Control systems with paraboloid nonholonomic constraints

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Definitions and problem statement 1/3

Conventions:

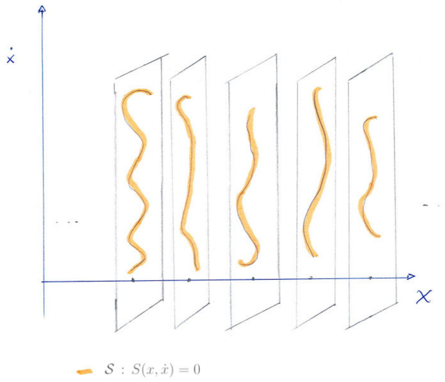
1. *smooth*: means C^∞ smooth;
2. We consider a smooth n-dimensional manifold $\mathcal{X}$, since all results are local we can take $\mathcal{X}$ as an open subset of $\mathbb{R}^n$, equipped with coordinates $x = (z, y_1, \ldots, y_{n-1})$;
3. $T\mathcal{X}$: the tangent bundle of $\mathcal{X}$, with coordinates $(x, \dot{x})$.

We consider a smooth $(2n-1)$-dimensional submanifold $\mathcal{S} \subset T\mathcal{X}$ given by

$$\mathcal{S} = \{(x, \dot{x}) \in T\mathcal{X},\ S(x, \dot{x}) = 0\},$$

where $S : T\mathcal{X} \to \mathbb{R}$ is a smooth scalar-valued map such that $\frac{\partial S}{\partial \dot{x}} \neq 0$ for all $(x, \dot{x}) \in \mathcal{S}$.

Definitions and problem statement 2/3



A submanifold $\mathcal{S}$ defines a nonholomic constraint, i.e. a curve $\gamma : [0, T] \to \mathcal{X}$ satisfies the nonholomic constraint given by $\mathcal{S}$ if we have $S(\gamma(t), \dot{\gamma}(t)) = 0$, for all t.

Definitions and problem statement 3/3

Definition (Equivalence of submanifolds)

We say that two submanifolds $\mathcal{S} = \{S(x, \dot{x}) = 0\}$ and $\tilde{\mathcal{S}} = \{\tilde{S}(\tilde{x}, \dot{\tilde{x}}) = 0\}$ of $T\mathcal{X}$ and $T\tilde{\mathcal{X}}$, respectively, are (locally)-equivalent if there exists a (local) diffeomorphism $\tilde{x} = \phi(x)$ and a nonvanishing scalar function $\delta(x, \dot{x})$ such that

$$\tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x, \dot{x})S(x, \dot{x}).$$

Purpose of our work: to characterise and classify the following category

$$\mathcal{S}_{p,q} = \{(x, \dot{x}) \in T\mathcal{X},\ \dot{z} = \dot{y}^t Q(x)\dot{y} + b(x)\dot{y} + c(x)\},$$

where $Q(x)$ is a smooth $(n-1)$ by $(n-1)$ symmetric matrix of full rank with signature (p, q), $b(x) = (b_1(x), \dots, b_{n-1}(x))$ is a smooth covector, and $c(x)$ is a smooth scalar function.

$\mathcal{S}_{p,q}$ is said to define a (p, q)-paraboloid nonholonomic constraint

Definitions and problem statement: Motivations

Before turning towards quadratic nonholomic constraints, a lot of work has been done for *linear* and *affine* Pfaffian equations

$$\omega(x)\dot{x} = 0 \quad \text{and} \quad \omega(x)\dot{x} + h(x) = 0.$$

☞ [Zhi92; Zhi95; JZ01; ZR98; Elk12], and many others.

Dubin's car

A simple model of the dynamics of a car, studied in [Dub57],

$$\begin{cases} \dot{z} & = \cos(\theta) \\ \dot{y} & = \sin(\theta) \end{cases}$$

corresponds to the constraints $(\dot{z})^2 + (\dot{y})^2 - 1 = 0$.

In [ANN15], the following relations $\dot{z} = \frac{1}{2}\sum_{i,j=1}^{m} Q_{i,j}\dot{y}_i\dot{y}_j$, where $\operatorname{sgn}(Q) = (p, q)$, appear as models whose Lie algebra of symmetries are isomorphic to $\mathfrak{so}(p+2, q+2)$.

