

$x_2 = 1$. By Lemma 6, it must then be the case that $\boldsymbol{\alpha}$ puts zero weight on good 2. But that simply means $\boldsymbol{\alpha} = (1, 0)$, which is a vertical learning strategy, contradicting the assumption that $(\boldsymbol{\alpha}, \mathcal{M})$ is a horizontal learning equilibrium.

Cases (A) and (B2). We now jointly consider the remaining two cases.

In both cases, $(\bar{x}, \bar{p}) \in O$ and $((1,1), p_{12}) \in O$, where $\bar{p}$ is the price of the rationing option and p_{12} the price of the grand bundle. The set O can include up to two other outcomes: $((1,0), p_1)$ for some p_1 and $((0,1), p_2)$ for some p_2. Note that if $((1,0), p_1) \in O$, then $p_1 > \min_s \theta_1(s; \boldsymbol{\alpha})$. That is, not all types are willing to buy good 1 by itself. If it were not true, then all types but the lowest would get a strictly positive payoff out of option $((1,0), p_1)$. Yet we know that a positive mass of them must get the rationing option and zero payoff under any optimal mechanism. Similarly, if $((0,1), p_2) \in O$, then $p_2 > \min_s \theta_2(s; \boldsymbol{\alpha})$. Finally, if both options are included in O, then we must have $p_1 + p_2 > p_{12}$. Indeed, by Lemma 13, we know that type $\bar{t}_0$ must be indifferent between $((0,1), p_2)$ and nothing, type $\bar{t}_1$ between $((1,1), p_{12})$ and nothing, and type $\bar{t}_2$ between $((1,0), p_1)$ and $((1,1), p_{12})$. Thus, we have

$$p_1 + p_2 - p_{12} = -a_2 \bar{t}_2 - b_2 + a_2 \bar{t}_0 + b_2 = -a_2(\bar{t}_2 - \bar{t}_0) > 0\,,$$

since $a_2 < 0$ and $\bar{t}_2 > \bar{t}_0$.

We show that, against menu O, the buyer strictly prefers the "flipped" vertical learning strategy $\hat{\boldsymbol{\alpha}} = (\alpha_1, -\alpha_2)$ to $\boldsymbol{\alpha}$.

As shown in the proof of Proposition 4, the two learning strategies lead to the same distribution of posterior expected values for good 1. That is, $\theta_1(s; \boldsymbol{\alpha})$ and $\theta_1(\hat{s}; \hat{\boldsymbol{\alpha}})$ follow the same distribution, which we denote by G. From now on, we index types under both learning strategies by θ_1. We can then write the value that a type θ_1-buyer has for good 2 as

$$\theta_2(\theta_1) = \mu_2 + \frac{\alpha_2 \sigma_2^2}{\alpha_1 \sigma_1^2}(\theta_1 - \mu_1) \quad \text{under strategy } \boldsymbol{\alpha}\,,$$
$$\hat{\theta}_2(\theta_1) = \mu_2 - \frac{\alpha_2 \sigma_2^2}{\alpha_1 \sigma_1^2}(\theta_1 - \mu_1) \quad \text{under strategy } \hat{\boldsymbol{\alpha}}\,.$$

Let $U_{\mathcal{M}}(\boldsymbol{\alpha})$ and $U_{\mathcal{M}}(\hat{\boldsymbol{\alpha}})$ denote the buyer's expected payoff under mechanism

$\mathcal{M}$ when he chooses learning strategy $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\alpha}}$, respectively. Define $U_O(\boldsymbol{\alpha})$ and $U_O(\hat{\boldsymbol{\alpha}})$ similarly. We want to show that $U_{\mathcal{M}}(\boldsymbol{\alpha}) < U_{\mathcal{M}}(\hat{\boldsymbol{\alpha}})$. By construction, $U_{\mathcal{M}}(\boldsymbol{\alpha}) = U_O(\boldsymbol{\alpha})$. Furthermore, since $O \subseteq \mathcal{M}$, $U_{\mathcal{M}}(\hat{\boldsymbol{\alpha}}) \geq U_O(\hat{\boldsymbol{\alpha}})$. Thus, it is enough to show that $U_O(\boldsymbol{\alpha}) < U_O(\hat{\boldsymbol{\alpha}})$.

