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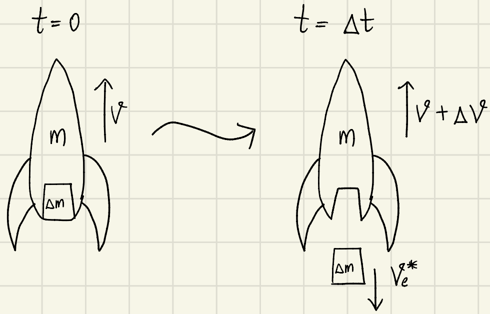
Tsiolkovsky Ideal Rocket Equation Derivation

Consider a rocket with initial mass $(m+\Delta m)$ where Δm is the amount of mass exhausted at time Δt:



← these graphics are relative to a hypothetical observer on the ground

We know from Newton's third law that the rocket gains velocity Δv as a result of expelling mass Δm.
The change in momentum of the system Δp is:

$$\begin{aligned}\Delta p &= p_2 - p_1\\ &= (m+\Delta m)v - m(v+\Delta v) - \Delta m v_e^*\\ &= \cancel{mv} + m\Delta v + \Delta m v_e^* - \cancel{mv} - \Delta m v\end{aligned}$$

v_e^* is the exhaust velocity as observed by an observer on the ground.
Taken relative to the rocket, the exhaust velocity is

$$v_e = v - v_e^*$$

← this is the exhaust velocity according to the observer, negative because its direction is opposite that of the rocket.

↑ think of this as the transformation per unit time from the reference point of the rocket to the r.p. of the observer

substituting $v_e^* = v - v_e$, we get

$$\Delta p = m\Delta v + \cancel{\Delta m v} - \Delta m v_e - \cancel{\Delta m v}$$

$$\Delta p = m\Delta v - \Delta m v_e$$

Working in infinitesimal time changes, since $\oplus \Delta m$ means an increase in rocket mass, conversely $-\Delta m$ indicates decrease of rocket mass (exhaust). $\oplus \Delta v$ means an increase in rocket velocity so we use it as-is

$$\therefore dm = -\Delta m \quad \text{and} \quad dv = \Delta v$$ for infinitesimal time change $\Delta t \to 0$

From Newton's 2nd Law of Motion, the rate of change in momentum is equal to the sum of external forces acting on the object:

$$\begin{aligned}\Sigma F &= \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}\\ &= m\frac{dv}{dt} + v_e \frac{dm}{dt}\end{aligned}$$

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With the assumption that there are no external forces acting on the rocket,

$$\sum F = 0 = m\frac{dv}{dt} + v_e\frac{dm}{dt}$$

$$m\frac{dv}{dt} = -v_e\frac{dm}{dt}$$

$$dv = -v_e\left(\frac{1}{m}\,dm\right)$$

integrate both sides from v_0 to v_f for the left side and from m_0 to m_f for the right side

Note: this is different from the earlier velocities and v and $v+\Delta v$, and masses m and $m+\Delta m$.

m_0 is the initial total mass of the rocket (including the propellant).

m_f is its final mass after all the propellants are exhausted.

v_0 is the velocity of the rocket before exhausting any propellant.

v_f is its velocity after exhausting all of its propellant.

$$\int_{v_0}^{v_f} dv = -v_e\int_{m_0}^{m_f}\frac{1}{m}\,dm$$

$$[v]_{v_0}^{v_f} = -v_e\left[\ln(m_f) - \ln(m_0)\right]$$

$$\underbrace{v_f - v_0} = -v_e\ln\left(\frac{m_f}{m_0}\right)$$

$$\boxed{\Delta v = v_e\ln\left(\frac{m_0}{m_f}\right)}$$

$m_0 \rightarrow$ initial total mass

$m_f \rightarrow$ final total mass

$v_e \rightarrow$ effective exhaust velocity taken relative to the rocket

$\Delta v \rightarrow$ maximum change of velocity by expulsion of propellant

Deriving relation with specific impulse

$$\Delta v = v_e\ln\left(\frac{m_0}{m_f}\right)$$

Specific impulse I_{sp} is defined as the thrust integrated over time per unit weight on Earth.