Definitions and problem statement: Methodology 1/2

To study the equivalence problem of submanifolds $\mathcal{S} = \{S(x, \dot{x}) = 0\}$ we proceed as follows.

$$\begin{aligned} \mathcal{S} &\longleftrightarrow \Xi_{\mathcal{S}} : \dot{x} = F(x, w), \quad S(x, F(x, w)) = 0, \forall\, w \in \mathcal{W} \subset \mathbb{R}^{n-1} \\ &\longleftrightarrow \Sigma_{\mathcal{S}} : \left\{ \begin{array}{ll} \dot{x} & = F(x, w) \\ \dot{w} & = u \end{array} \right., \quad \text{where} \end{aligned}$$

- $w \in \mathbb{R}^{n-1}$ is the control for $\Xi_{\mathcal{S}}$;
- $u \in \mathbb{R}^{n-1}$ is the control for $\Sigma_{\mathcal{S}}$, and $(x, w) \in \mathcal{M} = \mathcal{X} \times \mathcal{W}$ is the extended coordinate system.

	Base manifold	Dimension	Controls
$\mathcal{S}$	$\mathcal{X}$	n	
$\Xi_{\mathcal{S}}$	$\mathcal{X}$	n	$m := n - 1$
$\Sigma_{\mathcal{S}}$	$\mathcal{M}$	$2n - 1$	$m := n - 1$

Definitions and problem statement: Methodology 2/2

☞ To characterise $\mathcal{S}_{p,q}$ we will characterise the class of control-affine systems $\Sigma_{\mathcal{S}_{p,q}}$;
☞ To classify $\mathcal{S}_{p,q}$ we will classify the class of control-nonlinear systems $\Xi_{\mathcal{S}_{p,q}}$;

What is the notion of equivalence for control systems that makes this diagram commute ?

$$\begin{tikzcd}
\mathcal{S} \arrow[rr, leftrightarrow, "\text{parametrisation}"] \arrow[d, leftrightarrow, "{(\phi,\,\delta)}"'] & & \Xi_{\mathcal{S}} \arrow[rr, leftrightarrow, "\text{extension}"] \arrow[d, leftrightarrow, "?"] & & \Sigma_{\mathcal{S}} \arrow[d, leftrightarrow, "?"] \\
\mathcal{S}_{p,q} \arrow[rr, leftrightarrow, "\text{parametrisation}"] & & \Xi_{\mathcal{S}_{p,q}} \arrow[rr, leftrightarrow, "\text{extension}"] & & \Sigma_{\mathcal{S}_{p,q}}
\end{tikzcd}$$

Definitions and problem statement: Feedback equivalence

Feedback for control-nonlinear systems: We call $\Xi : \dot{x} = F(x, w)$ and $\tilde{\Xi} : \dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{w})$ *feedback equivalent* if there exists a diffeomorphism $\Phi : \mathcal{X} \times \mathcal{W} \to \tilde{\mathcal{X}} \times \tilde{\mathcal{W}}$ of the form

$$(\tilde{x}, \tilde{w}) = \Phi(x, w) = (\phi(x), \psi(x, w)),$$

which transforms the first system into the second, i.e.

$$D\phi(x)F(x, w) = \tilde{F}(\phi(x), \psi(x, w)).$$

Feedback for control-affine systems: For control-affine systems $\Sigma : \dot{\xi} = f(\xi) + \sum_{i=1}^{m} g_i(\xi)u_i$, feedback transformations are restricted to those of the form

$$\tilde{u} = \psi(\xi, u) = \alpha(\xi) + \beta(\xi)u,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)^t$ and $\beta = (\beta^i_j)$ are smooth functions depending on the state and satisfy $\beta(\cdot) \in GL_m(\mathbb{R})$.

Summary of the talk

Proposition

Equivalence of submanifolds of the tangent bundle corresponds to the equivalences of their parametrisations (nonlinear and control-affine) under feedback transformations.