Let O' be the menu constructed from removing the rationing options from O. Because the rationing options yield zero surplus to any type under strategy $\boldsymbol{\alpha}$, it must be that $U_{O'}(\boldsymbol{\alpha}) = U_O(\boldsymbol{\alpha})$ while $U_{O'}(\hat{\boldsymbol{\alpha}}) \leq U_O(\hat{\boldsymbol{\alpha}})$. Thus, it is enough to show that $U_{O'}(\boldsymbol{\alpha}) < U_{O'}(\hat{\boldsymbol{\alpha}})$.

In our proof, we consider a fictitious separate sales mechanism SS. If both $((1,0),p_1),((0,1),p_2) \in O$, then SS simply consists of good 1 at price p_1, good 2 at price p_2, and the grand bundle at $p_1 + p_2$. If there is no $(x,p) \in O$ with $x = (0,1)$, then define $p_2 := \max_{\theta_1} \theta_2(\theta_1)$. By construction, this ensures that if $x = (0,1)$ is not included in O, then the constructed $((0,1),p_2)$ is not purchased by any type under $\boldsymbol{\alpha}$. Note that this option is not purchased by any type under $\hat{\boldsymbol{\alpha}}$ either, since $\max_{\theta_1} \hat{\theta}_2(\theta_1) = \max_{\theta_1} \theta_2(\theta_1)$. Similarly, if there is no $(x,p) \in O$ with $x = (1,0)$, then define $p_1 := p_{12} - \min_{\theta_1} \theta_2(\theta_1)$. As before, the price p_1 is chosen so that no type ever purchases this option under both $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\alpha}}$.

Let $U_{SS}(\boldsymbol{\alpha})$ and $U_{SS}(\hat{\boldsymbol{\alpha}})$ under separate sales mechanism SS. Recall from Proposition 4 that learning strategies $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\alpha}}$ yield the same expected payoff to the buyer under any separate sales mechanisms. Thus, $U_{SS}(\boldsymbol{\alpha}) = U_{SS}(\hat{\boldsymbol{\alpha}})$. What we want to show ($U_{O'}(\boldsymbol{\alpha}) < U_{O'}(\hat{\boldsymbol{\alpha}})$) is then equivalent to:

$$U_{O'}(\hat{\boldsymbol{\alpha}}) - U_{SS}(\hat{\boldsymbol{\alpha}}) > U_{O'}(\boldsymbol{\alpha}) - U_{SS}(\boldsymbol{\alpha}).$$

For the buyer's payoff, the only difference between menus O' and SS is that the former sells the grand bundle at p_{12} while the latter sells the grand bundle at $p_1 + p_2 > p_{12}$. Indeed, if both $((1,0),p_1),((0,1),p_2) \in O'$, this is the only difference between menu O' and menu SS. If either option is not included in O', then SS includes it but its price is set such that the buyer never purchases it under either learning strategy. Therefore, it suffices to show that the buyer suffers more from an increase in the price of the grand bundle under the vertical learning strategy $\hat{\boldsymbol{\alpha}}$ than under the horizontal learning strategy $\boldsymbol{\alpha}$.

Let $\Theta_{\{1,2\}} := \left\{\theta_1 : \theta_1 + \theta_2(\theta_1) - p_{12} \geq \max\{0, \theta_1 - p_1, \theta_2(\theta_1) - p_2\}\right\}$ be the set of types who purchase the grand bundle under $\boldsymbol{\alpha}$. Only these types are potentially affected by an increase in the price of the grand bundle. Thus,

$$U_{O'}(\boldsymbol{\alpha}) - U_{SS}(\boldsymbol{\alpha}) = \int_{\theta_1 \in \Theta_{\{1,2\}}} \Delta(\theta_1) dG(\theta_1)\,,$$

where

$$\Delta(\theta_1) := \theta_1 + \theta_2(\theta_1) - p_{12} - (\theta_1 - p_1)_+ - (\theta_2(\theta_1) - p_2)_+ \geq 0$$

is the effect on a buyer with realized type $\theta_1 \in \Theta_{\{1,2\}}$. Set $\Delta(\theta_1) = 0$ for all $\theta_1 \notin \Theta_{\{1,2\}}$ since any such type is unaffected by an increase in the grand bundle price. Define $\hat{\Theta}_{\{1,2\}}$ and $\hat{\Delta}(\,\cdot\,)$ similarly.