$$I_{sp} = \frac{\int_0^{\Delta t} F_{thrust}\,dt}{\Delta W}$$

$\Delta W \rightarrow$ total weight of propellant used from $t=0$ to $t=\Delta t$

$$= \frac{\Delta p}{\Delta m \cdot g_0}$$

$$= \frac{\cancel{\Delta m}\cdot v_e}{\cancel{\Delta m}\cdot g_0}$$

$$I_{sp} = \frac{v_e}{g_0} \Rightarrow v_e = I_{sp}\,g_0$$

substituting this into the Tsiolkovsky equation,

$$\Delta v = v_e\ln\left(\frac{m_0}{m_f}\right)$$

$$= I_{sp}\,g_0\ln\left(\frac{m_0}{m_f}\right)$$

$$m_0 = \underset{\text{engine}}{m_E} + \underset{\text{propellant}}{m_P} + \underset{\text{Payload}}{m_{PL}}$$

$$m_f = m_E + m_{PL}$$

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$$= I_{sp}\, g_0 \ln\left(\boxed{\frac{m_E + m_P + m_{PL}}{m_E + m_{PL}}}\right)$$

initial to final mass ratio η (eta)

$$\Delta v = I_{sp}\, g_0 \ln \eta$$

another way to express η:

$$\eta = \frac{m_E + m_P + m_{PL}}{m_E + m_{PL}}$$

divide numerator and denominator by $m_E + m_P$:

$$\eta = \frac{\frac{m_E + m_P + m_{PL}}{m_E + m_P}}{\boxed{\frac{m_E}{m_E + m_P}} + \boxed{\frac{m_{PL}}{m_E + m_P}}}$$

structural ratio ε
(ratio of empty stage mass and initial stage mass)

payload ratio λ
(ratio between payload mass and initial stage mass)

$$\eta = \frac{1 + \frac{m_{PL}}{m_E + m_P}}{\varepsilon + \lambda}$$

$$\eta = \frac{1 + \lambda}{\varepsilon + \lambda}$$

Checking multistaging advantage

$$\Delta v = I_{sp}\, g_0 \ln\left(\frac{m_0}{m_f}\right)$$

Let's introduce two more ratios p and σ, where

$$p = \frac{m_{PL}}{m_0} \quad \text{and} \quad \sigma = \frac{m_E}{m_P}$$

it follows that $m_{PL} = p\, m_0$ and $m_E = \sigma m_P$.

$$m_0 = m_{PL} + m_E + m_P$$

$$m_0 = p m_0 + \sigma m_P + m_P$$

$$(1-p) m_0 = (\sigma + 1) m_P$$

$$m_0 = \frac{1+\sigma}{1-p} m_P$$

the mass ratio in Tsiolkovsky's equation can be rewritten in terms of p and σ:

$$\frac{m_0}{m_f} = \frac{\frac{1+\sigma}{1-p} m_P}{m_{PL} + m_P}$$

$$= \frac{\frac{1+\sigma}{1-p} m_P}{p m_0 + \sigma m_P}$$

$$= \frac{\frac{1+\sigma}{1-p} \cancel{m_P}}{p \frac{1+\sigma}{1-p} \cancel{m_P} + \sigma \cancel{m_P}}$$

$$= \frac{\frac{1+\sigma}{1-p}}{p \frac{1+\sigma}{1-p} + \sigma}$$

$$= \frac{1+\sigma}{p(1+\sigma) + \sigma(1-p)}$$

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$$= \frac{1+\sigma}{p+\cancel{p\sigma}+\sigma-\cancel{p\sigma}}$$

$$\frac{m_0}{m_f} = \frac{1+\sigma}{p+\sigma}$$

Substituting this back to the equation, we have

$$\Delta V = I_{sp}\, g_0 \ln\left(\frac{1+\sigma}{p+\sigma}\right)$$

For the i-th stage of a rocket, the ΔV is

$$\Delta V_i = I_{sp,i}\, g_0 \ln\left(\frac{1+\sigma_i}{p_i+\sigma_i}\right)$$

For the purposes of this exercise let's assume the stages are identical in engine-to-propellant ratio ($\sigma_i = \sigma$) and specific impulse ($I_{sp,i} = I_{sp}$). Then the equation becomes

$$\Delta V_i = I_{sp}\, g_0 \ln\left(\frac{1+\sigma}{p_i+\sigma}\right) \Rightarrow \Delta V_{tot} = \sum I_{sp}\, g_0 \ln\left(\frac{1+\sigma}{p_i+\sigma}\right)$$