1. Problem I: Characterise (p,q)-paraboloid nonholomic constraints $\mathcal{S}_{p,q}$;
   - ☞ Define a novel class of control-affine systems $\Sigma_{p,q}$, which corresponds to prolongation of parametrisations of $\mathcal{S}_{p,q}$;
   - ☞ Find invariants of the class $\Sigma_{p,q}$ and exploit them to obtain a characterisation that class;
2. Problem II: Classify (p,q)-paraboloid nonholomic constraints $\mathcal{S}_{p,q}$;
   - ☞ Study the nature of feedback transformations acting on $\Xi_{p,q}$;
   - ☞ Identify structures of $\Xi_{p,q}$ that transform nicely under feedback;
   - ☞ Use those structures to classify $\Xi_{p,q}$;

We will first solve both problem for the case $\dim \mathcal{X} = 2$ and then generalise to $\dim \mathcal{X} \geq 3$.

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We study the characterisation of $\mathcal{S}_{1,0} = \dot{z} - a(x)\dot{y}^2 - b(x)\dot{y} - c(x) = 0$.

Definition (Parametrisation)

We say that a control-affine system Σ is parabolisable if it is feedback equivaent to

$$\Sigma_{1,0} : \begin{cases} \dot{x} &= \mathsf{f}(x, w) \\ \dot{w} &= u \end{cases}, \quad \text{where} \quad \frac{\partial^3 \mathsf{f}}{\partial w^3} = 0$$

and $\left(\frac{\partial^2 \mathsf{f}}{\partial w^2} \wedge \frac{\partial \mathsf{f}}{\partial w}\right)(x_0, w_0) \neq 0$.

We have

$$\Sigma_{1,0} : \begin{cases} \dot{x} &= A(x)w^2 + B(x)w + C(x) \\ \dot{w} &= u \end{cases} \quad \text{with} \quad A \wedge B \neq 0.$$

Study of parabolic systems: Characterisation 2/3

Theorem (Characterisation of parabolic nonholomic constraint)

Let Σ be a control-affine system on a 3-dimensional smooth manifold given by vector fields f and g. Σ is, locally around ξ_0, feedback equivalent to $\Sigma_{1,0}$ if and only if

1. $g \wedge \mathrm{ad}_g f \wedge \mathrm{ad}_g^2 f(\xi_0) \neq 0$,
2. *The structure functions ρ and τ in the decomposition* $\mathrm{ad}_g^3 f = \rho\, \mathrm{ad}_g^2 f + \tau\, \mathrm{ad}_g f \mod \mathrm{span}\{g\}$ *satisfy*

$$\chi = 3L_g\rho - 2\rho^2 - 9\tau = 0.$$

NB: $\mathrm{ad}_g^k f = \left[g, \mathrm{ad}_g^{k-1} f\right]$, with $\mathrm{ad}_g f = [g, f]$ and $[\cdot, \cdot]$ is the Lie bracket of vector fields.

These conditions are checkable by algebraic operations and derivations.

Study of parabolic systems: Characterisation 3/3

Idea of the proof:

1. Check that $\Sigma_{1,0}$ satisfies the conditions,
2. Check that the conditions are invariant under feedback transformations,
3. Given Σ with ρ and τ, find a feedback (α, β) such that $\tilde{\rho} \equiv 0$, then applying a diffeomorphism ϕ satisfying $\phi_* g = \frac{\partial}{\partial w}$ we obtain $\Sigma_{1,0}$.

☞ Using the condition $\chi = 0$, we give a local normal form of all control-affine systems Σ that are equivalent to $\Sigma_{1,0}$

$$\left\{ \begin{array}{ll} \dot{z} & = 2a(x) \frac{w^2}{\left(\sqrt{e(x)w+1}+1\right)^2} + b(x)w + c(x) \\ \dot{y} & = w \\ \dot{w} & = u \end{array} \right. ,$$

where $a \neq 0$; see [SR21].

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Study of parabolic systems: Classification 1/5

We now turn to the classification problem of parabolic nonholonomic constraints

$$\mathcal{S}_{1,0} = \{\dot{z} = a(x)\dot{y}^2 + b(x)\dot{y} + c(x)\}, \quad x \in \mathbb{R}^2.$$

The forms that we characterise are the following:

weakly-flat parabolic	$\mathcal{S}'_{1,0} = \left\{\dot{z} = \dot{y}^2 + b(x)\dot{y} + c(x)\right\}$
strongly-flat parabolic	$\mathcal{S}''_{1,0} = \left\{\dot{z} = \dot{y}^2 + c(x)\right\}$
constant-form parabolic	$\mathcal{S}'''_{1,0} = \left\{\dot{z} = \dot{y}^2 + c\right\}$
null-form parabolic	$\mathcal{S}^0_{1,0} = \left\{\dot{z} = \dot{y}^2\right\}$

Table: Nomenclature of parabolic submanifolds.

