INTRODUCTION TO DIFFERENTIABLE MANIFOLDS

spring 2006

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Introduction to differentiable manifolds
Lecture notes version 1.1, May 26, 2006

This is a self contained set of lecture notes. The notes were written by Rob van der Vorst. We follow the book 'Introduction to Smooth Manifolds' by John M. Lee as a reference text.

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I. Manifolds

1. Topological manifolds

Basically an m-dimensional (topological) manifold is a topological space M which is locally homeomorphic to $\mathbb{R}^m$. A more precise definition is:

Definition 1.1. [1] A topological space M is called an ***m-dimensional (topological) manifold***, if the following conditions hold:

(i) M is a Hausdorff space,
(ii) for any $p \in M$ there exists a neighborhood[2] U of p which is homeomorphic to an open subset $V \subset \mathbb{R}^m$, and
(iii) M has a countable basis of open sets.

Axiom (ii) is equivalent to saying that $p \in M$ has a open neighborhood $U \ni p$ homeomorphic to the open disc D^m in $\mathbb{R}^m$. We say M is ***locally homeomorphic*** to $\mathbb{R}^m$. Axiom (iii) says that M can be covered by countably many of such neighborhoods.

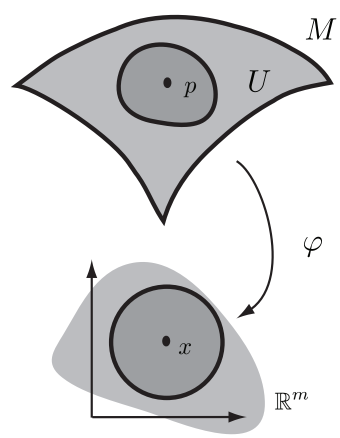

FIGURE 1. Coordinate charts (U, φ).

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[1] See Lee, pg. 3.

[2] Usually an open neighborhood U of a point $p \in M$ is an open set containing p. A neighborhood of p is a set containing an open neighborhood containing p. Here we will always assume that a neighborhood is an open set.

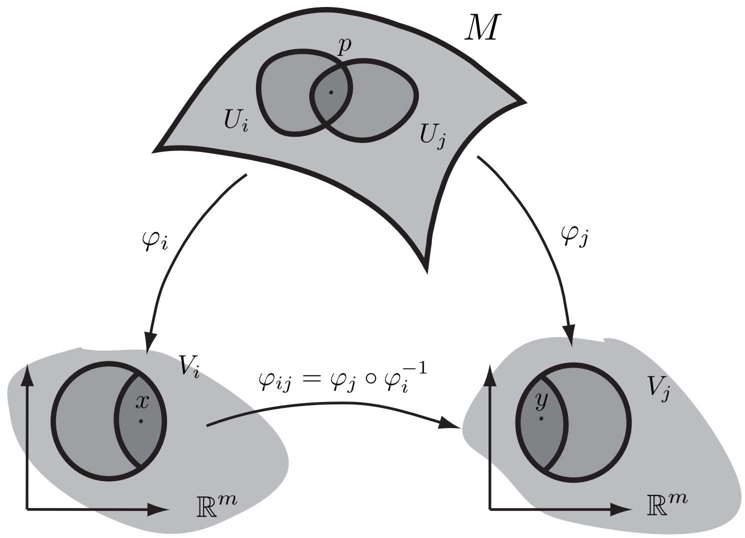

FIGURE 2. The transition maps φ_{ij}.

Recall some notions from topology: A topological space M is called ***Hausdorff*** if for any pair $p,q \in M$, there exist (open) neighborhoods $U \ni p$, and $U' \ni q$ such that $U \cap U' = \varnothing$. For a topological space M with topology τ, collection $\beta \subset \tau$ is a ***basis*** if and only if each $U \in \tau$ can be written as union of sets in β. A basis is called a ***countable basis*** if β is a countable set.

Figure 1 displays ***coordinate charts*** (U,φ), where U are coordinate neighborhoods, or charts, and φ are (coordinate) homeomorphisms. Transitions between different choices of coordinates are called ***transitions maps*** $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$, which are again homeomorphisms by definition. We usually write $x = \varphi(p)$, $\varphi : U \to V \subset \mathbb{R}^n$, as ***coordinates*** for U, see Figure 2, and $p = \varphi^{-1}(x)$, $\varphi^{-1} : V \to U \subset M$, as a ***parametrization*** of U. A collection $\mathcal{A} = \{(\varphi_i, U_i)\}_{i \in I}$ of coordinate charts with $M = \cup_i U_i$, is called an ***atlas*** for M.

The following Theorem gives a number of useful characteristics of topological manifolds.

Theorem 1.4. [3] *A manifold is locally connected, locally compact, and the union of countably many compact subsets. Moreover, a manifold is normal and metrizable.*

◀ **1.5** *Example.* $M = \mathbb{R}^m$; the axioms (i) and (iii) are obviously satisfied. As for (ii) we take $U = \mathbb{R}^m$, and φ the identity map. ▶

◀ **1.6** *Example.* $M = S^1 = \{p = (p_1, p_2) \in \mathbb{R}^2 \mid p_1^2 + p_2^2 = 1\}$; as before (i) and (iii) are satisfied. As for (ii) we can choose many different atlases.

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[3] Lee, 1.6, 1.8, and Boothby.

(a): Consider the sets

$$U_1 = \{p \in S^1 \mid p_2 > 0\}, \quad U_2 = \{p \in S^1 \mid p_2 < 0\},$$
$$U_3 = \{p \in S^1 \mid p_1 < 0\}, \quad U_4 = \{p \in S^1 \mid p_1 > 0\}.$$

The associated coordinate maps are $\varphi_1(p) = p_1$, $\varphi_2(p) = p_1$, $\varphi_3(p) = p_2$, and $\varphi_4(p) = p_2$. For instance

$$\varphi_1^{-1}(x) = \Big(x, \sqrt{1-x^2}\Big),$$

and the domain is $V_1 = (-1,1)$. It is clear that φ_i and φ_i^{-1} are continuous, and therefore the maps φ_i are homeomorphisms. With these choices we have found an atlas for S^1 consisting of four charts.

(b): (Stereographic projection) Consider the two charts

$$U_1 = S^1 \backslash \{(0,1)\}, \quad \text{and} \quad U_2 = S^1 \backslash \{(0,-1)\}.$$

The coordinate mappings are given by

$$\varphi_1(p) = \frac{2p_1}{1-p_2}, \quad \text{and} \quad \varphi_2(p) = \frac{2p_1}{1+p_2},$$

which are continuous maps from U_i to $\mathbb{R}$. For example

$$\varphi_1^{-1}(x) = \Big(\frac{4x}{x^2+4}, \frac{x^2-4}{x^2+4}\Big),$$

which is continuous from $\mathbb{R}$ to U_1.

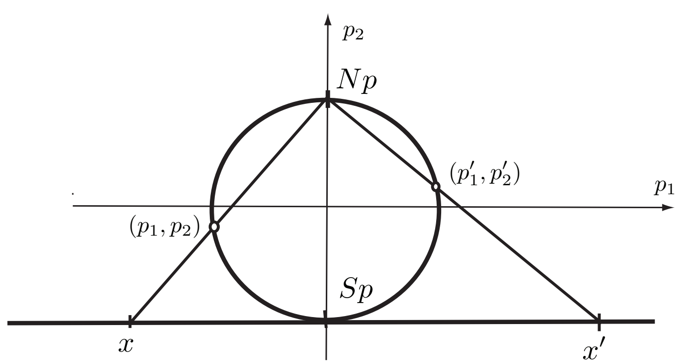

FIGURE 3. The stereographic projection describing φ_1.

(c): (Polar coordinates) Consider the two charts $U_1 = S^1 \backslash \{(1,0)\}$, and $U_2 = S^1 \backslash \{(-1,0)\}$. The homeomorphism are $\varphi_1(p) = \theta \in (0,2\pi)$ (polar angle counter clockwise rotation), and $\varphi_2(p) = \theta \in (-\pi,\pi)$ (complex argument). For example $\varphi_1^{-1}(\theta) = (\cos(\theta), \sin(\theta))$. ▶

◀ **1.8** *Example.* $M = S^n = \{p = (p_1, \cdots p_{n+1}) \in \mathbb{R}^{n+1} \mid |p|^2 = 1\}$. Obvious extension of the choices for S^1. ▶

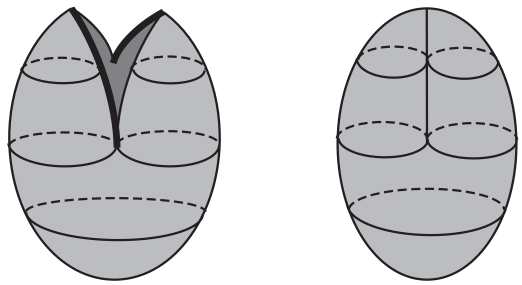

FIGURE 4. Identification of the curves indicated above yields an immersion of $P\mathbb{R}^2$ into $\mathbb{R}^3$.

◀ **1.9** *Example.* (see Lee) Let $U \subset \mathbb{R}^n$ be an open set and $g : U \to \mathbb{R}^m$ a continuous function. Define

$$M = \Gamma(g) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ x \in U, \ y = g(x)\},$$

endowed with the subspace topology. This topological space is an n-dimensional manifold. Indeed, define $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ as $\pi(x,y) = x$, and $\varphi = \pi|_{\Gamma(g)}$, which is continuous onto U. The inverse $\varphi^{-1}(x) = (x, g(x))$ is also continuous. Therefore $\{(\Gamma(g), \varphi)\}$ is an appropriate atlas. The manifold $\Gamma(g)$ is homeomorphic to U. ▶

◀ **1.10** *Example.* $M = P\mathbb{R}^n$, the real projective spaces. Consider the following equivalence relation on points on $\mathbb{R}^{n+1}\backslash\{0\}$: For any two $x, y \in \mathbb{R}^{n+1}\backslash\{0\}$ define

$$x \sim y \text{ if there exists a } \lambda \neq 0, \text{ such that } x = \lambda y.$$

Define $P\mathbb{R}^n = \{[x] \ : \ x \in \mathbb{R}^{n+1}\backslash\{0\}\}$ as the set of equivalence classes. One can think of $P\mathbb{R}^n$ as the set of lines through the origin in $\mathbb{R}^{n+1}$. Consider the natural map $\pi : \mathbb{R}^{n+1}\backslash\{0\} \to P\mathbb{R}^n$, via $\pi(x) = [x]$. A set $U \subset P\mathbb{R}^n$ is open if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1}\backslash\{0\}$. This makes $P\mathbb{R}^n$ a compact Hausdorff space. In order to verify that we are dealing with an n-dimensional manifold we need to describe an atlas for $P\mathbb{R}^n$. For $i = 1, \cdots n+1$, define $V_i \subset \mathbb{R}^{n+1}\backslash\{0\}$ as the set of points x for which $x_i \neq 0$, and define $U_i = \pi(V_i)$. Furthermore, for any $[x] \in U_i$ define

$$\varphi_i([x]) = \Big(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\Big),$$

which is continuous, and its continuous inverse is given by

$$\varphi_i(z_1, \cdots, z_n) = \big[(z_1, \cdots, z_{i-1}, 1, z_i, \cdots, z_n)\big].$$

These charts U_i cover $P\mathbb{R}^n$. In dimension $n = 1$ we have that $P\mathbb{R} \cong S^1$, and in the dimension $n = 2$ we obtain an immersed surface $P\mathbb{R}^2$ as shown in Figure 4. ▶

The examples 1.6 and 1.8 above are special in the sense that they are subsets of some $\mathbb{R}^m$, which, as topological spaces, are given a manifold structure.

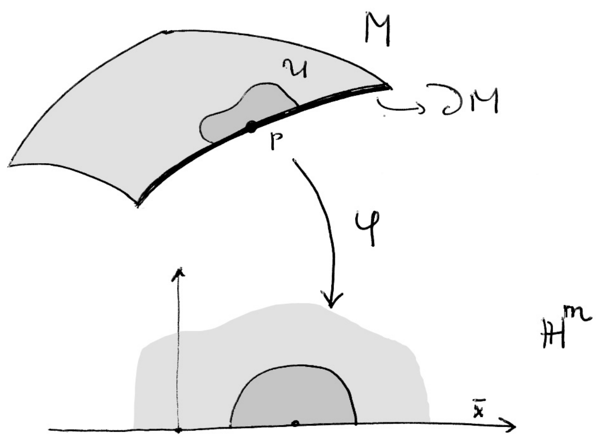

FIGURE 5. Coordinate maps for boundary points.

Define
$$\mathbb{H}^m = \{(x_1, \cdots, x_m) \mid x_m \geq 0\},$$
as the standard Euclidean half-space.

Definition 1.12. A topological space M is called an m***-dimensional (topological) manifold with boundary*** $\partial M \subset M$, if the following conditions hold:

(i) M is a Hausdorff space,
(ii) for any point $p \in M$ there exists a neighborhood U of p, which is homeomorphic to an open subset $V \subset \mathbb{H}^m$, and
(iii) M has a countable basis of open sets.

Axiom (ii) can be rephrased as follows, any point $p \in M$ is contained in a neighborhood U, which either homeomorphic to D^m, or to $D^n \cap \mathbb{H}^m$. The set M is locally homeomorphic to $\mathbb{R}^m$, or $\mathbb{H}^m$. The ***boundary*** $\partial M \subset M$ is a subset of M which consists of points p for which any neighborhood cannot be homeomorphic to an open subset of $\operatorname{int}(\mathbb{H}^m)$. Points $p \in \partial M$ are mapped to points on $\partial\mathbb{H}^m$ and ∂M is an $(m-1)$-dimensional topological manifold.

◀ **1.14** *Example.* Consider the bounded cone
$$M = C = \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1^2 + p_2^2 = p_3^2,\ 0 \leq p_3 \leq 1\},$$
with boundary $\partial C = \{p \in C \mid p_3 = 1\}$. We can describe the cone via an atlas consisting of three charts;
$$U_1 = \{p \in C \mid p_3 < 1\},$$
with $x = (x_1, x_2) = \varphi_1(p) = (p_1, p_2 + 1)$, and
$$\varphi_1^{-1}(x) = \left(x_1, x_2 - 1, \sqrt{x_1^2 + (x_2 - 1)^2}\right),$$

The other charts are given by

$$U_2 = \{p \in C \mid \frac{1}{2} < p_3 \le 1, \quad (p_1, p_2) \neq (0, p_3)\},$$
$$U_3 = \{p \in C \mid \frac{1}{2} < p_3 \le 1, \quad (p_1, p_2) \neq (0, -p_3)\}.$$

For instance φ_2 can be constructed as follows. Define

$$q = \psi(p) = \Big(\frac{p_1}{p_3}, \frac{p_2}{p_3}, p_3\Big), \quad \sigma(q) = \Big(\frac{2q_1}{1-q_2}, 1-q_3\Big),$$

and $x = \varphi_2(p) = (\sigma \circ \psi)(p)$, $\varphi_2(U_2) = \mathbb{R} \times [0, \frac{1}{2}) \subset \mathbb{H}^2$. The map φ_3 is defined similarly. The boundary is given by $\partial C = \varphi_2^{-1}(\mathbb{R} \times \{0\}) \cup \varphi_3^{-1}(\mathbb{R} \times \{0\})$, see Figure 6. ▶

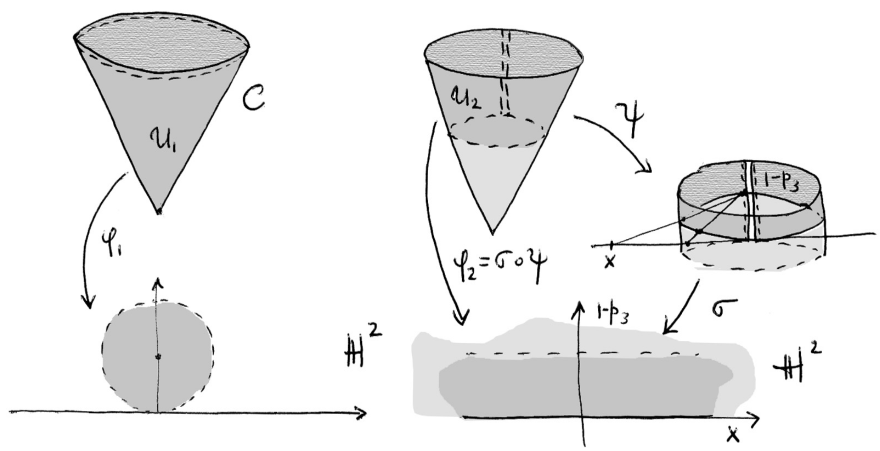

FIGURE 6. Coordinate maps for the cone C.

◀ **1.16** *Example.* The open cone $M = C = \{p = (p_1, p_2, p_3) \mid p_1^2 + p_2^2 = p_3^2,\ 0 \le p_3 < 1\}$, can be described by one coordinate chart, see U_1 above, and is therefore a manifold without boundary (non-compact), and C is homeomorphic to D^2, or $\mathbb{R}^2$, see definition below. ▶

So far we have seen manifolds and manifolds with boundary. A manifold can be either compact or non-compact, which we refer to as ***closed, or open*** manifolds respectively. Manifolds with boundary are also either compact, or non-compact. In both cases the boundary can be compact. Open subsets of a topological manifold are called ***open submanifolds***, and be given a manifold structure again.

Let N and M be manifolds, and let $f : N \to M$ be a continuous mapping. A mapping f is called a ***homeomorphism*** between N and M if f is continuous and has a continuous inverse $f^{-1} : M \to N$. In this case the manifolds N and M are said to be ***homeomorphic***. Using charts (U,φ), and (V,ψ) for N and M respectively, we can give a coordinate expression for f, i.e. $\tilde{f} = \psi \circ f \circ \varphi^{-1}$.

Recall the subspace topology. Let X be a topological space and let $S \subset X$ be any subset, then the ***subspace, or relative topology*** on S (induced by the topology on X) is defined as follows. A subset $U \subset S$ is open if there exists an open set $V \subset X$ such that $U = V \cap S$. In this case S is called a (topological) ***subspace*** of X. A ***(topological) embedding*** is a continuous injective mapping $f : X \to Y$, which is a homeomorphism onto its image $f(X) \subset Y$ with respect to the subspace topology. Let $f : N \to M$ be an embedding, then its image $f(N) \subset M$ is called a ***submanifold*** of M. Notice that an open submanifold is the special case when $f = i : U \hookrightarrow M$ is an inclusion mapping.

2. Differentiable manifolds and differentiable structures

A topological manifold M for which the transition maps $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ for all pairs φ_i, φ_j in the atlas are diffeomorphisms is called a ***differentiable, or smooth*** manifold. The transition maps are mappings between open subsets of $\mathbb{R}^m$. Diffeomorphisms between open subsets of $\mathbb{R}^m$ are C^∞-maps, whose inverses are also C^∞-maps. For two charts U_i and U_j the transitions maps are mappings:

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \ \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

and as such are homeomorphisms between these open subsets of $\mathbb{R}^m$.

Definition 2.1. A C^∞-atlas is a set $\mathcal{A} = \{(U, \varphi_i)\}_{i \in I}$ such that

(i) $M = \cup_{i \in I} U_i$,

(ii) the transition maps φ_{ij} are diffeomorphisms between $\varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$, for all $i \neq j$ (see Figure 2).

The charts in a C^∞-atlas are said to be C^∞***-compatible***. Two C^∞-atlases $\mathcal{A}$ and $\mathcal{A}'$ are equivalent if $\mathcal{A} \cup \mathcal{A}'$ is again a C^∞-atlas, which clearly defines a equivalence relation on C^∞-atlases. An equivalence class of this equivalence relation is called a ***differentiable structure*** $\mathcal{D}$ on M. The collection of all atlases associated with $\mathcal{D}$, denoted $\mathcal{A}_\mathcal{D}$, is called the ***maximal atlas*** for the differentiable structure. Figure 7 shows why compatibility of atlases defines an equivalence relation.

Definition 2.3. Let M be a topological manifold, and let $\mathcal{D}$ be a differentiable structure on M with maximal atlas $\mathcal{A}_\mathcal{D}$. Then the pair $(M, \mathcal{A}_\mathcal{D})$ is called a ***(C^∞-)differentiable manifold***.

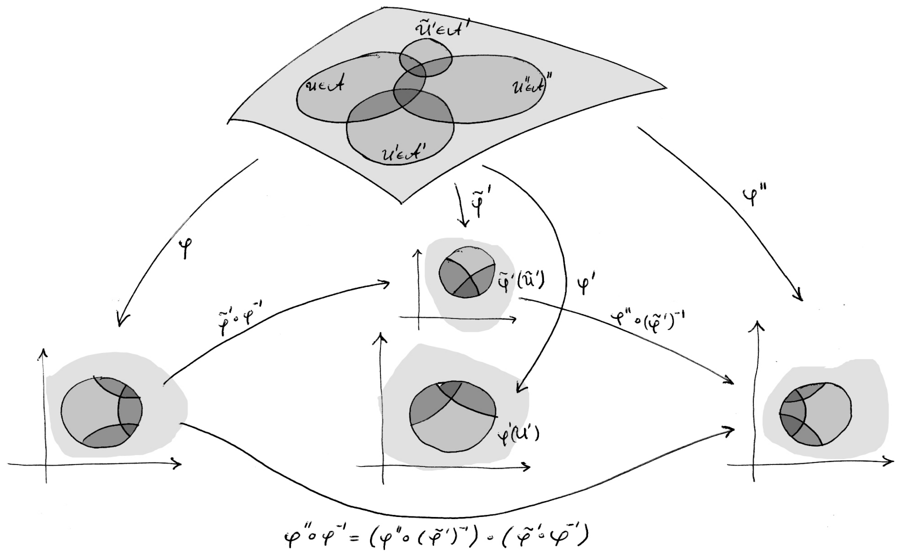

FIGURE 7. Differentiability of $\varphi'' \circ \varphi^{-1}$ is achieved via the mappings $\varphi'' \circ (\tilde{\varphi}')^{-1}$, and $\tilde{\varphi}' \circ \varphi^{-1}$, which are diffeomorphisms since $\mathcal{A} \sim \mathcal{A}'$, and $\mathcal{A}' \sim \mathcal{A}''$ by assumption. This establishes the equivalence $\mathcal{A} \sim \mathcal{A}''$.

Basically, a manifold M with a C^∞-atlas, defines a differentiable structure on M. The notion of a differentiable manifold is intuitively a difficult concept. In dimensions 1 through 3 all topological manifolds allow a differentiable structure (only one up to diffeomorphisms). Dimension 4 is the first occurrence of manifolds without a differentiable structure. Also in higher dimensions uniqueness of differentiable structures is no longer the case as the famous example by Milnor shows; S^7 has 28 different (non-diffeomorphic) differentiable structures. The first example of a manifold that allows many non-diffeomorphic differentiable structures occurs in dimension 4; exotic $\mathbb{R}^4$'s. One can also consider C^r-differentiable structures and manifolds. Smoothness will be used here for the C^∞-case here.

Theorem 2.4. [4] *Let M be a topological manifold with a C^∞-atlas $\mathcal{A}$. Then there exists a unique differentiable structure $\mathcal{D}$ containing $\mathcal{A}$, i.e. $\mathcal{A} \subset \mathcal{A}_\mathcal{D}$.*

◀ **2.5** *Remark.* In our definition of of n-dimensional differentiable manifold we use the local model over ***standard*** $\mathbb{R}^n$, i.e. on the level of coordinates we express differentiability with respect to the standard differentiable structure on $\mathbb{R}^n$, or diffeomorphic to the standard differentiable structure on $\mathbb{R}^n$. ▶

◀ **2.6** *Example.* The cone $M = C \subset \mathbb{R}^3$ as in the previous section is a differentiable manifold whose differentiable structure is generated by the one-chart atlas (C, φ)

[4] Lee, Lemma 1.10.

as described in the previous section. As we will see the cone is not a smoothly embedded submanifold. ▶

◀ **2.7** *Example.* $M = S^1$ (or more generally $M = S^n$). As in Sect. 1 consider S^1 with two charts via stereographic projection. We have the overlap $U_1 \cap U_2 = S^1 \backslash \{(0, \pm 1)\}$, and the transition map $\varphi_{12} = \varphi_2 \circ \varphi_1^{-1}$ given by $y = \varphi_{12}(x) = \frac{4}{x}$, $x \neq 0, \pm\infty$, and $y \neq 0, \pm\infty$. Clearly, φ_{12} and its inverse are differentiable functions from $\varphi_1(U_1 \cap U_2) = \mathbb{R} \backslash \{0\}$ to $\varphi_2(U_1 \cap U_2) = \mathbb{R} \backslash \{0\}$. ▶

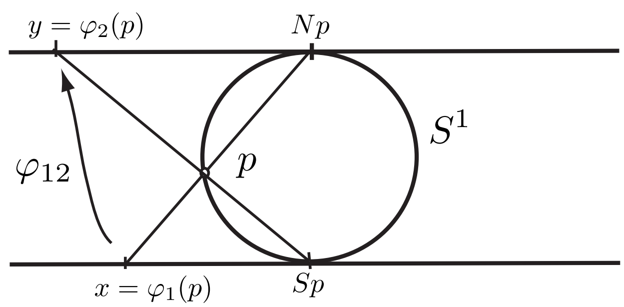

FIGURE 8. The transition maps for the stereographic projection.

◀ **2.9** *Example.* The real projective spaces $P\mathbb{R}^n$, see exercises Chapter VI. ▶

◀ **2.10** *Example.* The generalizations of projective spaces $P\mathbb{R}^n$, the so-called (k, n)-Grassmannians $G^k\mathbb{R}^n$ are examples of smooth manifolds. ▶

Let N and M be smooth manifolds (dimensions n and m respectively). Let $f : N \to M$ be a mapping from N to M.

Definition 2.11. A mapping $f : N \to M$ is said to be C^∞, or ***smooth*** if for every $p \in N$ there exist charts (U, φ) of p and (V, ψ) of $f(p)$, with $f(U) \subset V$, such that $\tilde{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is a C^∞-mapping (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

The above definition also holds true for mappings defined on open subsets of N, i.e. let $W \subset N$ is an open subset, and $f : W \subset N \to M$, then smoothness on W is defined as above by restricting to pionts $p \in W$. Such a map is called a ***local diffeomorphism***.

The definition of smooth mappings allows one to introduce the notion of differentiable homeomorphism, of ***diffeomorphism*** between manifolds. Let N and M be smooth manifolds. A C^∞-mapping $f : N \to M$, is called a diffeomorphism if it is a homeomorphism and also f^{-1} is a smooth mapping, in which case N and M are

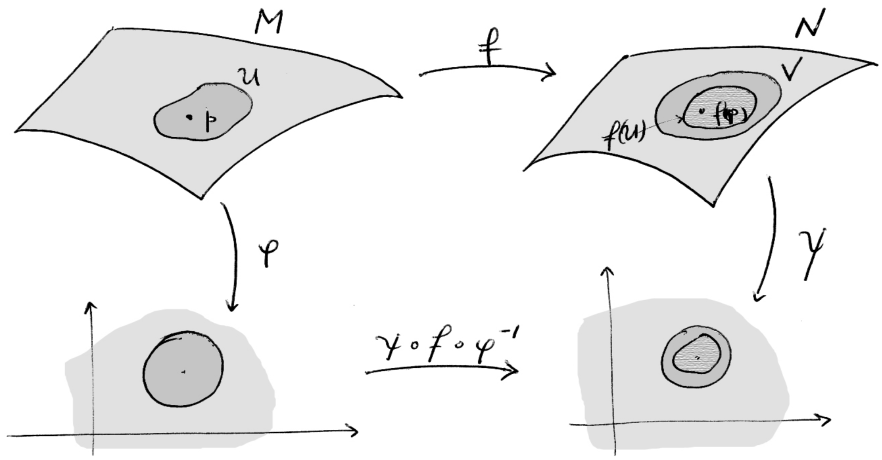

FIGURE 9. Coordinate representation for f, with $f(U) \subset V$.

said to be ***diffeomorphic***. The associated differentiable structures are also called diffeomorphic.

◀ **2.13** *Example.* Consider $N = \mathbb{R}$ with atlas $(\mathbb{R}, \varphi(p) = p)$, and $M = \mathbb{R}$ with atlas $(\mathbb{R}, \psi(q) = q^3)$. Clearly these define different differentiable structures. Between N and M we consider the mapping $f(p) = p^{1/3}$, which is a homeomorphism between N and M. The claim is that f is also a diffeomorphism. Take $U = V = \mathbb{R}$, then $\psi \circ f \circ \varphi^{-1}(p) = (p^{1/3})^3 = p$ is the identity and thus C^∞ on $\mathbb{R}$, and the same for $\varphi \circ f^{-1} \circ \psi^{-1}(q) = (q^3)^{1/3} = q$. The associated differentiable structures are diffeomorphic. In fact the above described differentiable structures correspond to defining the differential quotient via $\lim_{h\to 0} \frac{f^3(p+h) - f^3(p)}{h}$. ▶

Theorem 2.14. [5] *Let N, M be smooth manifolds with atlases $\mathcal{A}_N$ and $\mathcal{A}_M$ respectively.*

(i) *Given smooth maps $f_\alpha : U_\alpha \to M$, for all $U_\alpha \in \mathcal{A}_N$, with $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ for all α, β. Then there exists a unique smooth map $f : N \to M$ such that $f|_{U_\alpha} = f_\alpha$.*

(ii) *A smooth map $f : N \to M$ between smooth manifolds is continuous.*

(iii) *Let $f : N \to M$ be continuous, and suppose that the maps $\tilde{f}_{\alpha\beta} = \psi_\beta \circ f \circ \varphi_\alpha^{-1}$, for charts $(U_\alpha, \varphi_\alpha) \in \mathcal{A}_N$, and $(V_\alpha, \psi_\beta) \in \mathcal{A}_M$, are smooth on there domains for all α, β. Then f is smooth.*

[5]See Lee Lemmas 2.1, 2.2 and 2.3.

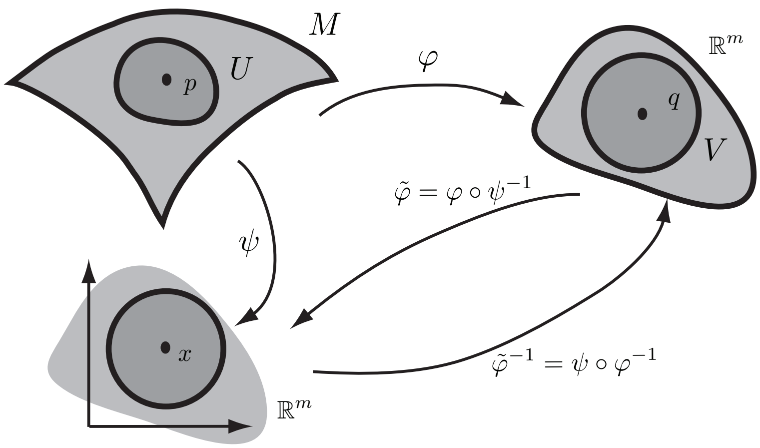

FIGURE 10. Coordinate diffeomorphisms and their local representation as smooth transition maps.

With this theorem at hand we can verify differentiability locally for continuous maps. In order to show that f is a diffeomorphism we need that f is a homeomorphism that satisfies the above local smoothness conditions, as well as for the inverse.

The coordinate maps φ in a chart (U,φ) are local diffeomorphisms. Indeed, $\varphi : U \subset M \to \mathbb{R}^n$, then if (V,ψ) is any other chart with $U \cap V \neq \varnothing$, then by the definition the transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth. For φ we then have $\tilde{\varphi} = \varphi \circ \psi^{-1}$ and $\tilde{\varphi}^{-1} = \psi \circ \varphi^{-1}$, which proves that φ is a local diffeomorphism, see Figure 10. This justifies the terminology ***smooth charts***.

◀ **2.16** *Example.* For arbitrary subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ a map $f : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ is said to be smooth at $p \in U$, if there exists an open set $U^\dagger \subset \mathbb{R}^m$ containing p, and a smooth map $f^\dagger : U^\dagger \to \mathbb{R}^m$, such that f and $f^\dagger$ coincide on $U \cap U^\dagger$. The latter is called an ***extension*** of f. A mapping $f : U \to V$ between arbitrary subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ is called a diffeomorphism if f is smooth (as just described above) and has a smooth inverse f^{-1}. In this case U and V are said to be diffeomorphic. ▶

◀ **2.17** *Example.* An important example of a class of differentiable manifolds are appropriately chosen subsets of Euclidean space that can be given a smooth manifold structure. Let $M \subset \mathbb{R}^\ell$ be a subset such that every $p \in M$ has a neighborhood $U \ni p$ in M (open in the subspace topology, see Section 3) which is diffeomorphic to an open subset $V \subset \mathbb{R}^m$ (or, equivalently an open disc $D^m \subset \mathbb{R}^m$). In this case the set M is a smooth m-dimensional manifold. Its topology is the subsapce topology and the smooth structure is inherited from the standard smooth structure on $\mathbb{R}^\ell$, which can be described as follows. By definition a coordinate map φ is a diffeo-

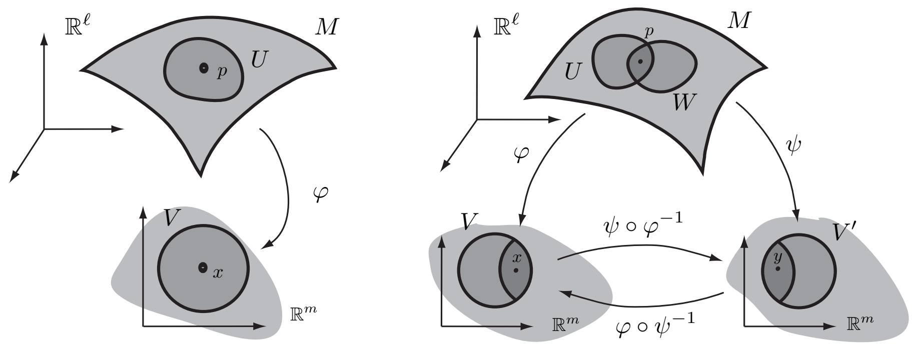

FIGURE 11. The transitions maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ smooth mappings establishing the smooth structure on M.

morphisms which means that $\varphi : U \to V = \varphi(U)$ is a smooth mapping (smoothness via a smooth map $\varphi^\dagger$), and for which also φ^{-1} is a smooth map. This then directly implies that the transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth mappings, see Figure 11. In some books this construction is used as the definition of a smooth manifold. In Section 4 we will refer to the above class of differentiable manifolds as manifolds in Euclidean space, which are smoothly embedded submanifolds in Euclidean space, see also Sections 3 and 4. ▶

◀ **2.19** *Example.* Let us consider the cone $M = C$ described in Example 1 (see also Example 2 in Section 1). We already established that C is manifold homeomrphic to $\mathbb{R}^2$, and moreover C is a differentiable manifold, whose smooth structure is defined via a one-chart atlas. However, C is not a smooth manifold with respect to the induced smooth structure as subset of $\mathbb{R}^3$. Indeed, following the definition in the above remark, we have $U = C$, and coordinate homeomorphism $\varphi(p) = (p_1, p_2) = x$. By the definition of smooth maps it easily follows that φ is smooth. The inverse is given by $\varphi^{-1}(x) = \left(x_1, x_2, \sqrt{x_1^2 + x_2^2}\right)$, which is clearly ***not*** differentiable at the cone-top $(0,0,0)$. The cone C is not a smoothly embedded submainfold of $\mathbb{R}^3$ (topological embedding). ▶

Let U, V and W be open subsets of $\mathbb{R}^n$, $\mathbb{R}^k$ and $\mathbb{R}^m$ respectively, and let $f : U \to V$ and $g : V \to W$ be smooth maps with $y = f(x)$, and $z = g(y)$. Then the ***Jacobians*** are

$$Jf|_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix}, \qquad Jg|_y = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_k} \end{pmatrix},$$

and $J(g \circ f)|_x = Jg|_{y=f(x)} \cdot Jf|_x$ (chain-rule). The commutative diagram for the maps f, g and $g \circ f$ yields a commutative diagram for the Jacobians:

$$\begin{tikzcd} & V & \\ U \arrow[ur, "f"] \arrow[rr, "f\circ g"'] & & W \arrow[ul, "g"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{R}^k & \\ \mathbb{R}^n \arrow[ur, "Jf|_x"] \arrow[rr, "J(f\circ g)|_x"'] & & \mathbb{R}^m \arrow[ul, "Jg|_y"'] \end{tikzcd}$$

For diffeomorphisms between open sets in $\mathbb{R}^n$ we have a number of important properties. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets, and let $f : U \to V$ is a diffeomorphism, then $n = m$, and $Jf|_x$ is invertible for any $x \in U$. Using the above described commutative diagrams we have that $f^{-1} \circ f$ yields that $J(f^{-1})|_{y=f(x)} \cdot Jf|_x$ is the identity on $\mathbb{R}^n$, and $f \circ f^{-1}$ yields that $Jf|_x \cdot J(f^{-1})|_{y=f(x)}$ is the identity on $\mathbb{R}^m$. Thus Jf_x has an inverse and consequently $n = m$. Conversely, if $f : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$, open, then we have the Inverse Function Theorem;

Theorem 2.20. *If $Jf|_x : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, then f is a diffeomorphism between sufficiently small neighborhoods U' and $f(U')$ of x and y respectively.*

3. Immersions, submersions and embeddings

Let N and M be smooth manifolds of dimensions n and m respectively, and let $f : N \to M$ be a smooth mapping. In local coordinates $\tilde{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$, with respects to charts (U, φ) and (V, ψ). The ***rank*** of f at $p \in N$ is defined as the rank of $\tilde{f}$ at $\varphi(p)$, i.e. $\mathrm{rk}(f)|_p = \mathrm{rk}(J\tilde{f})|_{\varphi(p)}$, where $J\tilde{f}|_{\varphi(p)}$ is the Jacobian of f at p:

$$J\tilde{f}|_{x=\varphi(p)} = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial x_1} & \cdots & \frac{\partial \tilde{f}_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{f}_m}{\partial x_1} & \cdots & \frac{\partial \tilde{f}_m}{\partial x_n} \end{pmatrix}$$

This definition is independent of the chosen charts, see Figure 12. Via the commutative diagram in Figure 12 we see that $\tilde{\tilde{f}} = (\psi' \circ \psi^{-1}) \circ \tilde{f} \circ (\varphi' \circ \varphi^{-1})^{-1}$, and by the chain rule $J\tilde{\tilde{f}}|_{x'} = J(\psi' \circ \psi^{-1})|_y \cdot J\tilde{f}|_x \cdot J(\varphi' \circ \varphi^{-1})^{-1}|_{x'}$. Since $\psi' \circ \psi^{-1}$ and $\varphi' \circ \varphi^{-1}$ are diffeomorphisms it easily follows that $\mathrm{rk}(J\tilde{f})|_x = \mathrm{rk}(J\tilde{\tilde{f}})|_{x'}$, which shows that our notion of rank is well-defined. If a map has constant rank for all $p \in N$ we simply write $\mathrm{rk}(f)$. These are called ***constant rank*** mappings. Let us now consider the various types of constant rank mappings between manifolds.