First, consider what happens when $\min\{\theta_1 : \theta_1 \in \Theta_{\{1,2\}}\} > \mu_1$, that is, when the lowest type who purchases the grand bundle under $\boldsymbol{\alpha}$ has a value for good 1 greater than the mean. Note that this implies $\hat{\theta}_2(\theta_1) > \theta_2(\theta_1)$ for all $\theta_1 \in \Theta_{\{1,2\}}$. We show that *(i)* $\Theta_{\{1,2\}} \subset \hat{\Theta}_{\{1,2\}}$, and that *(ii)* $\hat{\Delta}(\theta_1) \geq \Delta(\theta_1)$ for all $\theta_1 \in \Theta_{\{1,2\}}$, strictly for some. Together, these imply that $U_{O'}(\hat{\boldsymbol{\alpha}}) - U_{SS}(\hat{\boldsymbol{\alpha}}) > U_{O'}(\boldsymbol{\alpha}) - U_{SS}(\boldsymbol{\alpha})$.

To establish *(i)*, note that if a buyer who values good 1 at θ_1 and good 2 at $\theta_2(\theta_1)$ finds it optimal to purchase the grand bundle, then so does a buyer who values good 1 at θ_1 and good 2 at $\hat{\theta}_2(\theta_1) > \theta_2(\theta_1)$. For all $\theta_1 \in \Theta_{\{1,2\}}$, we thus have:

$$\begin{aligned}\hat{\Delta}(\theta_1) - \Delta(\theta_1) =& \theta_1 + \hat{\theta}_2(\theta_1) - p_{12} - (\theta_1 - p_1)_+ - (\hat{\theta}_2(\theta_1) - p_2)_+ \\ & - \theta_1 - \theta_2(\theta_1) + p_{12} + (\theta_1 - p_1)_+ + (\theta_2(\theta_1) - p_2)_+ \\ =& \hat{\theta}_2(\theta_1) - \theta_2(\theta_1) + (\theta_2(\theta_1) - p_2)_+ - (\hat{\theta}_2(\theta_1) - p_2)_+ \geq 0\,,\end{aligned}$$

where the inequality is strict if $\theta_2(\theta_1) < p_2$. Note that $\theta_2(\theta_1) < p_2$ must hold for a positive measure of $\theta_1 \in \Theta_{\{1,2\}}$ since $\boldsymbol{\alpha}$ is a horizontal learning strategy. This establishes *(ii)*.

Now consider what happens when $\min\{\theta_1 : \theta_1 \in \Theta_{\{1,2\}}\} \leq \mu_1$. This case is slightly more involved as $\hat{\theta}_2(\theta_1) < \theta_2(\theta_1)$ for some $\theta_1 \in \Theta_{\{1,2\}}$, such that the above argument is no longer sufficient. We can, however, leverage the symmetry of the type distribution around the mean: any type $\theta_1 < \mu_1$ has the same probability (under both $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\alpha}}$) as type $2\mu_1 - \theta_1$. We show that $\hat{\Delta}(\theta_1) + \hat{\Delta}(2\mu_1 - \theta_1) \geq \Delta(\theta_1) +$