Study of parabolic systems: Classification 2/5

The classification problem of $\mathcal{S}_{1,0}$ is treated via a classification of control-nonlinear systems of the form

$$\Xi_{1,0} \; : \; \dot{x} = A(x)w^2 + B(x)w + C(x).$$

where A, B, and C are smooth vector fields satisfying $A \wedge B \neq 0$. We denote $\Xi_{1,0} = (A, B, C)$.

$\mathcal{S}'_{1,0} = \{\dot{z} = \dot{y}^2 + b(x)\dot{y} + c(x)\}$	(A, B) commutative
$\mathcal{S}''_{1,0} = \{\dot{z} = \dot{y}^2 + c(x)\}$	(A, B) commutative and $A \wedge C = 0$
$\mathcal{S}'''_{1,0} = \{\dot{z} = \dot{y}^2 + c\}$	(A, B) commutative, and C constant

Table: Reflection of classification of parabolic submanifolds in properties of parabolic systems

Study of parabolic systems: Classification 3/5

Although $\Xi_{1,0}$ is a control-nonlinear system, feedback transformations that preserve its parabolic shape are of the form

$$\tilde{x} = \phi(x) \quad \text{and} \quad w = \psi(x, \tilde{w}) = \beta(x)\tilde{w} + \alpha(x),$$

with $\beta \neq 0$. And, feedback acts on $\Xi_{1,0} = (A, B, C)$ by

$$\tilde{A} = \beta^2 A, \quad \tilde{B} = 2A\alpha\beta + B\beta, \quad \tilde{C} = C + A\alpha^2 + B\alpha$$

Theorem (Existence of a commutative parabolic frame)

There always exists (α, β) such that $(\tilde{A}, \tilde{B})$ is a commutative parabolic frame. As a consequence, $\Xi_{1,0}$ always admits the following normal form

$$\Xi'_{1,0} : \begin{cases} \dot{z} &= w^2 + c_0(x) \\ \dot{y} &= w + c_1(x) \end{cases}$$

Next, we characterise the forms

$$\Xi''_{1,0} : \begin{cases} \dot{z} &= w^2 + c_0(x) \\ \dot{y} &= w \end{cases}, \qquad \Xi'''_{1,0} : \begin{cases} \dot{z} &= w^2 + c \\ \dot{y} &= w \end{cases}.$$

To this end, we define

$$\Gamma = c_0 + (c_1)^2,$$

which under feedback transformations behaves as $\Gamma = \beta^2\tilde{\Gamma}$, hence its sign is an invariant. With $c \in \mathbb{R}$ we have the following canonical forms

$$\Xi^{\pm}_{1,0} : \begin{cases} \dot{z} &= w^2 \pm 1 \\ \dot{y} &= w \end{cases}, \quad \text{and} \quad \Xi^{0}_{1,0} : \begin{cases} \dot{z} &= w^2 \\ \dot{y} &= w \end{cases}.$$

Theorem (Normalisation of parabolic systems)

Let $\Xi'_{1,0} = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y}, C\right)$ be a parabolic control system. Then the following statements hold.

1. *$\Xi'_{1,0}$ is feedback equivalent to $\Xi''_{1,0}$ if and only if $L_A^2 c_1 = 0$.*
2. *$\Xi'_{1,0}$ is feedback equivalent to $\Xi'''_{1,0}$ with $c \neq 0$ if and only if $\Gamma \neq 0$ and it holds*
$$L_A\Gamma = 0, \quad \textit{and} \quad L_B\Gamma + 2\Gamma L_A c_1 = 0.$$
3. *$\Xi_{1,0}$ is feedback equivalent to $\Xi'''_{1,0}$ with $c = 0$ if and only if $L_A^2 c_1 = 0$ holds and, additionally, $\Gamma \equiv 0$.*

☞ In [SR21], we express the conditions for a general parabolic parabolic system $\Xi_{1,0}$.