Definition 3.2. A mapping $f : N \to M$ is called an ***immersion*** if $\mathrm{rk}(f) = n$, and a ***submersion*** if $\mathrm{rk}(f) = m$. An immersion that is injective, or 1-1, and is a homeomorphism onto its image $f(N) \subset M$, with respect to the subspace topology, is called a ***smooth embedding***.

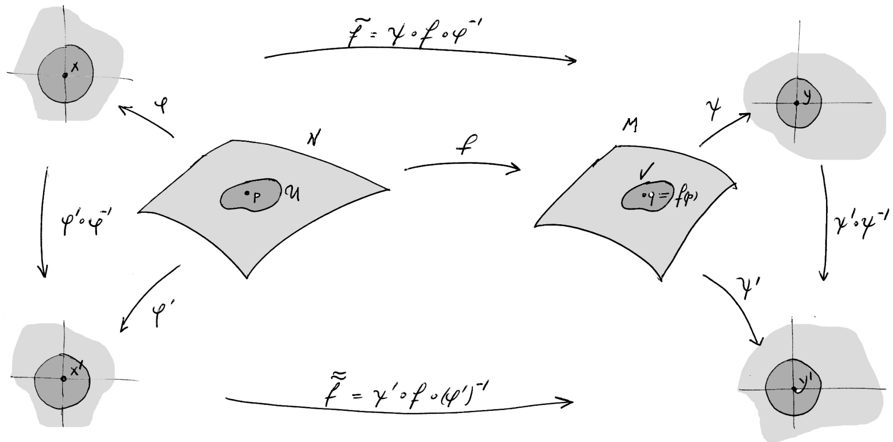

FIGURE 12. Representations of f via different coordinate charts.

The above described smooth embedding is an injective immersion that is a topological embedding.

◀ **3.3** *Example.* Let $N = (-\frac{\pi}{2}, \frac{3\pi}{2})$, and $M = \mathbb{R}^2$, and the mapping f is given by $f(t) = (\sin(2t), \cos(t))$. In Figure 13 we displayed the image of f. The Jacobian is given by

$$Jf|_t = \begin{pmatrix} 2\cos(2t) \\ -\sin(t) \end{pmatrix}$$

Clearly, $\mathrm{rk}(Jf|_t) = 1$ for all $t \in N$, and thus f is an injective immersion. Since N is an open manifold and $f(N) \subset M$ is a compact set with respect to the subspace topology, it follows f is not a homeomorphism onto $f(N)$, and therefore is not an embedding. ▶

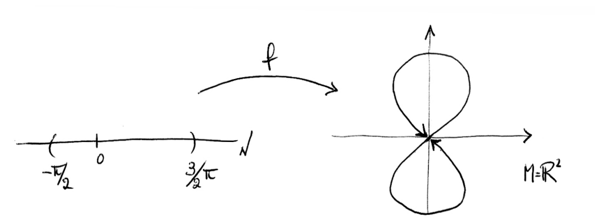

FIGURE 13. Injective parametrization of the figure eight.

◀ **3.5** *Example.* Let $N = S^1$ be defined via the atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$, with $\varphi_1^{-1}(t) = ((\cos(t), \sin(t))$, and $\varphi_2^{-1}(t) = ((\sin(t), \cos(t))$, and $t \in (-\frac{\pi}{2}, \frac{3\pi}{2})$. Furthermore, let $M = \mathbb{R}^2$, and the mapping $f : N \to M$ is given in local coordinates; in U_1 as in Example 1. Restricted to $S^1 \subset \mathbb{R}^2$ the map f can also be described by

$$f(x,y) = (2xy, x).$$

This then yields for U_2 that $\tilde{f}(t) = (\sin(2t), \sin(t))$. As before $\mathrm{rk}(f) = 1$, which shows that f is an immersion of S^1. However, this immersion is not injective at the origin in $\mathbb{R}^2$, indicating the subtle differences between these two examples, see Figures 13 and 14. ▶

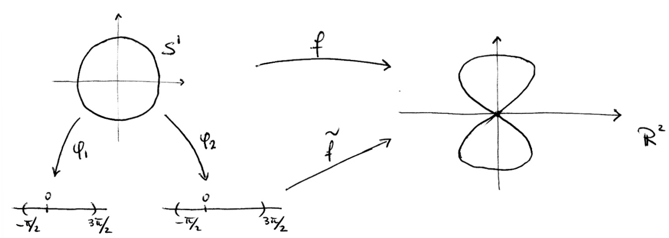

FIGURE 14. Non-injective immersion of the circle.

◀ **3.7** *Example.* Let $N = \mathbb{R}$, $M = \mathbb{R}^2$, and $f : N \to M$ defined by $f(t) = (t^2, t^3)$. We can see in Figure 15 that the origin is a special point. Indeed, $\mathrm{rk}(Jf)|_t = 1$ for all $t \neq 0$, and $\mathrm{rk}(Jf)|_t = 0$ for $t = 0$, and therefore f is not an immersion. ▶

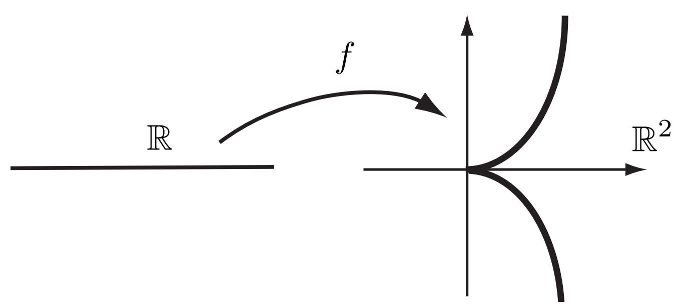

FIGURE 15. The map f fails to be an immersion at the origin.

◀ **3.9** *Example.* Consider $M = P\mathbb{R}^n$. We established $P\mathbb{R}^n$ as smooth manifolds. For $n = 1$ we can construct an embedding $f : P\mathbb{R} \to \mathbb{R}^2$ as depicted in Figure 16 For $n = 2$ we find an immersion $f : P\mathbb{R}^2 \to \mathbb{R}^3$ as depicted in Figure 17. ▶

◀ **3.12** *Example.* Let $N = \mathbb{R}^2$ and $M = \mathbb{R}$, then the projection mapping $f(x,y) = x$ is a submersion. Indeed, $Jf|_{(x,y)} = (1\ 0)$, and $\mathrm{rk}(Jf|_{(x,y)}) = 1$. ▶

◀ **3.13** *Example.* Let $N = M = \mathbb{R}^2$ and consider the mapping $f(x,y) = (x^2, y)^t$. The Jacobian is

$$Jf|_{(x,y)} = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix},$$

and $\mathrm{rk}(Jf)|_{(x,y)} = 2$ for $x \neq 0$, and $\mathrm{rk}(Jf)|_{(x,y)} = 1$ for $x = 0$. See Figure 18. ▶

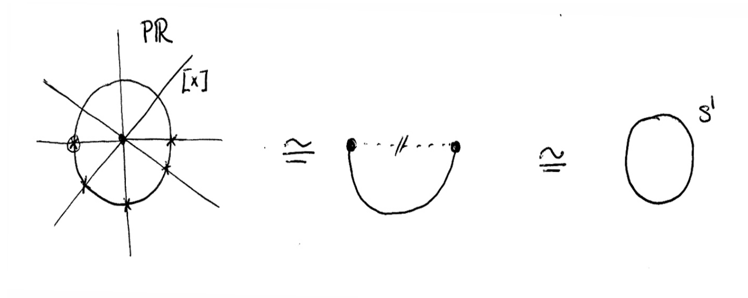

FIGURE 16. $P\mathbb{R}$ is diffeomorphic to S^1.

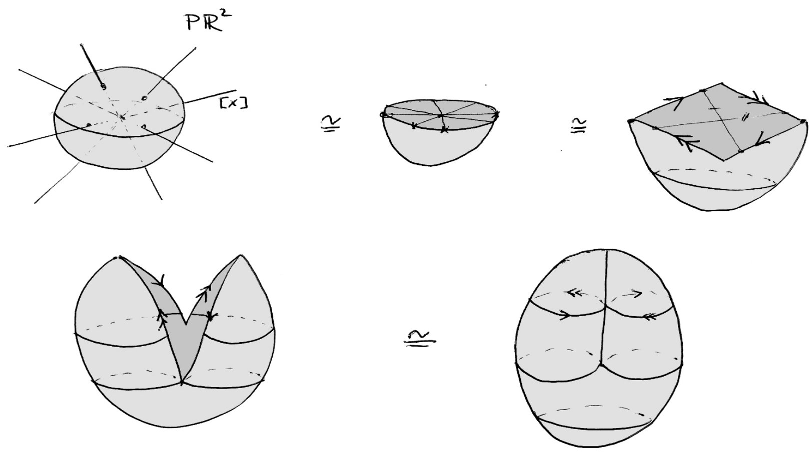

FIGURE 17. Identifications for $P\mathbb{R}^2$ giving an immersed non-orientable surface in $\mathbb{R}^3$.

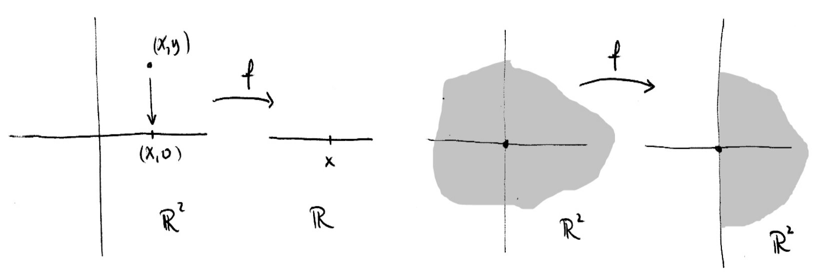

FIGURE 18. The projection (submersion), and the folding of the plane (not a submersion).

◀ **3.15** *Example.* We alter Example 2 slightly, i.e. $N = S^1 \subset \mathbb{R}^2$, and $M = \mathbb{R}^2$, and again use the atlas $\mathcal{A}$. We now consider a different map f, which, for instance on U_1, is given by $\tilde{f}(t) = (2\cos(t), \sin(t))$ (or globally by $f(x,y) = (2x,y)$). It is clear that f is an injective immersion, and since S^1 is compact it follows from Lemma 3.17 below that S^1 is homeomorphic to it image $f(S^1)$, which show that f is a smooth embedding (see also Appendix in Lee). ▶

◀ **3.16** *Example.* Let $N = \mathbb{R}$, $M = \mathbb{R}^2$, and consider the mapping $f(t) = (2\cos(t), \sin(t))$. As in the previous example f is an immersion, not injective however. Also $f(\mathbb{R}) = f(S^1)$ in the previous example. The manifold N is the universal covering of S^1 and the immersion f descends to a smooth embedding of S^1. ▶

Lemma 3.17. [6] *Let $f : N \to M$ be an injective immersion. If*

(i) *N is compact, or if*
(ii) *f is a proper map,*[7]

then f is a smooth embedding.

Let us start with defining the notion of embedded submanifolds.

Definition 3.18. *A subset $N \subset M$ is called a smooth* ***embedded n-dimensional submanifold*** *in M if for every $p \in N$, there exists a chart (U, φ) for M, with $p \in U$, such that*

$$\varphi(U \cap N) = \varphi(U) \cap \left(\mathbb{R}^n \times \{0\}\right) = \{x \in \varphi(U) \; : \; x_{n+1} = \cdots = x_m = 0\}.$$

The ***co-dimension*** *of N is defined as* $\operatorname{codim} N = \dim M - \dim N$.

The set $W = U \cap N$ in N is called a ***n-dimensional slice, or n-slice*** of U, see Figure 19, and (U, φ) a ***slice chart***. The associated coordinates $x(x_1, \cdots, x_n)$ are called ***slice coordinates***. Embedded submanifolds can be characterized in terms of embeddings.

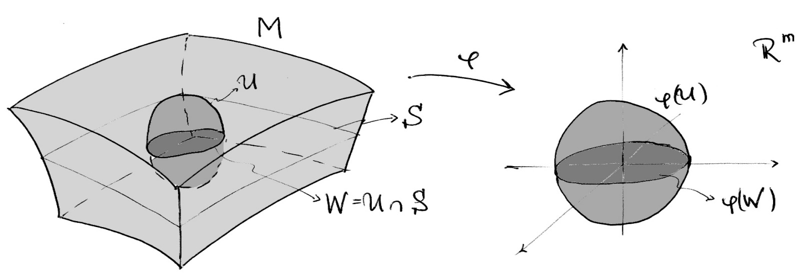

FIGURE 19. Take a k-slice W. On the left the image $\varphi(W)$ corresponds to the Eucliden subspace $\mathbb{R}^k \subset \mathbb{R}^m$.

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[6] Lee, Prop. 7.4, and pg. 47.

[7] A(ny) map $f : N \to M$ is called proper if for any compact $K \subset M$, it holds that also $f^{-1}(K) \subset N$ is compact. In particular, when N is compact, continuous maps are proper.

Theorem 3.20. [8] *Let $N \subset M$ be a smooth embedded n-submanifold. Endowed with the subspace topology, N is a n-dimensional manifold with a unique (induced) smooth structure such that the inclusion map $i : N \hookrightarrow M$ is an embedding (smooth).*

To get an idea let us show that N is a topological manifold. Axioms (i) and (iii) are of course satisfied. Consider the projection $\pi : \mathbb{R}^m \to \mathbb{R}^n$, an inclusion $j : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$\pi(x_1, \cdots, x_n, x_{n+1}, \cdots, x_m) = (x_1, \cdots, x_n),$$
$$j(x_1, \cdots, x_n) = (x_1, \cdots, x_n, 0, \cdots, 0).$$

Now set $Z = (\pi \circ \varphi)(W) \subset \mathbb{R}^n$, and $\bar{\varphi} = (\pi \circ \varphi)|_W$, then $\bar{\varphi}^{-1} = (\varphi^{-1} \circ j)|_Z$, and $\bar{\varphi} : W \to Z$ is a homeomorphism. Therefore, pairs $(W, \bar{\varphi})$ are charts for N, which form an atlas for N. The inclusion $i : N \hookrightarrow M$ is a topological embedding.

Theorem 3.21. [9] *The image of an embedding is a smooth embedded submanifold.*

An important special case is when $M = \mathbb{R}^m$, and we consider submanifolds N of $\mathbb{R}^m$, which is the subject of Section 4. We should point out that this case the coordinate mappings $\bar{\varphi}$ for N are smooth mappings, and the associated coordinate mappings φ can be chosen to be smooth mappings on $\mathbb{R}^m$. To be more precise:

Lemma 3.22. *For a smooth embedded manifold $N \hookrightarrow M = \mathbb{R}^m$ the maps $\bar{\varphi}$ are local diffeomorphisms, and the slice coordinates φ can chosen to be local diffeomorphisms on $\mathbb{R}^m$.*

Proof: To see this we argue as follows. Consider the diagram

$$\begin{array}{ccc} W \subset N & \xrightarrow{i} & M = \mathbb{R}^m \\ \bar{\varphi}\big\downarrow & & \big\downarrow\psi \\ Z \subset \mathbb{R}^n & \xrightarrow{\tilde{i}} & \mathbb{R}^m \end{array}$$

From Theorem 3.20 we have that the inclusion i is smooth, and by using slice coordinates the expression for $\tilde{i} = \psi \circ i \circ \bar{\varphi}^{-1}$ is: $\tilde{i}(x) = (x, 0) \in \mathbb{R}^m$ (i.e. taking $\psi = \varphi$). If we choose $\psi = \mathrm{id}$ we obtain that $\tilde{i} = i \circ \bar{\varphi}^{-1}$ is a smooth mapping from $Z \subset \mathbb{R}^n$ to $\mathbb{R}^n$, and which is an injective immersion. Consequently, $J\bar{\varphi}^{-1}|_x$ is an injective linear map, and

$$\mathbb{R}^m = J\bar{\varphi}^{-1}|_x(\mathbb{R}^n) \oplus \left[J\bar{\varphi}^{-1}|_x(\mathbb{R}^n)\right]^{\perp}.$$

Now let $L : \mathbb{R}^{m-n} \to \left[J\bar{\varphi}^{-1}|_x(\mathbb{R}^n)\right]^{\perp}$ be a surjective linear map, and define

$$G(x_1, \cdots, x_n, x_{n+1}, \cdots, x_m) = \bar{\varphi}(x_1, \cdots, x_n) + L(x_{n+1}, \cdots, x_m).$$

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[8] See Lee, Thm's 8.2

[9] See Lee, Thm's 8.3

By construction the Jacobian $JG|_x$ is invertible, and thus by the Inverse Function Theorem there exists a local diffeomorphism G^{-1} extending $\bar{\varphi}$. The extension is again denoted φ, and (U,φ) is a slice chart with a smooth coordinate mapping. ∎

From now on when we deal with smooth embedded manifolds in $\mathbb{R}^m$ we will always assume that the slice coordinates φ are local diffeomorphisms.

A famous result by Whitney says that considering embedding into $\mathbb{R}^m$ is not not really a restriction for defining smooth manifolds.

Theorem 3.23. [10] *Any smooth n-dimensional manifold M can be (smoothly) embedded into* $\mathbb{R}^{2n+1}$.

A subset $N \subset M$ is called an ***immersed submanifold*** if N is a smooth n-dimensional manifold, and the mapping $i : N \hookrightarrow M$ is a (smooth) immersion. This means that we can endow N with an appropriate manifold topology and smooth structure, such that the natural inclusion of N into M is an immersion. If $f : N \to M$ is an injective immersion we can endow $f(N)$ with a topology and unique smooth structure; a set $U \subset f(N)$ is open if and only if $f^{-1}(U) \subset N$ is open, and the (smooth) coordinate maps are taken to be $\varphi \circ f^{-1}$, where φ's are coordinate maps for N. This way $f : N \to f(N)$ is a diffeomorphism, and $i : f(N) \hookrightarrow M$ an injective immersion via the composition $f(N) \to N \to M$. This proves:

Theorem 3.24. [11] *Immersed submanifolds are exactly the images of injective immersions.*

We should point out that embedded submanifolds are examples of immersed submanifolds, but not the other way around. In this setting Whitney established some improvements of Theorem 3.23. Namely, for dimension $n > 0$, any smooth n-dimensional manifold can be embedded into $\mathbb{R}^{2n}$ (e.g. the embedding of curves in $\mathbb{R}^2$). Also, for $n > 1$ any smooth n-dimensional manifold can be immersed into $\mathbb{R}^{2n-1}$ (e.g. the Klein bottle). In this course we will often think of smooth manifolds as embedded submanifolds of $\mathbb{R}^m$. An important tool thereby is the general version of Inverse Function Theorem, which can easily be derived from the 'Euclidean' version 2.20.

Theorem 3.26. [12] *Let N, M be smooth manifolds, and $f : N \to M$ is a smooth mapping. If, at some point $p \in N$, it holds that $J\tilde{f}|_{\varphi(p)=x}$ is an invertible matrix, then there exist sufficiently small neighborhoods $U_0 \ni p$, and $V_0 \in f(p)$ such that $f|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

As a direct consequence of this result we have that if $f : N \to M$, with $\dim N = \dim M$, is an immersion, or submersion, then f is a local diffeomorphism. If f is a bijection, then f is a (global) diffeomorphism.

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[10] See Lee, Ch. 10.

[11] See Lee, Theorem 8.16.

[12] See Lee, Thm's 7.6 and 7.10.

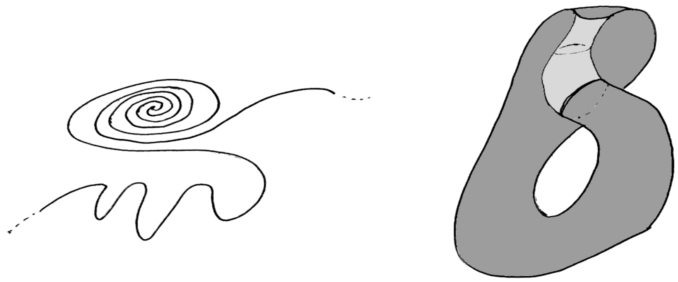

FIGURE 20. An embedding of $\mathbb{R}$ [left], and an immersion of S^2 [right] called the Klein bottle.

Theorem 3.27. *Let $f : N \to M$ be a constant rank mapping with $\mathrm{rk}(f) = k$. Then for each $q \in f(N)$, the level set $S = f^{-1}(q)$ is an embedded submanifold in N with co-dimension equal to k.*

In particular, when $f : N \to M$ is a submersion, then for each $q \in f(N)$, the level set $S = f^{-1}(q)$ is an embedded submanifold of co-dimension $\mathrm{codim}\, S = m = \dim M$. In the case of maximal rank this statement can be restricted to just one level. A point $p \in N$ is called a ***regular point*** if $\mathrm{rk}(f)|_p = m = \dim M$, otherwise a point is called a ***critical point***. A level $q \in f(N)$ is called a ***regular level*** if all points $p \in f^{-1}(q)$ are regular points, otherwise a level is called a ***critical level***.

Theorem 3.28. *Let $f : N \to M$ be a smooth map. If $q \in f(N)$ is a regular value, then $S = f^{-1}(q)$ is an embedded submanifold co-dimension equal to* $\dim M$.

Proof: Let us illustrate the last result for the important case $N = \mathbb{R}^n$ and $M = \mathbb{R}^m$. For any $p \in S = f^{-1}(q)$ the Jacobian $Jf|_p$ is surjective by assumption. Denote the kernel of $Jf|_p$ by $\ker Jf|_p \subset \mathbb{R}^n$, which has dimension $n - m$. Define

$$g : N = \mathbb{R}^n \to \mathbb{R}^{n-m} \times \mathbb{R}^m \cong \mathbb{R}^n,$$

by $g(\xi) = (L\xi, f(\xi) - q)^t$, where $L : N = \mathbb{R}^n \to \mathbb{R}^{n-m}$ is any linear map which is invertible on the subspace $\ker Jf|_p \subset \mathbb{R}^n$. Clearly, $Jg|_p = L \oplus Jf|_p$, which, by construction, is an invertible (linear) map on $\mathbb{R}^n$. Applying Theorem 2.20 (Inverse Function Theorem) to g we conclude that a sufficiently small neighborhood of U of p maps diffeomorphically onto a neighborhood V of $(L(p), 0)$. Since g is a diffeomorphism it holds that g^{-1} maps $\left(\mathbb{R}^{n-m} \times \{0\}\right) \cap V$ onto $f^{-1}(q) \cap U$ (the $0 \in \mathbb{R}^m$ corresponds to q). This exactly says that every point $p \in S$ allows an $(n-m)$-slice and is therefore an $(n-m)$-dimensional submanifold in $N = \mathbb{R}^n$ ($\mathrm{codim}\, S = m$). ■

◄ **3.30** *Example.* Let us start with an explicit illustration of the above proof. Let $N = \mathbb{R}^2$, $M = \mathbb{R}$, and $f(p_1, p_2) = p_1^2 + p_2^2$. Consider the regular value $q = 2$, then $Jf|_p = (2p_1 \quad 2p_2)$, and $f^{-1}(2) = \{p \;:\; p_1^2 + p_2^2 = 2\}$, the circle with radius $\sqrt{2}$.

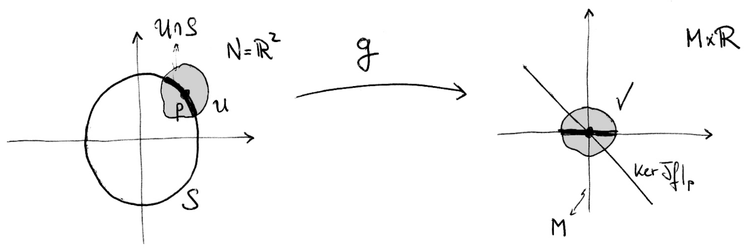

FIGURE 21. The map g yields 1-slices for the set S.

We have $\ker Jf|_p = \mathrm{span}\{(p_1, -p_2)^t\}$, and is always isomorphic to $\mathbb{R}$. For example fix the point $(1,1) \in S$, then $\ker Jf|_p = \mathrm{span}\{(1,-1)^t\}$ and define

$$g(\xi) = \begin{pmatrix} L(\xi) \\ f(\xi) - 2 \end{pmatrix} = \begin{pmatrix} \xi_1 - \xi_2 \\ \xi_1^2 + \xi_2^2 - 2 \end{pmatrix}, \quad Jg|_p = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$$

where the linear map is $L = (1 \quad -1)$. The map g is a local diffeomorphism and on $S \cap U$ this map is given by

$$g(\xi_1, \sqrt{2 - \xi_1^2}) = \begin{pmatrix} \xi_1 - \sqrt{2 - \xi_1^2} \\ 0 \end{pmatrix},$$

with $\xi_1 \in (1-\varepsilon, 1+\varepsilon)$. The first component has derivative $1 + \frac{\xi_1}{\sqrt{2-\xi_1^2}}$, and therefore $S \cap U$ is mapped onto a set of the form $\big(\mathbb{R} \times \{0\}\big) \cap V$. This procedure can be carried out for any point $p \in S$, see Figure 21. ▶

◀ **3.31** *Example.* Let $N = \mathbb{R}^2 \backslash \{(0,0)\} \times \mathbb{R} = \mathbb{R}^3 \backslash \{(0,0,\lambda)\}$, $M = (-1,\infty) \times (0,\infty)$, and

$$f(x,y,z) = \begin{pmatrix} x^2 + y^2 - 1 \\ 1 \end{pmatrix}, \quad \text{with} \quad Jf|_{(x,y,z)} = \begin{pmatrix} 2x & 2y & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We see immediately that $\mathrm{rk}(f) = 1$ on N. This map is not a submersion, but is of constant rank, and therefore for any value $q \in f(N) \subset M$ it holds that $S = f^{-1}(q)$ is a embedded submanifold, see Figure 22. ▶

◀ **3.33** *Example.* Let N, M as before, but now take

$$f(x,y,z) = \begin{pmatrix} x^2 + y^2 - 1 \\ z \end{pmatrix}, \quad \text{with} \quad Jf|_{(x,y,z)} = \begin{pmatrix} 2x & 2y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now $\mathrm{rk}(f) = 2$, and f is a submersion. For every $q \in M$, the set $S = f^{-1}(q)$ is a embedded submanifold, see Figure 23. ▶

◀ **3.35** *Example.* Let $N = \mathbb{R}^2$, $M = \mathbb{R}$, and $f(x,y) = \frac{1}{4}(x^2-1)^2 + \frac{1}{2}y^2$. The Jacobian is $Jf||_{(x,y)} = (x(x^2-1) \quad y)$. This is not a constant rank, nor a submersion. However, any $q > 0$ is a regular value since then the rank is equal to 1. Figure 24

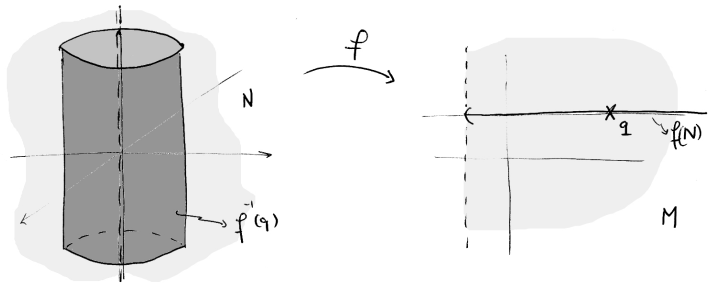

FIGURE 22. An embedding of a cylinder via a constant rank mapping.

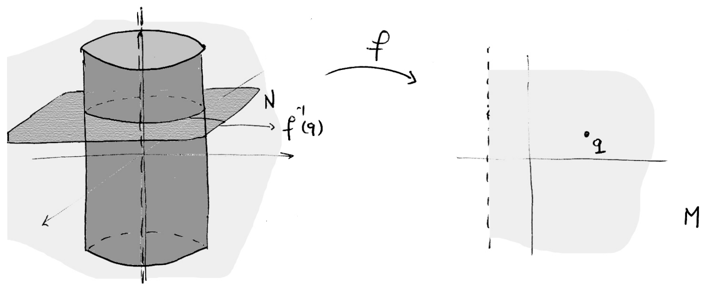

FIGURE 23. An embedding circle of an submersion.

shows the level set $f^{-1}(0)$ (not an embedded manifold), and $f^{-1}(1)$ (an embedded circle) ▶

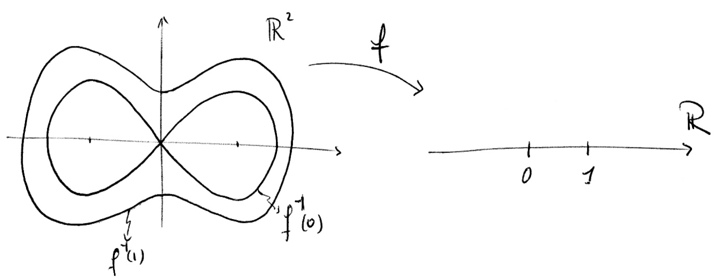

FIGURE 24. A regular and critical level set.

Theorem 3.37. [13] *Let $S \subset M$ be a subset of a smooth m-dimensional manifold M. Then S is a k-dimensional smooth embedded submanifold if and if for every $p \in S$*

[13]See Lee, Prop.8.12.

there exists a neighborhood $W \ni p$ such that $W \cap S$ is the level set of a submersion $f : U \to \mathbb{R}^{m-k}$.

4. Manifolds in Euclidean space

In the following sections and chapters we will mainly deal with smooth embedded submanifolds in $\mathbb{R}^\ell$ due to the Whitney result. We will refer to these manifolds in $\mathbb{R}^\ell$. For completeness we restate the definition again.

Definition 4.1. A subset $M \subset \mathbb{R}^\ell$ is called a smooth ***m-dimensional manifold in*** $\mathbb{R}^\ell$, if for every $p \in M$ there exists an neighborhood $U \subset M$ of p (open in the subspace topology), an open set $U' \subset \mathbb{R}^m$ and a diffeomorphism $\varphi : U \to U'$. Pairs (U, φ) are smooth charts for M.

In other words φ allows a smooth extension $\varphi^\dagger : U^\dagger \to \mathbb{R}^m$, with $\varphi^\dagger|_{U\cap U^\dagger} = \varphi|_{U\cap U^\dagger}$, and $\varphi^{-1} : U' \to \mathbb{R}^\ell$ smooth. In Example 2.17 we established that these are smooth manifolds. We want to show now that we that manifolds in $\mathbb{R}^\ell$ are smooth embedded manifolds in $\mathbb{R}^\ell$.

Start with a differentiable manifold M as in Definition 4.1 with charts (U, φ). By definition there exists a smooth function $\varphi^\dagger$ on a neighborhood $U^\dagger \subset \mathbb{R}^\ell$, with $\varphi^\dagger|_{U\cap U^\dagger} = \varphi|_{U\cap U^\dagger}$, Using the Inverse Function Theorem exactly as we did in the proof of Lemma 3.22 we can construct a diffeomorphism $\varphi^\dagger$. Choose U and $U^\dagger$ such that $U^\dagger \cap M = U$, then by definition

$$\varphi^\dagger(U^\dagger \cap M) = \varphi^\dagger(U^\dagger) \cap \big(\mathbb{R}^m \times \{0\}\big).$$

This shows that M as defined in Definition 4.1 is a smooth embedded m-dimensional submanifold of $\mathbb{R}^\ell$. In addition the coordinate maps are local diffeomorphisms.

Let us start with an embedded submanifold $M \subset \mathbb{R}^\ell$ with smooth coordinate maps. Consider the projection $\pi : \mathbb{R}^\ell \to \mathbb{R}^m$, and immersion $j : \mathbb{R}^m \to \mathbb{R}^\ell$ defined by

$$\pi(x_1, \cdots, x_m, x_{m+1}, \cdots, x_\ell) = (x_1, \cdots, x_m),$$
$$j(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0 \cdots, 0).$$

Then the composition $\pi \circ \varphi = \varphi^\dagger$ is a smooth coordinate map from $U \subset \mathbb{R}^\ell$ to $\mathbb{R}^m$. By definition it holds that $\varphi^\dagger|_{U\cap M} = \varphi|_{U\cap M} = \overline{\varphi}$ defined on $W = U \cap M$. If we restrict the inverse φ^{-1} to $\mathbb{R}^m \times \{0\}$, then

$$\overline{\varphi}^{-1} = (\varphi^{-1} \circ j)|_Z,$$

where $Z = \overline{\varphi}(W)$. showing that M is a manifold in $\mathbb{R}^\ell$ as defined in Definition 4.1, with charts $(W, \overline{\varphi})$. This can be summarized as follows.