By definition, the "true" or "total" payload-to-initial mass ratio p_{tot} is equal to the sequential multiplication of all the stages' payload ratios. For example, consider a simple TSTO rocket:

$$\left(p_{tot} = \prod p_i = p_1 \times p_2 \times \ldots \times p_N\right)$$

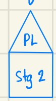
pi (notation for sequential multiplication)

Stage 1

not only PL but the mass of stage 2 also counts as payload of stage 1

$$p_1 = \frac{m_{PL}+m_2}{m_{PL}+m_2+m_1}$$

Stage 2



$$p_2 = \frac{m_{PL}}{m_{PL}+m_2}$$

if we multiply these together, we get

$$p_1 \times p_2 = \frac{\cancel{m_{PL}+m_2}}{m_0} \times \frac{m_{PL}}{\cancel{m_{PL}+m_2}}$$

$$= \frac{m_{PL}}{m_0}$$ (which is the total payload ratio, obviously).

hopefully it's obvious to see why p_{tot} is a sequential multiplication of the stages' payload-to-initial mass ratios. The initial mass of stage $i+1$ is the payload mass of stage i (the stage before it), so they cancel out and yield $\frac{m_{PL}}{m_0}$ at the end.

Again we will assume that $p_i = p$ for this exercise.

$$p_{tot} = \prod_1^N p_i$$

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$$p_i = \sqrt[N]{p_{tot}}$$ (since $p_i = p$)

Finally, substituting this back to the Tsiolkovsky equation,

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$$\Delta V_{tot} = \sum_{1}^{N} I_{sp} g_0 \ln\left(\frac{1+\sigma}{p_i+\sigma}\right)$$

$$\Delta V_{tot} = N \times I_{sp} g_0 \ln\left(\frac{1+\sigma}{\sqrt[N]{p_{tot}}+\sigma}\right) \quad //$$

as $N \to \infty$,

$$\Delta V_{tot_{N\to\infty}} = \lim_{N\to\infty} N \times I_{sp} g_0 \ln\left(\frac{1+\sigma}{\sqrt[N]{p_{tot}}+\sigma}\right)$$

$$= I_{sp} g_0 \underbrace{\lim_{N\to\infty} N \ln\left(\frac{1+\sigma}{\sqrt[N]{p_{tot}}+\sigma}\right)}$$

Focusing on this limit,

as $b \to \infty$, $(a)^{1/b} = e^{\ln(a)/b} \approx 1 + \frac{\ln a}{b}$.

$$\lim_{N\to\infty} N \ln\left(\frac{1+\sigma}{1+\frac{\ln(p_{tot})}{N}+\sigma}\right) = \lim_{N\to\infty} N \ln\left(\frac{1}{1+\frac{\ln(p_{tot})}{N(1+\sigma)}}\right)$$

$$= \lim_{N\to\infty} N \left\{\cancelto{0}{\ln(1)} - \ln\left(1+\frac{\ln(p_{tot})}{N(1+\sigma)}\right)\right\}$$

this goes to 0 as $N \to \infty$.

using property $\ln(1+\varepsilon) \approx \varepsilon$ as $\varepsilon \to 0$,

$$= \lim_{N\to\infty} \cancel{N} \times \left(-\frac{\ln(p_{tot})}{\cancel{N}(1+\sigma)}\right)$$

$$= -\frac{\ln(p_{tot})}{1+\sigma}$$

$$\therefore \Delta V_{tot_{N\to\infty}} = I_{sp} g_0 \left(-\frac{\ln(p_{tot})}{1+\sigma}\right) \quad //$$

Now let's try varying N with the same parameters for everything else.
Suppose $I_{sp} = 400$ [s], $g_0 = 9.8$ [ms^{-2}], $p_{tot} = 0.05$ and $\sigma = 10\%$.