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We now turn to the characterisation of

$$\mathcal{S}_Q = \{(x, \dot{x}) \in T\mathcal{X},\ \dot{z} = \dot{y}^t Q(x)\dot{y} + b(x)\dot{y} + c(x)\},$$

where $\dim \mathcal{X} \geq 3$.
☞ To characterise $\mathcal{S}_{p,q}$, we will characterise the class of their parametrisations $\Sigma_{p,q}$.

Major difference with the case $n = 2$

To characterise the class $\Sigma_{p,q}$, we will need relations between 2nd order Lie brackets only and not 3rd order relations.

Definition: (p, q)-parabolisable systems

We say that a control-affine system Σ on a $(2m+1)$-dimensional manifold $\mathcal{M}$ and with m controls, is (p, q)-*parabolisable* if it is feedback equivalent to

$$\Sigma_{p,q} : \begin{cases} \dot{x} &= A(x) w^t \mathtt{I}_{p,q} w + \sum_{i=1}^{m} B_i(x) w_i + C(x) \\ \dot{w} &= u \end{cases} ,$$

$(x, w) \in \mathcal{M},\ u \in \mathbb{R}^m$, where $\mathtt{I}_{p,q} = \begin{pmatrix} \mathsf{Id}_p & 0 \\ 0 & -\mathsf{Id}_q \end{pmatrix}$, A, $B = (B_1, \dots, B_m)$, and C are smooth vector fields satisfying $A \wedge B_1 \wedge \dots \wedge B_m \neq 0$.

Consider a control-affine system

$$\Sigma : \dot{\xi} = f(\xi) + \sum_{i=1}^{m} u_i g_i(\xi),$$

with state $\xi \in \mathcal{M}$, a $(2m+1)$-dimensional manifold, and control $u \in \mathbb{R}^m$. We attach to Σ the following distributions

$$\mathcal{D}^0 = \text{span}\left\{g_1, \ldots, g_m\right\} \quad \text{and} \quad \mathcal{D}^1 = \text{span}\left\{g_1, \ldots, g_m, \text{ad}_f g_1, \ldots, \text{ad}_f g_m\right\}.$$

First necessary conditions (for Σ to be feedback equivalent to $\Sigma_{p,q}$):

1. $\mathcal{D}^0$ is involutive and of constant rank m,
2. $\mathcal{D}^1$ has constant rank $2m$,

which encode the fact that Σ is a prolongation of a regular parametrisation of $\mathcal{S}$.

For any $\omega \in \operatorname{ann}\left(\mathcal{D}^1\right)$, we define

$$\begin{aligned} \Omega_\omega \ : \ \mathcal{D}^0 \times \mathcal{D}^0 &\longrightarrow \mathbb{R} \\ (g_i, g_j) &\longmapsto \omega\left([g_i, \mathrm{ad}_f g_j]\right) \end{aligned}$$

Properties of Ω

Ω_ω is a smooth symmetric $(0, 2)$-tensor on $\mathcal{D}^0$ and feedback transformations $f \mapsto f + g\alpha$ and $g \mapsto \tilde{g} = g\beta$ transform Ω_ω into

$$\tilde{\Omega}_\omega = \beta^t \Omega_\omega \beta$$

For $\Sigma_{p,q}$ we have $\operatorname{sgn}(\Omega_\omega) = (p, q)$ thus, third necessary condition

3. For Σ, Ω_ω has constant signature (p, q) with $p + q = m$.

Definition (Weak and strong quadratic frames)

We say that a frame $g = (g_1, \dots, g_m)$ of $\mathcal{D}^0$ is a *weak quadratic frame*, resp. *a strong quadratic frame*, if there exists a smooth vector field $Z \notin \mathcal{D}^1$ such that

$$[g_i, \mathsf{ad}_f g_j] = \mathtt{I}^i_j Z \mod \mathcal{D}^1, \quad \text{resp.} \quad [g_i, \mathsf{ad}_f g_j] = \mathtt{I}^i_j Z \mod \mathcal{D}^0,$$

☞ Weak quadratic frame correspond to $\Omega_\omega = \lambda \mathtt{I}_{p,q}$, for some smooth function $\lambda \neq 0$.