Lemma 4.2. *A subset $M \subset \mathbb{R}^\ell$ is a smooth m-dimensional manifold in $\mathbb{R}^\ell$ in the sense of Definition 4.1 if and only if for every $p \in M$ there exists an neighborhood $U \subset \mathbb{R}^\ell$ of p, an open set $V \subset \mathbb{R}^\ell$ and a diffeomorphism $\varphi : U \to V$, such that*

$$\varphi(U \cap M) = V \cap \mathbb{R}^m = \{x \in V \ : \ x_{m+1} = \cdots = x_\ell = 0\},$$

i.e. if $W = U \cap M$ is an m-slice of U.

This characterization can serve as a useful equivalent way of defining smooth manifolds in $\mathbb{R}^\ell$.

In Section 2 we introduced the notion of a differentiable function between differentiable manifolds. In the case of manifolds embedded in Euclidean space there is an equivalent notion of differentiability that is convenient in many practical cases.

Lemma 4.3. *Let $N \subset \mathbb{R}^k$ and $M \subset \mathbb{R}^\ell$ be manifolds. A mapping $f : N \to M$ is a smooth mapping (in the sense of Definition 2.11) if and only if for every $p \in N$ there exists a neighborhood $U \subset \mathbb{R}^k$ and smooth mapping $f^\dagger : U \to \mathbb{R}^\ell$ such that $f^\dagger|_{U\cap M} = f|_{U\cap M}$. The map $f^\dagger$ is called an **extension** of f.*

Proof: To prove this lemma we need to show that the existence of a smooth mapping f in the sense of Definition 2.11 yields the existence of a smooth extension $f^\dagger$. Via the charts (U,φ) and (V,ψ) (as in Definition 4.1) we have a smooth representation $\tilde{f} = \psi \circ f \circ \varphi^{-1} : U' \cap \mathbb{R}^k\} \to V' \cap \mathbb{R}^\ell$. Define the smooth map

$$\tilde{\tilde{f}}(x_1, \cdots x_n, x_{n+1}, \cdots, x_k) = \tilde{f}(x_1, \cdots x_n),$$

mapping from U' to V'. This trivial extension now allows us to define $f^\dagger : U \to \mathbb{R}^\ell$ by:

$$f^\dagger = \psi^{-1} \circ \tilde{\tilde{f}} \circ \varphi.$$

This extension clearly satisfies the requirements in the lemma. ∎

The notion of differentiable maps for manifolds in Euclidean space is therefore the usual notion of multi-variable differentiability. Due to Lemma 4.3 it makes sense to talk about the Jacobian of f at a point $p \in N$, i.e. given $p \in N$ define

$$Jf|_p = Jf^\dagger|_p,$$

for any extension $f^\dagger$. The commuting diagram below shows that $Jf|_p$ is independent of the chosen extension $f^\dagger$:

$$\begin{tikzcd} U \subset N \arrow[r, "f"] \arrow[d, "\varphi"'] & V \subset M \arrow[d, "\psi"] \\ U' \arrow[r, "\tilde{\tilde{f}}"] & V' \end{tikzcd}$$

In Section 1 we also introduced manifolds with boundary. The theory in this section immediately goes through for manifolds with boundary. For such object the same holds as for manifolds; manifolds with boundary can always be embedded into some $\mathbb{R}^\ell$. For this reason we also give the definition of a manifold with boundary in $\mathbb{R}^\ell$.

Definition 4.4. A subset $M \subset \mathbb{R}^\ell$ is called a smooth m***-dimensional manifold in*** $\mathbb{R}^\ell$ ***with boundary*** $\partial M \subset M$, if for every $p \in M$ there exists an neighborhood $U \subset M$ of p (open in the subspace topology), an open set $U' \subset \mathbb{H}^m$ and a diffeomorphism $\varphi : U \to U'$. The boundary ∂M are the points in M which corresponds to points on $\partial\mathbb{H}^m$ under the diffeomorphisms φ. Notation: $(M, \partial M)$.

As in the case of manifolds we have an equivalent characterization for manifolds with boundary as images of smooth embeddings.

Lemma 4.5. *A subset $M \subset \mathbb{R}^\ell$ is a smooth m-**dimensional manifold in** $\mathbb{R}^\ell$ **with boundary** $\partial M \subset M$ in the sense of Definition 4.4 if and only if for every $p \in M \backslash \partial M$ there exists an neighborhood $U \subset \mathbb{R}^\ell$ of p, an open set $V \subset \mathbb{R}^\ell$ and a diffeomorphism $\varphi : U \to V$, such that*

$$\varphi(U \cap M) = V \cap \mathbb{R}^m = \{x \in V \ : \ x_{m+1} = \cdots = x_\ell = 0\},$$

and for every $p \in \partial M$ there exists an neighborhood $U \subset \mathbb{R}^\ell$ of p, an open set $V \subset \mathbb{R}^\ell$ and a diffeomorphism $\varphi : U \to V$, such that

$$\varphi(U \cap M) = V \cap \mathbb{H}^m = \{x \in V \ : \ x_m \geq 0, \ x_{m+1} = \cdots = x_\ell = 0\},$$

and $\varphi_m(p) = 0$.

This idea can be further extended by considering manifolds with corner, see for instance Lee, pg. 363.

II. Tangent and cotangent spaces

5. Tangent spaces and vector fields

For a(n) (embedded) manifold $M \subset \mathbb{R}^\ell$ the tangent space T_pM at a point $p \in M$ can be pictured as a hyperplane tangent to M. In Figure 25 we consider the parametrizations $x + te_i$ in $\mathbb{R}^m$. These parametrizations yield curves $\gamma_i(t) = \varphi^{-1}(x + te_i)$ on M whose velocity vectors are given by

$$\gamma'(0) = \frac{d}{dt}\varphi^{-1}(x+te_i)\Big|_{t=0} = J\varphi^{-1}|_x(e_i).$$

The vectors $p + \gamma'(t)$ are tangent to M at p and span an m-dimensional affine linear subspace M_p of $\mathbb{R}^\ell$. Since the vectors $J\varphi^{-1}|_x(e_i)$ span T_pM the affine subspace is given by

$$M_p := p + T_pM \subset \mathbb{R}^\ell,$$

which is tangent to M at p.

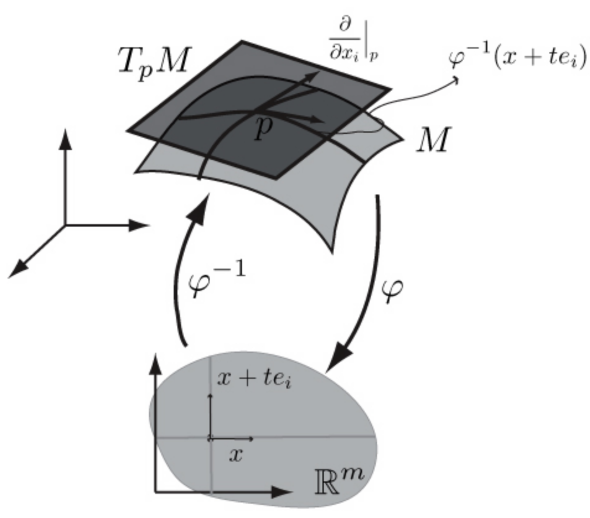

FIGURE 25. Velocity vectors of curves of M span the 'tangent space'.

The considerations are rather intuitive in the sense that we consider only embedded manifolds, and so the tangent spaces are tangent m-dimensional affine subspaces of $\mathbb{R}^\ell$. One can also define the notion of tangent space for abstract smooth manifolds. There are many ways to do this. Let us describe one possible way (see e.g. Lee, or Abraham, Marsden and Ratiu) which is based on the above considerations.

Let $a < 0 < b$ and consider a smooth mapping $\gamma : I = (a,b) \subset \mathbb{R} \to M$, such that $\gamma(0) = p$. This mapping is called a ***(smooth) curve*** on M, and is parametrized by $t \in I$. If the mapping γ (between the manifolds $N = I$, and M) is an immersion, then γ is called an immersed curve. For such curves the 'velocity vector' $J\tilde{\gamma}|_t = (\varphi\circ\gamma)'(t)$ in $\mathbb{R}^m$ is nowhere zero. We build the concept of tangent spaces in order to define the notion velocity vector to a curve γ.

Let (U,φ) be a chart at p. Then, two curves γ and $\tilde{\gamma}$ are equivalent, $\tilde{\gamma} \sim \gamma$, if

$$\tilde{\gamma}(0) = \gamma(0), \text{ and } (\varphi\circ\tilde{\gamma})'(0) = (\varphi\circ\gamma)'(0).$$

The equivalence class of a curve γ through $p \in M$ is denoted by $[\gamma]$.

Definition 5.2. [14] At a $p \in M$ define the tangent space T_pM as the space of all equivalence classes $[\gamma]$ of curves γ through p. A tangent vector X_p, as the equivalence class of curves, is given by

$$X_p := [\gamma] = \big\{\tilde{\gamma} \,:\, \tilde{\gamma}(0) = \gamma(0),\ (\varphi\circ\tilde{\gamma})'(0) = (\varphi\circ\gamma)'(0)\big\},$$

which is an element of T_pM.

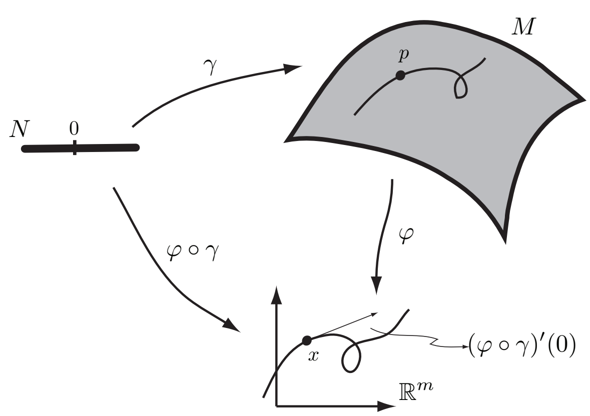

FIGURE 26. Immersed curves and velocity vectors in $\mathbb{R}^m$.

The above definition does not depend on the choice of charts at $p \in M$. Let (U',φ') be another chart at $p \in M$. Then, using that $(\varphi\circ\tilde{\gamma})'(0) = (\varphi\circ\gamma)'(0)$, for

[14] Lee, Ch. 3.

$(\varphi' \circ \gamma)'(0)$ we have

$$
\begin{aligned}
(\varphi' \circ \gamma)'(0) &= \Big[(\varphi' \circ \varphi^{-1}) \circ (\varphi \circ \gamma)\Big]'(0) \\
&= J(\varphi' \circ \varphi^{-1})\big|_x (\varphi \circ \gamma)'(0) \\
&= J(\varphi' \circ \varphi^{-1})\big|_x (\varphi \circ \tilde{\gamma})'(0) \\
&= \Big[(\varphi' \circ \varphi^{-1}) \circ (\varphi \circ \tilde{\gamma})\Big]'(0) = (\varphi' \circ \tilde{\gamma})'(0),
\end{aligned}
$$

which proves that the equivalence relation does not depend on the particular choice of charts at $p \in M$.

One can prove that $T_pM \cong \mathbb{R}^m$. Indeed, T_pM can be given a linear structure as follows; given two equivalence classes $[\gamma_1]$ and $[\gamma_2]$, then

$$
\begin{aligned}
[\gamma_1] + [\gamma_2] &:= \big\{\gamma \,:\, (\varphi \circ \gamma)'(0) = (\varphi \circ \gamma_1)'(0) + (\varphi \circ \gamma_2)'(0)\big\}, \\
\lambda[\gamma_1] &:= \big\{\gamma \,:\, (\varphi \circ \gamma)'(0) = \lambda(\varphi \circ \gamma_1)'(0)\big\}.
\end{aligned}
$$

The above argument shows that these operation are well-defined, i.e. independent of the chosen chart at $p \in M$. The mapping

$$
\tau_\varphi : T_pM \to \mathbb{R}^m, \qquad \tau_\varphi\big([\gamma]\big) = (\varphi \circ \gamma)'(0),
$$

is a linear isomorphism and $\tau_{\varphi'} = J(\varphi' \circ \varphi^{-1})|_x \circ \tau_\varphi$.

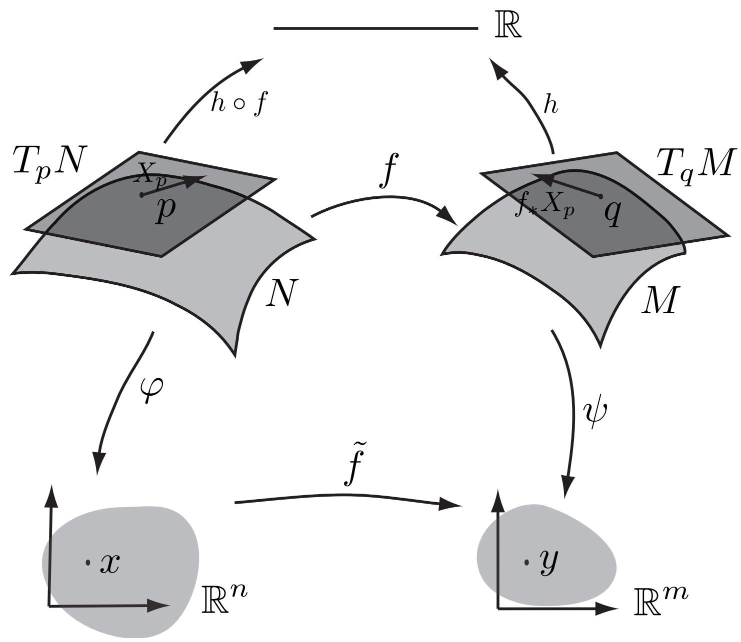

FIGURE 27. Tangent vectors in $X_p \in T_pN$ yield tangent vectors $f_*X_p \in T_qM$ under the pushforward of f.

Given a smooth mapping $f : N \to M$ we can define how tangent vectors in T_pN are mapped to tangent vectors in T_qM, with $q = f(p)$. Choose charts (U,φ) for $p \in N$, and (V,ψ) for $q \in M$. We define the ***tangent map*** or ***pushforward of*** f as follows, see Figure 27. For a given tangent vector $X_p = [\gamma] \in T_pN$,

$$d f_p = f_* : T_pN \to T_qM, \qquad f_*([\gamma]) = [f \circ \gamma].$$

The following commutative diagram shows that f_* is a linear map and its definition does not depend on the charts chosen at $p \in N$, or $q \in M$.

$$\begin{tikzcd} T_pN \arrow[r, "f_*"] \arrow[d, "\tau_\varphi"'] & T_qM \arrow[d, "\tau_\psi"] \\ \mathbb{R}^n \arrow[r, "J(\psi\circ f\circ\varphi^{-1})|_x"] & \mathbb{R}^m \end{tikzcd}$$

Indeed, a velocity vector $(\varphi\circ\gamma)'(0)$ is mapped to $(\psi\circ\gamma)'(0)$, and

$$(\psi\circ\gamma)'(0) = (\psi\circ f\circ\varphi^{-1}\circ\varphi\circ\gamma)'(0) = J\tilde{f}\big|_x\cdot(\varphi\circ\gamma)'(0).$$

If we apply the definition of pushforward to the coordinate mapping $\varphi : N \to \mathbb{R}^n$, then τ_φ can be identified with φ_*, and $J(\psi\circ f\circ\varphi^{-1})|_x$ with $(\psi\circ f\circ\varphi^{-1})_*$. Indeed, $\tau_\varphi([\gamma]) = (\varphi\circ\gamma)'(0)$ and $\varphi_*([\gamma]) = [\varphi\circ\gamma]$, and in $\mathbb{R}^n$ the equivalence class can be labeled by $(\varphi\circ\gamma)'(0)$. The labeling map is given as follows

$$\tau_{\rm id}([\varphi\circ\gamma]) = (\varphi\circ\gamma)'(0),$$

and is an isomorphism, and satisfies the relations

$$\tau_{\rm id}\circ\varphi_* = \tau_\varphi, \qquad \varphi_* = \tau_{\rm id}^{-1}\circ\tau_\varphi.$$

From now one we identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ by identifying φ_* and τ_φ. This justifies the notation

$$\varphi_*([\gamma]) = [\varphi\circ\gamma] := (\varphi\circ\gamma)'(0).$$

Properties of the pushforward can be summarized as follows:

Lemma 5.5. [15] *Let $f : N \to M$, and $g : M \to P$ be smooth mappings, and let $p \in M$, then*

(i) $f_* : T_pN \to T_{f(p)}M$*, and* $g_* : T_{f(p)}M \to T_{(g\circ f)(p)}P$ *are linear maps (homomorphisms),*
(ii) $(g\circ f)_* = g_*\cdot f_* : T_pN \to T_{(g\circ f)(p)}P$*,*
(iii) $(\mathrm{id})_* = \mathrm{id} : T_pN \to T_pN$*,*
(iv) *if f is a diffeomorphism, then the pushforward f_* is a isomorphism from T_pN to $T_{f(p)}M$.*

[15] Lee, Lemma 3.5.

A parametrization $\varphi^{-1} : \mathbb{R}^m \to M$ coming from a chart (U,φ) is a local diffeomorphism, and can be used to find a canonical basis for T_pM. Choosing local coordinates $x = (x_1,\cdots,x_n) = \varphi(p)$, and the standard basis vectors e_i for $\mathbb{R}^m$, we define

$$\frac{\partial}{\partial x_i}\Big|_p := \varphi_*^{-1}(e_i).$$

By definition $\frac{\partial}{\partial x_i}\big|_p \in T_pM$, and since the vectors e_i form a basis for $\mathbb{R}^m$, the vectors $\frac{\partial}{\partial x_i}\big|_p$ form a basis for T_pM. An arbitrary tangent vector $X_p \in T_pM$ can now be written with respect to the basis $\{\frac{\partial}{\partial x_i}\big|_p\}$:

$$X_p = \varphi_*^{-1}(X_i e_i) = X_i \frac{\partial}{\partial x_i}\Big|_p.$$

where the notation $X_i \frac{\partial}{\partial x_i}\big|_p = \sum_i X_i \frac{\partial}{\partial x_i}\big|_p$ denotes the Einstein summation convention, and (X_i) is a vector in $\mathbb{R}^m$!

The union of tangent spaces

$$TM := \bigcup_{p\in M} T_pM$$

is called the ***tangent bundle*** of M. Points in TM are pairs of the form (p,X_p), with $p \in M$ and $X_p \in T_pM$.

Theorem 5.6. *The tangent bundle TM is smooth $2m$-dimensional manifold, and the projection $\pi : TM \to M$, defined by $\pi(p,X_p) = p$, is a smooth mapping.*

Smooth functions from M to TM lead to the following definition.

Definition 5.7. A ***smooth (tangent) vector field*** is a smooth mapping

$$X : M \to TM,$$

with the property that $\pi \circ X = \mathrm{id}_M$. A mapping with this property is also called a ***(cross) section*** in the vector bundle TM, see Figure 28. The space of smooth vector fields on M is denoted by $\mathcal{F}(M)$.

For a chart (U,φ) a vector field X can be expressed as follows

$$X = X_i \frac{\partial}{\partial x_i}\Big|_p,$$

where $X_i : U \to \mathbb{R}$. Smoothness of vector fields can be described in terms of the component functions X_i.

Lemma 5.9. *A vector field X is smooth at $p \in U$ if and only if the coordinate functions $X_i : U \to \mathbb{R}$ are smooth.*

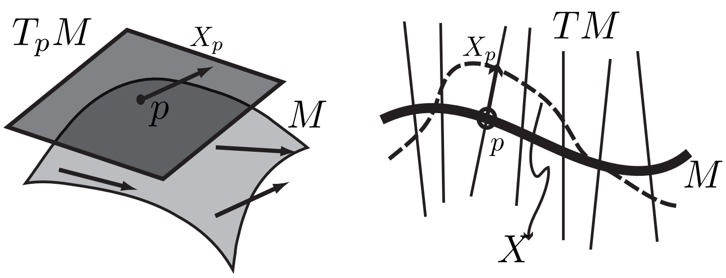

FIGURE 28. A smooth vector field X on a manifold M [right], and as a 'curve', or section in the vector bundle TM [left].

We now define the directional derivative of a smooth function $h : M \to \mathbb{R}$ in the direction of $X_p \in T_pM$ by

$$X_p h := h_* X_p = [h \circ \gamma].$$

In fact we have that $X_p h = (h \circ \gamma)'(0)$, with $X_p = [\gamma]$. For the basis vectors of T_pM this yields

$$X_p h = \big((h \circ \varphi^{-1}) \circ (\varphi \circ \gamma)\big)'(0) = \frac{\partial \tilde{h}}{\partial x_i} \cdot (\varphi \circ \gamma)'(0) = X_i \frac{\partial \tilde{h}}{\partial x_i},$$

in local coordinates, which explains the notation for tangent vectors. In particular,

$$\frac{\partial}{\partial x_i}\Big|_p h = \frac{\partial \tilde{h}}{\partial x_i}. \tag{2}$$

Let go back to the curve $\gamma : N = (a,b) \to M$ and express velocity vectors. Consider the chart (N, id) for N, with coordinates t, then

$$\frac{d}{dt}\Big|_{t=0} := \mathrm{id}_*^{-1}(1).$$

We have the following commuting diagrams:

$$\begin{array}{ccc} N & \xrightarrow{\gamma} & M \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{R} & \xrightarrow{\tilde{\gamma}=\varphi\circ\gamma} & \mathbb{R}^m \end{array} \qquad \begin{array}{ccc} T_tN & \xrightarrow{\gamma_*} & T_pM \\ {\scriptstyle \mathrm{id}_*}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \mathbb{R} & \xrightarrow{\tilde{\gamma}_*} & \mathbb{R}^m \end{array}$$

Using the idenfication of the tangent spaces we have that $\tilde{\gamma}_* \xi = (\varphi \circ \gamma)'(0)\xi$. We now define

$$\gamma'(0) = \gamma_*\Big(\frac{d}{dt}\Big|_{t=0}\Big) = \varphi_*^{-1}\big((\varphi \circ \gamma)'(0)\big) \in T_pM,$$

by using the second commuting diagram.

If we take a closer look at Figure 31 we can see that using different charts at $p \in M$, or equivalently, considering a change of coordinates leads to the following relation. For the charts (U, φ) and (U', φ') we have local coordinates $x = \varphi(p)$ and $x' = \varphi'(p)$, and $p \in U \cap U'$. This yields two different basis for T_pM, namely

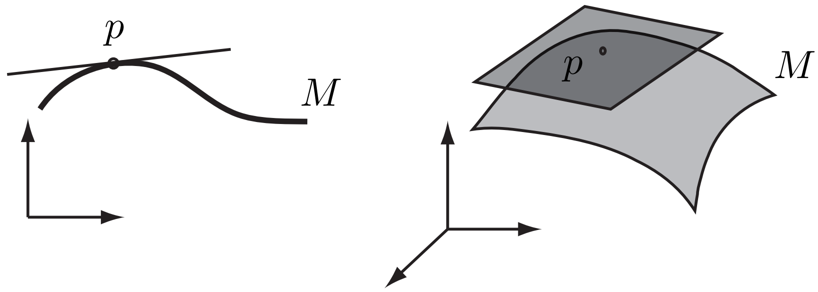

FIGURE 29. Tangent spaces to embedded manifolds.

$\left\{\frac{\partial}{\partial x_i}\big|_p\right\}$, and $\left\{\frac{\partial}{\partial x_i'}\big|_p\right\}$. Consider the identity mapping $f = \mathrm{id} : M \to M$, and the push-forward yields the identity on T_pM. If we use the different choices of coordinates as described above (see Figure ?) we obtain

$$\left(\mathrm{id}_* \frac{\partial}{\partial x_i}\Big|_p\right) h = (\varphi' \circ \varphi^{-1})_* \frac{\partial \tilde{h}}{\partial y_j} = \frac{\partial y_j}{\partial x_i} \frac{\partial \tilde{h}}{\partial y_j}.$$

In terms of the different basis for T_pM this gives

$$\frac{\partial}{\partial x_i}\Big|_p = \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}\Big|_p.$$

We remark that the definition of f_* also allows use to restate definitions about the rank of a map in terms of the differential, or pushforward f_* at a point $p \in N$. Under submersions and immersion a vector fields $X : N \to TN$ does not necessarily push forward under $f : N \to M$ to a vector field on N, see Figure ?. If f is a diffeomorphism, then $f_*X := (f(p), f_*X_p) = Y$ is a vector field on M.

6. Tangent spaces for embedded manifolds

Intuitively the tangent space at a point on a smooth (embedded) curve is the line tangent to the curve at that point, and similarly, for an embedded surface it is the tangent plane to the surface at the given point, see Figure 29. For embedded manifolds M we can define the notion of tangent space at a point $p \in M$ as follows. Consider an m-dimensional manifold $M \subset \mathbb{R}^\ell$. Given a chart (φ, U), consider the (smooth) parametrization

$$\varphi^{-1} : V = \varphi(U) \subset \mathbb{R}^m \to U \subset M \subset \mathbb{R}^\ell,$$

where V is a neighborhood of a point $x \in \mathbb{R}^m$, such that $\varphi^{-1}(x) = p$. Since φ^{-1} is smooth the Jacobian $J\varphi^{-1}|_x : \mathbb{R}^m \to \mathbb{R}^\ell$ is well-defined. For embedded manifolds we now use the following definition of tangent space at a point $p \in M$:

$$T_pM := J\varphi^{-1}|_x(\mathbb{R}^m) = \mathrm{range}\, J\varphi^{-1}|_x.$$

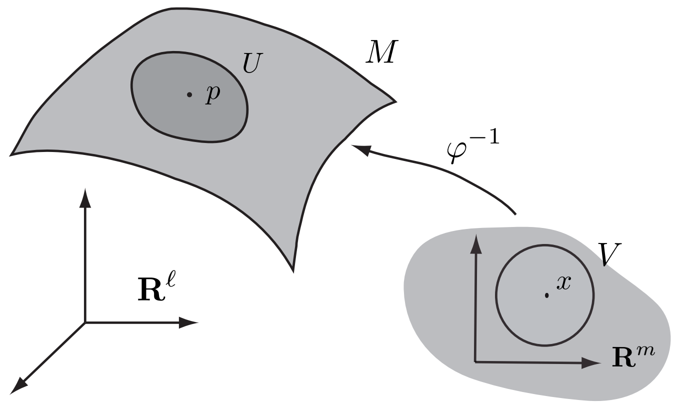

FIGURE 30. Parametrization of an embedded manifold.

It is straightforward to see that this definition does not depend on the chosen parametrization. Let (φ', U') be another chart such that $p \in U'$ and let $V' = \varphi'(U)$, and $\varphi'^{-1}(x') = p$. Figure 31 shows that $J(\varphi'\varphi^{-1})|_x$ is an isomorphism and thus

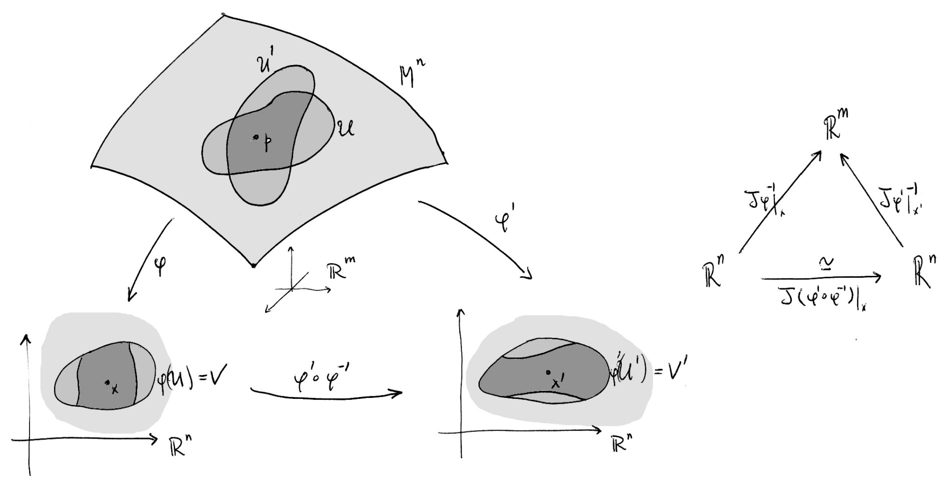

FIGURE 31. The tangent space is independent of the chosen parametrization.

$J\varphi^{-1}|_x(\mathbb{R}^m) = J\varphi'^{-1}|_x(\mathbb{R}^m)$, which proves our assertion. For an m-dimensional manifold M, the tangent space T_pM at a point $p \in M$ is an m-dimensional subspace of $\mathbb{R}^\ell$. In order to prove this we use an essential property of smooth maps between embedded manifolds. The commuting diagrams below show that the commuting diagrams yield that rank of $J\varphi^{-1}|_x$ is equal m. It holds that $\varphi : U \to V = \varphi(U)$, has an extension $\varphi^\dagger : U^\dagger \to \mathbb{R}^m$, and φ and $\varphi^\dagger$ coincide on $U \cap U^\dagger$, which proves that T_pM is well-defined.

$$
\begin{tikzcd}
 & U^\dagger & \\
V \arrow[ur, "\varphi^{-1}"] \arrow[rr, "i"'] & & \mathbb{R}^m \arrow[ul, "\varphi^\dagger"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{R}^\ell & \\
\mathbb{R}^m \arrow[ur, "J\varphi^{-1}|_x"] \arrow[rr, "\mathrm{id}"'] & & \mathbb{R}^m \arrow[ul, "J\varphi^\dagger|_p"']
\end{tikzcd}
$$

◀ **6.4** *Example.* Consider the standard circle $S^1 = \{(p_1, p_2) \in \mathbb{R}^2 \ : \ p_1^2 + p_2^2 = 1\}$, and the parametrization $\varphi^{-1} : (0, 2\pi) \to S^1 \subset \mathbb{R}^2$ given by $\varphi^{-1}(x) = (\cos(x), \sin(x))^t$. We compute T_pS^1 at $p = (\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$. The point p corresponds to $x = \frac{\pi}{4}$. We have that $J\varphi^{-1}|_x = (-\sin(x), \cos(x))^t : \mathbb{R} \to \mathbb{R}^2$, and thus $J\varphi^{-1}|_{\frac{\pi}{4}} = (-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})^t$, which yields that $T_pS^1 = \big[(-1,1)^t\big] \cong \mathbb{R}$. ▶

By identifying T_pM with a pair $\{p\} \times T_pM$, tangent spaces are given the property that $T_pM \cap T_qM = \varnothing$, whenever $p \neq q$. The union of these tangent spaces $TM := \cup_{p\in M}\big(\{p\} \times T_pM\big) \subset \mathbb{R}^\ell \times \mathbb{R}^\ell$ gives the tangent bundle of M. In this definition TM is smooth embedded manifold of $\mathbb{R}^{2\ell}$ (see Exercises). A tangent vector at $p \in M$ will be denoted by $X_p \in T_pM$. Smooth functions from M to TM $X : \ M \to TM$ a defined by $X(p) = (p, X_p)$, for every $p \in M$ (i.e. $X_p \in T_pM$), are vector fields. Of course one can also regard X as a mapping $X : M \to \mathbb{R}^\ell$.

◀ **6.5** *Example.* In the above example $M = S^1$ the mapping

$$S^1 \ni (p_1, p_2) \mapsto (p_1, p_2) \times (-p_2, p_1),$$

is a smooth vector field on S^1. Indeed, for a given p described by the chart $U = S^1 \backslash \{(1,0)\}$ we have that $p_1 = \cos(x)$ and $p_2 = \sin(x)$, and therefore $X_p = (-\sin(x), \cos(x))^t = J\varphi^{-1}|_x(1) \in T_pM$, which proves that the mapping X takes values in TM. ▶

A parametrization φ^{-1} which comes from with a chart (U, φ) can be used to find a canonical basis for T_pM. Choosing local coordinates $x = (x_1, \cdots, x_n) = \varphi(p)$, and the standard basis vectors e_i, we define

$$\frac{\partial}{\partial x_i}\Big|_p := J\varphi^{-1}|_x(e_i).$$

By definition $\frac{\partial}{\partial x_i}\big|_p \in T_pM$, and since the vectors e_i form a basis for $\mathbb{R}^m$, the vectors $\frac{\partial}{\partial x_i}\big|_p$ form a basis for T_pM. An arbitrary tangent vector $X_p \in T_pM$ can now be written with respect to the basis $\{\frac{\partial}{\partial x_i}\big|_p\}$:

$$X_p = J\varphi^{-1}|_x(X_ie_i) = X_i \frac{\partial}{\partial x_i}\Big|_p.$$

where the notation $X_i \frac{\partial}{\partial x_i}\Big|_p = \sum_i X_i \frac{\partial}{\partial x_i}\Big|_p$ denotes the Einstein summation convention, and (X_i) is a vector in $\mathbb{R}^m$! This notation can easily be justified via the following elementary calculus exercise. Consider a smooth function $h : M \to \mathbb{R}$, then the Jacobian $Jh|_p = (\frac{\partial h}{\partial p_i})_{i=1}^n$ is well-defined by virtue of Lemma 4.3. Given a tangent vector $X_p \in T_pM$ we can define the directional derivative of h in the direction of X_p

$$X_p h := \langle Jh|_p, X_p \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^\ell$. Now, by setting $\tilde{h} = h \circ \varphi^{-1}$,

$$X_p h = \langle Jh|_p, X_p \rangle = Jh|_p \cdot J\varphi^{-1}|_x (X_i e_i) = X_i \frac{\partial}{\partial x_i}(h \circ \varphi^{-1}) = X_i \frac{\partial \tilde{h}}{\partial x_i},$$

justifies the notation of tangent vector as a representation in local coordinates. In

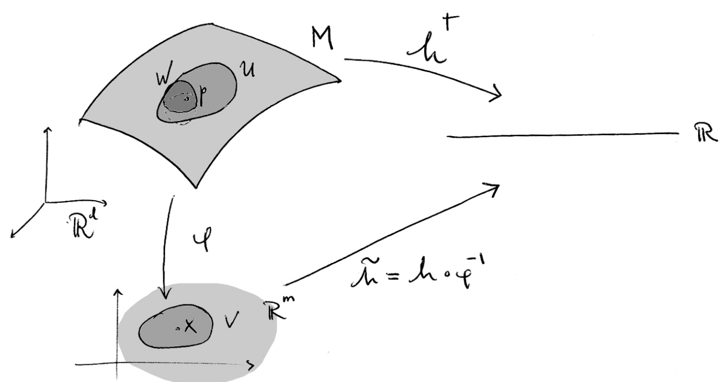

FIGURE 32. The function h and $h^\dagger$ coincide on $W \cap U$.

particular we have

$$\frac{\partial}{\partial x_i}\Big|_p h = \frac{\partial \tilde{h}}{\partial x_i}\Big|_{\varphi(p)=x}.$$

◀ **6.7** *Example.* Let us go back to the above example of $M = S^1$, and consider a point $p \in S^1 \subset \mathbb{R}^2$. Since S^1 is 1-dimensional, $e_1 = (1)$, and $x = x_1$. We have that $\frac{\partial}{\partial x}\Big|_p = J\varphi^{-1}|_x(1) = (-\sin(x), \cos(x))^t$. Let $h : S^1 \to \mathbb{R}$ be given by $f(p_1, p_2) = p_1^2$, and $X_p = (-p_2, p_1)^t$. Then

$$X_p h = (2p_1, 0) \cdot (-p_2, p_1)^t = -2p_1 p_2.$$

In local coordinates this reads: $X_p h = -2\cos(x)\sin(x)$. If we express f in local coordinates we obtain $\tilde{h}(x) = (h \circ \varphi^{-1})(x) = \cos^2(x)$, and $\frac{\partial}{\partial x}(h \circ \varphi^{-1})(x) = -2\cos(x)\sin(x) = ((X_p h) \circ \varphi^{-1})(x)$. ▶

◀ **6.8** *Example.* Consider the embedded torus $\mathbb{T}^2 = \{(p_1, \cdots p_4) \in \mathbb{R}^4 \ : \ p_1^2 + p_2^2 = 1, \ p_3^2 + p_4^2 = 1\}$. Consider the chart (φ, U), with $U = (0, 2\pi) \times (0, 2\pi)$, and parametrization $\varphi^{-1}(x_1, x_2) = (\cos(x_1), \sin(x_1), \cos(x_2), \sin(x_2))$. Consider the function $h(p) = p_1^2 + p_3^2$, which is a mapping from $\mathbb{T}^2 \subset \mathbb{R}^4 \to \mathbb{R}$. It follows that

$Jh|_p = (2p_1, 0, 2p_3, 0) : \mathbb{R}^4 \to \mathbb{R}$. Let us compute the directional derivative of h in the direction $X_p = (-p_2, p_1, -p_4, p_3)^t$. Then

$$X_p h = (2p_1, 0, 2p_3, 0) \cdot (-p_2, p_1, -p_4, p_3)^t = -2p_1p_2 - 2p_3p_4.$$

Let us now use the local coordinates $x = (x_1, x_2)$ to compute $\frac{\partial}{\partial x_i}\big|_p$, $i = 1, 2$. It follows that

$$\begin{aligned} \frac{\partial}{\partial x_1}\Big|_p &= J\varphi^{-1}|_x(e_1) = (-\sin(x_1), \cos(x_1), 0, 0)^t, \\ \frac{\partial}{\partial x_2}\Big|_p &= J\varphi^{-1}|_x(e_2) = (0, 0, -\sin(x_2), \cos(x_2))^t. \end{aligned}$$

Therefore the tangent vector X_p is given by $X_p = \frac{\partial}{\partial x_1}\big|_p + \frac{\partial}{\partial x_2}\big|_p$. In local coordinates h reads: $\tilde{h}(x_1, x_2) = (f \circ \varphi^{-1})(x) = \cos^2(x_1) + \cos^2(x_2)$. We now apply the differential operator $X_p \circ \varphi^{-1} = \frac{\partial}{\partial x_1}\big|_p + \frac{\partial}{\partial x_2}\big|_p$ to the function $\tilde{h} = h \circ \varphi^{-1}$:

$$\Big(\frac{\partial}{\partial x_1}\Big|_p + \frac{\partial}{\partial x_2}\Big|_p\Big)\tilde{h} = -2\cos(x_1)\sin(x_1) - 2\cos(x_2)\sin(x_2),$$

which agrees with the direct calculation in p. ▶

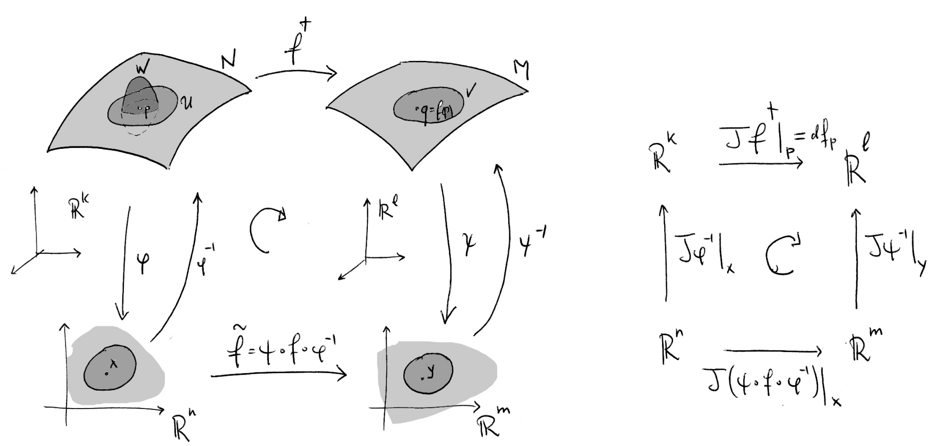

FIGURE 33. Commutative diagrams.