The theoretical limit when $N = \infty$ is

$$\Delta V_{tot_{N\to\infty}} = 400 \times 9.8 \times \left(-\frac{\ln(0.05)}{1+0.1}\right)$$

$$\Delta V_{tot_{N\to\infty}} = 10{,}675.70047 \text{ m/s}.$$

Let's see how close we can get with more feasible # of stages. We can use numpy to calculate the respective ΔV_{tot}'s and MatPlotLib to display the results as a scatter plot.

continued on the next page

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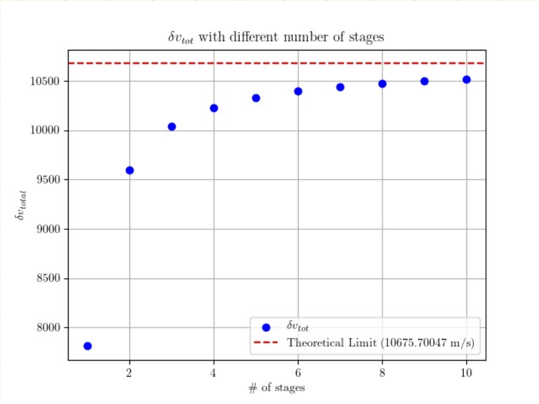
```python
def main():
    setup()

    specific_impulse = 400
    surface_grav_accel = 9.8
    payload_ratio_tot = 0.05
    construct_ratio = 0.1
    # Number of stages
    n = np.arange(1,11)

    delta_v_tot = n * specific_impulse * surface_grav_accel * np.log((1+construct_ratio)/(pow(payload_ratio_tot, 1/n) + construct_ratio))
    theoretical_max = specific_impulse * surface_grav_accel * (-(np.log(payload_ratio_tot))/(1 + construct_ratio))

    print(f"The theoretical limit of delta v_tot is {theoretical_max} m/s.")
    for n_i, delta_v_i in zip(n, delta_v_tot):
        print(f"For n = {n_i}, delta v_tot = {round(delta_v_i, 5)} m/s ({round(delta_v_i / theoretical_max * 100, 2)}% of theoretical limit).")

    # Scatter plot
    plt.grid(True)
    plt.scatter(n, delta_v_tot, color = 'blue', label='$\delta v_{tot}$')
    plt.xlabel('\# of stages')
    plt.ylabel('$\delta v_{total}$')
    plt.title("$\delta v_{tot}$ with different number of stages")

    # Line showing theoretical max
    plt.axhline(y=theoretical_max, color='red', linestyle='--', label=f'Theoretical Limit ({round(theoretical_max, 5)} m/s)')

    plt.legend()
    plt.show()
```

```
python main.py
(.venv) max@tuf-a14 ~/Desktop/temp-report-2 $ python main.py
/home/max/Desktop/temp-report-2/main.py:29: SyntaxWarning: invalid escape sequence '\d'
  plt.scatter(n, delta_v_tot, color = 'blue', label='$\delta v_{tot}$')
/home/max/Desktop/temp-report-2/main.py:30: SyntaxWarning: invalid escape sequence '\#'
  plt.xlabel('\# of stages')
/home/max/Desktop/temp-report-2/main.py:31: SyntaxWarning: invalid escape sequence '\d'
  plt.ylabel('$\delta v_{total}$')
/home/max/Desktop/temp-report-2/main.py:32: SyntaxWarning: invalid escape sequence '\d'
  plt.title("$\delta v_{tot}$ with different number of stages")
The theoretical limit of delta v_tot is 10675.700465756041 m/s.
For n = 1, delta v_tot = 7810.32625 m/s (73.16% of theoretical limit).
For n = 2, delta v_tot = 9592.52433 m/s (89.85% of theoretical limit).
For n = 3, delta v_tot = 10039.93656 m/s (94.04% of theoretical limit).
For n = 4, delta v_tot = 10229.71416 m/s (95.82% of theoretical limit).
For n = 5, delta v_tot = 10333.13928 m/s (96.79% of theoretical limit).
For n = 6, delta v_tot = 10397.91484 m/s (97.4% of theoretical limit).
For n = 7, delta v_tot = 10442.20378 m/s (97.81% of theoretical limit).
For n = 8, delta v_tot = 10474.36496 m/s (98.11% of theoretical limit).
For n = 9, delta v_tot = 10498.76598 m/s (98.34% of theoretical limit).
For n = 10, delta v_tot = 10517.90658 m/s (98.52% of theoretical limit).
```



We see that just with 2 stages, we can theoretically obtain 90% of the theoretical limit of performance. As we increase the number of stages further, the increase of performance obtained gets smaller and smaller (diminishing returns). This is why most rockets are only two or three-stage-to-orbit. The increased complexity in manufacturing, assembly, operation and control systems, in addition to the decrease in reliability, are disadvantages that engineers weigh against the advantages of multi-staging.