☞ Under assumptions (1-3), a weak quadratic frame of $\mathcal{D}^0$ always exists and can be constructed explicitly;

Study of (p, q)-paraboloid systems: Characterisation 6/6

Theorem (Characterisation of $\Sigma_{p,q}$)

Under assumptions (1-3), Σ is feedback equivalent to $\Sigma_{p,q}$ if and only if there exists a strong quadratic frame.

☞ Existence of a strong quadratic frame can be tested on structure functions defined with the help of any weak quadratic frame attached to Σ.
☞ A equivalent definition of strong quadratic frames is

$$[g_i, \mathrm{ad}_f g_i]\, \mathtt{I}_i^i - [g_j, \mathrm{ad}_f g_j]\, \mathtt{I}_j^j = 0 \mod \mathcal{D}^0 \quad \text{for all } i, j = 1, \ldots, m,$$
$$\text{and} \quad [g_i, \mathrm{ad}_f g_j] = 0 \mod \mathcal{D}^0 \quad \text{if } i \neq j,$$

whose meaning on

$$\Sigma_{p,q} : \begin{cases} \dot{x} &= A(x) w^t \mathtt{I}_{p,q} w + \sum_{i=1}^m B_i(x) w_i + C(x) \\ \dot{w} &= u \end{cases}$$

is clear.

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Study of (p,q)-paraboloid systems: Classification 1/5

We now turn to the classification problem of paraboloid nonholonomic constraints

$$\mathcal{S}_{p,q} = \{\dot{z} = \dot{y}^t Q(x)\dot{y} + b(x)\dot{y} + c(x)\}, \quad x \in \mathbb{R}^n.$$

The forms that we characterise are the following:

diagonal paraboloid	$\mathcal{S}^d_{p,q} = \left\{\dot{z} = \dot{y}^t D(x)\dot{y} + b(x)\dot{y} + c(x)\right\}$
weakly-flat paraboloid	$\mathcal{S}'_{p,q} = \left\{\dot{z} = \dot{y}^t \mathtt{I}_{p,q}\dot{y} + b(x)\dot{y} + c(x)\right\}$
strongly-flat paraboloid	$\mathcal{S}''_{p,q} = \left\{\dot{z} = \dot{y}^t \mathtt{I}_{p,q}\dot{y} + c(x)\right\}$
constant-form paraboloid	$\mathcal{S}'''_{p,q} = \left\{\dot{z} = \dot{y}^t \mathtt{I}_{p,q}\dot{y} + c\right\}$
null-form paraboloid	$\mathcal{S}^0_{p,q} = \left\{\dot{z} = \dot{y}^t \mathtt{I}_{p,q}\dot{y}\right\}$

Table: Nomenclature of paraboloid submanifolds.

The classification problem of $\mathcal{S}_{p,q}$ is treated via a classification of control-nonlinear systems of the form

$$\Xi_{p,q} \ : \ \dot{x} = A(x) w^t \mathtt{I}_{p,q} w + \sum_{i=1}^{m} B_i(x) w_i + C(x), \quad n = m+1,$$

where A, $B_1, \ldots, B_m$, and C are smooth vector fields satisfying $A \wedge B_1 \wedge \ldots \wedge B_m \neq 0$, and $\mathtt{I}_{p,q} = \begin{pmatrix} \mathsf{Id}_p & 0 \\ 0 & -\mathsf{Id}_q \end{pmatrix}$. We denote $\Xi_{p,q} = (A, B, C)$, where $B = (B_1, \ldots, B_m)$.