Consider two embedded manifolds N and M in $\mathbb{R}^k$ and $\mathbb{R}^\ell$ respectively, and let $f : N \to M$ be a differentiable mapping. We now introduce the notion of the ***differential*** of f at a point $p \in N$. We define for any $X_p \in T_pN$

$$f_* X_p = df_p \cdot X_p := Jf|_p(X_p) \in \mathbb{R}^\ell.$$

The commutative diagrams in Figure 33 show that $Jf|_p \cdot J\varphi^{-1}|_x = J\psi^{-1}|_y \cdot J(\psi \circ f \circ \varphi^{-1})|_x$, which implies that $Jf|_p$ maps $T_pN = \mathrm{range}\, J\varphi^{-1}|_x$ into $\mathrm{range}\, J\psi^{-1}|_y = T_qM$. This does not depend on the choice the extension $f^\dagger$, since by the 'bottum part' in the commutative diagram in Figure 33, i.e. $J\psi^{-1}|_y \cdot J(\psi\circ f\circ\varphi^{-1})|_x \cdot (J\varphi^{-1}|_x)^{-1} = Jf|_p$, we obtain the same linear mapping.

Another way of defining f_* is via

$$(f_*X_p)h := X_p(h\circ f),$$

where $h : M \to \mathbb{R}$ arbitrary smooth function. Figure 27 gives a schematic account of this construction. Let $q = f(p)$ and (V,ψ) is a chart for M containing q. The vectors $\frac{\partial}{\partial y_j}\big|_q$ form a basis for T_qM. If we write $X_p = X_i\frac{\partial}{\partial x_i}\big|_p$, then for the basis vectors $\frac{\partial}{\partial x_i}\big|_p$ we have

$$\begin{aligned}\Big(f_*\frac{\partial}{\partial x_i}\Big|_p\Big)h &= \frac{\partial}{\partial x_i}\Big|_p(h\circ f) = \frac{\partial}{\partial x_i}\big(\widetilde{h\circ f}\big) = \frac{\partial}{\partial x_i}\big(h\circ f\circ\varphi^{-1}\big)\\ &= \frac{\partial}{\partial x_i}\big(\tilde{h}\circ\psi\circ f\circ\varphi^{-1}\big) = \frac{\partial}{\partial x_i}\big(\tilde{h}\circ\tilde{f}\big)\\ &= \frac{\partial\tilde{h}}{\partial y_j}\frac{\partial\tilde{f}_j}{\partial x_i} = \Big(\frac{\partial\tilde{f}_j}{\partial x_i}\frac{\partial}{\partial y_j}\Big|_q\Big)h\end{aligned}$$

which implies that T_pN is mapped to T_qM under the map f_*, and that f_* is given by $Jf^\dagger|_p$. In local coordinates this is exactly the Jacobian of $\tilde{f}$. The homomorphism $f_* = df_p$ is also called the ***pushforward*** of f at p. These considerations show that f_* is well-defined (independent of the extension $f^\dagger$), and

$$f_* = df_p :\ T_pN \to T_qM.$$

It is clear from the above definition of pushforward that this construction also works in the abstract case, i.e. the definition via $(f_*X_p)h := X_p(h\circ f)$. The terminology pushforward comes from the fact that under f a tangent vector $X_i\frac{\partial}{\partial x_i}\big|_p$ is pushed forward to a tangent vector

$$Y_j\frac{\partial}{\partial y_j}\Big|_q = \Big[\frac{\partial\tilde{f}_j}{\partial x_i}X_i\Big]\frac{\partial}{\partial y_j}\Big|_q \in T_qM, \tag{3}$$

expressed in local coordinates.

7. Cotangent spaces and differential 1-forms

In linear algebra it is often useful to study the space of linear functions on a given vector space V. This space is denoted by V^* and called the ***dual vector space*** to V — again a linear vector space. So the elements of V^* are linear functions $\theta : V \to \mathbb{R}$. As opposed to vectors $v \in V$, the elements, or vectors in V^* are called ***covectors***.

Lemma 7.1. *Let V be a n-dimensional vector space with basis $\{v_1,\cdots,v_n\}$, then the there exist covectors $\{\theta^1,\cdots,\theta^n\}$ such that*

$$\theta^i(v_j) = \delta^i_j, \quad \text{Kronecker delta},$$

and the covectors $\{\theta^1,\cdots,\theta^n\}$ form a basis for V^*.

This procedure can also be applied to the tangent spaces T_pM described in the previous chapter.

Definition 7.2. Let M be a smooth m-dimensional manifold, and let T_pM be the tangent space at some $p \in M$. The the ***cotangent space*** T_p^*M is defined as the dual vector space of T_pM, i.e.

$$T_p^*M := (T_pM)^*.$$

By definition the cotangent space T_p^*M is also m-dimensional and it has a canonical basis as described in Lemma 7.1. As before have the canonical basis vectors $\frac{\partial}{\partial x_i}|_p$ for T_pM, the associated basis vectors for T_p^*M are denoted $dx^i|_p$. Let us now describe this dual basis for T_p^*M and explain the notation.

The covectors $dx^i|_p$ are called ***differentials*** and we show now that these are indeed related dh_p. Let $h : M \to \mathbb{R}$ be a smooth function and $h_* \in T_p^*M$. Since the differentials $dx^i|_p$ form a basis of T_p^*M we have that $h_* = \lambda_i dx^i|_p$, and therefore

$$h_* \frac{\partial}{\partial x_i}\Big|_p = \lambda_j dx^j|_p \cdot \frac{\partial}{\partial x_i}\Big|_p = \lambda^j \delta^i_j = \lambda_i = \frac{\partial \tilde{h}}{\partial x_i},$$

and thus

$$dh_p = h_* = \frac{\partial \tilde{h}}{\partial x_i} dx^i\Big|_p. \tag{4}$$

Choose h such that h_* satisfies the identity in Lemma 7.1, i.e. let $\tilde{h} = x_i$ ($h = x_i \circ \varphi$ = $\langle \varphi, e_i \rangle$). These linear functions h of course span T_p^*M, and

$$h_* = (x_i \circ \varphi)_* = d(x_i \circ \varphi)_p = dx^i|_p.$$

Cotangent vectors are of the form

$$\theta_p = \theta^i dx^i\Big|_p.$$

The pairing between a tangent vector X_p and a cotangent vector θ_p is expressed component wise as follows:

$$\theta_p \cdot X_p = \theta^i X_j \delta^i_j = \theta^i X_i.$$

In the case of tangent spaces a mapping $f : N \to M$ pushes forward to a linear $f_*; T_pN \to T_qM$ for each $p \in N$. For cotangent spaces one expect a similar construction. Let $q = f(p)$, then for a given cotangent vector $\theta_q \in T_q^*M$ define

$$(f^*\theta_q) \cdot X_p = \theta_q \cdot (f_* X_p) \in T_p^*N,$$

for any tangent vector $X_p \in T_pN$. The homomorphism $f^* : T_q^*M \to T_p^*N$, defined by $\theta_q \mapsto f^*\theta_q$ is called the ***pullback*** of f at p. It is a straightforward consequence from linear algebra that f^* defined above is indeed the dual homomorphism of f_*, also called the adjoint, or transpose (see Lee, Ch. 6, for more details, and compare the

the definition of the transpose of a matrix). If we expand the definition of pullback in local coordinates, using (3), we obtain

$$\begin{aligned}\Big(f^*dy^j\big|_q\Big)\cdot X_i\frac{\partial}{\partial x_i}\Big|_p &= dy^j\big|_q\cdot f_*\Big(X_i\frac{\partial}{\partial x_i}\Big|_p\Big)\\ &= dy^j\big|_q\Big[\frac{\partial \tilde{f}_j}{\partial x_i}X_i\Big]\frac{\partial}{\partial y_j}\Big|_q = \frac{\partial \tilde{f}_j}{\partial x_i}X_i\end{aligned}$$

Using this relation we obtain that

$$\Big(f^*\sigma^j dy^j\big|_q\Big)\cdot X_i\frac{\partial}{\partial x_i}\Big|_p = \sigma^j\frac{\partial \tilde{f}_j}{\partial x_i}X_i = \sigma^j\frac{\partial \tilde{f}_j}{\partial x_i}dx^i\big|_p X_i\frac{\partial}{\partial x_i}\Big|_p,$$

which produces the local formula

$$f^*\sigma^j dy^j\big|_q = \Big[\sigma^j\frac{\partial \tilde{f}_j}{\partial x_i}\Big]dx^i\big|_p = \Big[\sigma^j\circ\tilde{f}\frac{\partial \tilde{f}_j}{\partial x_i}\Big]_x dx^i\big|_p. \tag{5}$$

Lemma 7.3. *Let $f: N\to M$, and $g: M\to P$ be smooth mappings, and let $p\in M$, then*

(i) *$f^*: T_{f(p)}M\to T_pN$, and $g^*: T_{(g\circ f)(p)}P\to T_{f(p)}M$ are linear maps (homomorphisms),*
(ii) *$(g\circ f)^* = f^*\cdot g^*: T_{(g\circ f)(p)}P\to T_pN$,*
(iii) *$(\mathrm{id})^* = \mathrm{id}: T_pN\to T_pN$,*
(iv) *if f is a diffeomorphism, then the pullback f^* is a isomorphism from $T_{f(p)}M$ to T_pN.*

Now consider the coordinate mapping $\varphi: U\subset M\to\mathbb{R}^m$, which is local diffeomorphism. Using Lemma 7.3 we then obtain an isomorphism $\varphi^*:\mathbb{R}^m\to T_p^*M$, which justifies the notation

$$dx^i|_p = \varphi^*(e_i).$$

The union

$$\bigcup_{p\in M} T_p^*M = T^*M$$

is called ***cotangent bundle*** of M.

Theorem 7.4. *The cotangent bundle T^*M is a smooth $2m$-dimensional manifold, and the projection $\pi: T^*M\to M$, defined by $(\pi(p,\theta_p) = p$, is a smooth mapping.*

The differential $dh: M\to T^*M$, defined by $dh(p) = (p, dh_p)$ is an example of a smooth function. The above consideration give the coordinate wise expression for dh.

Definition 7.5. A ***smooth covector field*** is a smooth mapping

$$\theta: \ M \to T^*M,$$

with the property that $\pi \circ \theta = \mathrm{id}_M$. The space of smooth covector fields on M is denoted by $\mathcal{F}^*(M)$. Usually the space $\mathcal{F}^*(M)$ is denoted by $\Gamma^1(M)$, and covector fields are referred to a ***differential 1-form*** on M.

For a chart (U, φ) a covector field θ can be expressed as follows

$$\theta = \theta^i dx^i\big|_p,$$

where $\theta^i : U \to \mathbb{R}$. Smoothness of a covector fields can be described in terms of the component functions θ^i.

Lemma 7.6. *A covector field θ is smooth at $p \in U$ if and only if the coordinate functions $\theta^i : U \to \mathbb{R}$ are smooth.*

We saw in the previous section that for arbitrary mappings $f : N \to M$, a vector field $X \in \mathcal{F}(N)$ does not necessarily push forward to a vector on M under f_*. The reason is that surjectivity and injectivity are both needed to guarantee this, which requires f to be a diffeomorphism. In the case of covector fields or differential 1-forms the situation is completely opposite, because the pullback acts in the opposite direction and is therefore onto the target space and uniquely defined. To be mote precise; given a 1-form $\theta \in \Lambda^1(M)$ we define a a 1-form $f^*\theta \in \Lambda^1(N)$ as follows

$$(f^*\theta)_p = f^*\theta_{f(p)}.$$

Theorem 7.7. [16] *The above defined pullback $f^*\theta$ of θ under a smooth mapping $f : N \to M$ is a smooth covector field, or differential 1-form on N.*

The following identities are useful in various situations. Let $f : N \to M$, $g : N \to \mathbb{R}$, both smooth, and $\theta \in \Lambda^1(N)$, then

$$f^*dg = d(g \circ f), \qquad f^*(g\theta) = (g \circ f) f^*\theta. \tag{6}$$

The above identities suggest that the differential of a smooth function $g : N \to M$ defines a differential 1-form. Indeed, dg, defined by $dg(p) = (p, dg_p)$ is a smooth covector field on N. We have from Section 6 that $dg_p : T_pN \to \mathbb{R}$, and thus $dg_p \in T_p^*N$. Since dg_p depends smoothly on p, $dg : N \to T^*N$ is a smooth mapping.

Using the formulas in (6) we can also obtain (5) in a rather straightforward way. Let $g = \psi_j = \langle \psi, e_j \rangle = y_j$, and $\omega = dg = dy^j|_q$ in local coordinates, then

$$f^*\big((\sigma^j \circ \psi)\omega\big) = \big(\sigma^j \circ \psi \circ f\big) f^*dg = \big(\sigma^j \circ \psi \circ f\big) d(g \circ f) = \Big[\sigma^j \circ \tilde{f} \frac{\partial \tilde{f}_j}{\partial x_i}\Big]_x dx^i\big|_p,$$

where the last step follows from (4).

[16] Lee, Proposition 6.13.

Definition 7.8. A differential 1-form $\theta \in \Lambda^1(N)$ is called an ***exact*** 1-form if there exists a smooth function $g : N \to \mathbb{R}$, such that $\theta = dg$.

The notation for tangent vectors was motivated by the fact that functions on a manifold can be differentiated in tangent directions. The notation for the cotangent vectors was partly motivated as the 'reciprocal' of the partial derivative. The introduction of line integral will give an even better motivation for the notation for cotangent vectors. Let $N = \mathbb{R}$, and θ a 1-form on N given in local coordinates by $\theta_t = h(t)dt$, which can be identified with a function h. The notation makes sense because θ can be integrated over any interval $[a,b] \subset \mathbb{R}$:

$$\int_{[a,b]} \theta := \int_a^b h(t)dt.$$

Let $M = \mathbb{R}$, and consider a mapping $f : M = \mathbb{R} \to N = \mathbb{R}$, which satisfies $f'(t) > 0$. Then $t = f(s)$ is an appropriate change of variables. Let $[c,d] = f([a,b])$, then

$$\int_{[c,d]} f^*\theta = \int_c^d h(f(s))f'(s)ds = \int_a^b h(t)dt = \int_{[a,b]} \theta,$$

which is the change of variables formula for integrals. We can use this now to define the line integral over a curve γ on a manifold N.

Definition 7.9. Let $\gamma : [a,b] \subset \mathbb{R} \to N$, and let θ be a 1-form on N and $\gamma^*\theta$ the pullback of θ, which is a 1-form on $\mathbb{R}$. Denote the image of γ in N also by γ, then

$$\int_\gamma \theta := \int_{[a,b]} \gamma^*\theta.$$

If γ' is nowhere zero then the map $\gamma : [a,b] \to N$ is either an immersion or embedding. For example in the embedded case this gives an embedded submanifold $\gamma \subset N$ with boundary $\partial\gamma = \{\gamma(a), \gamma(b)\}$. Let $\theta = dg$ be an exact 1-form, then

$$\int_\gamma dg = g\big|_{\partial\gamma} = g(\gamma(b)) - g(\gamma(a)).$$

Indeed,

$$\int_\gamma dg = \int_{[a,b]} \gamma^* dg = \int_{[a,b]} d(g\circ\gamma) = \int_a^b (g\circ\gamma)'(t)dt = g(\gamma(b)) - g(\gamma(a)).$$

This identify is called the ***Fundamental Theorem for Line Integrals*** and is a spacial case of the Stokes Theorem.

8. Cotangent spaces for embedded manifolds

For a function $h : M \to \mathbb{R}$ we have[17]

$$\frac{\partial}{\partial x_i}\Big|_p h = h_* \frac{\partial}{\partial x_i}\Big|_p = \langle Jh|_p, \frac{\partial}{\partial x_i}\Big|_p \rangle = \frac{\partial \tilde{h}}{\partial x_i},$$

where $\tilde{h} = h \circ \varphi^{-1}$, $h^* : T_pM \to \mathbb{R}$ is an element of T_p^*M. We now seek a function h such that h_* satisfies the identity in Lemma 7.1. In order to do so we choose $\tilde{h} = x_i$, i.e. $h = x_i \circ \varphi = \langle \varphi, e_i \rangle$. This then yields

$$dx^i\big|_p := J\langle \varphi, e_i \rangle\big|_p = (e_i)^t \cdot J\varphi\big|_p.$$

This definition gives the following relation

$$dx^i\big|_p \cdot \frac{\partial}{\partial x_j}\Big|_p = \delta^i_j,$$

which shows that $\{dx^i|_p\}$ is a basis for T_p^*M. Since f_* is also given by $Jf^\dagger|_p$ we have that $Jf^\dagger|_p = J(\tilde{f} \circ \varphi)|_x = J\tilde{f}|_x \cdot J\varphi|_p = J\tilde{f}|_x(e_i) \cdot (e_i)^t J\varphi|_p$ again justifies our definitions. An alternative direct calculation gives

$$f_* \frac{\partial}{\partial x_i}\Big|_p = \frac{\partial \tilde{f}}{\partial x_i} = \frac{\partial f \circ \varphi^{-1}}{\partial x_i} = Jf^\dagger|_p \cdot J\varphi^{-1}|_x(e_i) = Jf^\dagger|_p \frac{\partial}{\partial x_i}\Big|_p,$$

and thus $Jf^\dagger|_p = \frac{\partial \tilde{f}}{\partial x_i} dx^i\big|_p$. In a general a cotangent vector at $p \in M$ is denoted by $\theta_p \in T_p^*M$. As in the case of tangent space one can identify the spaces T_p^*M with $\{p\} \times T_p^*M$, so that they are all mutually disjoint when $p \neq q$. The union $\cup_{p\in M}\big(\{p\} \times T_p^*M\big) = T^*M$ is can the ***cotangent bundle*** of M, As before also T^*M is a smooth embedded manifold in $\mathbb{R}^{2\ell}$. The differential $df : M \to T^*M$, defined by $df(p) = (p, df_p)$ is an example of a smooth function. The above consideration give the coordinate wise expression for df.

- examples

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[17] In the relation $(h^*X_p)g = X_p(g \circ h)$ we can take $h, g : M \to \mathbb{R}$, with $g = \mathrm{id}$, which then $h_*X_p = X_p h$.

III. Tensors and differential forms

9. Tensors and tensor products

In the previous chapter we encountered linear functions on vector spaces, linear functions on tangent spaces to be precise. In this chapter we extend to notion of linear functions on vector spaces to multilinear functions.

Definition 9.1. *Let $V_1, \cdots, V_r$, and W be real vector spaces. A mapping $T : V_1 \times \cdots V_r \to W$ is called a **multilinear mapping** if*

$$T(v_1, \cdots, \lambda v_i + \mu v_i', \cdots, v_r) = \lambda T(v_1, \cdots, v_i, \cdots v_r) + \mu T(v_1, \cdots, v_i', \cdots v_r), \quad \forall i,$$

and for all $\lambda, \mu \in \mathbb{R}$ i.e. f is linear in each variable v_i separately.

Now consider the special case that $W = \mathbb{R}$, then T becomes a ***multilinear function, or form***, and a generalization of linear functions. If in addition $V_1 = \cdots V_r = V$, then

$$T : V \times \cdots \times V \to \mathbb{R},$$

is a multilinear function on V, and is called a ***covariant r-tensor*** on V. The number of copies r is called the rank of T. The space of covariant r-tensors on V is denoted by $T^r(V)$, which clearly is a real vector space using the multilinearity property in Definition 9.1. In particular we have that $T^0(V) \cong \mathbb{R}$, $T^1(V) = V^*$, and $T^2(V)$ is the space of bilinear forms on V. If we consider the case $V_1 = \cdots = V_r = V^*$, then

$$T : V^* \times \cdots \times V^* \to \mathbb{R},$$

is a multilinear function on V^*, and is called a ***contravariant r-tensor*** on V. The space of contravariant r-tensors on V is denoted by $T_r(V)$. Here we have that $T_0(V) \cong \mathbb{R}$, and $T_1(V) = (V^*)^* \cong V$.

◀ **9.2** *Example.* The cross product on $\mathbb{R}^3$ is an example of a multilinear (bilinear) function mapping not to $\mathbb{R}$ to $\mathbb{R}^3$. Let $x, y \in \mathbb{R}^3$, then

$$T(x,y) = x \times y \in \mathbb{R}^3,$$

which clearly is a bilinear function on $\mathbb{R}^3$. ▶

Since multilinear functions on V can be multiplied, i.e. given vector spaces V, W and tensors $T \in T^r(V)$, and $S \in T^s(W)$, the multilinear function

$$R(v_1, \cdots v_r, w_1, \cdots, w_s) = T(v_1, \cdots v_r) S(w_1, \cdots, w_s)$$

is well defined and is a multilnear function on $V^r \times W^s$. This brings us to the following definition. Let $T \in T^r(V)$, and $S \in T^s(W)$, then

$$T \otimes S : V^r \times W^s \to \mathbb{R},$$

is given by

$$T \otimes S(v_1, \cdots, v_r, w_1, \cdots w_s) = T(v_1, \cdots, v_r)S(w_1, \cdots w_s).$$

This product is called the ***tensor product***. By taking $V = W$, $T \otimes S$ is a covariant $(r+s)$-tensor on V, which is a element of the space $T^{r+s}(V)$ and $\otimes : T^r(V) \times T^s(V) \to T^{r+s}(V)$.

Lemma 9.3. *Let $T \in T^r(V)$, $S, S' \in T^s(V)$, and $R \in T^t(V)$, then*

(i) $(T \otimes S) \otimes R = T \otimes (S \otimes R)$ *(associative),*
(ii) $T \otimes (S + S') = T \otimes S + T \otimes S'$ *(distributive),*
(iii) $T \otimes S \neq S \otimes T$ *(non-commutative).*

The tensor product is also defined for contravariant tensors and mixed tensors. As a special case of the latter we also have the product between covariant and contravariant tensors.

◀ **9.4** *Example.* The last property can easily be seen by the following example. Let $V = \mathbb{R}^2$, and $T, S \in T^1(\mathbb{R}^2)$, given by $T(v) = v_1 + v_2$, and $S(w) = w_1 - w_2$, then

$$T \otimes S(1,1,1,0) = 2 \neq 0 = S \otimes T(1,1,1,0),$$

which shows that $\otimes$ is not commutative in general. ▶

The following theorem shows that the tensor product can be used to build the tensor space $T^r(V)$ from elementary building blocks.

Theorem 9.5. [18] *Let $\{v_1, \cdots, v_n\}$ be a basis for V, and let $\{\theta^1, \cdots, \theta^n\}$ be the dual basis for V^*. Then the set*

$$\mathcal{B} = \left\{\theta^{i_1} \otimes \cdots \otimes \theta^{i_r} \ : \ 1 \leq i_1, \cdots, i_r \leq n\right\},$$

is a basis for the n^r-dimensional vector space $T^r(V)$.

By using the multilinearity of tensors we obtain from the above theorem that a tensor T can be expanded in the basis $\mathcal{B}$ as follows;

$$T = T_{i_1 \cdots i_r} \theta^{i_1} \otimes \cdots \otimes \theta^{i_r},$$

where $T_{i_1 \cdots i_r} = T(v_{i_1}, \cdots, v_{i_r})$, the components of the tensor T.

◀ **9.6** *Example.* Consider the the 2-tensors $T(x,y) = x_1y_1 + x_2y_2$, and $T' = x_1y_1 + x_2y_2 + x_1y_2$ on $\mathbb{R}^2$. With respect to the standard bases $\theta^1(x) = x_1$, $\theta^2(x) = x_1$, and

$$\theta^1 \otimes \theta^1(x,y) = x_1y_1, \quad \theta^1 \otimes \theta^2(x,y) = x_1y_2,$$
$$\theta^1 \otimes \theta^2(x,y) = x_2y_1, \quad \text{and} \quad \theta^2 \otimes \theta^2(x,y) = x_2y_2.$$

[18] See Lee, Prop. 11.2.

Using this the components of T are given by $T_{11}=1$, $T_{12}=0$, $T_{21}=0$, and $T_{22}=1$. Also notice that $T'=S\otimes S'$, where $S(x)=x_1+x_2$, and $S'(y)=y_2$. Observe that not every tensor $T\in T^2(\mathbb{R}^2)$ is of the form $T=S\otimes S'$. ▶

In Lee, Ch. 11, the notion of tensor product between arbitrary vector spaces is explained. Here we will discuss a simplified version of the abstract theory. Let V and W be two (finite dimensional) real vector spaces, with bases $\{v_1,\cdots v_n\}$ and $\{w_1,\cdots w_m\}$ respectively, and for their dual spaces V^* and W^* we have the dual bases $\{\theta^1,\cdots,\theta^n\}$ and $\{\sigma^1,\cdots,\sigma^m\}$ respectively. If we use the identification $\{V^*\}^*\cong V$, and $\{W^*\}^*\cong W$ we can define $V\otimes W$ as follows:

Definition 9.7. The tensor product of V and W is the real vector space of (finite) linear combinations

$$V\otimes W := \left\{\lambda^{ij}v_i\otimes v_j \;:\; \lambda^{ij}\in\mathbb{R}\right\} = \left[\left\{v_i\otimes w_j\right\}_{i,j}\right],$$

where $v_i\otimes w_j(v^*,w^*) := v^*(v_i)w^*(w_j)$, using the identification $v_i(v^*) := v^*(v_i)$, and $w_j(w^*) := w^*(w_j)$, with $(v^*,w^*)\in V^*\times W^*$.

To get a feeling of what the tensor product of two vector spaces represents consider the tensor product of the dual spaces V^* and W^* we obtain the real vector space of (finite) linear combinations

$$V^*\otimes W^* := \left\{\lambda_{ij}\theta^i\otimes\sigma^j \;:\; \lambda_{ij}\in\mathbb{R}\right\} = \left[\left\{\theta^i\otimes\sigma^j\right\}_{i,j}\right],$$

where $\theta^i\otimes\sigma^j(v,w)=\theta^i(v)\sigma^j(w)$ for any $(v,w)\in V\times W$. One can show that $V^*\otimes W^*$ is isomorphic to space of bilinear maps from $V\times W$ to $\mathbb{R}$. In particular elements $v^*\otimes w^*$ all lie in $V^*\otimes W^*$, but not all elements in $V^*\otimes W^*$ are of this form. The isomorphism is easily seen as follows. Let $v=\xi_i v_i$, and $w=\eta_j w_j$, then for a given bilinear form b it holds that $b(v,w)=\xi_i\eta_j b(v_i,w_j)$. By definition of dual basis we have that $\xi_i\eta_j=\theta^i(v)\sigma^j(w)=\theta^i\otimes\sigma^j(v,w)$, which shows the isomorphism by setting $\lambda_{ij}=b(v_i,w_j)$.

In the case $V^*\otimes W$ the tensors represent linear maps from V to W. Indeed, from the previous we know that elements in $V^*\otimes W$ represent bilinear maps from $V\times W^*$ to $\mathbb{R}$. For an element $b\in V^*\otimes W$ this means that $b(v,\cdot):W^*\to\mathbb{R}$, and thus $b(v,\cdot)\in(W^*)^*\cong W$.

◀ **9.8** *Example.* Consider vectors $a\in V$ and $b^*\in W$, then $a^*\otimes(b^*)^*$ can be identified with a matrix, i.e $a^*\otimes(b^*)^*(v,\cdot)=a^*(v)(b^*)^*(\cdot)\cong a^*(v)b$. For example let $a^*(v)=a_1v_1+a_2v_2+a_3v_3$, and

$$Av=a^*(v)b=\begin{pmatrix} a_1b_1v_1+a_2b_1v_2+a_3b_1v_3 \\ a_1b_2v_1+a_2b_2v_2+a_3b_2v_3\end{pmatrix}=\begin{pmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2\end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}.$$

Symbolically we can write

$$A = a^* \otimes b = \begin{pmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

which shows how a vector and covector can be 'tensored' to become a matrix. Note that it also holds that $A = (a \cdot b^*)^* = b \cdot a^*$. ▶

Lemma 9.9. *We have that*

(i) $V \otimes W$ *and* $W \otimes V$ *are isomorphic;*
(ii) $(U \otimes V) \otimes W$ *and* $U \otimes (V \otimes W)$ *are isomorphic.*

With the notion of tensor product of vector spaces at hand we now conclude that the above describe tensor spaces $T^r(V)$ and $V_r(V)$ are given as follows;

$$T^r(V) = \underbrace{V^* \otimes \cdots \otimes V^*}_{r \text{ times}}, \qquad T_r(V) = \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}}.$$

By considering tensor products of V's and V^*'s we obtain the tensor space of mixed tensors;

$$T_s^r(V) := \underbrace{V^* \otimes \cdots \otimes V^*}_{r \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{s \text{ times}}.$$

Elements in this space are called (r,s)-***mixed tensors*** on V — r copies of V^*, and s copies of V. Of course the tensor product described above is defined in general for tensors $T \in T_s^r(V)$, and $S \in T_{s'}^{r'}(V)$:

$$\otimes : \ T_s^r(V) \times T_{s'}^{r'}(V) \to T_{s+s'}^{r+r'}(V).$$

The analogue of Theorem 9.5 can also be established for mixed tensors. In the next sections we will see various special classes of covariant, contravariant and mixed tensors.

◀ **9.10** *Example.* The inner product on a vector space V is an example of a covariant 2-tensor. This is also an example of a symmetric tensor. ▶

◀ **9.11** *Example.* The determinant of n vectors in $\mathbb{R}^n$ is an example of covariant n-tensor on $\mathbb{R}^n$. The determinant is skew-symmetric, and an example of an alternating tensor. ▶

If $f : V \to W$ is a linear mapping between vector spaces and T is an covariant tensor on W we can define concept of pullback of T. Let $T \in T^r(W)$, then $f^*T \in T^r(V)$ is defined as follows:

$$f^*T(v_1, \cdots v_r) = T(f(v_1), \cdots, f(v_r)),$$

and $f^* : T^r(W) \to T^r(V)$ is a linear mapping. If we represent f by a matrix A with respect to bases $\{v_i\}$ and $\{w_j\}$ for V and W respectively, then the matrix for the linear f^* is given by

$$\underbrace{A^* \otimes \cdots \otimes A^*}_{r \text{ times}},$$

with respect to the bases $\{\theta^{i_1}\otimes\cdots\otimes\theta^{i_r}\}$ and $\{\sigma^{j_1}\otimes\cdots\otimes\sigma^{j_r}\}$ for $T^r(W)$ and $T^r(V)$ respectively.

◀ **9.12** *Remark.* The direct sum

$$T^*(V)=\bigoplus_{r=0}^{\infty}T^r(V),$$

consisting of finite sums of covariant tensors is called the ***covariant tensor algebra*** of V with multiplication given by the tensor product $\otimes$. Similarly, one defines the contravariant tensor algebra

$$T_*(V)=\bigoplus_{r=0}^{\infty}T_r(V).$$

For mixed tensors we have

$$T(V)=\bigoplus_{r,s=0}^{\infty}T_s^r(V),$$

which is called the ***tensor algebra of mixed tensor*** of V. Clearly, $T^*(V)$ and $T_*(V)$ subalgebras of $T(V)$. ▶

10. Symmetric and alternating tensors

There are two special classes of tensors which play an important role in the analysis of differentiable manifolds. The first class we describe are symmetric tensors. We restrict here to covariant tensors.

Definition 10.1. A covariant r-tensor T on a vector space V is called ***symmetric*** if

$$T(v_1,\cdots,v_i,\cdots,v_j,\cdots,v_r)=T(v_1,\cdots,v_j,\cdots,v_i,\cdots,v_r),$$

for any pair of indices $i\le j$. The set of symmetric covariant r-tensors on V is denoted by $\Sigma^r(V)\subset T^r(V)$, which is a (vector) subspace of $T^r(V)$.

If $\mathbf{a}\in S_r$ is a permutation, then define

$$^{\mathbf{a}}T(v_1,\cdots,v_r)=T(v_{\mathbf{a}(1)},\cdots,v_{\mathbf{a}(r)}),$$

where $\mathbf{a}(\{1,\cdots,r\})=\{\mathbf{a}(1),\cdots,\mathbf{a}(r)\}$. From this notation we have that for two permutations $\mathbf{a},\mathbf{b}\in S_r$, $^{\mathbf{b}}(^{\mathbf{a}}T)={}^{\mathbf{ba}}T$. Define

$$\mathrm{Sym}\,T=\frac{1}{r!}\sum_{\mathbf{a}\in S_r}{}^{\mathbf{a}}T.$$

It is straightforward to see that for any tensor $T \in T^r(V)$, Sym T is a symmetric. Moreover, a tensor T is symmetric if and only if Sym $T = T$. For that reason Sym T is called the ***(tensor) symmetrization***.