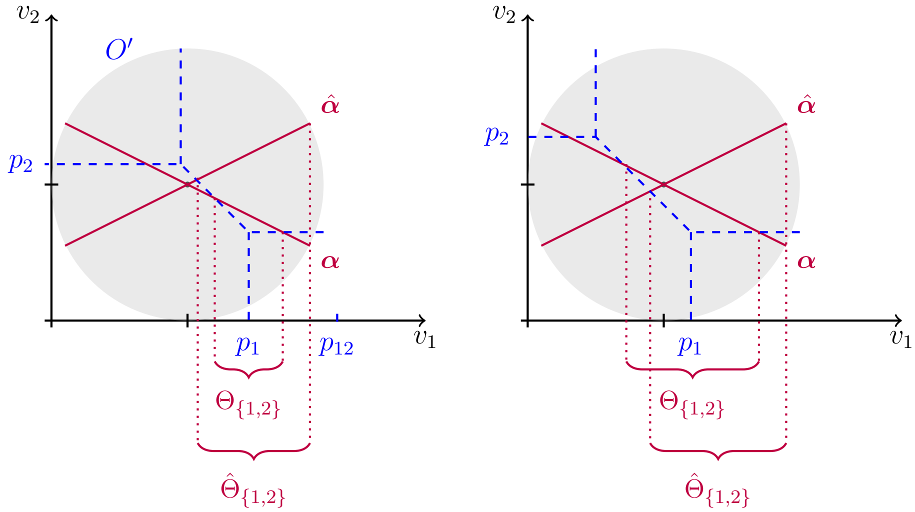

Figure 10: Illustration of the two cases for the comparison of $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\alpha}}$

$\Delta(2\mu_1-\theta_1)$ for all $\theta_1 \in \Theta_{\{1,2\}}$ where $\theta_1 < \mu_1$, and strictly so for a positive measure of these types. This then implies that $U_{O'}(\hat{\boldsymbol{\alpha}}) - U_{SS}(\hat{\boldsymbol{\alpha}}) > U_{O'}(\boldsymbol{\alpha}) - U_{SS}(\boldsymbol{\alpha})$. In particular, this argument covers all $\theta_1 \in \Theta_{\{1,2\}}$ if for any $\theta_1' > \mu_1$ where $\theta_1' \in \Theta_{\{1,2\}}$, we have $2\mu_1 - \theta_1' \in \Theta_{\{1,2\}}$. Otherwise, for any "unmatched type" $\theta_1' > \mu_1$ where $\theta_1' \in \Theta_{\{1,2\}}$, it is easy to see that $\hat{\Delta}(\theta_1') \geq \Delta(\theta_1')$ by the previous argument since $\hat{\theta}_2(\theta_1') > \theta_2(\theta_1')$.

Now, there are two subcases: either $\theta_1 \in \hat{\Theta}_{\{1,2\}}$ or $\theta_1 \notin \hat{\Theta}_{\{1,2\}}$. Consider the first subcase first, such that type θ_1 purchases the grand bundle under both the vertical and horizontal learning strategy. Then, as above:

$$\hat{\Delta}(\theta_1) - \Delta(\theta_1) = \hat{\theta}_2(\theta_1) - \theta_2(\theta_1) + (\theta_2(\theta_1) - p_2)_+ - (\hat{\theta}_2(\theta_1) - p_2)_+ .$$

Note that, by construction, $\hat{\theta}_2(2\mu_1 - \theta_1) = \theta_2(\theta_1)$ and $\theta_2(2\mu_1 - \theta_1) = \hat{\theta}_2(\theta_1)$. Thus, $2\mu_1 - \theta_1 \in \hat{\Theta}_{\{1,2\}}$ and $2\mu_1 - \theta_1 \in \Theta_{\{1,2\}}$, and, as above:

$$\begin{aligned}\hat{\Delta}(2\mu_1 - \theta_1) - \Delta(2\mu_1 - \theta_1) &= \theta_2(\theta_1) - \hat{\theta}_2(\theta_1) + (\hat{\theta}_2(\theta_1) - p_2)_+ - (\theta_2(\theta_1) - p_2)_+ \\ &= \Delta(\theta_1) - \hat{\Delta}(\theta_1) .\end{aligned}$$

Thus, $\hat{\Delta}(\theta_1) + \hat{\Delta}(2\mu_1 - \theta_1) = \Delta(\theta_1) + \Delta(2\mu_1 - \theta_1)$.

Now consider the latter subcase where type θ_1 does not purchase the grand bundle under the vertical learning strategy $\hat{\boldsymbol{\alpha}}$. Note that there must exist a positive mass of such types since otherwise almost all types under $\hat{\boldsymbol{\alpha}}$ would have strictly positive payoff, which implies that all types under $\boldsymbol{\alpha}$ would have a strictly positive payoff by consuming the grand bundle. But that is impossible given that at least a positive measure of types must have payoff 0 in the equilibrium. Also, note that for these types, we must have either $\theta_1 + \hat{\theta}_2(\theta_1) < p_{12}$ or $\hat{\theta}_2(\theta_1) < p_{12} - p_1$.