$\mathcal{S}'_{p,q} = \left\{ \dot{z} = \dot{y}^t \mathtt{I}_{p,q} \dot{y} + b(x) \dot{y} + c(x) \right\}$	(A, B) commutative
$\mathcal{S}''_{p,q} = \left\{ \dot{z} = \dot{y}^t \mathtt{I}_{p,q} \dot{y} + c(x) \right\}$	(A, B) commutative and $A \wedge C = 0$
$\mathcal{S}'''_{p,q} = \left\{ \dot{z} = \dot{y}^t \mathtt{I}_{p,q} \dot{y} + c \right\}$	(A, B) commutative, and C constan

Study of (p, q)-paraboloid systems: Classification 3/5

Proposition: Equivalence of (p, q)-paraboloid control-nonlinear systems

If two (p, q)-system $\Xi_{p,q} = (A, B, C)$ and $\tilde{\Xi}_{p,q} = (\tilde{A}, \tilde{B}, \tilde{C})$ are feedback equivalent via a diffeomorphism $\tilde{x} = \phi(x)$ and an invertible feedback transformation $w = \psi(x, \tilde{w})$, then $\psi(x, \tilde{w}) = \alpha(x) + \beta(x)\tilde{w}$ where $\alpha \in C^\infty(\mathcal{X}, \mathbb{R}^m)$ and $\beta \in C^\infty(\mathcal{X}, GO(p, q))$, i.e. $\beta^t \mathtt{I}_{p,q} \beta = \lambda\, \mathtt{I}_{p,q}$ with λ a smooth function satisfying $\lambda(\cdot) \neq 0$. Moreover, we have

$$\tilde{A} = \phi_*(\lambda A), \quad \tilde{B} = \phi_*(2A\,\alpha^t \mathtt{I}_{p,q}\beta + B\beta),$$
$$\text{and} \quad \tilde{C} = \phi_*(C + A\,\alpha^t \mathtt{I}_{p,q}\alpha + B\alpha).$$

- ☞ Feedback acts locally in x and globally in w;
- ☞ Distribution $\mathcal{A} = \text{span}\,\{A\}$ is invariantly related to $\Xi_{p,q}$;

Study of (p, q)-paraboloid systems: Classification 4/5

We now present our characterisation of the following normal form

$$\Xi'_{p,q} : \dot{x} = w^t \mathtt{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + C(x);$$

To this end, we define two subclasses of (p, q)-frame:

Definition (p, q)-frame

We say that a (p, q)-frame (A, B) if

1. *pseudo-commutative* if $[A, B_j] = 0 \mod \mathcal{A}$;
2. *commutative* if $[A, B_j] = [B_i, B_j] = 0$;

Moreover, for any almost-commutative frame (A, B), we define

$$\begin{aligned} \pi_* : T\mathcal{X} &\longrightarrow T\mathcal{X}/\mathcal{A} \\ B_i &\longmapsto \pi_* B_i. \end{aligned}$$

And we set the pseudo-Riemannian metric $\mathtt{g}_B$ such that

$$\mathtt{g}\left(\pi_* B_i, \pi_* B_j\right) = \mathtt{I}_j^i,$$

i.e. the vector fields $\pi_* B_i$ are mutually orthonormal with respect to the quadratic form $\mathtt{I}_{p,q}$.

Theorem (Existence of a commutative (p,q)-frame)

Consider a (p,q)-paraboloid nonlinear system $\Xi_{p,q} = (A, B, C)$ with its (p,q)-frame (A, B). Then, the following statements are locally equivalent,

1. *$\Xi_{p,q}$ is feedback equivalent to $\Xi'_{p,q}$,*
2. *There exists (α, β) such that $(\tilde{A}, \tilde{B})$ is a commutative (p,q)-frame.*
3. *There exists (α, β) such that $(\tilde{A}, \tilde{B})$ is a pseudo-commutative (p,q)-frame and the pseudo-Riemannian metric $\mathtt{g}_{\tilde{B}}$ is conformally flat.*

☞ Existence of a pseudo-commutative (p,q)-frame can be tested with the help of well defined structure functions attached to (A, B).

Study of (p,q)-paraboloid systems: Classification 1/2

Normalisation of paraboloid systems

We have algebraic conditions to characterise the systems

$$\Xi''_{p,q} : \dot{x} = w^t \mathrm{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + c_0(x) \frac{\partial}{\partial z},$$

$$\Xi'''_{p,q} : \dot{x} = w^t \mathrm{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + c_0 \frac{\partial}{\partial z}.$$

☞ Those are very complicated to interpret. We have additional conditions because, to obtain $A \wedge C = 0$ we use α and we are left with only β to construct a commutative frame.