◀ **10.2** *Example.* Let $T, T' \in T^2(\mathbb{R}^2)$ be defined as follows: $T(x,y) = x_1 y_2$, and $T'(x,y) = x_1 y_1$. Clearly, T is not symmetric and T' is. We have that

$$\begin{aligned} \text{Sym } T(x,y) &= \frac{1}{2}T(x,y) + \frac{1}{2}T(y,x) \\ &= \frac{1}{2}x_1 y_2 + \frac{1}{2}y_1 x_2, \end{aligned}$$

which clearly is symmetric. If we do the same thing for T' we obtain:

$$\begin{aligned} \text{Sym } T'(x,y) &= \frac{1}{2}T'(x,y) + \frac{1}{2}T'(y,x) \\ &= \frac{1}{2}x_1 y_1 + \frac{1}{2}y_1 x_1 = T'(x,y), \end{aligned}$$

showing that operation Sym applied to symmetric tensors produces the same tensor again. ▶

Using symmetrization we can define the ***symmetric product***. Let $S \in \Sigma^r(V)$ and $T \in \Sigma^s(V)$ be symmetric tensors, then

$$S \cdot T = \text{Sym } (S \otimes T).$$

The symmetric product of symmetric tensors is commutative which follows directly from the definition:

$$S \cdot T(v_1, \cdots, v_{r+s}) = \frac{1}{(r+s)!} \sum_{\mathbf{a} \in S_{r+s}} S(v_{\mathbf{a}(1)}, \cdots, v_{\mathbf{a}(r)}) T(v_{\mathbf{a}(r+1)}, \cdots, v_{\mathbf{a}(r+s)}).$$

◀ **10.3** *Example.* Consider the 2-tensors $S(x) = x_1 + x_2$, and $T(y) = y_2$. Now $S \otimes T(x,y) = x_1 y_2 + x_2 y_2$, and $T \otimes S(x,y) = x_2 y_1 + x_2 y_2$, which clearly gives that $S \otimes T \neq T \otimes S$. Now compute

$$\begin{aligned} \text{Sym } (S \otimes T)(x,y) &= \frac{1}{2}x_1 y_2 + \frac{1}{2}x_2 y_2 + \frac{1}{2}y_1 x_2 + \frac{1}{2}x_2 y_2 \\ &= \frac{1}{2}x_1 y_2 + \frac{1}{2}x_2 y_1 + x_2 y_2 = S \cdot T(x,y). \end{aligned}$$

Similarly,

$$\begin{aligned} \text{Sym } (T \otimes S)(x,y) &= \frac{1}{2}x_2 y_1 + \frac{1}{2}x_2 y_2 + \frac{1}{2}y_2 x_1 + \frac{1}{2}x_2 y_2 \\ &= \frac{1}{2}x_1 y_2 + \frac{1}{2}x_2 y_1 + x_2 y_2 = T \cdot S(x,y), \end{aligned}$$

which gives that $S \cdot T = T \cdot S$. ▶

Lemma 10.4. *Let* $\{v_1, \cdots, v_n\}$ *be a basis for* V*, and let* $\{\theta^1, \cdots, \theta^n\}$ *be the dual basis for* V^**. Then the set*

$$\mathcal{B}_\Sigma = \left\{\theta^{i_1} \cdots \theta^{i_r} \ : \ 1 \leq i_1 \leq \cdots \leq i_r \leq n\right\},$$

is a basis for the (sub)space $\Sigma^r(V)$ *of symmetric* r*-tensors. Moreover,* $\dim \Sigma^r(V) = \left(\begin{array}{c} n+r-1 \\ r \end{array} \right) = \frac{(n+r-1)!}{r!(n-1)!}$.

Another important class of tensors are alternating tensors and are defined as follows.

Definition 10.5. A covariant r-tensor T on a vector space V is called ***alternating*** if

$$T(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_r) = -T(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_r),$$

for any pair of indices $i \leq j$. The set of alternating covariant r-tensors on V is denoted by $\Lambda^r(V) \subset T^r(V)$, which is a (vector) subspace of $T^r(V)$.

As before we define

$$\mathrm{Alt}\ T = \frac{1}{r!} \sum_{\mathbf{a} \in S_r} (-1)^{\mathbf{a}}\ {}^{\mathbf{a}}T,$$

where $(-1)^{\mathbf{a}}$ is $+1$ for even permutations, and -1 for odd permutations. We say that Alt T is the ***alternating projection*** of a tensor T, and Alt T is of course a alternating tensor.

◀ **10.6** *Example.* Let $T, T' \in T^2(\mathbb{R}^2)$ be defined as follows: $T(x,y) = x_1y_2$, and $T'(x,y) = x_1y_2 - x_2y_1$. Clearly, T is not alternating and $T'(x,y) = -T'(y,x)$ is alternating. We have that

$$\begin{aligned} \mathrm{Alt}\ T(x,y) &= \frac{1}{2}T(x,y) - \frac{1}{2}T(y,x) \\ &= \frac{1}{2}x_1y_2 - \frac{1}{2}y_1x_2 = \frac{1}{2}T'(x,y), \end{aligned}$$

which clearly is alternating. If we do the same thing for T' we obtain:

$$\begin{aligned} \mathrm{Alt}\ T'(x,y) &= \frac{1}{2}T'(x,y) - \frac{1}{2}T'(y,x) \\ &= \frac{1}{2}x_1y_2 - \frac{1}{2}x_2y_1 - \frac{1}{2}y_1x_2 + \frac{1}{2}y_2x_1 = T'(x,y), \end{aligned}$$

showing that operation Alt applied to alternating tensors produces the same tensor again. Notice that $T'(x,y) = \det(x,y)$. ▶

This brings us to the fundamental product of alternating tensors called the ***wedge product***. Let $S \in \Lambda^r(V)$ and $T \in \Lambda^s(V)$ be symmetric tensors, then

$$S \wedge T = \frac{(r+s)!}{r!s!}\mathrm{Alt}\ (S \otimes T).$$

The wedge product of alternating tensors is anti-commutative which follows directly from the definition:

$$S \wedge T(v_1, \cdots, v_{r+s}) = \frac{1}{r!s!} \sum_{\mathbf{a} \in S_{r+s}} (-1)^{\mathbf{a}} S(v_{\mathbf{a}(1)}, \cdots, v_{\mathbf{a}(r)}) T(v_{\mathbf{a}(r+1)}, \cdots, v_{\mathbf{a}(r+s)}).$$

In the special case of the wedge of two covectors $\theta, \omega \in V^*$ gives

$$\theta \wedge \omega = \theta \otimes \omega - \omega \otimes \theta.$$

In particular we have that

(i) $(T \wedge S) \wedge R = T \wedge (S \wedge R)$;
(ii) $(T + T') \wedge S = T \wedge S + T' \wedge S$;
(iii) $T \wedge S = (-1)^{rs} S \wedge T$, for $T \in \Lambda^r(V)$ and $S \in \Lambda^s(V)$;
(iv) $T \wedge T = 0$.

The latter is a direct consequence of the definition of Alt . In order to prove these properties we have the following lemma.

Lemma 10.7. *Let $T \in T^r(V)$ and $S \in T^s(V)$, then*

$$\mathrm{Alt}\,(T \otimes S) = \mathrm{Alt}\,((\mathrm{Alt}\,T) \otimes S) = \mathrm{Alt}\,(T \otimes \mathrm{Alt}\,S).$$

Proof: Let $G \cong S_r$ be the subgroup of S_{r+s} consisting of permutations that only permute the element $\{1, \cdots, r\}$. For $\mathbf{a} \in G$, we have $\mathbf{a}' \in S_r$. Now ${}^{\mathbf{a}}(T \otimes S) = {}^{\mathbf{a}}T \otimes S$, and thus

$$\frac{1}{r!} \sum_{\mathbf{a} \in G} (-1)^{\mathbf{a}}\, {}^{\mathbf{a}}(T \otimes S) = (\mathrm{Alt}\,T) \otimes S.$$

For the right cosets $\{\mathbf{ba} \ : \ \mathbf{a} \in G\}$ we have

$$\begin{aligned} \sum_{\mathbf{a} \in G} (-1)^{\mathbf{ba}}\, {}^{\mathbf{ba}}(T \otimes S) &= (-1)^{\mathbf{b}}\, {}^{\mathbf{b}}\Big(\sum_{\mathbf{a} \in G} (-1)^{\mathbf{a}}\, {}^{\mathbf{a}}(T \otimes S) \Big) \\ &= r!(-1)^{\mathbf{b}}\, {}^{\mathbf{b}}\Big((\mathrm{Alt}\,T) \otimes S \Big). \end{aligned}$$

Taking the sum over all right cosets with the factor $\frac{1}{(r+s)!}$ gives

$$\begin{aligned} \mathrm{Alt}\,(T \otimes S) &= \frac{1}{(r+s)!} \sum_{\mathbf{b}} \sum_{\mathbf{a} \in G} (-1)^{\mathbf{ba}}\, {}^{\mathbf{ba}}(T \otimes S) \\ &= \frac{r!}{(r+s)!} \sum_{\mathbf{b}} (-1)^{\mathbf{b}}\, {}^{\mathbf{b}}\Big((\mathrm{Alt}\,T) \otimes S \Big) = \mathrm{Alt}\,((\mathrm{Alt}\,T) \otimes S), \end{aligned}$$

where the latter equality is due to the fact that $r!$ terms are identical under the definition of $\mathrm{Alt}\,((\mathrm{Alt}\,T) \otimes S)$. ∎

Property (i) can now be proved as follows. Clearlt $\mathrm{Alt}\,(\mathrm{Alt}\,(T \otimes S) - T \otimes S) = 0$, and thus from Lemma 10.7 we have that

$$0 = \mathrm{Alt}\,((\mathrm{Alt}\,(T \otimes S) - T \otimes S) \otimes R) = \mathrm{Alt}\,(\mathrm{Alt}\,(T \otimes S) \otimes R) - \mathrm{Alt}\,(T \otimes S \otimes R).$$

By definition

$$
\begin{aligned}
(T \wedge S) \wedge R &= \frac{(r+s+t)!}{(r+s)!t!} \mathrm{Alt}\,((T \wedge S) \otimes R) \\
&= \frac{(r+s+t)!}{(r+s)!t!} \mathrm{Alt}\,\Big(\Big(\frac{(r+s)!}{r!s!} \mathrm{Alt}\,(T \otimes S)\Big) \otimes R\Big) \\
&= \frac{(r+s+t)!}{r!s!t!} \mathrm{Alt}\,(T \otimes S \otimes R).
\end{aligned}
$$

The same formula holds for $T \wedge (S \wedge R)$, which prove associativity. More generally it holds that for $T_i \in \Lambda^{r_i}(V)$

$$
T_1 \wedge \cdots \wedge T_k = \frac{(r_1 + \cdots + r_k)!}{r_1! \cdots r_k!} \mathrm{Alt}\,(T_1 \otimes \cdots \otimes T_k).
$$

◀ **10.8** *Example.* Consider the 2-tensors $S(x) = x_1 + x_2$, and $T(y) = y_2$. As before $S \otimes T(x,y) = x_1 y_2 + x_2 y_2$, and $T \otimes S(x,y) = x_2 y_1 + x_2 y_2$. Now compute

$$
\begin{aligned}
\mathrm{Alt}\,(S \otimes T)(x,y) &= \frac{1}{2} x_1 y_2 + \frac{1}{2} x_2 y_2 - \frac{1}{2} y_1 x_2 - \frac{1}{2} x_2 y_2 \\
&= \frac{1}{2} x_1 y_2 - \frac{1}{2} x_2 y_1 = 2\Big(S \wedge T(x,y)\Big).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathrm{Alt}\,(T \otimes S)(x,y) &= \frac{1}{2} x_2 y_1 + \frac{1}{2} x_2 y_2 - \frac{1}{2} y_2 x_1 - \frac{1}{2} x_2 y_2 \\
&= -\frac{1}{2} x_1 y_2 + \frac{1}{2} x_2 y_1 = -2\Big(T \wedge S(x,y)\Big),
\end{aligned}
$$

which gives that $S \wedge T = -T \wedge S$. Note that if $T = e_1^*$, i.e. $T(x) = x_1$, and $S = e_2^*$, i.e. $S(x) = x_2$, then

$$
T \wedge S(x,y) = x_1 y_2 - x_2 y_1 = \det(x,y).
$$

▶

◀ **10.9** *Remark.* Some authors use the more logical definition

$$
S \bar{\wedge} T = \mathrm{Alt}\,(S \otimes T),
$$

which is in accordance with the definition of the symmetric product. This definition is usually called the alt convention for the wedge product, and our definition is usually referred to as the determinant convention. For computational purposes the determinant convention is more appropriate. ▶

If $\{e_1^*, \cdots, e_n^*\}$ is the standard dual basis for $(\mathbb{R}^n)^*$, then for vectors $a_1, \cdots, a_n \in \mathbb{R}^n$,

$$
\det(a_1, \cdots, a_n) = e_1^* \wedge \cdots \wedge e_n^*(a_1, \cdots, a_n).
$$

The alternating tensor $\det = e_1^* \wedge \cdots \wedge e_n^*$ is called the determinant function on $\mathbb{R}^n$. If $f: V \to W$ is a linear map between vector spaces then the pullback $f^*T \in \Lambda^r(V)$ of any alternating tensor $T \in \Lambda^r(W)$ is given via the relation:

$$f^*T(v_1, \cdots, v_r) = T\big(f(v_1), \cdots, f(v_r)\big), \qquad f^*: \Lambda^r(W) \to \Lambda^r(V).$$

In particular, $f^*(T \wedge S) = (f^*T) \wedge f^*(S)$. As a special case we have that if $f: V \to V$, linear, and $\dim V = n$, then

$$f^*T = \det(f) T, \tag{7}$$

for any alternating tensor $T \in \Lambda^n(V)$. This can be seen as follows. By multilinearity we verify the above relation for the vectors $\{e_i\}$. We have that

$$\begin{aligned} f^*T(e_1, \cdots, e_n) &= T(f(e_1), \cdots, f(e_n)) \\ &= T(f_1, \cdots, f_n) = c \det(f_1, \cdots, f_n) = c \det(f), \end{aligned}$$

where we use the fact that $\Lambda^n(V) \cong \mathbb{R}$ (see below). On the other hand

$$\begin{aligned} \det(f) T(e_1, \cdots e_n) &= \det(f) c \cdot \det(e_1, \cdots, e_n) \\ &= c \det(f), \end{aligned}$$

which proves (7).

Lemma 10.10. *Let $\{\theta^1, \cdots, \theta^n\}$ be a basis for V^*, then the set*

$$\mathcal{B}_\Lambda = \big\{\theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \ : \ 1 \le i_1 < \cdots < i_r \le n\big\},$$

is a basis for $\Lambda^r(V)$, and $\dim \Lambda^r(V) = \frac{n!}{(n-r)!r!}$. In particular, $\dim \Lambda^r(V) = 0$ for $r > n$.

Proof: From Theorem 9.5 we know that any alternating tensor $T \in \Lambda^r(V)$ can be written as

$$T = T_{j_1 \cdots j_r} \theta^{j_1} \otimes \cdots \otimes \theta^{j_r}.$$

We have that Alt $T = T$, and so

$$T = T_{j_1 \cdots j_r} \mathrm{Alt}\, (\theta^{j_1} \otimes \cdots \otimes \theta^{j_r}) = \frac{1}{r!} T_{j_1 \cdots j_r} \theta^{j_1} \wedge \cdots \wedge \theta^{j_r}$$

In the expansion the terms with $j_k = j_\ell$ are zero since $\theta^{j_k} \wedge \theta^{j_\ell} = 0$. Of we order the indices in increasing order we obtain

$$T = \pm \frac{1}{r!} T_{i_1 \cdots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r},$$

which show that $\mathcal{B}_\Lambda$ spans $\Lambda^r(V)$.

Linear independence can be proved as follows. Let $0 = \lambda_{i_1 \cdots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}$, and thus $\lambda_{i_1 \cdots i_r} = \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}(v_{i_1}, \cdots, v_{i_r}) = 0$, which proves linear independence.

It is immediately clear that $\mathcal{B}_\Lambda$ consists of $\begin{pmatrix} n \\ r \end{pmatrix}$ elements. ∎

As we mentioned before the operation Alt is called the alternating projection. As a matter of fact Sym is also a projection.

Lemma 10.11. *Some of the basic properties can be listed as follows;*

(i) Sym *and* Alt *are projections on* $T^r(V)$, *i.e.* $\mathrm{Sym}^2 = \mathrm{Sym}$, *and* $\mathrm{Alt}^2 = \mathrm{Alt}$*;*
(ii) T *is symmetric if and only if* $\mathrm{Sym}\, T = T$, *and* T *is alternating if and only if* $\mathrm{Alt}\, T = T$*;*
(iii) $\mathrm{Sym}(T^r(V)) = \Sigma^r(V)$, *and* $\mathrm{Alt}(T^r(V)) = \Lambda^r(V)$*;*
(iv) $\mathrm{Sym} \circ \mathrm{Alt} = \mathrm{Alt} \circ \mathrm{Sym} = 0$, *i.e. if* $T \in \Lambda^r(V)$, *then* $\mathrm{Sym}\, T = 0$, *and if* $T \in \Sigma^r(V)$, *then* $\mathrm{Alt}\, T = 0$*;*
(v) *let* $f : V \to W$, *then* Sym *and* Alt *commute with* $f^* : T^r(W) \to T^r(V)$, *i.e.* $\mathrm{Sym} \circ f^* = f^* \circ \mathrm{Sym}$, *and* $\mathrm{Alt} \circ f^* = f^* \circ \mathrm{Alt}$.

11. Tensor bundles and tensor fields

Generalizations of tangent spaces and cotangent spaces are given by the tensor spaces

$$T^r(T_pM), \qquad T_s(T_pM), \quad \text{and} \qquad T^r_s(T_pM),$$

where $T^r(T_pM) = T_r(T^*_pM)$. As before we can introduce the tensor bundles:

$$T^rM = \bigcup_{p \in M} T^r(T_pM),$$
$$T_sM = \bigcup_{p \in M} T_s(T_pM),$$
$$T^r_sM = \bigcup_{p \in M} T^r_s(T_pM),$$

called the ***covariant*** r***-tensor bundle, contravariant*** s***-tensor bundle***, and the ***mixed*** (r,s)***-tensor bundle*** on M. As for the tangent and cotangent bundle the tensor bundles are also smooth manifolds. In particular, $T^1M = T^*M$, and $T_1M = TM$. Recalling the symmetric and alternating tensors as introduced in the previous section we also define the tensor bundles Σ^rM and Λ^rM.

On tensor bundles we also have the natural projection

$$\pi : T^r_sM \to M,$$

defined by $\pi(p,T) = p$. A ***smooth section*** in T^r_sM is a smooth mapping

$$\sigma : M \to T^r_sM,$$

such that $\pi \circ \sigma = \mathrm{id}_M$. The space of smooth sections in T^r_sM is denoted by $\mathcal{F}^r_s(M)$. For the co- and contravariant tensors these spaces are denoted by $\mathcal{F}^r(M)$ and $\mathcal{F}_s(M)$ respectively. Smooth sections in these tensor bundles are also called ***smooth tensor fields***. Clearly, vector fields and 1-forms are examples of tensor fields. Sections in the above described tensor fields can be expressed in coordinates as follows:

$$\sigma = \begin{cases} \sigma_{i_1\cdots i_r} dx^{i_1} \otimes \cdots \otimes dx^{i_r}, & \sigma \in \mathcal{F}^r(M), \\ \sigma^{j_1\cdots j_s} \frac{\partial}{\partial x_{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{j_s}}, & \sigma \in \mathcal{F}_s(M), \\ \sigma^{j_1\cdots j_s}_{i_1\cdots i_r} dx_{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{j_s}}, & \sigma \in \mathcal{F}^r_s(M). \end{cases}$$

Tensor fields are often denoted by the component functions. The tensor and tensor fields in this course are, except for vector fields, all covariant tensors and covariant tensor fields. Smoothness of covariant tensor fields can be described in terms of the component functions $\sigma_{i_1\cdots i_r}$.

Lemma 11.1. [19] *A covariant tensor field σ is smooth at $p \in U$ if and only if*

(i) *the coordinate functions $\sigma_{i_1\cdots i_r} : U \to \mathbb{R}$ are smooth;*
(ii) *for smooth vector fields $X_1, \cdots, X_r$ defined on any open set $U \subset M$, then the function $\sigma(X_1, \cdots, X_r) : U \to \mathbb{R}$, given by*

$$\sigma(X_1, \cdots, X_r)(p) = \sigma_p(X_1(p), \cdots, X_r(p)),$$

is smooth.

The same equivalences hold for contravariant and mixed tensor fields.

◀ **11.2** *Example.* Let $M = \mathbb{R}^2$ and let $\sigma = dx^1 \otimes dx^1 + x_1^2 dx^2 \otimes dx^2$. If $X = \xi^1(x)\frac{\partial}{\partial x_1} + \xi^2(x)\frac{\partial}{\partial x_2}$ and $Y = \eta^1(x)\frac{\partial}{\partial x_1} + \eta^2(x)\frac{\partial}{\partial x_2}$ are arbitrary smooth vector fields on $T_x\mathbb{R}^2 \cong \mathbb{R}^2$, then

$$\sigma(X,Y) = \xi_1(x)\eta_1(x) + x_1^2\xi_2(x)\eta_2(x),$$

which clearly is a smooth function in $x \in \mathbb{R}^2$. ▶

For covariant tensors we can also define the notion of pullback of a mapping f between smooth manifolds. Let $f : N \to M$ be a smooth mappings, then the pullback $f^* : T^r(T_{f(p)}M) \to T^r(T_pN)$ is defined as

$$(f^*T)(X_1, \cdots X_r) := T(f_*X_1, \cdots, f_*X_r),$$

where $T \in T^r(T_{f(p)}M)$, and $X_1, \cdots, X_r \in T_pN$. We have the following properties.

Lemma 11.3. [20] *Let $f : N \to M$, $g : M \to P$ be smooth mappings, and let $p \in N$, $S \in T^r(T_{f(p)}M)$, and $T \in T^{r'}(T_{f(p)}M)$, then:*

(i) *$f^* : T^r(T_{f(p)}M) \to T^r(T_pN)$ is linear;*
(ii) *$f^*(S \otimes T) = f^*S \otimes f^*T$;*
(iii) *$(g \circ f)^* = f^* \circ g^* : T^r(T_{(g\circ f)(p)}P) \to T^r(T_pM)$;*
(iv) *$\mathrm{id}_M^* S = S$;*
(v) *$f^* : T^rM \to T^rN$ is a smooth bundle map.*

[19] See Lee, Lemma 11.6.
[20] See Lee, Proposition 11.8.

◀ **11.4** *Example.* Let us continue with the previous example and let $N = M = \mathbb{R}^2$. Consider the mapping $f : N \to M$ defined by

$$f(x) = (2x_1, x_2^3 - x_1).$$

For a given tangent vectors $X, Y \in T_xN$ we can compute the pushforward

$$f_* = \begin{pmatrix} 2 & 0 \\ -1 & 3x_2^2 \end{pmatrix},$$

$$f_*X = 2\xi_1 \frac{\partial}{\partial x_1} + (-\xi_1 + 3x_2^2\xi_2)\frac{\partial}{\partial x_2},$$
$$f_*Y = 2\eta_1 \frac{\partial}{\partial x_1} + (-\eta_1 + 3x_2^2\eta_2)\frac{\partial}{\partial x_2}.$$

Let σ be given by $\sigma = dy^1 \otimes dy^1 + y_1^2 dy^2 \otimes dy^2$. This then yields

$$\begin{aligned} \sigma(f_*X, f_*Y) &= 4\xi_1\eta_1 + 4x_1^2(\xi_1 - 3x_1^2\xi_2)(\eta_1 - 3x_1^2\eta_2) \\ &= 4(1 + x_1^2)\xi_1\eta_1 - 12x_1^2x_2^2\xi_1\eta_2 \\ &\quad -12x_1^2x_2^2\xi_2\eta_1 + 36x_1^2x_2^4x_1^2\xi_2\eta_2. \end{aligned}$$

We have to point out here that f_*X and f_*Y are tangent vectors in T_xN and not necessarily vector fields, although we can use this calculation to compute $f^*\sigma$, which clearly is a smooth 2-tensor field on N. ▶

◀ **11.5** *Example.* A different way of computing $f^*\sigma$ is via a local representation of σ directly. We have $\sigma = dy^1 \otimes dy^1 + y_1^2 dy^2 \otimes dy^2$, and

$$\begin{aligned} f^*\sigma &= d(2x_1) \otimes d(2x_1) + 4x_1^2 d(x_2^3 - x_1) \otimes d(x_2^3 - x_1) \\ &= 4dx^1 \otimes dx^1 + 4x_1^2\big(3x_2^2 dx^2 - dx^1\big) \otimes \big(3x_2^2 dx^2 - dx^1\big) \\ &= 4(1 + x_1^2)dx^1 \otimes dx^1 - 12x_1^2x_2^2 dx^1 \otimes dx^2 \\ &\quad -12x_1^2x_2^2 dx^2 \otimes dx^1 + 36x_1^2x_2^4x_1^2 dx^2 \otimes dx^2. \end{aligned}$$

Here we used the fact that computing the differential of a mapping to $\mathbb{R}$ produces the pushforward to a 1-form on N. ▶

◀ **11.6** *Example.* If we perform the previous calculation for an arbitrary 2-tensor

$$\sigma = a_{11}dy^1 \otimes dy^1 + a_{12}dy^1 \otimes dy^2 + a_{21}dy^2 \otimes dy^1 + a_{22}dy^2 \otimes dy^2.$$

Then,

$$\begin{aligned} f^*\sigma &= (4a_{11} - 2a_{12} - 2a_{21} + a_{22})dx^1 \otimes dx^1 + (6a_{12} - 3a_{22})x_2^2 dx^1 \otimes dx^2 \\ &\quad +(6a_{21} - 3a_{22})x_2^2 dx^2 \otimes dx^1 + 9x_2^4 a_{22} dx^2 \otimes dx^2, \end{aligned}$$

which produces the following matrix if we identify $T^2(T_xN)$ and $T^2(T_yM)$ with $\mathbb{R}^4$:

$$f^* = \begin{pmatrix} 4 & -2 & -2 & 1 \\ 0 & 6x_2^2 & 0 & -3x_2^2 \\ 0 & 0 & 6x_2^2 & -3x_2^2 \\ 0 & 0 & 0 & 9x_2^4 \end{pmatrix},$$

which clearly equal to the tensor product of the matrices $(Jf)^*$, i.e.

$$f^* = (Jf)^* \otimes (Jf)^* = \begin{pmatrix} 2 & -1 \\ 0 & 3x_2^2 \end{pmatrix} \otimes \begin{pmatrix} 2 & -1 \\ 0 & 3x_2^2 \end{pmatrix}.$$

This example show how to interpret $f^* : T^2(T_yM) \to T^2(T_xN)$ as a linear mapping. ▶

As for smooth 1-forms this operation extends to smooth covariant tensor fields: $(f^*\sigma)_p = f^*(\sigma_{f(p)})$, $\sigma \in \mathcal{T}^r(N)$, which in coordinates reads

$$(f^*\sigma)_p(X_1, \cdots X_r) := \sigma_{f(p)}(f_*X_1, \cdots, f_*X_r),$$

for tangent vectors $X_1, \cdots, X_r \in T_pN$.

Lemma 11.7. [21] *Let $f : N \to M$, $g : M \to P$ be smooth mappings, and let $h \in C^\infty(M)$, $\sigma \in \mathcal{T}^r(M)$, and $\tau \in \mathcal{T}^r(N)$, then:*

(i) $f^* : \mathcal{T}^r(M) \to \mathcal{T}^r(N)$ *is linear;*
(ii) $f^*(h\sigma) = (h \circ f) f^*\sigma$*;*
(iii) $f^*(\sigma \otimes \tau) = f^*\sigma \otimes f^*\tau$*;*
(iv) $f^*\sigma$ *is a smooth covariant tensor field;*
(v) $(g \circ f)^* = f^* \circ g^*$*;*
(vi) $\mathrm{id}_M^*\sigma = \sigma$*;*

12. Differential forms

A special class of covariant tensor bundles and associated bundle sections are the so-called alternating tensor bundles. Let $\Lambda^r(T_pM) \subset T^r(T_pM)$ be the space alternating tensors on T_pM. We know from Section 10 that a basis for $\Lambda^r(T_pM)$ is given by

$$\Big\{ dx^{i_1} \wedge \cdots \wedge dx^{i_r} \ : \ 1 \le i_1, \cdots, i_r \le m \Big\},$$

and $\dim \Lambda^r(T_pM) = \frac{m!}{r!(m-r)!}$. Smooth sections in $\Lambda^r M$ are called ***differential** r-**forms***, and the space of smooth sections is denoted by $\Gamma^r(M) \subset \mathcal{T}^r(M)$. In particular $\Gamma^0(M) = C^\infty(M)$, and $\Gamma^1(M) = \mathcal{T}^*(M)$. In terms of components a differential r-form, or r-form for short, is given by

$$\sigma_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

[21] See Lee, Proposition 11.9.

and the components $\sigma_{i_1\cdots i_r}$ are smooth functions. An r-form σ acts on vector fields $X_1,\cdots,X_r$ as follows:

$$\begin{aligned}\sigma(X_1,\cdots,X_r) &= \sum_{\mathbf{a}\in S_r}(-1)^{\mathbf{a}}\sigma_{i_1\cdots i_r}dx_{i_1}(X_{\mathbf{a}(1)})\cdots dx_{i_r}(X_{\mathbf{a}(r)})\\ &= \sum_{\mathbf{a}\in S_r}(-1)^{\mathbf{a}}\sigma_{i_1\cdots i_r}X^{i_1}_{\mathbf{a}(1)}\cdots X^{i_r}_{\mathbf{a}(r)}.\end{aligned}$$

◀ **12.1** *Example.* Let $M=\mathbb{R}^3$, and $\sigma = dx\wedge dz$. Then for vector fields

$$X_1 = X_1^1\frac{\partial}{\partial x}+X_1^2\frac{\partial}{\partial y}+X_1^3\frac{\partial}{\partial z},$$

and

$$X_2 = X_2^1\frac{\partial}{\partial x}+X_2^2\frac{\partial}{\partial y}+X_2^3\frac{\partial}{\partial z},$$

we have that

$$\sigma(X_1,X_2) = X_1^1X_2^3 - X_1^3X_2^1.$$

▶

An important notion that comes up in studying differential forms is the notion of contracting an r-form. Given an r-form $\sigma\in\Gamma^r(M)$ and a vector field $X\in\mathcal{F}(M)$, then

$$i_X\sigma := \sigma(X,\cdots),$$

is called the ***contraction with*** X, and is a differential $(r-1)$-form on M. Another notation for this is $i_X\sigma = X\lrcorner\sigma$. Contraction is a linear mapping

$$i_X:\Gamma^r(M)\to\Gamma^{r-1}(M).$$

The contraction mapping is also linear in X, i.e. for vector fields X,Y it holds that

$$i_{X+Y}\sigma = i_X\sigma+i_Y\sigma,\qquad i_{\lambda X}\sigma=\lambda\cdot i_X\sigma.$$

Lemma 12.2. [22] *Let $\sigma\in\Gamma^r(M)$ and $X\in\mathcal{F}(M)$ a smooth vector field, then*

(i) $i_X\sigma\in\Gamma^{r-1}(M)$ *(smooth $(r-1)$-form);*
(ii) $i_X\circ i_X=0$*;*
(iii) i_X *is an **anti-derivation**, i.e. for $\sigma\in\Gamma^r(M)$ and $\omega\in\Gamma^s(M)$,*

$$i_X(\sigma\wedge\omega) = (i_X\sigma)\wedge\omega+(-1)^r\sigma\wedge(i_X\omega).$$

A direct consequence of (iii) is that if $\sigma=\sigma_1\wedge\cdots\wedge\sigma_r$, where $\sigma_i\in\Gamma^1(M)=\mathcal{F}^*(M)$, then

$$i_X\sigma = (-1)^{i-1}\sigma_i(X)\sigma_1\wedge\cdots\widehat{\sigma}_i\wedge\cdots\wedge\sigma_r, \tag{10}$$

[22] See Lee, Lemma 13.11.

where the hat indicates that σ_i is to be omitted, and we use the summation convention.

◀ **12.3** *Example.* Let $\sigma = x_2 dx^1 \wedge dx^3$ be a 2-form on $\mathbb{R}^3$, and $X = x_1^2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + (x_1+x_2)\frac{\partial}{\partial x_3}$ a given vector field on $\mathbb{R}^3$. If $Y = Y^1 \frac{\partial}{\partial y_1} + Y^2 \frac{\partial}{\partial y_2} + Y^3 \frac{\partial}{\partial y_3}$ is an arbitrary vector fields then

$$\begin{aligned}(i_X\sigma)(Y) = \sigma(X,Y) &= dx^1(X)dx^3(Y) - dx^1(Y)dx^3(X)\\ &= x_1^2 Y^3 - (x_1+x_2)Y^1,\end{aligned}$$

which gives that

$$i_X\sigma = x_1^2 dx^3 - (x_1+x_2)dx^1.$$

▶

Since σ is a 2-form the calculation using X,Y is still doable. For higher forms this becomes to involved. If we use the multilinearity of forms we can give a simple procedure for computing $i_X\sigma$ using Formula (10).

◀ **12.4** *Example.* Let $\sigma = dx^1 \wedge dx^2 \wedge dx^3$ be a 3-form on $\mathbb{R}^3$, and X the vector field as given in the previous example. By linearity

$$i_X\sigma = i_{X_1}\sigma + i_{X_2}\sigma + i_{X_3}\sigma,$$

where $X_1 = x_1^2 \frac{\partial}{\partial x_1}$, $X_2 = x_3 \frac{\partial}{\partial x_2}$, and $X_3 = (x_1+x_2)\frac{\partial}{\partial x_3}$. This composition is chosen so that X is decomposed in vector fields in the basis directions. Now

$$\begin{aligned}i_{X_1}\sigma &= dx^1(X_1)dx^2 \wedge dx^3 = x_1^2 dx^2 \wedge dx^3,\\ i_{X_2}\sigma &= -dx^2(X_2)dx^1 \wedge dx^3 = -x_3 dx^1 \wedge dx^3,\\ i_{X_3}\sigma &= dx^2(X_3)dx^1 \wedge dx^2 = (x_1+x_2)dx^1 \wedge dx^2,\end{aligned}$$

which gives

$$i_X\sigma = x_1^2 dx^2 \wedge dx^3 - x_3 dx^1 \wedge dx^3 + (x_1+x_2)dx^1 \wedge dx^2,$$

a 2-form on $\mathbb{R}^3$. One should now verify that the same answer is obtained by computing $\sigma(X,Y,Z)$. ▶

For completeness we recall that for a smooth mapping $f : N \to M$, the pullback of a r-form σ is given by

$$(f^*\sigma)_p(X_1,\cdots,X_r) = f^*\sigma_{f(p)}(X_1,\cdots,X_r) = \sigma_{f(p)}(f_*X_1,\cdots,f_*X_r).$$

We recall that for a mapping $h : M \to \mathbb{R}$, then pushforward, or differential of h $dh_p = h_* \in T_p^*M$. In coordinates $dh_p = \frac{\partial \tilde{h}}{\partial x_i} dx^i|_p$, and thus the mapping $p \mapsto dh_p$ is a smooth section in $\Lambda^1(M)$, and therefore a differential 1-form, with component $\sigma_i = \frac{\partial \tilde{h}}{\partial x_i}$ (in local coordinates).

