Mathematical Logic:
A Precursor to Mathematical Proof

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1 Elements of Mathematical Logic

What is a proof? What is knowledge? What is truth? These are questions of central importance in our study of computer modeling; for computers are merely machines that manipulate information and understanding the relationship between pieces of information and their importance is necessary in order to use these information machines to our advantage.

1.1 Mathematical/logic Symbols

In order to develop some facility with using computers at this level, some understanding of mathematical logic is necessary. The following symbols will be used.

Mathematical/Logic Symbols	
Symbol	Meaning / Interpretation
$\neg$	negation
$\wedge$	and
$\vee$	or
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
$\in$	an element of
$\exists$	there exists a ..., there are some ...
$\forall$	for all ...
$\ni$	such that ...
$\subset$	contained in ...
$\subseteq$	a subset of ...
$\emptyset$	the null or empty set

Table 1: Mathematical/Logic Symbols

We will be using the information in Table 1.1 extensively.

1.2 Truth Tables

Another important element in our exploration of logic involves the use of **truth tables**. Such tables can facilitate an understanding of the "truth" of a statement. In symbolic logic a statement may be referred to using a simple letter such as A. This is often used to represent statements or the *hypothesis* of some set of statements. Another statement representing the *conclusion* can be denoted symbolically by B. Various logical relationships and statements can then be based on the relationships between the statements A and B. The right-hand column is used to denote whether the statement at the top of the column is *consistent* and therefore *true* with the *truth-values* of the statements in previous columns. If A is a true statement, it is often symbolized by placing a 1 in the table; if it is false then a 0 is placed in the table. (The use of 1s and 0s can be used to our advantage in the computer world.) Thus, if A is true (1), then $\neg A$ is false (0) and vice versa (sometimes the negation of a statement A is symbolized by $\overline{A}$). The following truth table represents the truth of the statements: If A is true **and** B is true, then the statement $A \wedge B$ is true. This is shown in the following truth table which in essence defines the statement $A \wedge B$.

Truth Table		
A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

Note that the last row is the only row in which the statement $A \wedge B$ is consistent with the truth values in the previous two columns. This table therefore contains all the logic relationships between these three statements. Any other truth table that has the same truth values in the same positions is said to be *logically equivalent* to this truth table.

One of the most important statement in proofs is the inference statement A implies B. Although this can be understood in various forms of the English language, it is less ambiguous (usually) to state it symbolically and indicate its truth table. This statement can also be put as follows: **If** A is **true**, **then** B **must** be true. This is often shortened to "If A then B. Yet another way of stating this is "A is **sufficient** for B or "B is a necessary consequence of A. Symbolically, it is often written using the following symbols: $A \Rightarrow B$. This *implication* operator has the following truth table.

Truth Table		
A	B	$A \Rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

While it may seem surprising that the last column for rows 1, 2 and 4 show that the implication statement is true or is consistent with the truth values of A and B in those rows, upon further examination we see that indeed, this makes sense. Afterall, the implication statement says nothing about what value B *must* have except in cases where A is true. Thus, the only row that is inconsistent here is the 3rd row, hence, has the value of 0.

Regardless of how we might be tempted to argue or justify which cells have 0s and 1s in them, the truth table essentially defines the meaning of a logical relationship. In this case, by definition, the statement $A \Rightarrow B$ is to be regarded as true *except* in cases where A is true *and* B is false. The truth table completely describes all the possibilities for this relational operator.

1.3 Compound Statements

Statements can be made up of other statements—that is, a compound statement can be concocted which has its own truth values. For example, the statement $A \wedge B$ can itself be used as one of the hypotheses of a larger statement. Using the logic of the previous truth table, it is possible to generate the following table . Can you see the logic?

Truth Table			
A	B	$A \wedge B$	$A \Rightarrow A \wedge B$
0	0	0	1
0	1	0	1
1	0	0	0
1	1	1	1

Here the last column is based on the truth values of column 1 and column 3 using our $A \Rightarrow B$ truth table as a guide for the truth value of the last column. Note also that the last column is logically equivalent to the 3rd column in the $A \Rightarrow B$ truth table. Thus, we can conclude that $A \Rightarrow B$ is *logically equivalent* to $A \Rightarrow (A \wedge B)$, or $A \Rightarrow B \equiv A \Rightarrow (A \wedge B)$. Both statements have the same truth values given the same truth values of their constituent parts. Thus, when one statement is true or false, we know right away whether the other statement is true or false. When you think about it for these statements, it makes sense.

Another important compound statement is based on the following truth table.

Truth Table				
A	B	$A \Rightarrow B$	$B \Rightarrow A$	$(A \Rightarrow B) \wedge (B \Rightarrow A)$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

The statement $(A \Rightarrow B) \wedge (B \Rightarrow A)$ is often written as $A \Leftrightarrow B$. Thus, if A is true then B must be true *and* if B is true then A must be true. Another common way of stating this is: A if and only if B. In this statement, B is a *necessary* consequence of A and B is *sufficient* for A. This gives rise to a commonly used description of necessary and sufficient conditions. Finally, note that the statement $A \Leftrightarrow B$ indicates that if either A or B is true (false) then the other is also true (false).

1.4 Tautologies

It can be shown that when two statements S_1 and S_2 are logically equivalent (*i.e.*,$S_1 \equiv S_2$, then the statement $S_1 \Leftrightarrow S_2$ is a *tautology* which means that it is *always true* regardless of the truth values of each statements constituent parts. It is easy to see that when two statements always have the same relative truth values, then obviously both have either 0s or 1s. In this case, the implication operator is true regardless of whether $A \Rightarrow B$ or $B \Rightarrow A$. More generally, if $S_1 \equiv S_2$ then $S_1 \Rightarrow S_2 \wedge S_2 \Rightarrow S_1$ is always true, hence a tautology. Can you show this by a truth table?