Study of (p, q)-paraboloid systems: Classification 2/2

Canonical form of constant paraboloid systems

When $c_0 \in \mathbb{R}$ is constant we have the following canonical forms, which depends on the invariant sign of a suitable function $\Gamma_{p,q}$,

$$\Xi^0_{p,q} \; : \; \dot{x} = w^t \mathrm{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + 0, \quad \text{or}$$

$$\Xi^\varepsilon_{p,q} \; : \; \dot{x} = w^t \mathrm{I}_{p,q} w \frac{\partial}{\partial z} + \sum_{i=1}^{m} w_i \frac{\partial}{\partial y_i} + \varepsilon_{p,q} \frac{\partial}{\partial z},$$

with $\varepsilon_{p,q} = \begin{cases} \pm 1 & \text{if} \quad p \neq q \\ 1 & \text{if} \quad p = q \end{cases}$. Moreover, $\Xi_{p,q}$ is equivalent to the former if and only if $\Gamma_{p,q} \equiv 0$ and to the latter if and only if $\Gamma_{p,q} > 0$ or $\Gamma_{p,q} < 0$ when $p \neq q$, or $\Gamma_{p,q} \neq 0$ when $p = q$.

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Conclusion and perspectives 1/2

Summary of our contributions

1. Study of the link between nonholomic constraints and control systems;
2. To characterise and classify paraboloid nonholomic constraints $\mathcal{S}_{p,q}$, we introduce a new class of control-nonlinear systems $\Xi_{p,q}$ together with their extensions $\Sigma_{p,q}$;
3. We give a complete characterisation of (p,q)-paraboloid control systems $\Sigma_{p,q}$ in any dimension;
4. We propose a classification of (p,q)-paraboloid systems $\Xi_{p,q}$;

Conclusion and perspectives 2/2

Perspectives

1. Obtain better interpretations of our conditions;
2. Study the problem of equivalence of general quadratic nonholomic constraints (done for $n = 2$);

$$S_q(x, \dot{x}) = \dot{x}^t \mathrm{g}(x)\dot{x} + 2\omega(x)\dot{x} + h(x) = 0$$

3. Generalise our results to polynomials of any degree in $\dot{y}$:

$$\dot{z} - \sum_{i=0}^{d} a_i(x)\dot{y}^i = 0.$$

4. Generalise to corank $k \geq 2$ quadratic submanifolds;

Thank you for your attention!
Any questions?

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References 1/2

[ANN15] Ian Anderson, Zhaohu Nie, and Pawel Nurowski. "Non-Rigid Parabolic Geometries of Monge Type". *Advances in Mathematics* 277 (2015), pp. 24–55. doi: 10.1016/j.aim.2015.01.021.

[Dub57] Lester E. Dubins. "On Curves of Minimal Length with a Constraint on Average Curvature, and with Prescribed Initial and Terminal Positions and Tangents". *American Journal of Mathematics* 79.3 (1957), p. 497. doi: 10.2307/2372560.

[Elk12] Vladimir I. Elkin. *Reduction of Nonlinear Control Systems: A Differential Geometric Approach*. Vol. 472. Springer Science & Business Media, 2012.

[JZ01] Bronislaw Jakubczyk and Michail Zhitomirskii. "Local Reduction Theorems and Invariants for Singular Contact Structures". *Annales de l'institut Fourier* 51.1 (2001), pp. 237–295. doi: 10.5802/aif.1823.

[SR21] Timothée Schmoderer and Witold Respondek. *Conic Nonholonomic Constraints on Surfaces and Control Systems*. 2021. url: http://arxiv.org/abs/2106.08635.

[Zhi92] Michail Zhitomirskii. *Typical Singularities of Differential 1-Forms and Pfaffian Equations*. Vol. 113. American Mathematical Soc., 1992.

[Zhi95] Michail Zhitomirskii. "Singularities and Normal Forms of Smooth Distributions". *Banach Center Publications* 32.1 (1995), pp. 395–409. doi: 10.4064/-32-1-395-409.

References 2/2

[ZR98] Michail Zhitomirskii and Witold Respondek. "Simple Germs of Corank One Affine Distributions". *Banach Center Publications* 44.1 (1998), pp. 269–276. doi: 10.4064/-44-1-269-276.