If $f : N \to M$ is a mapping between m-dimensional manifolds with charts (U,φ), and (V,ψ) respectively, and $f(U) \subset V$. Set $x = \varphi(p)$, and $y = \psi(q)$, then

$$f^*\big(\sigma dy^1 \wedge \cdots \wedge dy^m\big) = (\sigma \circ f)\det\big(J\tilde{f}|_x\big)dx^1 \wedge \cdots \wedge dx^m. \tag{11}$$

This can be proved as follows. As a consequence of this a change of coordinates yields

$$dy^1 \wedge \cdots \wedge dy^m = \det\big(J\tilde{f}|_x\big) dx^1 \wedge \cdots \wedge dx^m. \tag{12}$$

◀ **12.5** *Example.* Consider $\sigma = dx \wedge dy$ on $\mathbb{R}^2$, and mapping f given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The map f the identity mapping that maps $\mathbb{R}^2$ in Cartesian coordinates to $\mathbb{R}^2$ in Polar coordinates (consider the chart $U = \{(r,\theta) \,:\, r > 0,\ 0 < \theta < 2\pi\}$). As before we can compute the pullback of σ to $\mathbb{R}^2$ with Polar coordinates:

$$\begin{aligned}
\sigma = dx \wedge dy &= d(r\cos(\theta)) \wedge d(r\sin(\theta)) \\
&= (\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta) \\
&= r\cos^2(\theta)dr \wedge d\theta - r\sin^2(\theta)d\theta \wedge dr \\
&= rdr \wedge d\theta.
\end{aligned}$$

Of course the same can be obtained using (12). ▶

◀ **12.6** *Remark.* If we define

$$\Gamma(M) = \bigoplus_{r=0}^{\infty} \Gamma^r(M),$$

which is an associative, anti-commutative graded algebra, then $f^* : \Gamma(N) \to \Gamma(M)$ is a algebra homomorphism. ▶

13. Orientations

In order to explain orientations on manifolds we first start with orientations of finite-dimensional vector spaces. Let V be a real m-dimensional vector space. Two ordered basis $\{v_1, \cdots, v_m\}$ and $\{v_1', \cdots, v_m'\}$ are said to ***consistently oriented*** if the transition matrix $A = (a_{ij})$, defined by the relation

$$v_i = a_{ij} v_j',$$

has positive determinant. This notion defines an equivalence relation on ordered bases, and there are exactly two equivalence classes. An ***orientation*** for V is a choice of an equivalence class of order bases. Given an ordered basis $\{v_1, \cdots, v_m\}$ the orientation is determined by the class $\mathcal{O} = [v_1, \cdots, v_m]$. The pair $(V, \mathcal{O})$ is called an ***oriented vector space***. For a given an orientation any ordered basis that has the same orientation is called ***positively oriented***, and otherwise ***negatively oriented***.

Lemma 13.1. *Let $0 \neq \theta \in \Lambda^m(V)$, then the set of all ordered basis $\{v_1, \cdots, v_m\}$ for which $\theta(v_1, \cdots, v_m) > 0$ is an orientation for V.*

Proof: Obvious from the fact that $\theta(v_1,\cdots,v_m) = c\cdot\det(v_1,\cdots,v_m) > 0$. ∎

Let us now describe orientations for smooth manifolds M. We will assume that $\dim M = m \geq 1$ here. For each point $p \in M$ we can choose an orientation $\mathcal{O}_p$ for the tangent space T_pM, making $(T_pM,\mathcal{O}_p)$ an oriented vector space.

Definition 13.2. A smooth m-dimensional manifold M is ***oriented*** if for each point $p \in M$ there exists a neighborhood U of p, a diffeomorphism φ mapping from U to an open subset of $\mathbb{R}^m$, and an orientation $\mathcal{O}_p = \big[X_1,\cdots,X_m\big]$ of T_pM, such that

$$\big[\varphi_*(X_1),\cdots,\varphi_*(X_m)\big] = [e_1,\cdots,e_m]. \tag{13}$$

The orientation $[e_1,\cdots,e_m]$ is the standard orientation of $\mathbb{R}^m$.

A smooth manifold M with orientations $\mathcal{O}_p$ for the tangent spaces T_pM is called ***orientable*** if the orientations $\mathcal{O}_p$ can be chosen as to satisfy (13). A consistent choice of $\{\mathcal{O}_p\}_{p\in M}$ is called an ***orientation*** on M, and is denoted by $\mathcal{O} = \{\mathcal{O}_p\}_{p\in M}$. If a consistent choice of orientations $\mathcal{O}_p$ does not exist we say that a manifold is ***non-orientable***.

◀ **13.3** *Remark.* If we look at a single chart (U,φ) we can choose orientations $\mathcal{O}_p$ that are consistently oriented for $p \in \mathcal{O}_p$. Choose $X_i = \varphi_*^{-1}(e_i) = \frac{\partial}{\partial x_i}|_p$ are an ordered basis for T_pM. Then by definition $\varphi_*(X_i) = e_i$, and thus by this choice we obtain a consistent orientation for all $p \in U$. This procedure can be repeated for each chart in an atlas for M. In order to get a globally consistent ordering we need to worry about the overlaps between charts. ▶

◀ **13.4** *Example.* Let $M = S^1$ be the circle in $\mathbb{R}^2$, i.e. $M = \{p = (p_1,p_2) : p_1^2+p_2^2 = 1\}$. The circle is an orientable manifold and we can find a orientation as follows. Consider the stereographic charts $U_1 = S^1\backslash Np$ and $U_2 = S^1\backslash Sp$, and the associated mappings

$$\varphi_1(p) = \frac{2p_1}{1-p_2}, \qquad \varphi_2(p) = \frac{2p_1}{1+p_2}$$

$$\varphi_1^{-1}(x) = \Big(\frac{4x}{x^2+4},\frac{x^2-4}{x^2+4}\Big), \qquad \varphi_2^{-1}(x) = \Big(\frac{4x}{x^2+4},\frac{4-x^2}{x^2+4}\Big).$$

Let us start with a choice of orientation and verify its validity. Choose $X(p) = (X_1(p),X_2(p)) = (-p_2,p_1)$, then

$$(\varphi_1)_*(X) = J\varphi_1|_p(X) = \Big(\frac{2}{1-p_2} \;\; \frac{2p_1}{(1-p_2)^2}\Big)\begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} = \frac{2}{1-p_2} > 0,$$

on U_1 (standard orientation). If we carry out the same calculation for U_2 we obtain $(\varphi_2)_*(X) = J\varphi_2|_p(X) = \frac{-2}{1+p_2} < 0$ on U_2, with corresponds to the opposite orientation. Instead by choosing $\tilde{\varphi}_2(p) = \varphi_2(-p_1,p_2)$ we obtain

$$(\tilde{\varphi}_2)_*(X) = J\tilde{\varphi}_2|_p(X) = \frac{2}{1+p_2} > 0,$$

which chows that $X(p)$ is defines an orientation $\mathcal{O}$ on S^1. The choice of vectors $X(p)$ does not have to depend continuously on p in order to satisfy the definition of orientation, as we will explain now.

For $p \in U_1$ choose the canonical vectors $X(p) = \frac{\partial}{\partial x}|_p$, i.e.

$$\frac{\partial}{\partial x}\Big|_p = J\varphi_1^{-1}|_p(1) = \begin{pmatrix} \frac{16-4x^2}{(x^2+4)^2} \\ \frac{16x}{(x^2+4)^2} \end{pmatrix}.$$

In terms of p this gives $X(p) = \frac{1}{2}(1+p_2)\begin{pmatrix} -p_2 \\ p_1 \end{pmatrix}$. By definition $(\varphi_1)_*(X) = 1$ for $p \in U_1$. For $p = Np$ we choose $X(p) = (-2 \quad 0)^t$, so $X(p)$ is defined for $p \in S^1$ and is not a continuous function of p! It remains to verify that $(\tilde{\varphi}_2)_*(X) > 0$ for some neighborhood of p. First,

$$J\tilde{\varphi}_2|_p \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \frac{4}{1+p_2}2 > 0.$$

Secondly,

$$J\tilde{\varphi}_2|_p(X(p)) = \frac{1}{2}(1+p_2)\Big(\frac{-2}{1+p_2} \ \ \frac{2p_1}{(1+p_2)^2}\Big)\begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} = 1,$$

for all $p \in U_2$. ▶

What the above example shows us is that if we choose $X(p) = \frac{\partial}{\partial x_i}|_p$ for all charts, it remains to be verified if we have the proper collection of charts, i.e. does $\left[\{J\varphi_\beta \circ J\varphi_\alpha^{-1}(e_i)\}\right] = [\{e_i\}]$ hold? This is equivalent to having $\det(J\varphi_\beta \circ J\varphi_\alpha^{-1}) > 0$. These observations lead to the following theorem.

Theorem 13.5. *A smooth manifold M is orientable if and only if there exists an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha\}$ such that for any $p \in M$*

$$\det(J\varphi_{\alpha\beta}|_x) > 0, \qquad \varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1},$$

for any pair α, β for which $x = \varphi_\alpha(p)$, and $p \in U_\alpha \cap U_\beta$.

As we have seen for vector spaces m-form can be used to define an orientation on a vector space. This concept can also be use for orientations on manifolds.

Theorem 13.6. [23] *Let M be a smooth m-dimensional manifold. A nowhere vanishing differential m-form $\theta \in \Gamma^m(M)$ determines a unique orientation $\mathcal{O}$ on M for which θ_p is positively orientated at each T_pM. Conversely, given an orientation $\mathcal{O}$ on M, then there exists a nowhere vanishing m-form $\theta \in \Gamma^m(M)$ that is positively oriented at each T_pM.*

[23] See Lee, Prop. 13.4.

This theorem implies in particular that non-orientable m-dimensional manifolds do not admit a nowhere vanishing m-form, or volume form.

◀ **13.7** *Example.* Consider the Möbius strip M. The Möbius strip is an example of a non-orientable manifold. Let us parametrize the Möbius strip as an embedded manifold in $\mathbb{R}^3$:

$$g:\mathbb{R}\times(-1,1)\to\mathbb{R}^3, \quad g(\theta,r)=\begin{pmatrix} r\sin(\theta/2)\cos(\theta)+\cos(\theta) \\ r\sin(\theta/2)\sin(\theta)+\sin(\theta) \\ r\cos(\theta/2)\end{pmatrix},$$

where g is a smooth embedding when regarded as a mapping from $\mathbb{R}/2\pi\mathbb{Z}\times(-1,1)\to\mathbb{R}^3$. Let us assume that the Möbius strip M is oriented, then by the above theorem there exists a nonwhere vanishing 2-form σ which can be given as follows

$$\sigma = a(x,y,z)dy\wedge dz + b(x,y,z)dz\wedge dx + c(x,y,z)dx\wedge dy,$$

where $(x,y,z)=g(\theta,r)$. Since, g is a smooth embedding (parametrization) the pullback form $g^*\sigma=\rho(\theta,r)d\theta\wedge dr$ is a nowhere vanishing 2-form on $\mathbb{R}/2\pi\mathbb{Z}\times(-1,1)$. In particular, this means that ρ is 2π-periodic in θ. Notice that $\sigma = i_X dx\wedge dy\wedge dz$, where

$$X = a(x,y,z)\frac{\partial}{\partial x}+b(x,y,z)\frac{\partial}{\partial y}+c(x,y,z)\frac{\partial}{\partial z}.$$

For vector fields $\xi=\xi_1\frac{\partial}{\partial x}+\xi_2\frac{\partial}{\partial y}+\xi_3\frac{\partial}{\partial z}$ and $\eta=\eta_1\frac{\partial}{\partial x}+\eta_2\frac{\partial}{\partial y}+\eta_3\frac{\partial}{\partial z}$ we then have that

$$\sigma(\xi,\eta)=i_X dx\wedge dy\wedge dz(\xi,\eta)=X\cdot(\xi\times\eta),$$

where $\xi\times\eta$ is the cross product of ξ and η. The condition that θ is nowhere vanishing can be to $\mathbb{R}^2$ as follows:

$$g^*\sigma(e_1,e_2)=\sigma(g_*(e_1),g_*(e_2)).$$

Since $g_*(e_1)\times g_*(e_2)|_{\theta=0}=-g_*(e_1)\times g_*(e_2)|_{\theta=2\pi}$, the pullback g$^*\sigma$ form cannot be nowhere vanishing on $\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}$, which is a contradiction. This proves that the Möbius band is a non-orientable manifold. ▶

Let N,M be oriented manifolds, and let $f:N\to M$ be a smooth mapping. Such a mapping is called ***orientation preserving*** if for each $p\in N$, f_* maps positively oriented bases of T_pN to positively oriented bases in $T_{f(p)}M$. A mapping is called ***orientation reversing*** if f_* maps positively oriented bases of T_pN to negatively oriented bases in $T_{f(p)}M$.

Let us now look at manifolds with boundary; $(M,\partial M)$. For a point $p\in\partial M$ we distinguish three types of tangent vectors:

(i) tangent boundary vectors $X\in T_p(\partial M)\subset T_pM$, which form an $(m-1)$-dimensional subspace of T_pM;

(ii) outward vectors; let $\varphi^{-1} : W \subset \mathbb{H}^m \to M$, then $X \in T_pM$ is called an outward vector if $\varphi_*^{-1}(Y) = X$, for some $Y = (y_1, \cdots, y_m)$ with $y_1 < 0$;

(iii) inward vectors; let $\varphi^{-1} : W \subset \mathbb{H}^m \to M$, then $X \in T_pM$ is called an inward vector if $\varphi_*^{-1}(Y) = X$, for some $Y = (y_1, \cdots, y_m)$ with $y_1 > 0$.

Using this concept we can now introduce the notion of induced orientation on ∂M. Let $p \in \partial M$ and choose a basis $\{X_1, \cdots, X_m\}$ for T_pM such that $[X_1, \cdots, X_m] = \mathcal{O}_p$, $\{X_2, \cdots, X_m\}$ are tangent boundary vectors, and X_1 is an outward vector. In this case $[X_2, \cdots, X_m] = (\partial\mathcal{O})_p$ determines an orientation for $T_p(\partial M)$, which is consistent, and therefore $\partial\mathcal{O} = \{(\partial\mathcal{O})_p\}_{p\in\partial M}$ is an orientation on ∂M induced by $\mathcal{O}$. Thus for an oriented manifold M, with orientation $\mathcal{O}$, ∂M has an orientation $\partial\mathcal{O}$, called the ***induced orientation on*** ∂M.

◀ **13.8** *Example.* Any open set $M \subset \mathbb{R}^m$ (or $\mathbb{H}^m$) is an orientable manifold. ▶

◀ **13.9** *Example.* Consider a smooth embedded co-dimension 1 manifold

$$M = \{p \in R^{m+1} \ : \ f(p) = 0\}, \qquad f : \mathbb{R}^{m+1} \to \mathbb{R}, \quad \mathrm{rk}(f)|_p, \ p \in M.$$

Then M is an orientable manifold. Indeed, $M = \partial N$, where $N = \{p \in \mathbb{R}^{m+1} \ : \ f(p) > 0\}$, which is an open set in $\mathbb{R}^{m+1}$ and thus an oriented manifold. Since $M = \partial N$ the manifold M inherits an orientation from M and hence it is orientable. ▶

IV. Integration on manifolds

14. Integrating m-forms on $\mathbb{R}^m$

We start off integration of m-forms by considering m-forms on $\mathbb{R}^m$.

Definition 14.1. A subset $D \subset \mathbb{R}^m$ is called a domain of integration if

(i) D is bounded, and
(ii) ∂D has m-dimensional Lebesgue measure $d\mu = dx_1 \cdots dx_m$ equal to zero.

In particular any finite union or intersection of open or closed rectangles is a domain of integration. Any bounded[24] continuous function f on D is integrable, i.e.

$$-\infty < \int_D f dx_1 \cdots dx_m < \infty.$$

Since $\Lambda^m(\mathbb{R}^m) \cong \mathbb{R}$, a smooth m-form on $\mathbb{R}^m$ is given by

$$\omega = f(x_1, \cdots, x_m) dx^1 \wedge \cdots \wedge dx^m,$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a smooth function. For a given (bounded) domain of integration D we define

$$\begin{aligned} \int_D \omega \quad &:= \quad \int_D f(x_1, \cdots, x_m) dx_1 \cdots dx_m = \int_D f d\mu \\ &= \quad \int_D \omega_x(e_1, \cdots, e_m) d\mu. \end{aligned}$$

An m-form ω is compactly supported if $\operatorname{supp}(\omega) = \operatorname{cl}\{x \in \mathbb{R}^m \;:\; \omega(x) \neq 0\}$ is a compact set. The set of compactly supported m-forms on $\mathbb{R}^m$ is denoted by $\Gamma_c^m(\mathbb{R}^m)$, and is a linear subspace of $\Gamma^m(\mathbb{R}^m)$. Similarly, for any open set $U \subset \mathbb{R}^m$ we can define $\omega \in \Gamma_c^m(U)$. Clearly, $\Gamma_c^m(U) \subset \Gamma_c^m(\mathbb{R}^m)$, and can be viewed as a linear subspace via zero extension to $\mathbb{R}^m$. For any open set $U \subset \mathbb{R}^m$ there exists a domain of integration D such that $U \supset D \supset \operatorname{supp}(\omega)$ (see Exercises).

Definition 14.2. Let $U \subset \mathbb{R}^m$ be open and $\omega \in \Gamma_c^m(U)$, and let D be a domain of integration D such that $U \supset D \supset \operatorname{supp}(\omega)$. We define the integral

$$\int_U \omega := \int_D \omega.$$

If $U \subset \mathbb{H}^m$ open, then

$$\int_U \omega := \int_{D \cap \mathbb{H}^m} \omega.$$

---

[24] Boundedness is needed because the rectangles are allowed to be open.

The next theorem is the first step towards defining integrals on m-dimensional manifolds M.

Theorem 14.3. *Let $U, V \subset \mathbb{R}^m$ be open sets, $f : U \to V$ an orientation preserving diffeomorphism, and let $\omega \in \Gamma_c^m(V)$. Then,*

$$\int_V \omega = \int_U f^*\omega.$$

If f is orientation reversing, then $\int_U \omega = -\int_V f^*\omega$.

Proof: Assume that f is an orientation preserving diffeomorphism from U to V. Let E be a domain a domain of integration for ω, then $D = f^{-1}(E)$ is a domain of integration for $f^*\omega$. We now prove the theorem for the domains D and E. We use coordinates $\{x_i\}$ and $\{y_i\}$ on D and E respectively. We start with $\omega = g(y_1, \cdots, y_m)dy^1 \wedge \cdots \wedge dy^m$. Using the change of variables formula for integrals and the pullback formula in (11) we obtain

$$\begin{aligned} \int_E \omega &= \int_E g(y) dy_1 \cdots dy_m \text{ (Definition)} \\ &= \int_D (g \circ f)(x) \det(J\tilde{f}|_x) dx^1 \cdots dx^m \\ &= \int_D (g \circ f)(x) \det(J\tilde{f}|_x) dx^1 \wedge \cdots \wedge dx^m \\ &= \int_D f^*\omega \text{ (Definition)}. \end{aligned}$$

One has to introduce a $-$ sign in the orientation reversing case. ∎

15. Partitions of unity

We start with introduce the notion partition of unity for smooth manifolds. We should point out that this definition can be used for arbitrary topological spaces.

Definition 15.1. Let M be smooth m-manifold with atlas $\mathcal{A} = \{(\varphi_i, U_i)\}_{i \in I}$. A ***partition of unity subordinate to*** $\mathcal{A}$ is a collection for smooth functions $\{\lambda_i : M \to \mathbb{R}\}_{i \in I}$ satisfying the following properties:

(i) $0 \leq \lambda_i(p) \leq 1$ for all $p \in M$ and for all $i \in I$;
(ii) $\text{supp}(\lambda_i) \subset U_i$;
(iii) the set of supports $\{\text{supp}(\lambda_i)\}_{i \in I}$ is locally finite;
(iv) $\sum_{i \in I} \lambda_i(p) = 1$ for all $p \in M$.

Condition (iii) says that every $p \in M$ has a neighborhood $U \ni p$ such that only finitely many λ_i's are nonzero in U. As a consequence of this the sum in Condition (iv) is always finite.

Theorem 15.2. *For any smooth m-dimensional manifold M with atlas $\mathcal{A}$ there exists a partition of unity $\{\lambda_i\}$ subordinate to $\mathcal{A}$.*

In order to prove this theorem we start with a series of auxiliary results and notions.

Lemma 15.3. *There exists a smooth function $h : \mathbb{R}^m \to \mathbb{R}$ such that $0 \le h \le 1$ on $\mathbb{R}^m$, and $h|_{B_1(0)} \equiv 1$ and $\mathrm{supp}(h) \subset B_2(0)$.*

Proof: Define the function $f_1 : \mathbb{R} \to \mathbb{R}$ as follows

$$f_1(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

One can easily prove that f_1 is a C^∞-function on $\mathbb{R}$. If we set

$$f_2(t) = \frac{f_1(2-t)}{f_1(2-t) + f_1(t-1)}.$$

This function has the property that $f_2(t) \equiv 1$ for $t \le 1$, $0 < f_2(t) < 1$ for $1 < t < 2$, and $\mathrm{f}_2(t) \equiv 0$ for $t \ge 2$. Moreover, f_2 is smooth. To construct f we simply write $f(x) = f_2(|x|)$ for $x \in \mathbb{R}^m \backslash \{0\}$. Clearly, f is smooth on $\mathbb{R}^m \backslash \{0\}$, and since $f|_{B_1(0)} \equiv 1$ it follows that f is smooth on $\mathbb{R}^m$. ∎

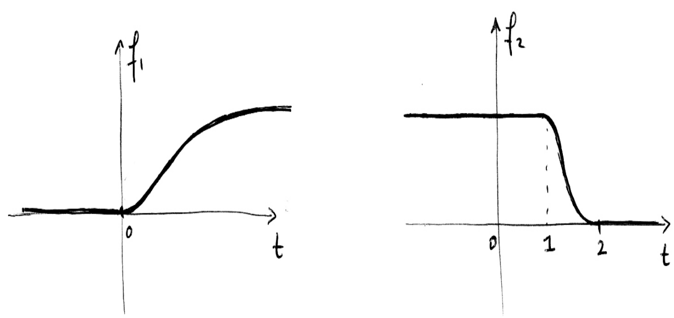

FIGURE 34. The functions f_1 and f_2.

An atlas $\mathcal{A}$ gives an open covering for M. The set U_i in the atlas need not be compact, nor locally finite. We say that a covering $\mathcal{U} = \{U_i\}$ of M is ***locally finite*** if every point $p \in M$ has a neighborhood that intersects only finitely $U_i \in \mathcal{U}$. If there exists another covering $\mathcal{V} = \{V_j\}$ such that every $V_j \in \mathcal{V}$ is contained in some $V_j \subset U_i \in \mathcal{U}$, then $\mathcal{V}$ is called a ***refinement of*** $\mathcal{U}$. A topological space X for which each open covering admits a locally finite refinement is called ***paracompact***.

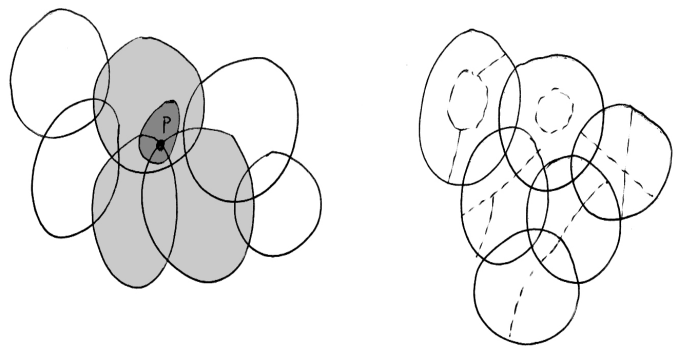

FIGURE 35. A locally finite covering [left], and a refinement [right].

Lemma 15.6. *Any topological manifold M allows a countable, locally finite covering by precompact open sets.*

Proof: We start with a countable covering of open balls $\mathcal{B} = \{B_i\}$. We now construct a covering $\mathcal{U}$ that satisfies

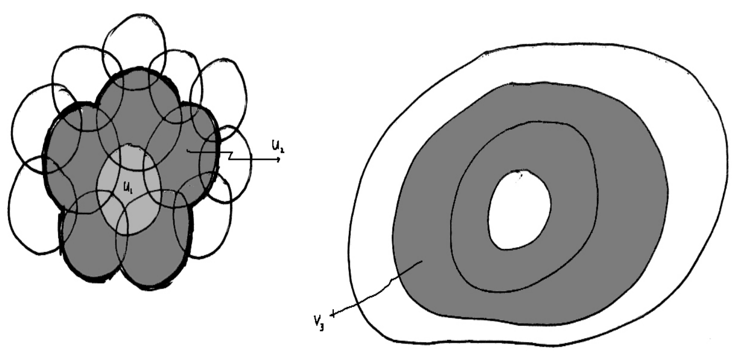

FIGURE 36. Constructing a nested covering $\mathcal{U}$ from a covering with balls $\mathcal{B}$ [left], and a locally finite covering obtained from the previous covering [right].

(i) all sets $U_i \in \mathcal{U}$ are precompact open sets in M;
(ii) $\overline{U_{i-1}} \subset U_i$, $i > 1$;
(iii) $B_i \subset U_i$.

We build the covering $\mathcal{U}$ from $\mathcal{B}$ using an inductive process. Let $U_1 = B_1$, and assume $U_1, \cdots, U_k$ have been constructed satisfying (i)-(iii). Then

$$\overline{U_k} \subset B_{i_1} \cup \cdots \cup B_{i_k},$$

where $B_{i_1} = B_1$. Now set

$$U_{k+1} = B_{i_1} \cup \cdots \cup B_{i_k}.$$

Choose i_k large enough so that $i_k \geq k+1$ and thus $B_{k+1} \subset U_{k+1}$.

From the covering $\mathcal{U}$ we can now construct a locally finite covering $\mathcal{V}$ by setting $V_i = U_i \backslash \overline{U_{i-2}}$, $i > 1$. ∎

Next we seek a special locally finite refinement $\{W_j\} = \mathcal{W}$ of $\mathcal{V}$ which has special properties with respect to coordinate charts of M;

(i) $\mathcal{W}$ is a countable and locally finite;
(ii) each $W_j \subset \mathcal{W}$ is in the domain of some smooth coordinate map $\varphi_j :_j \to \mathbb{R}^m$ for M;
(iii) the collection $\mathcal{Z} = \{Z_j\}$, $Z_j = \varphi_j^{-1}(B_1(0))$ covers M.

The covering $\mathcal{W}$ is called ***regular***.

Lemma 15.8. *For any open covering $\mathcal{U}$ for a smooth manifold M there exists a regular refinement. In particular M is paracompact.*

Proof: Let $\mathcal{V}$ be a countable, locally finite covering as described in the previous lemma. Since $\mathcal{V}$ is locally finite we can find a neighborhood W_p for each $p \in M$ that intersects only finitely many set $V_j \in \mathcal{V}$. We want to choose W_p in a smart way. First we replace W_p by $W_p \cap \{V_j \ : \ p \in V_j\}$. Since $p \in U_i$ for some i we then replace W_p by $W_p \cap V_i$. Finally, we replace W_p by a small coordinate ball $B_r(p)$ so that W_p is in the domain of some coordinate map φ_p. This provides coordinate charts (W_p, φ_p). Now define $Z_p = \varphi_p^{-1}(B_1(0))$.

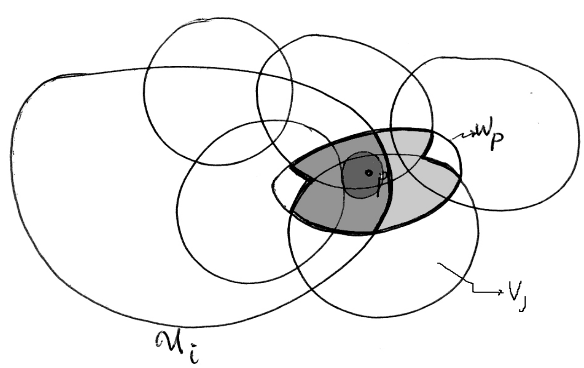

FIGURE 37. The different stages of constructing the sets W_p colored with different shades of grey.

For every k, $\{Z_p \ : \ p \in \overline{V_k}\}$ is an open covering of $\overline{V_k}$, which has a finite subcovering, say $Z_k^1, \cdots, Z_k^{m_k}$. The sets Z_k^i are sets of the form Z_{p_i} for some $p_i \in \overline{V_k}$. The associated coordinate charts are $(W_k^1, \varphi_k^1), \cdots, (W_k^{m_k}, \varphi_k^{m_k})$. Each W_k^i is obtained via the construction above for $p_i \in \overline{V_k}$. Clearly, $\{W_k^i\}$ is a countable open covering of

M that refines $\mathcal{U}$ and which, by construction, satisfies (ii) and (iii) in the definition of regular refinement. It is clear from compactness that $\{W_k^i\}$ is locally finite. Let us relabel the sets $\{W_k^i\}$ and denote this covering by $\mathcal{W} = \{W_i\}$. As a consequence M is paracompact. ∎

Proof of Theorem 15.2: From Lemma 15.8 there exists a regular refinement $\mathcal{W}$ for $\mathcal{A}$. By construction we have $Z_i = \varphi_i^{-1}(B_1(0))$, and we define

$$\widehat{Z}_i := \varphi_i^{-1}(B_2(0)).$$

Using Lemma 15.3 we now define the functions $\mu_i M \to \mathbb{R}$;

$$\mu_i = \begin{cases} f_2 \circ \varphi_i & \text{on } W_i \\ 0 & \text{on } M \backslash \overline{V}_i. \end{cases}$$

These functions are smooth and $\mathrm{supp}(\mu_i) \subset W_i$. The quotients

$$\widehat{\lambda} = \frac{\mu_j(p)}{\sum_i \mu_i(p)},$$

are, due to the local finiteness of W_i and the fact that the denominator is positive for each $p \in M$, smooth functions on M. Moreover, $0 \le \widehat{\lambda}_j \le 1$, and $\sum_j \widehat{\lambda}_j \equiv 1$.

Since $\mathcal{W}$ is a refinement of $\mathcal{A}$ we can choose an index k_j such that $W_j \subset U_{k_j}$. Therefore we can group together some of the function $\widehat{\lambda}_j$:

$$\lambda_i = \sum_{j\,:\,k_j=i} \widehat{\lambda}_j,$$

which give us the desired partition functions with $0 \le \lambda_i \le 1$, $\sum_i \lambda_i \equiv 1$, and $\mathrm{supp}(\lambda_i) \subset U_i$. ∎

Some interesting byproducts of the above theorem using partitions of unity are. In all these case M is assumed to be an smooth m-dimensional manifold.

Theorem 15.10. [25] *For any close subset $A \subset M$ and any open set $U \supset A$, there exists a smooth function $f : M \to \mathbb{R}$ such that*

(i) $0 \le f \le 1$ *on* M;
(ii) $f^{-1}(1) = A$;
(iii) $\mathrm{supp}(f) \subset U$.

Such a function f is called a ***bump function for*** A ***supported in*** U.

By considering functions $g = c(1-f) \ge 0$ we obtain smooth functions for which $g^{-1}(0)$ can be an arbitrary closed subset of M.

[25] See Lee, Prop. 2.26.

Theorem 15.11. [26] *Let $A \subset M$ be a closed subset, and $f : A \to \mathbb{R}^k$ be a smooth mapping.*[27] *Then for any open subset $U \subset M$ containing A there exists a smooth mapping $f^\dagger : M \to \mathbb{R}^k$ such that $f^\dagger|_A = f$, and* $\mathrm{supp}(f^\dagger) \subset U$.

16. Integration on of m-forms on m-dimensional manifolds.

In order to introduce the integral of an m-form on M we start with the case of forms supported in a single chart. In what follows we assume that M is an oriented manifold with an oriented atlas $\mathcal{A}$, i.e. a consistent choice of smooth charts such that the transitions mappings are orientation preserving.

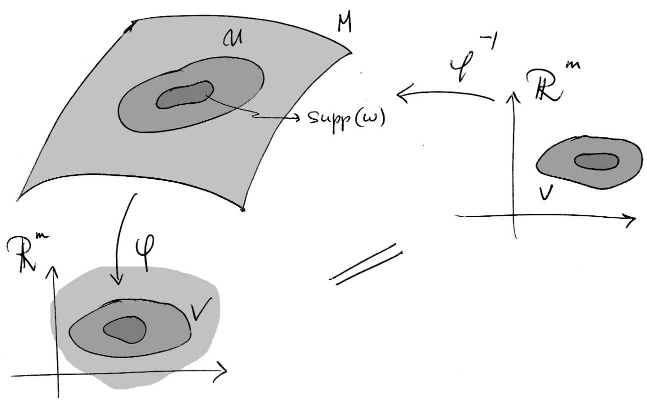

FIGURE 38. An m-form supported in a single chart.

Let $\omega \in \Gamma_c^m(M)$, with $\mathrm{supp}(\omega) \subset U$ for some chart U in $\mathcal{A}$. Now define the integral of ω over M as follows:

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega,$$

where the pullback form $(\varphi^{-1})^*\omega$ is compactly supported in $V = \varphi(U)$ (φ is a diffeomorphism). The integral over V of the pullback form $(\varphi^{-1})^*\omega$ is defined in Definition 14.2. It remains to show that the integral $\int_M \omega$ does not depend on the particular chart. Consider a different chart U', possibly in a different oriented atlas

[26] See Lee, Prop. 2.26.

[27] A mapping from a closed subset $A \subset M$ is said to be smooth if for every point $p \in A$ there exists an open neighborhood $W \subset M$ of p, and a smooth mapping $f^\dagger : W \to \mathbb{R}^k$ such that $f^\dagger|_{W \cap A} = f$.

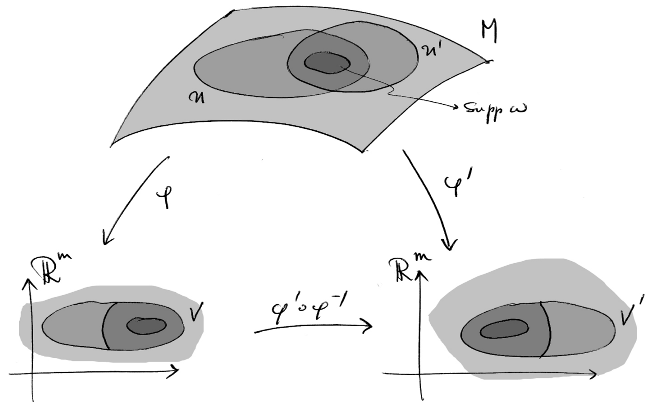

FIGURE 39. Different charts U and U' containing $\mathrm{supp}(\omega)$, i.e. $\mathrm{supp}(\omega) \subset U \cap U'$.

$\mathcal{A}'$ for M (same orientation), then

$$\begin{aligned}
\int_{V'} (\varphi'^{-1})^*\omega &= \int_{\varphi'(U\cap U')} (\varphi'^{-1})^*\omega = \int_{\varphi(U\cap U')} (\varphi' \circ \varphi^{-1})^*(\varphi'^{-1})^*\omega \\
&= \int_{\varphi(U\cap U')} (\varphi^{-1})^*(\varphi')^*(\varphi'^{-1})^*\omega = \int_{\varphi(U\cap U')} (\varphi^{-1})^*\omega \\
&= \int_{V} (\varphi^{-1})^*\omega,
\end{aligned}$$

which show that the definition is independent of the chosen chart. We crucially do use the fact that $\mathcal{A} \cup \mathcal{A}'$ is an oriented atlas for M.

By choosing a partition of unity as described in the previous section we can now define the integral over M of arbitrary m-forms $\omega \in \Gamma^m_c(M)$.

Definition 16.3. *Let $\omega \in \Gamma^m_c(M)$ and let $\mathcal{A}_I = \{(U_i, \varphi_i\}_{i\in I} \subset \mathcal{A}$ be a finite subcovering of* $\mathrm{supp}(\omega)$ *coming from an oriented atlas $\mathcal{A}$ for M. Let $\{\lambda_i\}$ be a partition of unity subordinate to $\mathcal{A}_I$. Then the* ***integral of** ω **over** M is defined as*

$$\int_M \omega := \sum_i \int_M \lambda_i\omega,$$

where the integrals $\int_M \lambda_i\omega$ are integrals of form that have support in single charts as defined above.