For all such types, we have $\hat{\Delta}(\theta_1) = 0$ and

$$\Delta(\theta_1) - \hat{\Delta}(\theta_1) = \theta_1 + \theta_2(\theta_1) - p_{12} - (\theta_1 - p_1)_+ - (\theta_2(\theta_1) - p_2)_+ \,.$$

As before, $2\mu_1 - \theta_1 \in \hat{\Theta}_{\{1,2\}}$. If $2\mu_1 - \theta_1 \notin \Theta_{\{1,2\}}$, then $\Delta(2\mu_1 - \theta_1) = 0$, and

$$\begin{aligned}
\hat{\Delta}(2\mu_1 - \theta_1) - &\Delta(2\mu_1 - \theta_1) \\
&= 2\mu_1 - \theta_1 + \theta_2(\theta_1) - p_{12} - (2\mu_1 - \theta_1 - p_1)_+ - (\theta_2(\theta_1) - p_2)_+ \\
&= \Delta(\theta_1) - \hat{\Delta}(\theta_1) + 2(\mu_1 - \theta_1) + (\theta_1 - p_1)_+ - (2\mu_1 - \theta_1 - p_1)_+ \\
&\geq \Delta(\theta_1) - \hat{\Delta}(\theta_1) \,,
\end{aligned}$$

where the inequality is due to that $2\mu_1 - \theta_1 > \theta_1$, and is strict if $\theta_1 < p_1$. Moreover, note that there exists a positive measure of such θ_1 with $\theta_1 < p_1$ since, under $\boldsymbol{\alpha}$, there exists a type who is indifferent between consuming nothing and consuming the bundle and there exists a positive measure of types consuming the bundle. Now, if $2\mu_1 - \theta_1 \in \Theta_{\{1,2\}}$, then

$$\begin{aligned}
\hat{\Delta}(2\mu_1 - \theta_1) - \Delta(2\mu_1 - \theta_1) &= \theta_2(\theta_1) - \hat{\theta}_2(\theta_1) + (\hat{\theta}_2(\theta_1) - p_2)_+ - (\theta_2(\theta_1) - p_2)_+ \\
&= \Delta(\theta_1) - \hat{\Delta}(\theta_1) + p_{12} - \theta_1 - \hat{\theta}_2(\theta_1) + (\hat{\theta}_2(\theta_1) - p_2)_+ + (\theta_1 - p_1)_+ \\
&> \Delta(\theta_1) - \hat{\Delta}(\theta_1) \,,
\end{aligned}$$

since either $\theta_1 + \hat{\theta}_2(\theta_1) < p_{12}$ or $\hat{\theta}_2(\theta_1) < p_{12} - p_1$.

Nested Bundling. By the previous parts, we know that every equilibrium must have vertical learning. Now, fix any equilibrium. Then, the effective types (after adjusting for costs) can be written as: for each i,

$$\theta_i = \tilde{a}_i t + \tilde{b}_i$$

where $\tilde{a}_i \geq 0$, and $t \in [0, 1]$. The only difference compared to Section 4 is that $\tilde{b}_i$ may be negative. The proof of the nested bundling part of Theorem 1 uses the fact that $a_i \Phi(t) + b_i$ is strictly single-crossing under positive a_i, b_i (Lemma 2). However, as noted before, since we have assumed the distribution $\mathbf{v}$ is log-concave, we have that θ_i is log-concave by the linear projection property of elliptical distribution and Prékopa's Theorem, and hence $a_i \Phi(t) + b_i$ is strictly increasing and hence strictly single-crossing. The rest of the proof is identical.

A.10 Proof of Proposition 9

Existence. We construct a weak equilibrium. For any learning strategy $\boldsymbol{\alpha}$, normalize types as in the proof of Theorem 1 such that type t values good i at $a_i t + b_i$, and let $\Phi(t)$ be the associated virtual value function. Fix any vertical learning strategy $\boldsymbol{\alpha}$ such that $a_i > 0$ for all i, and $b_i/a_i \neq b_j/a_j$ for all $i \neq j$. There must exist such an $\boldsymbol{\alpha}$ since the set of vertical learning strategies is a non-empty convex $(K-1)$-dimensional set, and learning strategies that fail to satisfy these conditions have measure-zero in that set.