The following truth table gives an example of two statements which are logically equivalent and includes the truth values of the statement $S_1 \Leftrightarrow S_2$. Do you know what “law” of logic this represents?

Truth Table				
A	B	$\neg A \vee \neg B$	$\neg(A \wedge B)$	$(\neg A \vee \neg B) \Leftrightarrow \neg(A \wedge B)$
0	0	1	1	1
0	1	1	1	1
1	0	1	1	1
1	1	0	0	1

Other examples of logic statements are:
$A \vee \neg A$
$A \Rightarrow A \vee B$
$A \Rightarrow A \wedge B$
$(A \Rightarrow B) \Rightarrow (A \wedge B)$
$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$.

Can you determine which ones are tautologies?

1.5 Contradictions

The concept of *contradiction* is similar to that of tautology: contradictions are essentially the opposite of a tautology. Thus, a contradiction is a statement that is *always false* regardless of the truth value of its constituent parts. The most obvious example of a contradiction is $A \Rightarrow \neg A$. Another example is the statement: A and B and not A or B. In symbols this can be stated as: $(A \wedge B) \wedge \neg(A \vee B)$. This is always false, *i.e.,*has truth value 0. To see this, we build the following truth table:

Truth Table				
A	B	$A \wedge B$	$\neg(A \vee B)$	$(A \wedge B) \wedge \neg(A \vee B)$
0	0	0	1	0
0	1	0	0	0
1	0	0	0	0
1	1	1	0	0

In this case, all possible combinations of the truth of A and B lead to a false truth value in the right-hand column. This makes sense because the right-hand column is simply the negation of the earlier tautology.

1.6 Rules of Inference

The "kernel" of mathematical and logical deduction are *rules of inference* in which it is possible to infer the truth of a statement based on the truths of other statements.

This is the heart of deductive logic. For example, it is fairly obvious that if $A \Rightarrow B$ and $B \Rightarrow C$, then $A \Rightarrow C$. To verify this, lets build the following truth table:

Truth Table							
A	B	C	$A \Rightarrow B$	$B \Rightarrow C$	$A \Rightarrow C$	$A \Rightarrow B \wedge B \Rightarrow C$	$(A \Rightarrow B \wedge B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	1	0	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	1	1	1
1	1	1	1	1	1	1	1

Thus, the statement $(A \Rightarrow B \wedge B \Rightarrow C) \Rightarrow (A \Rightarrow C)$ is tautological. Statements of the form $S_1 \wedge S_2 \wedge \ldots \wedge S_n \Rightarrow D$ that are tautological are called *rules of inference* and are the basic tool in deductive reasoning. Some examples of rules of inference are expressed below.

Rules of Inference	
Statements	Inference
$A \wedge (\neg B \Rightarrow \neg A)$	B
$A \wedge (A \Rightarrow B)$	B
$A \Rightarrow B \wedge B \Rightarrow C$	C
$A \wedge B$	$A \vee B$
$A \Rightarrow B \wedge B \Rightarrow A$	$A \Leftrightarrow B$

2 Mathematical Statements

So far, all of the statements we have considered have been very abstract using the letters A, B, etc. In mathematics and other realms of logic, statements have parts which are referred to as *quantifiers* and *predicates* and are described below.

2.1 Quantifiers

We have seen quantifiers before, *i.e.,*the *universal* quantifier $\forall$ and the *existential* quantifier $\exists$. These symbols represent the set of entities to which the *predicate* applies such that the predicate is either clearly true or clearly false. For example, quantifiers could refer to all values of x that aregreater than 2 and this is symbolized as $\forall x > 2$.

2.2 Predicates

These are parts of mathematical statements which have truth values that are functionally dependent on some set of entities specified by the quantifier. For instance, the statement $x^2 - x > 0$ can be true or false depending upon the value of x.

Used together, quantifiers and predicates specify statements that are either true or false. Thus, the statement $\{\forall x > 1, x^2 - x > 0\}$ is a true statement , whereas the statement $\{\forall x > 0, x^2 - x > 0\}$ is false because it is not true that the predicate is true *for all* values of $x > 0$. In this case, the predicate encompasses values of x between 0 and 1 in which case the predicate is false. **Remember, a statement that is not always true is false.**

Since the statement $\{\forall x > 0, x^2 - x > 0\}$ is false, its negation must be true. How can this be symbolized? One way is to simply *negate* the entire statement. Thus, $\neg\{\forall x > 0, x^2 - x > 0\}$ should be true. But how can this be rewritten?

The negation operator can be distributed through the entire statement as follows: $\neg\{\forall x > 0, x^2 - x > 0\} \equiv \{\neg\forall x > 0, \neg(x^2 - x > 0)\}$ and this should be true. But the negation of the universal quantifier is the existential quantifier. Thus, $\neg\forall x > 0$ means the same as $\exists x > 0$. This is because the opposite of *all* is *some*.

The negation $\neg(x^2 - x > 0)$ means the same as $x^2 - x \leq 0$. Thus, the statement can now be written as $\{\exists x > 0 : x^2 - x \leq 0\}$. Thus, in english, there exists an x greater than 0 such that the predicate is true. This statement (as a whole) is true because *there is* some x, namely $x = \frac{1}{2}$ such that $x^2 - x \leq 0$.

These are the types of statements (and more complex ones) that are used in making inferences in mathematical proofs. Just like above, certain rules of inference can be deduced and used in other contexts so that new truths can be ascertained and verified.