We need to show now that the integral is indeed well-defined, i.e. the sum is finite and independent of the atlas and partition of unity. Since $\mathrm{supp}(\omega)$ is compact $\mathcal{A}_I$ exists by default, and thus the sum is finite.

Lemma 16.4. *The above definition is independent of the chosen partition of unity and covering $\mathcal{A}_I$.*

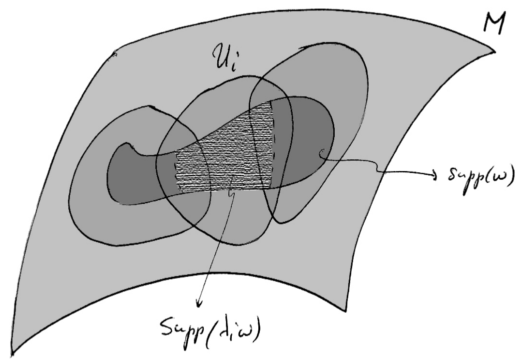

FIGURE 40. Using a partition of unity we can construct m-forms which are all supported in one, but possibly different charts U_i.

Proof: Let $\mathcal{A}'_J \subset \mathcal{A}'$ be another finite covering of supp(ω), where $\mathcal{A}'$ is a compactible oriented atlas for M, i.e. $\mathcal{A} \cup \mathcal{A}'$ is a oriented atlas. Let $\{\lambda'_j\}$ be a partition of unity subordinate to $\mathcal{A}'_J$. We have

$$\int_M \lambda_i \omega = \int_M \Big(\sum_j \lambda'_j\Big)\lambda_i\omega = \sum_j \int_M \lambda'_j \lambda_i \omega.$$

By summing over i we obtain $\sum_i \int_M \lambda_i\omega = \sum_{i,j}\int_M \lambda'_j\lambda_i\omega$. Each term $\lambda'_j\lambda_i\omega$ is supported in some U_i and by previous independent of the coordinate mappings. Similarly, if we interchange the i's and j's, we obtain that $\sum_j \int_M \lambda'_j\omega = \sum_{i,j}\int_M \lambda'_j\lambda_i\omega$, which proves the lemma. ■

◀ **16.6** *Remark.* If M is a compact manifold that for any $\omega \in \Gamma^m(M)$ it holds that supp$(\omega) \subset M$ is a compact set, and therefore $\Gamma^m_c(M) = \Gamma^m(M)$ in the case of compact manifolds. ▶

So far we considered integral of m-forms over M. One can of course integrate n-forms on M over n-dimensional immersed or embedded submanifolds $N \subset M$. Given $\omega \in \Gamma^n(M)$ for which the restriction to N, $\omega|_N$ has compact support in N, then

$$\int_N \omega := \int_N \omega|_N.$$

As a matter of fact we have $i : N \to M$, and $\omega|_N = i^*\omega$, so that $\int_N \omega := \int_N \omega|_N = \int_N i^*\omega$. If M is compact with boundary ∂M, then $\int_{\partial M}\omega$ is well-defined for any $(m-1)$-form $\omega \in \Gamma^{m-1}(M)$.

Theorem 16.7. *Let* $\omega \in \Gamma^m_c(M)$*, and* $f : N \to M$ *is a diffeomorphism. Then*

$$\int_M \omega = \int_N f^*\omega.$$

Proof: By the definition of integral above we need to prove the above statement for the terms

$$\int_M \lambda_i \omega = \int_N f^* \lambda_i \omega.$$

Therefore it suffices to prove the theorem for forms ω whose support is in a single chart U.

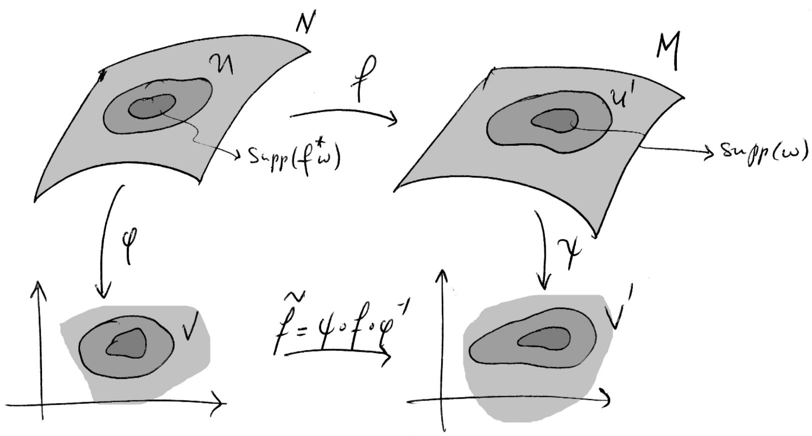

FIGURE 41. The pullback of an m-form.

$$\begin{aligned}
\int_U f^*\omega &= \int_{\varphi(U)} (\varphi^{-1})^* f^*\omega = \int_{\varphi(U)} (\varphi^{-1})^* f^* \psi^* (\psi^{-1})^* \omega \\
&= \int_{\psi(U')} (\psi^{-1})^* \omega = \int_{U'} \omega.
\end{aligned}$$

By applying this to $\lambda_i \omega$ and summing we obtain the desired result. ∎

For sake of completeness we summarize the most important properties of the integral.

Theorem 16.9. [28] *Let N, M be oriented manifolds (with or without boundary) of dimension m , and $\omega, \eta \in \Gamma_c^m(M)$, are m-forms. Then*

(i) $\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$;

(ii) *if $-\mathcal{O}$ is the opposite orientation to $\mathcal{O}$, then*

$$\int_{M,-\mathcal{O}} \omega = -\int_{M.\mathcal{O}} \omega;$$

(iii) *if ω is an orientation form, then $\int_M \omega > 0$;*

(iv) *if $f : N \to M$ is a diffeomorphism, then*

$$\int_M \omega = \int_N f^*\omega.$$

[28] See Lee Prop. 14.6.

For practical situations the definition of the integral above is not convenient since constructing partitions of unity is hard in practice. If the support of a differential form ω can be parametrized be appropriate parametrizations, then the integral can be easily computed from this. Let $D_i \subset \mathbb{R}^m$ — finite set of indices i — be compact domains of integration, and $g_i : D_i \to M$ are smooth mappings satisfying

(i) $E_i = g_i(D_i)$, and $g_i : \mathrm{int}\,(D_i) \to \mathrm{int}\,(E_i)$ are orientation preserving diffeomorphism;
(ii) $E_i \cap E_j$ intersect only along their boundaries, for all $i \neq j$.

Theorem 16.10. [29] *Let $\{(g_i, D_i)\}$ be a finite set of parametrizations as defined above. Then for any $\omega \subset \Gamma^m_c(M)$ such that $\mathrm{supp}(\omega) \subset \cup_i E_i$ it holds that*

$$\int_M \omega = \sum_i \int_{D_i} g_i^* \omega.$$

Proof: As before it suffices the prove the above theorem for a single chart U,

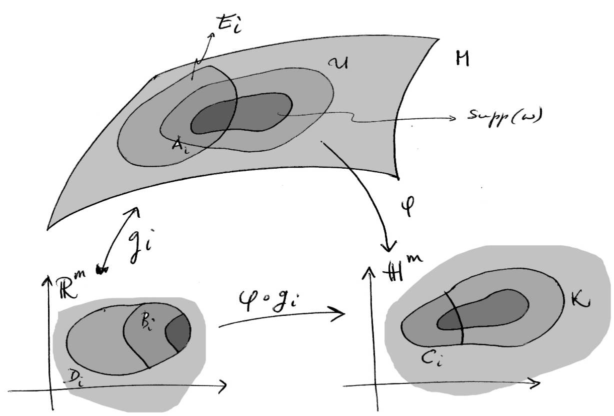

FIGURE 42. Carving up $\mathrm{supp}(\omega)$ via domains of integration for parametrizations g_i for M.

i.e. $\mathrm{supp}(\omega) \subset U$. One can choose U to have a boundary ∂U so that $\varphi(\partial U)$ has measure zero, and φ maps $\mathrm{cl}(U)$ to a compact domain of integration $K \subset \mathbb{H}^m$. Now set

$$A_i = \mathrm{cl}(U) \cap E_i, \quad B_i = g_i^{-1}(A_i), \quad C_i = \varphi_i(A_i).$$

We have

$$\int_{C_i} (\varphi^{-1})^* \omega = \int_{B_i} (\varphi \circ g_i)^* (\varphi^{-1})^* \omega = \int_{B_i} g_i^* \omega.$$

[29]See Lee Prop. 14.7.

Since the interiors of the sets C_i (and thus A_i) are disjoint it holds that

$$\begin{aligned}\int_M \omega &= \int_K (\varphi^{-1})^*\omega = \sum_i \int_{C_i} (\varphi^{-1})^*\omega \\ &= \sum_i \int_{B_i} g_i^*\omega = \sum_i \int_{D_i} g_i^*\omega,\end{aligned}$$

which proves the theorem. ■

◀ **16.12** *Remark.* From the previous considerations computing $\int_M \omega$ boils down to computing $(\varphi^{-1})^*\omega$, or $g^*\omega$ for appropriate parametrizations, and summing the various contrubutions. Recall from Section 14 that in order to integrate one needs to evaluate $(\varphi^{-1})^*\omega_x(e_1,\cdots,e_m)$, which is given by the formula

$$\big((\varphi^{-1})^*\omega\big)_x(e_1,\cdots,e_m) = \omega_{\varphi^{-1}(x)}(\varphi_*^{-1}(e_1),\cdots,\varphi_*^{-1}(e_m)).$$

For a single chart contribution this then yields the formula

$$\begin{aligned}\int_U \omega &= \int_{\varphi(U)} \omega_{\varphi^{-1}(x)}(\varphi_*^{-1}(e_1),\cdots,\varphi_*^{-1}(e_m))d\mu \\ &= \int_{\varphi(U)} \omega_{\varphi^{-1}(x)}\Big(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_1}\Big)d\mu\end{aligned}$$

or in the case of a parametrization $g : D \to M$:

$$\int_{g(D)} \omega = \int_D \omega_{g(x)}(g_*(e_1),\cdots,g_*(e_m))d\mu.$$

These expression are useful for computing integrals. ▶

◀ **16.13** *Example.* Consider the 2-sphere parametrized by the mapping $g : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ given as

$$g(\phi,\theta) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix}.$$

This mapping can be viewed as a covering map. From this expression we derive various charts for S^2. Given the 2-form $\omega = zdx\wedge dz$ let us compute the pullback form $g^*\omega$ on $\mathbb{R}^2$. We have that

$$\begin{aligned}g_*(e_1) &= \cos(\phi)\cos(\theta)\frac{\partial}{\partial x} + \cos(\phi)\sin(\theta)\frac{\partial}{\partial y} - \sin(\phi)\frac{\partial}{\partial z}, \\ g_*(e_2) &= -\sin(\phi)\sin(\theta)\frac{\partial}{\partial x} + \sin(\phi)\cos(\theta)\frac{\partial}{\partial y},\end{aligned}$$

and therefore

$$g^*\omega(e_1,e_2) = \omega_{g(x)}(g_*(e_1),g_*(e_2)) = -\cos(\phi)\sin^2(\phi)\sin(\theta),$$

and thus

$$g^*\omega = -\cos(\phi)\sin^2(\phi)\sin(\theta)d\phi\wedge d\theta.$$

The latter gives that $\int_S^2 \omega = -\int_0^{2\pi}\int_0^{\pi} \cos(\phi)\sin^2(\phi)\sin(\theta)d\phi d\theta = 0$, which that ω is not a volume form on S^2.

If we perform the same calculations for $\omega = xdy\wedge dz + ydz\wedge dx + zdx\wedge dy$, then

$$\int_{S^2}\omega = \int_0^{2\pi}\int_0^{\pi}\sin(\phi)d\phi d\theta = 4\pi.$$

The pullback form $g^*\omega = \sin(\phi)d\phi\wedge d\theta$, which shows that ω is a volume form on S^2. ▶

17. The exterior derivative

The ***exterior derivative*** d of a differential form is an operation that maps an k-form to a $(k+1)$-form. Write a k-form on $\mathbb{R}^m$ in the following notation

$$\omega = \omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \omega_I dx^I, \qquad I = (i_1\cdots i_k),$$

then we define

$$d\omega = d\omega_I\wedge dx^I. \tag{14}$$

Written out in all its differentials this reads

$$d\omega = d\Big(\omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\Big) = \frac{\partial\omega_I}{\partial x_i}dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Of course the Einstein summation convention is again applied here. This is the definition that holds for all practical purposes, but it is a local definition. We need to prove that d extends to differential forms on manifolds.

◀ **17.1** *Example.* Consider $\omega_0 = f(x,y)$, a 0-form on $\mathbb{R}^2$, then

$$d\omega_0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

For a 1-form $\omega_1 = f_1(x,y)dx + f_2(x,y)dy$ we obtain

$$\begin{aligned} d\omega_1 &= \frac{\partial f_1}{\partial x}dx\wedge dx + \frac{\partial f_1}{\partial y}dy\wedge dx + \frac{\partial f_2}{\partial x}dx\wedge dy + \frac{\partial f_2}{\partial y}dy\wedge dy \\ &= \frac{\partial f_1}{\partial y}dy\wedge dx + \frac{\partial f_2}{\partial x}dx\wedge dy = \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)dx\wedge dy. \end{aligned}$$

Finally, for a 2-form $\omega_2 = g(x,y)dx\wedge dy$ the d-operation gives

$$d\omega_2 = \frac{\partial g}{\partial x}dx\wedge dx\wedge dy + \frac{\partial g}{\partial y}dy\wedge dx\wedge dy = 0.$$

The latter shows that d applied to a top-form always gives 0. ▶

◀ **17.2** *Example.* In the previous example $d\omega_0$ is a 1-form, and $d\omega_1$ is a 2-form. Let us now apply the d operation to these forms:

$$d(d\omega_0) = \Big(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\Big)dx\wedge dy = 0,$$

and

$$d(d\omega_1) = 0,$$

since d acting on a 2-form always gives 0. These calculations seem to suggest that in general $d \circ d = 0$. ▶

As our examples indicate $d^2\omega = 0$. One can also have forms ω for which $d\omega = 0$, but $\omega \neq d\sigma$. We say that ω is a ***closed form***, and when $\omega = d\sigma$, then ω is called an ***exact form***. Clearly, closed forms form a possibly larger class than exact forms. In the next chapter on De Rham cohomology we are going to come back to this phenomenon in detail. On $\mathbb{R}^m$ one will not be able to find closed forms that are not exact, i.e. on $\mathbb{R}^m$ all closed forms are exact. However, if we consider different manifold we can examples where this is not the case.

◀ **17.3** *Example.* Let $M = \mathbb{R}^2 \backslash \{(0,0)\}$, and consider the 1-form

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

which clearly is a smooth 1-form on M. It holds that $d\omega = 0$, and thus ω is a closed form on M. Let $\gamma : [0, 2\pi) \to M$, $t \mapsto (\cos(t), \sin(t))$ be an embedding of S^1 into M, then

$$\int_\gamma \omega = \int_0^{2\pi} \gamma^*\omega = \int_0^{2\pi} dt = 2\pi.$$

Assume that ω is an exact 1-form on M, then $\omega = df$, for some smooth function $f : M \to \mathbb{R}$. In Section 7 we have showed that

$$\int_\gamma df = \int_0^{2\pi} \gamma^* df = (f \circ \gamma)'(t)dt = f(1,0) - f(1,0) = 0,$$

which contradicts the fact that $\int_\gamma \omega = \int_\gamma df = 2\pi$. This proves that ω is not exact. ▶

For the exterior derivative in general we have the following theorem.

Theorem 17.4. [30] *Let M be a smooth m-dimensional manifold. Then for all $k \geq 0$ there exist unique linear operations $d : \Gamma^k(M) \to \Gamma^{k+1}(M)$ such that;*

(i) *for any 0-form $\omega = f$, $f \in C^\infty(M)$ it holds that*

$$d\omega(X) = df(X) = Xf, \qquad X \in \mathfrak{F}(M);$$

(ii) *if $\omega \in \Gamma^k(M)$ and $\eta \in \Gamma^\ell(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta;$$

(iii) $d \circ d = 0$*;*

(iv) *if $\omega \in \Gamma^m(M)$, then $d\omega = 0$.*

*This operation d is called the **exterior derivative** on differential forms, and is a unique anti-derivation (of degree 1) on $\Gamma(M)$ with $d^2 = 0$.*

---

[30] See Lee Thm. 12.14

Proof: Let us start by showing the existence of d on a chart $U \subset M$. We have local coordinates $x = \varphi(p)$, $p \in U$, and we define

$$d_U : \Gamma^k(U) \to \Gamma^{k+1}(U),$$

via (14). Let us write d instead of d_U for notational convenience. We have that $d(f dx^I) = df \wedge dx^I$. Due to the linearity of d it holds that

$$\begin{aligned} d(f dx^I \wedge g dx^J) &= d(fg dx^I \wedge dx^J) = d(fg) dx^K \\ &= g df dx^K + f dg dx^K \\ &= (df \wedge dx^I) \wedge (g dx^J) + f dg \wedge dx^I \wedge dx^J \\ &= (df \wedge dx^I) \wedge (g dx^J) + (-1)^k f dx^I \wedge dg \wedge dx^J \\ &= d(f dx^I) \wedge (g dx^J) + (-1)^k (f dx^I) \wedge d(g dx^J), \end{aligned}$$

which proves (ii). As for (iii) we argue as follows. In the case of a 0-form we have that

$$\begin{aligned} d(df) &= d\Big(\frac{\partial f}{\partial x_i} dx^i = \frac{\partial^2 f}{\partial x_j \partial x_i}\Big) dx^j \wedge dx^i \\ &= \sum_{i<j} \Big(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\Big) dx^i \wedge dx^j = 0. \end{aligned}$$

Given a k-form $\omega = \omega_I dx^I$, then, using (ii), we have

$$\begin{aligned} d(d\omega) &= d\Big(d\omega_I \wedge dx^I\Big) \\ &= d(d\omega_I) \wedge dx^I + (-1)^{k+1} d\omega^I \wedge d(dx^I) = 0, \end{aligned}$$

since ω_I is a 0-form, and $d(dx^I) = (-1)^j dx^{i_1} \wedge \cdots \wedge d(dx^{i_j}) \wedge \cdots \wedge dx^{i_k} = 0$. The latter follows from $d(dx^{i_j}) = 0$. The latter also implies (iv), finishing the existence proof in the one chart case. The operation $d = d_U$ is well-defined, satisfying (i)-(iv), for any chart (U, φ).

The operation d_U is unique, for if there exists yet another exterior derivative $\widetilde{d}_U$, which satisfies (i)-(iv), then for $\omega = \omega_I dx^I$,

$$\widetilde{d}\omega = \widetilde{d}\omega_I \wedge dx^I + \omega^I \widetilde{d}(dx^I),$$

where we used (ii). From (ii) it also follows that $\widetilde{d}(dx^I) = (-1)^j dx^{i_1} \wedge \cdots \wedge \widetilde{d}(dx^{i_j}) \wedge \cdots \wedge dx^{i_k} = 0$. By (i) $d(\varphi(p)_{i_j}) = dx^{i_j}$, and thus by (iv) $\widetilde{d}(dx^{i_j}) = \widetilde{d} \circ \widetilde{d}(\varphi(p)_{i_j}) = 0$, which proves the latter. From (i) it also follows that $\widetilde{d}\omega_I = d\omega_I$, and therefore

$$\widetilde{d}\omega = \widetilde{d}\omega_I \wedge dx^I + \omega^I \widetilde{d}(dx^I) = d\omega_I \wedge dx^I = d\omega,$$

which proves the uniqueness of d_U.

Before giving the defining d for $\omega \in \Gamma^k(M)$ we should point out that d_U trivially satisfies (i)-(iii) of Theorem 17.5 (use (14). Since we have a unique operation d_U

for every chart U, we define for $p \in U$

$$(d\omega)_p = (d_U \omega|_U)_p,$$

as the exterior derivative for forms on M. If there is a second chart $U' \ni p$, then by uniqueness it holds that on

$$d_U(\omega|_{U\cap U'}) = d_{U\cap U'}(\omega|_{U\cap U'}) = d_{U'}(\omega|_{U\cap U'}),$$

which shows that the above definition is independent of the chosen chart.

The last step is to show that d is also uniquely defined on $\Gamma^k(M)$. Let $p \in U$, a coordinate chart, and consider the restriction

$$\omega|_U = \omega_I dx^I,$$

where $\omega_I \in C^\infty(U)$. Let $W \subset U$ be an open set containing p, with the additional property that $\mathrm{cl}(W) \subset U$ is compact. By Theorem 15.10 we can find a bump-function $g \in C^\infty(M)$ such that $g|_W = 1$, and $\mathrm{supp}(g) \subset U$. Now define

$$\widetilde{\omega} = g\omega_I d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k}).$$

Using (i) we have that $d(gx_i)|_W = dx^i$, and therefore $\tilde{\omega}|_W = \omega|_W$. Set $\eta = \widetilde{\omega} - \omega$, then $\eta|_W = 0$. Let $p \in W$ and $h \in C^\infty(M)$ satisfying $h(p) = 1$, and $\mathrm{supp}(h) \subset W$. Thus, $h\omega \equiv 0$ on M, and

$$0 = d(h\omega) = dh \wedge \omega + hd\omega.$$

This implies that $(d\omega)_p = -(df \wedge \omega)_p = 0$, which proves that $(d\widetilde{\omega})|_W = (d\omega)|_W$.

If we use (ii)-(iii) then

$$\begin{aligned} d\widetilde{\omega} &= d\big(g\omega_I d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k})\big) \\ &= d(g\omega_I) \wedge d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k}) + g\omega_I d\big(d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k})\big) \\ &= d(g\omega_I) \wedge d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k}). \end{aligned}$$

It now follows that since $g|_W = 1$, and since $(d\widetilde{\omega})|_W = (d\omega)|_W$, that

$$(d\omega)|_W = \frac{\partial \omega_I}{\partial x_i} dx^i \wedge dx^I,$$

which is exactly (14). We have only used properties (i)-(iii) the derive this expression, and since p is arbitrary it follows that $d : \Gamma^k(M) \to \Gamma^{k+1}(M)$ is uniquely defined. ■

The exterior derivative has other important properties with respect to restrictions and pullbacks that we now list here.

Theorem 17.5. *Let M be a smooth m-dimensional manifold, and let $\omega \in \Gamma^k(M)$, $k \geq 0$. Then*

(i) *in each chart (U, φ) for M, $d\omega$ in local coordinates is given by (14);*
(ii) *if $\omega = \omega'$ on some open set $U \subset M$, then also $d\omega = d\omega'$ on U;*
(iii) *if $U \subset M$ is open, then $d(\omega|_U) = (d\omega)|_U$;*

(iv) *if* $f : N \to M$ *is a smooth mapping, then*

$$f^*(d\omega) = d(f^*\omega),$$

i.e. $f^* : \Gamma^k(M) \to \Gamma^k(N)$, *and* d *commute as operations.*

Proof: Let us restrict ourselves here to proof of (iv). It suffices to prove (iv) in a chart U, and $\omega = \omega_I dx^I$. Let us first compute $f^*(d\omega)$:

$$\begin{aligned} f^*(d\omega) &= f^*\big(d\omega_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\big) \\ &= d(\omega_I \circ f) \wedge d(\varphi \circ f)_{i_1} \wedge \cdots \wedge d(\varphi \circ f)_{i_k}. \end{aligned}$$

Similarly,

$$\begin{aligned} d(f^*\omega) &= d\big((\omega_I \circ f) \wedge d(\varphi \circ f)_{i_1} \wedge \cdots \wedge d(\varphi \circ f)_{i_k}\big) \\ &= d(\omega_I \circ f) \wedge d(\varphi \circ f)_{i_1} \wedge \cdots \wedge d(\varphi \circ f)_{i_k}, \end{aligned}$$

which proves the theorem. ∎

18. Stokes' Theorem

The last section of this chapter deals with a generalization of the Fundamental Theorem of Integration: Stokes' Theorem. The Stokes' Theorem allows us to compute $\int_M d\omega$ in terms of a boundary integral for ω. In a way the $(m-1)$-form act as the 'primitive', 'anti-derivative' of the $d\omega$. Therefore if we are interested in $\int_M \sigma$ using Stokes' Theorem we need to first 'integrate' σ, i.e. write $\sigma = d\omega$. This is not always possible as we saw in the previous section.

Stokes' Theorem can be now be phrased as follows.

Theorem 18.1. *Let* M *be a smooth* m*-dimensional manifold with or without boundary* ∂M, *and* $\omega \in \Gamma_c^{m-1}(M)$. *Then,*

$$\int_M d\omega = \int_{\partial M} j^*\omega, \tag{15}$$

where $j : \partial M \hookrightarrow M$ *is the natural embedding of the boundary* ∂M *into* M.

In order to prove this theorem we start with the following lemma.

Lemma 18.2. *Let* $\omega \in \Gamma_c^{m-1}(\mathbb{H}^m)$ *be given by*

$$\omega = \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m.$$

Then,

(i) $\int_{\mathbb{H}^M} d\omega = 0$, *if* $\operatorname{supp}(\omega) \subset \operatorname{int}(\mathbb{H}^m)$;

(ii) $\int_{\mathbb{H}^m} d\omega = \int_{\partial\mathbb{H}^m} j^*\omega$, *if* $\operatorname{supp}(\omega) \cap \partial\mathbb{H}^m \neq \varnothing$,

where $j : \partial\mathbb{H}^m \hookrightarrow \mathbb{H}^m$ *is the canonical inclusion.*

Proof: We may assume without loss of generality that $\mathrm{supp}(\omega) \subset [0,R]\times\cdots\times[0,R] = I_R^m$. Now,

$$\begin{aligned}\int_{\mathbb{H}^m} d\omega &= (-1)^{i+1}\int_{I_R^m}\frac{\partial\omega_i}{\partial x_i}dx_1\cdots dx_m\\ &= (-1)^{i+1}\int_{I_R^{m-1}}\Big(\omega_i|_{x_i=R}-\omega_i|_{x_i=0}\Big)dx_1\cdots\widehat{dx_i}\cdots dx_m.\end{aligned}$$

If $\mathrm{supp}(\omega)\subset \mathrm{int}\, I_R^m$, then $\omega_i|_{x_i=R}-\omega_i|_{x_i=0}=0$ for all i, and

$$\int_{\mathbb{H}^m} d\omega = 0.$$

If $\mathrm{supp}(\omega)\cap\partial\mathbb{H}^m$, then $\omega_i|_{x_i=R}-\omega_i|_{x_i=0}=0$ for all $i\le m-1$, and $\omega_m|_{x_m=R}=0$. Therefore,

$$\begin{aligned}\int_{\mathbb{H}^m} d\omega &= (-1)^{i+1}\int_{I_R^m}\frac{\partial\omega_i}{\partial x_i}dx_1\cdots dx_m\\ &= (-1)^{i+1}\int_{I_R^{m-1}}\Big(\omega_i|_{x_i=R}-\omega_i|_{x_i=0}\Big)dx_1\cdots\widehat{dx_i}\cdots dx_m\\ &= (-1)^{m+1}\int_{I_R^{m-1}}\Big(-\omega_i|_{x_m=0}\Big)dx_1\cdots dx_{m-1}\\ &= (-1)^{m}\int_{I_R^{m-1}}\Big(\omega_i|_{x_m=0}\Big)dx_1\cdots dx_{m-1}.\end{aligned}$$

The mapping $j:(x_1,\cdots,x_{m-1})\mapsto(x_1,\cdots,x_{m-1},0)$ is orientation preserving if m is even and orientation reversing if m is odd (see Section 13, Example **??**). Under the mapping j we have that

$$j_*(e_i)=\mathbf{e}_{i+1},\qquad i=1,\cdots,m-1,$$

where the e_i's on the left hand side are the unit vectors in $\mathbb{R}^{m-1}$, and the bold face $\mathbf{e}_k$ are the unit vectors in $\mathbb{R}^m$. We have that the induced orientation for $\partial\mathbb{H}^m$ is obtained by the rotation $\mathbf{e}_1\to-\mathbf{e}_m$, $\mathbf{e}_m\to\mathbf{e}_1$, and therefore

$$\partial\mathcal{O}=[\mathbf{e}_m,\mathbf{e}_2,\cdots,\mathbf{e}_{m-1}]=[e_{m-1},e_1,\cdots,e_{m-2}].$$

Then pullback form on $\partial\mathbb{H}^m$, using the induced orientation on $\partial\mathbb{H}^m$, is given by

$$\begin{aligned}(j^*\omega)_{(x_1,\cdots,x_{m-1})} & (e_{m-1},e_1,\cdots,e_{m-2})\\ &= \omega_{(x_1,\cdots,x_{m-1},0)}\big(j_*(e_{m-1}),j_*(e_1),\cdots,j_*(e_{m-2})\big)\\ &= (-1)^m\omega_m(x_1,\cdots,x_{m-1},0),\end{aligned}$$

where we used the fact that

$$\begin{aligned}dx_2\wedge\cdots\wedge dx_m & \big(j_*(e_{m-1}),j_*(e_1),\cdots,j_*(e_{m-2})\big)\\ &= dx_2\wedge\cdots\wedge dx_m(\mathbf{e}_m,\mathbf{e}_2,\cdots,\mathbf{e}_{m-1})\\ &= (-1)^m dx_m\wedge dx_2\wedge\cdots\wedge dx_{m-1}(\mathbf{e}_m,\mathbf{e}_2,\cdots,\mathbf{e}_{m-1})=(-1)^m.\end{aligned}$$

Combing this with the integral over $\mathbb{H}^m$ we finally obtain

$$\int_{\mathbb{H}^m} d\omega = \int_{\partial\mathbb{H}^m} j^*\omega,$$

which completes the proof. ∎

Proof of Theorem 18.1: Let us start with the case that $\mathrm{supp}(\omega) \subset U$, where (U,φ) is an oriented chart for M. Then by the definition of the integral we have that

$$\int_M d\omega = \int_{\varphi(U)} (\varphi^{-1})^*(d\omega) = \int_{\mathbb{H}^m} (\varphi^{-1})^*(d\omega) = \int_{\mathbb{H}^m} d\Big((\varphi^{-1})^*\omega\Big),$$

where the latter equality follows from Theorem 17.5. It follows from Lemma 18.2 that if $\mathrm{supp}(\omega) \subset \mathrm{int}\,(\mathbb{H}^m)$, then the latter integral is zero and thus $\int_M \omega = 0$. Also, using Lemma 18.2, it follows that if $\mathrm{supp}(\omega) \cap \partial\mathbb{H}^m \neq \varnothing$, then

$$\begin{aligned} \int_M d\omega &= \int_{\mathbb{H}^m} d\Big((\varphi^{-1})^*\omega\Big) = \int_{\partial\mathbb{H}^m} j'^*(\varphi^{-1})^*\omega \\ &= \int_{\partial\mathbb{H}^m} (\varphi^{-1}\circ j')^*\omega = \int_{\partial M} j^*\omega, \end{aligned}$$

where $j' : \partial\mathbb{H}^m \to \mathbb{H}^m$ is the canonical inclusion as used Lemma 18.2, and $j = \varphi^{-1}\circ j' : \partial M \to M$ is the inclusion of the boundary of M into M.

Now consider the general case. As before we choose a finite covering $\mathcal{A}_I \subset \mathcal{A}$ of $\mathrm{supp}(\omega)$ and an associated partition of unity $\{\lambda_i\}$ subordinate to $\mathcal{A}_I$. Consider the forms $\lambda_i\omega$, and using the first part we obtain

$$\begin{aligned} \int_{\partial M} j^*\omega &= \sum_i \int_{\partial M} j^*\lambda_i\omega = \sum_i \int_M d(\lambda_i\omega) \\ &= \sum_i \int_M d\lambda_i \wedge \omega + \lambda_i d\omega \\ &= \int_M d\Big(\sum_i \lambda_i\Big) \wedge \omega + \int_M \Big(\sum_i \lambda_i\Big) d\omega \\ &= \int_M d\omega, \end{aligned}$$

which proves the theorem. ∎

◀ **18.3** *Remark.* If M is a closed (compact, no boundary), oriented manifold manifold, then by Stokes' Theorem for any $\omega \in \Gamma^{m-1}(M)$,

$$\int_M d\omega = 0,$$

since $\partial M = \varnothing$. As a consequence volume form σ cannot be exact, since $\int_M \sigma > 0$, and if σ were exact, then $\sigma = d\omega$, which implies that $\int_M \sigma = d\omega \in \Gamma^m(M) = 0$, a contradiction. ▶

Let us start with the obvious examples of Stokes' Theorem in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

◀ **18.4** *Example.* Let $M = [a,b] \subset \mathbb{R}$, and consider the 0-form $\omega = f$, then, since M is compact, ω is compactly supported 0-form. We have that $d\omega = f'(x)dx$, and

$$\int_a^b f'(x)dx = \int_{\{a,b\}} f = f(b) - f(a),$$

which is the fundamental theorem of integration. Since on M any 1-form is exact the theorem holds for 1-forms. ▶

◀ **18.5** *Example.* Let $M = \gamma \subset \mathbb{R}^2$ be a curve parametrized given by $\gamma : [a,b] \to \mathbb{R}^2$. Consider a 0-form $\omega = f$, then

$$\int_\gamma f_x(x,y)dx + f_y(x,y)dy = \int_{\{\gamma(a),\gamma(b)\}} f = f(\gamma(b)) - f(\gamma(a)).$$

Now let $M = \Omega \subset \mathbb{R}^2$, a closed subset of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, and consider a 1-form $\omega = f(x,y)dx + g(x,y)dy$. Then, $d\omega = \big(g_x - f_y\big)dx \wedge dy$, and Stokes' Theorem yields

$$\int_\Omega \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)dxdy = \int_{\partial\Omega} fdx + gdy,$$

also known as Green's Theorem in $\mathbb{R}^2$. ▶

◀ **18.6** *Example.* In the case of a curve $M = \gamma \subset \mathbb{R}^3$ we obtain the line-integral for 0-forms as before:

$$\int_\gamma f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z) = f(\gamma(b)) - f(\gamma(a)).$$

If we write the vector field

$$F = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z},$$

then

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z},$$

and the above expression can be rewritten as

$$\int_\gamma \nabla f \cdot ds = f|_{\partial\gamma}.$$

Next let $M = S \subset \mathbb{R}^3$ be an embedded or immersed hypersurface, and let $\omega = fdx + gdy + hdz$ be a 1-form. Then,

$$d\omega = \Big(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\Big)dy \wedge dz + \Big(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\Big)dz \wedge dx + \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)dx \wedge dy.$$

Write the vector fields

$$\text{curl } F = \nabla \times F = \Big(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\Big)\frac{\partial}{\partial x} + \Big(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\Big)\frac{\partial}{\partial y} + \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)\frac{\partial}{\partial z}.$$

Furthermore set

$$ds = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \qquad dS = \begin{pmatrix} dydz \\ dzdx \\ dxdy \end{pmatrix},$$

then from Stokes' Theorem we can write the following surface and line integrals:

$$\int_S \nabla \times F \cdot dS = \int_{\partial S} F \cdot ds,$$

whixh is usually referred to as the classical Stokes' Theorem in $\mathbb{R}^3$. The version in Theorem 18.1 is the general Stokes' Theorem. Finally let $M = \Omega$ a closed subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, and consider a a 2-form $\omega = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$. Then

$$d\omega = \Big(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\Big) dx \wedge dy \wedge dz.$$

Write

$$\text{div}\, F = \nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}, \quad dV = dxdydz,$$

then from Stokes' Theorem we obtain

$$\int_\Omega \nabla \cdot F dV = \int_{\partial\Omega} F \cdot dS,$$

which is referred to as the Gauss Divergence Theorem. ▶

V. De Rham cohomology

19. **Definition of De Rham cohomology**

In the previous chapters we introduced and integrated m-forms over manifolds M. We recall that k-form $\omega \in \Gamma^k(M)$ is closed if $d\omega = 0$, and a k-form $\omega \in \Gamma^k(M)$ is exact if there exists a $(k-1)$-form $\sigma \in \Gamma^{k-1}(M)$ such that $\omega = d\sigma$. Since $d^2 = 0$, exact forms are closed. We define

$$
\begin{aligned}
Z^k(M) &= \left\{\omega \in \Gamma^k(M) \,:\, d\omega = 0\right\} = \mathrm{Ker}(d), \\
B^k(M) &= \left\{\omega \in \Gamma^k(M) \,:\, \exists\, \sigma \in \Gamma^{k-1}(M) \ni \omega = d\sigma\right\} = \mathrm{Im}(d),
\end{aligned}
$$

and in particular

$$
B^k(M) \subset Z^k(M).
$$

The sets Z^k and B^k are real vector spaces, with B^k a vector subspace of Z^k. This leads to the following definition.