Consider the following direct revelation mechanism $\mathcal{M}$:

$$x_i(t) = \mathbb{1}\{a_i \Phi(t) + b_i \geq 0\}$$
$$p(t) = \sum_i [a_i t + b_i] x_i(t) - \int_0^t \sum_i a_i x_i(s) ds \,.$$

We know from the proof of Theorem 1 that this mechanism is optimal against $\boldsymbol{\alpha}$. We now argue that $\boldsymbol{\alpha}$ is $\mathcal{M}$-Blackwell undominated.

Let $t_i := \min\{t : x_i(t) = 1\}$ be the lowest type who is allocated good i, and label goods such that $t_1 \leq t_2 \leq \cdots \leq t_K$. We show that these inequalities can be strengthened to $t_1 < t_2 < \cdots < t_K < 1$. By Lemma 2, we know that $\Phi(t) > 0$ for any $t > 0.5$, which means $x_i(t) = 1$ for any $t > 0.5$, and thus $t_K < 1$. Lemma 2 also shows that $\Phi(t) \to -\infty$ as $t \to 0$. Thus, it can only be optimal to allocate some good i to the lowest type if $a_i = 0$. However, by construction, $a_i > 0$ for all i. Similarly, $t_i = t_j$ if and only if $b_i/a_i = b_j/a_j$, which is precluded by construction. Thus, the optimal mechanism constructed above allocates all the following bundles with positive probability: $\varnothing$, $\{1\}$, $\{1,2\}$, $\{1,2,3\}$, …, $\{1,2,\dots,K\}$. Furthermore, any type $t \in (t_l, t_{l+1})$ finds it *strictly* optimal to buy bundle $\{1,2,\dots,l\}$. Thus, any selection of optimal reports M^* must include all the above bundles, and $(U_m)_{m\in M^*} = (0, v_1, v_1+v_2, v_1+v_2+v_3, \dots, \sum_k v_k)$. Then, for another strategy $\boldsymbol{\alpha}'$ to $\mathcal{M}$-Blackwell dominate $\boldsymbol{\alpha}$, it must be strictly Blackwell more informative than $\boldsymbol{\alpha}$ about $(U_m)_{m\in M^*}$, in the sense that

$$\Big(\sum_{l=1}^{k}\theta_l(s;\boldsymbol{\alpha})\Big)_{k=1\dots K} \preceq_{\text{cx}} \Big(\sum_{l=1}^{k}\theta_l(s;\boldsymbol{\alpha}')\Big)_{k=1\dots K}.$$

This implies that, for any weights $(\lambda_1,\dots,\lambda_K)\in\mathbb{R}^K$, we have

$$\boldsymbol{\lambda}\cdot\boldsymbol{\theta}(s;\boldsymbol{\alpha}) \preceq_{\text{cx}} \boldsymbol{\lambda}\cdot\boldsymbol{\theta}(s;\boldsymbol{\alpha}')\,,$$

since the convex order implies the linear convex order. However, we know that signal $\boldsymbol{\alpha}\neq\mathbf{0}$ induces the most dispersed distribution of $\boldsymbol{\theta}$ along some line in $\mathbb{R}^K$. In particular, note that for $\boldsymbol{\lambda}^* = \boldsymbol{\alpha}$, we must have

$$\boldsymbol{\lambda}^*\cdot\boldsymbol{\theta}(s;\boldsymbol{\alpha}') \preceq_{\text{cx}} \boldsymbol{\lambda}^*\cdot\boldsymbol{\theta}(s;\boldsymbol{\alpha}) \quad \text{in addition to} \quad \boldsymbol{\lambda}^*\cdot\boldsymbol{\theta}(s;\boldsymbol{\alpha}) \preceq_{\text{cx}} \boldsymbol{\lambda}^*\cdot\boldsymbol{\theta}(s;\boldsymbol{\alpha}')\,,$$

since the signal $\boldsymbol{\alpha}\cdot\mathbf{v}$ fully reveals the state $\boldsymbol{\lambda}^*\cdot\mathbf{v}$. The above can only be possible if signals $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are identical, in the sense that $\boldsymbol{\alpha}' = c\cdot\boldsymbol{\alpha}$ for some constant c. But then $\boldsymbol{\alpha}'$ cannot $\mathcal{M}$-Blackwell dominate $\boldsymbol{\alpha}$.