Definition 19.1. Let M be a smooth m-dimensional manifold then the ***de Rham cohomology groups*** are defined as

$$
H^k_{dR}(M) := Z^k(M)/B^k(M), \qquad k = 0, \cdots m, \tag{16}
$$

where $B^0(M) := 0$.

It is immediate from this definition that $Z^0(M)$ are smooth functions on M that are constant on each connected component of M. Therefore, when M is connected, then $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$. Since $\Gamma^k(M) = \{0\}$, for $k > m = \dim\ M$, we have that $\mathrm{H}^k_{dR}(M) = 0$ for all $k > m$. For $k < 0$, we set $H^k_{dR}(M) = 0$.

◀ **19.2** *Remark.* The de Rham groups defined above are in fact real vector spaces, and thus groups under addition in particular. The reason we refer to de Rham cohomology groups instead of de Rham vector spaces is because (co)homology theories produce abelian groups. ▶

An equivalence class $[\omega] \in H^k_{\mathrm{dR}}(M)$ is called a ***cohomology class***, and two form $\omega, \omega' \in Z^k(M)$ are ***cohomologous*** if $[\omega] = [\omega']$. This means in particular that ω and ω' differ by an exact form, i.e.

$$
\omega' = \omega + d\sigma.
$$

Now let us consider a smooth mapping $f : N \to M$, then we have that the pullback f^* acts as follows: $f^* : \Gamma^k(M) \to \Gamma^k(N)$. From Theorem 17.5 it follows that

$d \circ f^* = f^* \circ d$ and therefore f^* descends to homomorphism in cohomology. This can be seen as follows:

$$d f^*\omega = f^* d\omega = 0, \quad \text{and} \quad f^* d\sigma = d(f^*\sigma),$$

and therefore the closed forms $Z^k(M)$ get mapped to $Z^k(N)$, and the exact form $B^k(M)$ get mapped to $B^k(N)$. Now define

$$f^*[\omega] = [f^*\omega],$$

which is well-defined by

$$f^*\omega' = f^*\omega + f^* d\sigma = f^*\omega + d(f^*\sigma)$$

which proves that $[f^*\omega'] = [f^*\omega]$, whenever $[\omega'] = [\omega]$. Summarizing, f^* maps cohomology classes in $H^k_{\mathrm{dR}}(M)$ to classes in $H^k_{\mathrm{dR}}(N)$:

$$f^* : H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(N),$$

Theorem 19.3. *Let $f : N \to M$, and $g : M \to K$, then*

$$g^* \circ f^* = (f \circ g)^* : \; H^k_{\mathrm{dR}}(K) \to H^k_{\mathrm{dR}}(N),$$

Moreover, id* is the identity map on cohomology.

Proof: Since $g^* \circ f^* = (f \circ g)^*$ the proof follows immediately. ■

As a direct consequence of this theorem we obtain the invariance of de Rham cohomology under diffeomorphisms.

Theorem 19.4. *If $f : N \to M$ is a diffeomorphism, then $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$.*

Proof: We have that $\mathrm{id} = f \circ f^{-1} = f^{-1} \circ f$, and by the previous theorem

$$\mathrm{id}^* = f^* \circ (f^{-1})^* = (f^{-1})^* \circ f^*,$$

and thus f^* is an isomorphism. ■

20. Homotopy invariance of cohomology

We will prove now that the de Rham cohomology of a smooth manifold M is even invariant under homeomorphisms. As a matter of fact we prove that the de Rham cohomology is invariant under homotopies of manifolds.

Definition 20.1. Two smooth mappings $f, g : N \to M$ are said to be ***homotopic*** if there exists a continuous map $H : N \times [0,1] \to M$ such that

$$\begin{aligned} H(p,0) &= f(p) \\ H(p,1) &= g(p), \end{aligned}$$

for all $p \in N$. Such a mapping is called a ***homotopy*** from/between f to/and g. If in addition H is smooth then f and g are said to be ***smoothly homotopic***, and H is called a ***smooth homotopy***.

Using the notion of smooth homotopies we will prove the following crucial property of cohomology:

Theorem 20.2. *Let $f, g : N \to M$ be two smoothly homotopic maps. Then for $k \geq 0$ it holds for $f^*, g^* : H^k_{\rm dR}(M) \to H^k_{\rm dR}(N)$, that*

$$f^* = g^*.$$

◄ **20.3** *Remark.* It can be proved in fact that the above results holds for two homotopic (smooth) maps f and g. This is achieved by constructing a smooth homotopy from a homotopy between maps. ►

Proof of Theorem 20.2: A map $\mathbf{h} : \Gamma^k(M) \to \Gamma^{k-1}(N)$ is called a ***homotopy map*** between f^* and g^* if

$$d\mathbf{h}(\omega) + \mathbf{h}(d\omega) = g^*\omega - f^*\omega, \quad \omega \in \Gamma^k(M). \tag{17}$$

Now consider the embedding $i_t : N \to N \times I$, and the trivial homotopy between i_0 and i_1 (just the identity map). Let $\omega \in \Gamma^k(N \times I)$, and define the mapping

$$\mathbf{h}(\omega) = \int_0^1 i_{\frac{\partial}{\partial t}} \omega dt,$$

which is a map from $\Gamma^k(N \times I) \to \Gamma^{k-1}(N)$. Choose coordinates so that either

$$\omega = \omega_I(x,t)dx^I, \quad \text{or} \quad \omega = \omega_{I'}(x,t)dt \wedge d^x I'.$$

In the first case we have that $i_{\frac{\partial}{\partial t}} \omega = 0$ and therefore $d\mathbf{h}(\omega) = 0$. On the other hand

$$\begin{aligned}
\mathbf{h}(d\omega) &= \mathbf{h}\Big(\frac{\partial \omega_I}{\partial t} dt \wedge dx^I + \frac{\partial \omega_I}{\partial x_i} dx^i \wedge dx^I\Big) \\
&= \Big(\int_0^1 \frac{\partial \omega_I}{\partial t} dt\Big) dx^I \\
&= \big(\omega_I(x,1) - \omega_I(x,0)\big) dx^I = i_1^*\omega - i_0^*\omega,
\end{aligned}$$

which prove (17) for i_0^* and i_1^*, i.e.

$$d\mathbf{h}(\omega) + \mathbf{h}(d\omega) = i_1^*\omega - i_0^*\omega.$$

In the second case we have

$$\begin{aligned}
\mathbf{h}(d\omega) &= \mathbf{h}\Big(\frac{\partial \omega_I}{\partial x_i} dx^i \wedge dt \wedge dx^{I'}\Big) \\
&= \int_0^1 \frac{\partial \omega_I}{\partial dx_i} i_{\frac{\partial}{\partial x_i}} \big(dx^i \wedge dt \wedge dx^{I'}\big) dt \\
&= -\Big(\int_0^1 \frac{\partial \omega_I}{\partial x_i} dt\Big) dx^i \wedge dx^{I'}.
\end{aligned}$$

On the other hand

$$\begin{aligned} d\mathbf{h}(\omega) &= d\Big(\Big(\int_0^1 \omega_{I'}(x,t)dt\Big)dx^{I'}\Big) \\ &= \frac{\partial}{\partial x_i}\Big(\int_0^1 \omega_{I'}(x,t)dt\Big)dx^i\wedge dx^{I'} \\ &= \Big(\int_0^1 \frac{\partial \omega_I}{\partial x_i}dt\Big)dx^i\wedge dx^{I'} \\ &= -\mathbf{h}(d\omega). \end{aligned}$$

This gives the relation that

$$d\mathbf{h}(\omega)+\mathbf{h}(d\omega)=0,$$

and since $i_1^*\omega = i_0^*\omega = 0$ in this case, this then completes the argument in both cases, and $\mathbf{h}$ as defined above is a homotopy map between i_0^* and i_1^*.

By assumption we have a smooth homotopy $H: N\times[0,1]\to M$ bewteen f and g, with $f = H\circ i_0$, and $g = H\circ i_1$. Consider the composition $\widetilde{\mathbf{h}} = \mathbf{h}\circ H^*$. Using the relations above we obtain

$$\begin{aligned} \widetilde{\mathbf{h}}(d\omega)+d\widetilde{\mathbf{h}}(\omega) &= \mathbf{h}(H^*d\omega)+d\mathbf{h}(H^*\omega) \\ &= \mathbf{h}(d(H^*\omega))+d\mathbf{h}(H^*\omega) \\ &= i_1^*H^*\omega - i_0^*H^*\omega \\ &= (H\circ i_1)^*\omega-(H\circ i_0)^*\omega \\ &= g^*\omega - f^*\omega. \end{aligned}$$

If we assume that ω is closed then

$$g^*\omega - f^*\omega = d\mathbf{h}(H^*\omega),$$

and thus

$$0=[d\mathbf{h}(H^*\omega)] = [g^*\omega-f^*\omega] = g^*[\omega]-f^*[\omega],$$

which proves the theorem. ■

◀ **20.4** *Remark.* Using the same ideas as for the Whitney embedding theorem one can prove, using approximation by smooth maps, that Theorem 20.2 holds true for continuous homotopies between smooth maps. ▶

Definition 20.5. Two manifolds N and M are said to be ***homotopy equivalent***, if there exist smooth maps $f: N\to M$, and $g: M\to N$ such that

$$g\circ f\cong \mathrm{id}_N, \qquad f\circ g\cong \mathrm{id}_M \quad \text{(homotopic maps)}.$$

We write $N\sim M$., The maps f and g are ***homotopy equivalences*** are each other ***homotopy inverses*** . If the homotopies involved are smooth we say that N and M ***smoothly homotopy equivalent***.

◀ **20.6** *Example.* Let $N = S^1$, the standard circle in $\mathbb{R}^2$, and $M = \mathbb{R}^2\backslash\{(0,0)\}$. We have that $N \sim M$ by considering the maps

$$f = i : S^1 \hookrightarrow \mathbb{R}^2\backslash\{(0,0)\}, \qquad g = \mathrm{id}/|\cdot|.$$

Clearly, $(g\circ f)(p) = p$, and $(f\circ g)(p) = p/|p|$, and the latter is homotopic to the identity via $H(p,t) = tp + (1-t)p/|p|$. ▶

Theorem 20.7. *Let N and M be smoothly homotopically equivalent manifold, $N \sim M$, then*

$$H^k_{\mathrm{dR}}(N) \cong H^k_{\mathrm{dR}}(M),$$

and the homotopy equivalences f g between N and M, and M and N respectively are isomorphisms.

As before this theorem remains valid for continuous homotopy equivalences of manifolds.

VI. Exercises

A number of the exercises given here are taken from the Lecture notes by J. Bochnak.

Manifolds

Topological Manifolds

1 Given the function $g : \mathbb{R} \to \mathbb{R}^2$, $g(t) = (\cos(t), \sin(t))$. Show that $f(\mathbb{R})$ is a manifold.

2 Given the set $\mathbb{T}^2 = \{p = (p_1, p_2, p_3) \mid 16(p_1^2 + p_2^2) = (p_1^2 + p_2^2 + p_3^2 + 3)^2\} \subset \mathbb{R}^3$, called the ***2-torus***. (***i***) Consider the product manifold $S^1 \times S^1 = \{q = (q_1, q_2, q_3, q_4) \mid q_1^2 + q_2^2 = 1,\ q_3^2 + q_4^2 = 1\}$, and the mapping $f : S^1 \times S^1 \to \mathbb{T}^2$, given by

$$f(q) = \big(q_1(2 + q_3), q_2(2 + q_3), q_4\big).$$

Show that f is onto and $f^{-1}(p) = \left(\frac{p_1}{r}, \frac{p_2}{r}, r - 2, p_3\right)$, where $r = \frac{|p|^2+3}{4}$.

(***ii***) Show that f is a homeomorphism between $S^1 \times S^1$ and $\mathbb{T}^2$.

3 (***i***) Find an atlas for $\mathbb{T}^2$ using the mapping f in 2.

(***ii***) Give a parametrization for $\mathbb{T}^2$.

4 Show that

$$A_{4,4} = \{(p_1, p_2) \in \mathbb{R}^2 \mid p_1^4 + p_2^4 = 1\},$$

is a manifold and $A_{4,4} \cong S^1$ (homeomorphic).

5 (***i***) Show that an open subset $U \subset M$ of a manifold M is again a manifold.

(***ii***) Let N and M be manifolds. Show that their cartesian production $N \times M$ is also a manifold.

6 Show that

(***i***) $\{A \in M_{2,2}(\mathbb{R}) \mid \det(A) = 1\}$ is a manifold.

(***ii***) $Gl(n, \mathbb{R}) = \{A \in M_{n,n}(\mathbb{R}) \mid \det(A) \neq 0\}$ is a manifold.

(***iii***) Determine the dimensions of the manifolds in (a) and (b).

7 Construct a simple counterexample of a set that is not a manifold.

8 Show that in Definition 1.1 an open set $U' \subset \mathbb{R}^n$ can be replaced by an open disc $D^n \subset \mathbb{R}^n$.

9 Show that $P\mathbb{R}^n$ is a Hausdorff space and is compact.

10 Define the Grassmann manifold $G^k\mathbb{R}^n$ as the set of all k-dimensional linear subspaces in $\mathbb{R}^n$. Show that $G^k\mathbb{R}^n$ is a manifold.

11 Consider X to be the parallel lines $\mathbb{R} \times \{0 \cup \mathbb{R} \times \{1$. Define the equivalence relation $(x,0) \sim (x,1)$ for all $x \neq 0$. Show that $M = X/\sim$ is a topological space that satisfies (ii) and (iii) of Definition 1.1.

12 Let M be an uncountable union of copies of $\mathbb{R}$. Show that M is a topological space that satisfies (i) and (ii) of Definition 1.1.

13 Let $M = \{(x,y) \in \mathbb{R}^2 \,:\, xy = 0\}$. Show that M is a topological space that satisfies (i) and (iii) of Definition 1.1.

Differentiable manifolds

14 Show that cartesian products of differentiable manifolds are again differentaible manifolds.

15 Show that $P\mathbb{R}^n$ is a smooth manifold.

16 Which of the atlases for S^1 given in Example 2, Sect. 1, are compatible. If not compatible, are they diffeomorphic?

17 Show that the standard torus $\mathbb{T}^2 \subset \mathbb{R}^3$ is diffeomorphic to $S^1 \times S^1$, where $S^1 \subset \mathbb{R}^2$ is the standard circle in $\mathbb{R}^2$.

Immersion, submersion and embeddings

18 Show that the torus the map f defined in Exer. 2 of 1.1 yields a smooth embedding of the torus $\mathbb{T}^2$ in $\mathbb{R}^3$.

19 Let k, m, n be positive integers. Show that the set

$$A_{k,m,n} := \{(x,y,z) \in \mathbb{R}^3 \,:\, x^k + y^m + z^n = 1\},$$

is a smooth embedded submanifold in $\mathbb{R}^3$, and is diffeomorphic to S^2 when these numbers are even.

20 (Lee) Consider the mapping $f(x,y,s) = (x^2 + y, x^2 + y^2 + s^2 + y)$ from $\mathbb{R}^3$ to $\mathbb{R}^2$. Show that $q = (0,1)$ is a regular value, and $f^{-1}(q)$ is diffeomorphic to S^1 (standard).

21 Prove Theorem 3.24 (Hint: prove the steps indicated above Theorem 3.24).

22 Prove Theorem 3.26.

23 Let N, M be two smooth manifolds of the same dimension, and $f : N \to M$ is a smooth mapping. Show (using the Inverse Function Theorem) that if f is a bijection then it is a diffeomorphism.

24 Prove Theorem 3.28.

25 (Boothby) Show that the map $f : S^n \to P\mathbb{R}^n$ defined by $f(x_1, \cdots, x_{n+1}) = [(x_1, \cdots, x_{n+1}]$ is smooth and everywhere of rank n.

26 (Lee) Let $a \in \mathbb{R}$, and $M_a = \{(x,y) \in \mathbb{R}^2 \,:\, y^2 = x(x-1)(x-a)\}$. (***i***) For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? (***ii***) For which values of a can M_a be given a topology and smooth structure so that M_a is an immersed submanifold?

27 Prove Lemma **??**.

28 Prove Theorem 3.37.

Manifolds in Euclidean space

29 Let M be a m-dimensional manifold with boundary ∂M. Show that ∂M is an $(m-1)$-dimensional manifold.

30 Show that an m-dimensional subspace in $\mathbb{R}^\ell$ is an m-dimensional manifold.

Tangent and cotangent spaces

Tangent spaces and vector fields

31 Let $U \subset \mathbb{R}^k$, $y \in \mathbb{R}^m$, and $f : U \to \mathbb{R}^m$ a smooth function. By Theorem **??** $M = f^{-1}(y) = \{p \in \mathbb{R}^k \,:\, f(p) = y\}$ is a smooth embedded manifold in $\mathbb{R}^m$ if $\mathrm{rk}(Jf)|_p = m$ for all $p \in M$. Show that

$$T_pM = \{X_p \in \mathbb{R}^k \,:\, X_p \cdot \partial_{p_i} f = 0,\ i = 1, \cdots, m\} = \ker Jf|_p,$$

i.e. the tangent space is the kernel of the Jacobian.

32 Given the set $M = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \,:\, p_1^3 + p_2^3 + p_3^3 - 3p_1p_2p_3 = 1\}$.

(***i***) Show that M is smooth embedded manifold in $\mathbb{R}^3$ of dimension 2.

(***ii***) Compute T_pM, at $p = (0,0,1)$.

33 Given the set $M = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \,:\, p_1^2 - p_2^2 + p_1p_3 - 2p_2p_3 = 0,\ 2p_1 - p_2 + p_3 = 3\}$.

(***i***) Show that M is smooth embedded manifold in $\mathbb{R}^3$ of dimension 1.

(***ii***) Compute T_pM, at $p = (1,-1,0)$.

34 Let $M \subset \mathbb{R}^\ell$ be a manifold in $\mathbb{R}^\ell$. Show that TM as defined in Section 6 is a smooth embedded submanifold of $\mathbb{R}^{2\ell}$.

35 Prove Lemma 5.5.

36 Let $S^1 \subset \mathbb{R}^2$ be the standard unit circle. Show that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

37 Express the following planar vector fields in polar coordinates:

(***i***) $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$;

(***ii***) $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$;

(***iii***) $X = (x^2 + y^2)\frac{\partial}{\partial x}$.

38 Find a vector field on S^2 that vanishes at exactly one point.

Cotangent spaces and differential 1-forms

39 Let $M \subset \mathbb{R}^\ell$ be a manifold in $\mathbb{R}^\ell$. Show that T^*M as defined in Section 8 is a smooth embedded submanifold of $\mathbb{R}^{2\ell}$.

40 (Lee) Let $f: \mathbb{R}^3 \to \mathbb{R}$ be given by $f(p_1, p_2, p_3) = |p|^2$, and $g: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$g(x_1, x_2) = \Big(\frac{2x_1}{|x|^2+1}, \frac{2x_2}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1}\Big).$$

Compute both g^*df, and $d(f \circ g)$ and compare the two answers.

41 (Lee) Compute df in coordinates, and determine the set points where df vanishes.

(***i***) $M = \{(p_1, p_2) \in \mathbb{R}^2 \ : \ p_2 > 0\}$, and $f(p) = \frac{p_1}{|p|^2}$ — standard coordinates in $\mathbb{R}^2$.

(***ii***) As the previous, but in polar coordinates.

(***iii***) $M = S^2 = \{p \in \mathbb{R}^3 \ : \ |p| = 1\}$, and $f(p) = p_3$ — stereographic coordinates.

(***iv***) $M = \mathbb{R}^n$, and $f(p) = |p|^2$ — standard coordinates in $\mathbb{R}^n$.

42 (Lee) Let $f: N \to M$ (smooth), $\omega \in \Lambda^1(N)$, and $\gamma: [a,b] \to N$ is a smooth curve. Show that

$$\int_\gamma f^*\omega = \int_{f\circ\gamma} \omega.$$

43 Given the following 1-forms on $\mathbb{R}^3$:

$$\alpha = -\frac{4z}{(x^2+1)^2}dx + \frac{2y}{y^2+1}dy + \frac{2x}{x^2+1}dz,$$
$$\omega = -\frac{4xz}{(x^2+1)^2}dx + \frac{2y}{y^2+1}dy + \frac{2}{x^2+1}dz.$$

(***i***) Let $\gamma(t) = (t,t,t)$, $t \in [0,1]$, and compute $\int_\gamma \alpha$ and $\int_\gamma \omega$.

(***ii***) Let γ be a piecewise smooth curve going from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$, and compute the above integrals.

(***iii***) Which of the 1-forms α and ω is exact.

(***iv***) Compare the answers.

Tensors

Tensors and tensor products

44 Describe the standard inner product on $\mathbb{R}^n$ as a covariant 2-tensor.

45 Construct the determinant on $\mathbb{R}^2$ and $\mathbb{R}^3$ as covariant tensors.

46 Given the vectors $a = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, and $b = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Compute $a^* \otimes b$ and $b^* \otimes a$.

47 Given finite dimensional vector spaces V and W, prove that $V \otimes W$ and $W \otimes V$ are isomorphic.

48 Given finite dimensional vector spaces U, V and W, prove that $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are isomorphic.

49 Show that $V \otimes \mathbb{R} \simeq V \simeq \mathbb{R} \otimes V$.

Symmetric and alternating tensors

50 Show that for $T \in T^s(V)$ the tensor $\mathrm{Sym}\, T$ is symmetric.

51 Prove that a tensor $T \in T^s(V)$ is symmetric if and only if $T = \mathrm{Sym}\, T$.

52 Show that the algebra

$$\Sigma^*(V) = \bigoplus_{k=0}^{\infty} \Sigma^k(V),$$

is a commutative algebra.

53 Prove that for any $T \in T^s(V)$ the tensor $\mathrm{Alt}\, T$ is alternating.

54 Show that a tensor $T \in T^s(V)$ is alternating if and only if $T = \mathrm{Alt}\, T$.

55 Given the vectors $a = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, and $c = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Compute $a \wedge b \wedge c$, and compare this with $\det(a, b, c)$.

56 Given vectors $a_1, \cdots, a_n$ show that $a_1 \wedge \cdots \wedge a_n = \det(a_1, \cdots, a_n)$.

57 Prove Lemma 10.11.

Tensor bundles and tensor fields

58 Let $M \subset \mathbb{R}^\ell$ be an embedded m-dimensional manifold. Show that $T^r M$ is a smooth manifold in $\mathbb{R}^{\ell+\ell^r}$.

59 Similarly, show that $T_s M$ is a smooth manifold in $\mathbb{R}^{\ell+\ell^s}$, and $T_s^r M$ is a smooth manifold in $\mathbb{R}^{\ell+\ell^r+\ell^s}$.

60 Prove that the tensor bundles introduced in Section 9 are smooth manifolds.

61 One can consider symmetric tensors in $T_p M$ as defined in Section 10. Define and describe $\Sigma^r(T_p M) \subset T^r(T_p M)$ and $\Sigma^r M \subset T^r M$.

62 Describe a smooth covariant 2-tensor field in $\Sigma^2 M \subset T^2 M$. How does this relate to an (indefinite) inner product on $T_p M$?

63 Prove Lemma 11.1.

64 Given the manifolds $N = M = \mathbb{R}^2$, and the smooth mapping

$$q = f(p) = (p_1^2 - p_2, p_1 + 2p_2^3),$$

acting from N to M. Consider the tensor spaces $T^2(T_p N) \cong T^2(T_q M) \cong T^2(\mathbb{R}^2)$ and compute the matrix for the pullback f^*.

65 In the above problem consider the 2-tensor field σ on M, given by

$$\sigma = dy^1 \otimes dy^2 + q_1 dy^2 \otimes dy^2.$$

(***i***) Show that $\sigma \in \mathcal{F}^2(M)$.

(***ii***) Compute the pulback $f^*\sigma$ and show that $f^*\sigma \in \mathcal{F}^2(N)$.

(***iii***) Compute $f^*\sigma(X,Y)$, where $X, Y \in \mathcal{F}(N)$.

Differential forms

66 Given the differential form $\sigma = dx^1 \wedge d^{x2} - dx^2 \wedge dx^3$ on $\mathbb{R}^3$. For each of the following vector fields X compute $i_X\sigma$:

(***i***) $X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}$;

(***ii***) $X = x_1 \frac{\partial}{\partial x_1} - x_2^2 \frac{\partial}{\partial x_3}$;

(***iii***) $X = x_1 x_2 \frac{\partial}{\partial x_1} - \sin(x_3) \frac{\partial}{\partial x_2}$;

67 Given the mapping

$$f(p) = (\sin(p_1 + p_2), p_1^2 - p_2),$$

acting from $\mathbb{R}^2$ to $\mathbb{R}^2$ and the 2-form $\sigma = p_1^2 dx^1 \wedge dx^2$. Compute the pullback form $f^*\sigma$.

68 Prove Lemma 12.2.

69 Derive Formula (10).

Orientations

70 Show that S^n is orientable and give an orientation.

71 Show that the standard n-torus $\mathbb{T}^n$ is orientable and find an orientation.

72 Prove that the Klein bottle and the projective space $P\mathbb{R}^2$ are non-orientable.

73 Give an orientation for the projective space $P\mathbb{R}^3$.

74 Prove that the projective spaces $P\mathbb{R}^n$ are orientable for $n = 2k+1$, $k \geq 0$.

75 Show that the projective spaces $P\mathbb{R}^n$ are non-orientable for $n = 2k$, $k \geq 1$.

Integration on manifolds

Integrating m-forms on $\mathbb{R}^m$

76 Let $U \subset \mathbb{R}^m$ be open and let $K \subset U$ be compact. Show that there exists a domain of integration D such that $K \subset D \subset \mathbb{R}^m$.

77 Show that Definition 14.2 is well-posed.

Partitions of unity

78 Show that the function f_1 defined in Lemma 15.3 is smooth.

79 If $\mathcal{U}$ is open covering of M for which each set U_i intersects only finitely many other sets in $\mathcal{U}$, prove that $\mathcal{U}$ is locally finite.

80 Give an example of uncountable open covering $\mathcal{U}$ of the interval $[0,1] \subset \mathbb{R}$, and a countable refinement $\mathcal{V}$ of $\mathcal{U}$.

Integration of m-forms on m-dimensional manifolds

81 Let $S^2 = \partial B^3 \subset \mathbb{R}^3$ oriented via the standard orientation of $\mathbb{R}^3$, and consider the 2-form

$$\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

Given the parametrization

$$F(\phi, \theta) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix},$$

for S^2, compute $\int_{S^2} \omega$.

82 Given the 2-form $\omega = x dy \wedge dz + z dy \wedge dx$, show that

$$\int_{S^2} \omega = 0.$$

83 Consider the circle $S^1 \subset \mathbb{R}^2$ parametrized by

$$F(\theta) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}.$$

Compute the integral over S^1 of the 1-form $\omega = x dy - y dx$.

84 If in the previous problem we consider the 1-form

$$\omega = \frac{x}{\sqrt{x^2+y^2}}dy - \frac{y}{\sqrt{x^2+y^2}}dx.$$

Show that $\int_{S^1}\omega$ represents the induced Euclidian length of S^1.

85 Consider the embedded torus

$$\mathbb{T}^2 = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \ : \ x_1^2+x_2^2 = 1, \quad x_3^2+x_4^2 = 1\}.$$

Compute the integral over $\mathbb{T}^2$ of the 2-form

$$\omega = x_1^2 dx_1 \wedge dx_4 + x_2 dx_3 \wedge dx_1.$$

86 Consider the following 3-manifold M parametrized by $g : [0,1]^3 \to \mathbb{R}^4$,

$$\begin{pmatrix} r \\ s \\ t \end{pmatrix} \mapsto \begin{pmatrix} r \\ s \\ t \\ (2r-t)^2 \end{pmatrix}.$$

Compute

$$\int_M x_2 dx_2 \wedge dx_3 \wedge dx_4 + 2x_1x_3 dx_1 \wedge dx_2 \wedge dx_3.$$

The exterior derivative

87 Let $(x,y,z) \in R^3$. Compute the exterior derivative $d\omega$, where ω is given as:

(***i***) $\omega = e^{xyz}$;

(***ii***) $\omega = x^2 + z\sin(y)$;

(***iii***) $\omega = xdx + ydy$;

(***iv***) $\omega = dx + xdy + (z^2 - x)dz$;

(***v***) $\omega = xydx \wedge dz + zdx \wedge dy$;

(***vi***) $\omega = dx \wedge dz$.

88 Which of the following forms on $\mathbb{R}^3$ are closed?

(***i***) $\omega = xdx \wedge dy \wedge dz$;

(***ii***) $\omega = zdy \wedge dx + xdy \wedge dz$;

(***iii***) $\omega = xdx + ydy$;

(***iv***) $\omega = zdx \wedge dz$.

89 Verify which of the following forms ω on $\mathbb{R}^2$ are exact, and if so write $\omega = d\sigma$:

(***i***) $\omega = xdy - ydx$;

(***ii***) $\omega = xdy + ydx$;

(***iii***) $\omega = dx \wedge dy$;

(***iv***) $\omega = (x^2 + y^3)dx \wedge dy$;

90 Verify which of the following forms ω on $\mathbb{R}^3$ are exact, and if so write $\omega = d\sigma$:

(***i***) $\omega = xdx + ydy + zdz$;

(***ii***) $\omega = x^2 dx \wedge dy + z^3 dx \wedge dz$;

(***iii***) $\omega = x^2 ydx \wedge dy \wedge dz$.

91 Verify that on $\mathbb{R}^2$ and $\mathbb{R}^3$ all closed k-forms, $k \geq 1$, are exact.

92 Find a 2-form on $\mathbb{R}^3 \backslash \{(0,0,0)\}$ which is closed but not exact.

Stokes' Theorem

93 Let $\Omega \subset \mathbb{R}^3$ be a parametrized 3-manifold,i.e. a solid, or 3-dimensional domain. Show that the standard volume of Ω is given by

$$\mathrm{Vol}(M) = \frac{1}{3}\int_{\partial\Omega} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

94 Let $\Omega \subset \mathbb{R}^n$ be an n-dimensional domain. Prove the analogue of the previous problem.

95 Prove the 'integration by parts' formula

$$\int_M f d\omega = \int_{\partial M} f\omega - \int_M df \wedge \omega,$$

where f is a smooth function and ω a k-form.

96 Compute the integral

$$\int_{S^2} x^2 ydx \wedge dz + x^3 dy \wedge dz + (z - 2x^2)dx \wedge dy,$$

where $S^2 \subset \mathbb{R}^3$ is the standard 2-sphere.

97 Use the standard polar coordinates for the $S^2 \subset \mathbb{R}^3$ with radius r to compute

$$\int_{S^2} \frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}},$$

and use the answer to compute the volume of the r-ball B_r.

Extra's

98 Use the examples in Section 18 to show that

$$\text{curl grad } f = 0, \qquad \text{and} \qquad \text{div curl } F = 0,$$

where $f : \mathbb{R}^3 \to \mathbb{R}$, and $F : \mathbb{R}^3 \to \mathbb{R}^3$.

99 Given the mapping $f : \mathbb{R}^2 \to \mathbb{R}^3$, $\omega = dx \wedge dz$, and

$$f(s,t) = \begin{pmatrix} \cos(s)\sin(t) \\ \sqrt{s^2+t^2} \\ st \end{pmatrix}.$$

Compute $f^*\omega$.

100 Given the mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$, $\omega = xydx \wedge dy \wedge dz$, and

$$f(s,t,u) = \begin{pmatrix} s\cos(t) \\ s\sin(t) \\ u \end{pmatrix}.$$

Compute $f^*\omega$.

101 Let $M = \mathbb{R}^3 \backslash \{(0,0,0)\}$, and define the following 2-form on M:

$$\omega = \frac{1}{(x^2+y^2+z^2)^{3/2}} \big(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy\big).$$

(***i***) Show that ω is closed ($d\omega = 0$) (Hint: compute $\int_S^2 \omega$).

(***ii***) Prove that ω is not exact on M!

On $N = \mathbb{R}^3 \backslash \{x = y = 0\} \supset M$ consider the 1-form:

$$\eta = \frac{-z}{(x^2+y^2+z^2)^{1/2}} \frac{xdy - ydx}{x^2+y^2}.$$

(***iii***) Show that ω is exact as 2-form on N, and verify that $d\eta = \omega$.

102 Let $M \subset \mathbb{R}^4$ be given by the parametrization

$$(s,t,u) \mapsto \begin{pmatrix} s \\ t+u \\ t \\ s-u \end{pmatrix}, \qquad (s,t,u) \in [0,1]^3.$$

(***i***) Compute $\int_M dx_1 \wedge dx_2 \wedge dx_4$.

(***ii***) Compute $\int_M x_1 x_3 dx_1 \wedge dx_2 \wedge dx_3 + x_3^2 dx_2 \wedge dx_3 \wedge dx_4$.

103 Let $C = \partial\Delta$ be the boundary of the triangle OAB in $\mathbb{R}^2$, where $O = (0,0)$, $A = (\pi/2, 0)$, and $B = (\pi/2, 1)$. Orient the traingle by traversing the boundary counter clock wise. Given the integral

$$\int_C (y - \sin(x))dx + \cos(y)dy.$$

(***i***) Compute the integral directly.

(***ii***) Compute the integral via Green's Theorem.

104 Compute the integral $\int_\Sigma F \cdot dS$, where $\Sigma = \partial[0,1]^3$, and

$$F(x,y,z) = \begin{pmatrix} 4xz \\ -y^2 \\ yz \end{pmatrix}$$

(Hint: use the Gauss Divergence Theorem).

De Rham cohomology

Definition of De Rham cohomology

105 Prove Theorem 19.3.

106 Let $M = \bigcup_j M_j$ be a (countable) disjoint union of smooth manifolds. Prove the isomorphism

$$H^k_{\mathrm{dR}}(M) \cong \prod_j H^k_{\mathrm{dR}}(M_j).$$

107 Show that the mapping $\cup : H^k_{\mathrm{dR}}(M) \times H^\ell_{\mathrm{dR}}(M) \to H^{k+\ell}_{\mathrm{dR}}(M)$, called the ***cup-product***, and defined by

$$[\omega] \cup [\eta] := [\omega \wedge \eta],$$

is well-defined.

108 Let $M = S^1$, show that

$$H^0_{\mathrm{dR}}(S^1) \cong \mathbb{R}, \quad H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}, \quad H^k_{\mathrm{dR}}(S^1) = 0,$$

for $k \geq 2$.

109 Show that

$$H^0_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{(0,0)\}) \cong \mathbb{R}, \quad H^1_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{(0,0)\}) \cong \mathbb{R}, \quad H^k_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{(0,0)\}) = 0,$$

for $k \geq 2$.

110 Find a generator for $H^1_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{(0,0)\})$.

111 Compute the de Rham cohomology of the n-torus $M = \mathbb{T}^n$.

112 Compute the de Rham cohomology of $M = S^2$.