Thus, strategy $\boldsymbol{\alpha}$ is $\mathcal{M}$-Blackwell undominated and $\mathcal{M}$ is revenue-maximizing given $\boldsymbol{\alpha}$. The strategy profile forms a weak equilibrium.

Vertical learning and nested bundling. We first show that every weak equilibrium has vertical learning and is outcome-equivalent to a nested bundling equilibrium. The proof is virtually identical to that of Theorem 1. By contradiction, suppose that there exists a weak equilibrium with horizontal learning $(\boldsymbol{\alpha},\mathcal{M})$. The characterization of the seller's best response is identical since weak equilibrium imposes the same restriction on the seller's behavior as Nash equilibrium. Thus, Lemma 4 holds. Using the same arguments as in the proof of Theorem 1, we can then show a contradiction. Indeed, Lemma 6, Lemma 7, Lemma 8, and Lemma 9 only rely on the buyer not choosing a signal that is Blackwell dominated by some other signal in the relevant payoff subspace, which is precisely what the notion of weak equilibrium requires. In particular, by the proof of Lemma 8, information must be strictly valuable in any weak equilibrium. By the proofs of Lemma 4 and Lemma 6, any $\boldsymbol{\alpha}$ such that $\alpha_i \neq 0$ for some negative good i is $\mathcal{M}$-Blackwell dominated (by selecting M^* to be the induced outcomes in the equilibrium). By the proofs of Lemma 4 and Lemma 7, any $\boldsymbol{\alpha}$ such that $\alpha_i \neq \alpha_j$ for some negative good i and some positive balancing good j is $\mathcal{M}$-Blackwell dominated (by selecting M^* to be the induced outcomes in the equilibrium except replacing the rationing options that yield 0 payoff to all buyer types with the empty set $\varnothing$). Therefore, in the weak equilibrium, we must have $\alpha_i = 0$ for all negative goods and all positive balancing goods i, which leads to a sign contradiction as before. Thus, every weak equilibrium has vertical learning.

The proof of outcome-equivalence to a nested bundling equilibrium is also unaffected by the weaker solution concept—in particular, for any weak equilibrium, removing options in the seller's menu that are not chosen by any equilibrium type continues to sustain a weak equilibrium where the buyer chooses the original signal.

Ordering of log-scale posterior variance. Fix any weak equilibrium $(\boldsymbol{\alpha}, \mathcal{M})$. We know from the previous part that $\boldsymbol{\alpha}$ must be a vertical learning strategy. The proof of Proposition 1 applies verbatim as the argument only leverages the fact that $\boldsymbol{\alpha}$ is a vertical learning strategy and that $\mathcal{M}$ is revenue-maximizing against $\boldsymbol{\alpha}$. Since the concept of weak equilibrium still requires that the seller chooses a revenue-maximizing mechanism, the result follows.

A.11 Proof of Proposition 10

We construct a pure bundling equilibrium. Consider learning the full bundle $\boldsymbol{\alpha} = \mathbf{1}$. Such a learning strategy leads to the following mapping between signal realizations and types: for all i,

$$\theta_i(s;\boldsymbol{\alpha}) := \mu_i + \frac{\sigma_i^2 + \rho\sigma_i \sum_{j\neq i}\sigma_j}{\sum_k \sigma_k^2 + \rho\sum_{l\neq k}\sigma_l\sigma_k}\left(s - \sum_j \mu_j\right),$$

with $s = \sum_i v_i$. By the proof of Theorem 1, we know that pure bundling is optimal in response to such comonotonic type distribution if the segment on which types are supported can be extended to cross the origin—i.e., there exists some $s_0 \in \mathbb{R}$ such that $\theta_i(s_0;\boldsymbol{\alpha}) = 0$ for all i. We can substitute out s_0 and rewrite θ_i as a function of any θ_j as follows:

$$\theta_i(s;\boldsymbol{\alpha}) = \mu_i + \frac{\sigma_i^2 + \rho\sigma_i \sum_{k\neq i}\sigma_k}{\sigma_j^2 + \rho\sigma_j\sum_{k\neq j}\sigma_k}[\theta_j(s;\boldsymbol{\alpha}) - \mu_j].$$

Thus, there exists s_0 such that $\theta_i(s_0;\boldsymbol{\alpha}) = 0$ for all i if and only if we have the following condition:

$$\mu_i = \frac{\sigma_i^2 + \rho\sigma_i \sum_{k\neq i}\sigma_k}{\sigma_j^2 + \rho\sigma_j\sum_{k\neq j}\sigma_k}\mu_j \quad \forall i \neq j.$$

Now, under this condition, since the extended posterior mean line must connect $\boldsymbol{\mu}$ and $\mathbf{0}$ and $\boldsymbol{\mu} > 0$, the strategy $\boldsymbol{\alpha} = \mathbf{1}$ must be a vertical learning strategy. Hence, the seller finds it optimal to offer only the grand bundle at some price. Then, the buyer, of course, finds it optimal to learn fully the grand bundle value and nothing else, irrespective of how many signals he can acquire.

A.12 Proof of Proposition 11

We first show that the optimal monopoly price against a symmetric unimodal value distribution must lie weakly below the mean of the distribution. This fact will prove useful in the proof of Proposition 11.

Lemma 14. *Let F be a symmetric unimodal distribution supported on $[\underline{v}, \overline{v}]$ where $\underline{v} \geq 0$. Then, there exists a unique optimal monopoly price $p_F \leq \mathbb{E}[v]$.*

Proof. This follows by the proof of Lemma 2. □

Fix any nested bundling equilibrium of our main model. Without loss of generality, suppose that good 1 is the base good, such that the equilibrium menu sells good 1 at price p_1 and the bundle at price $p_1 + p_\delta$. Let U^* be the buyer's equilibrium payoff, i.e., his payoff when he acquires only one signal. If the buyer acquires a second signal, he can become fully informed and achieve his full information payoff, which we denote by U_{FI}^{NB}. Thus, if the associated cost $c \geq U_{FI}^{NB} - U^*$, then the buyer finds it optimal not to acquire the second signal.

We first bound the buyer's equilibrium payoff U^* from below. A strategy available to the buyer is to fully learn his value for the upgrade good v_2 and to learn nothing about the base. Since $p_1 \leq \mu_1$ (by Lemma 14 since the base good will be priced at the usual monopoly price), doing so leads the buyer to always buy good 1 and to buy the upgrade if and only if $v_2 \geq p_\delta$. This learning strategy then yields an expected payoff of $U_2^{NB} = \mathbb{E}[\max\{v_2 - p_\delta, 0\}] + \mu_1 - p_1$. His equilibrium payoff must then be weakly higher: $U^* \geq U_2^{NB}$.

We now bound the full information payoff U_{FI}^{NB} from above. Note that the payoff from learning good 2 fully is unchanged under the separate sales mechanism that sells good 1 at price p_1 and good 2 at price p_δ since $p_1 \leq \mu_1$. So $U_2^{NB} = U_2^{SS}$. The full information payoff is always weakly greater under the separate sales mechanism than under the nested bundling mechanism since the separate sales mechanism expands the choice set of the buyer: $U_{FI}^{SS} \geq U_{FI}^{NB}$.

Combining the above bounds, we have

$$U_{FI}^{NB} - U^* \leq U_{FI}^{NB} - U_2^{NB} = U_{FI}^{NB} - U_2^{SS} \leq U_{FI}^{SS} - U_2^{SS} = \mathbb{E}\big[\max\{v_1 - p_1, 0\}\big] - (\mu_1 - p_1)\,.$$

Thus, for any

$$c \geq \mathbb{E}\big[\max\{v_1 - p_1, 0\}\big] - (\mu_1 - p_1)\,,$$

we have that the buyer prefers not to acquire the second signal under the one-signal nested bundling equilibrium, proving the result.