Integration

formula

i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

ii) $\int dx = x + c$

iii) $\int \frac{1}{x} dx = \log|x| + c$

iv) $\int a^x dx = \frac{a^x}{\log 2} + c$

v) $\int e^x dx = e^x + c$

vi) $\int \sin x\, dx = -\cos x + c$

vii) $\int \cos x\, dx = \sin x + c$

viii) $\int \sec^2 x\, dx = \tan x + c$

ix) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

x) $\int \sec x . \tan x = \sec x + c$

xi) $\int \operatorname{cosec} x . \cot x\, dx = -\operatorname{cosec} x + c$

xi) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

xii) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

xiii) $\int \frac{1}{\sqrt{1-x^2}} = -\cos^{-1} x + c$

xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$

xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + c$

- $\int (u.v) dx = u \int v dx - \int \left(\frac{du}{dx} \cdot \int v dx \right) dx + c$

- $\int dx / x^2 + a^2 = \frac{1}{a} \tan^{-1} \left[\frac{x}{a} \right] + c$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x + \sqrt{x^2+a^2}| + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2-a^2}| + c$
- $\int \sqrt{x^2-a^2}\, dx = \frac{1}{2} x \sqrt{x^2-a^2} + \frac{a^2}{2} \log|x + \sqrt{x^2-a^2}| + c$
- $\int \sqrt{x^2+a^2}\, dx = \frac{1}{2} x \sqrt{x^2+a^2} + \frac{a^2}{2} \log|x + \sqrt{x^2+a^2}| + c$
- $\int \sqrt{a^2-x^2}\, dx = \frac{1}{2} x \sqrt{a^2-x^2} + \frac{a^3}{2} \sin^{-1} \frac{x}{a} + c$

UNIT - 4 Multiple Integral

→ Double integral

→ Triple integral

Q $\int_0^1\int_0^{x^2} e^{y/x}\,dy\,dx$

Sol. $= \int_0^1 \left[\int_{y=0}^{x^2} e^{y/x}\,dy\right] dx$

$= \int_0^1 \left[\dfrac{e^{y/x}}{1/x}\right]_0^{x^2} \cdot dx$

$= \int_0^1 [x.e^{y/x}]_0^{x^2}$

$= \int_0^1 [xe^x - xe^0]\, dx$

$= \int_0^1 (xe^x - x)\,dx$

$= \int_0^1 \overset{I}{x}\,(\overset{II}{e^x - 1})\,dx$

$= x\int(e^x-1)dx - \int_0^1 1\,(e^x - x)dx.$

$\int \mathrm{I}\int \mathrm{II} - \int \dfrac{d(\mathrm{I})}{dx}\int \mathrm{II}$

$= x[e^x - x]_0^1 - \left[e^x - \dfrac{x^2}{2}\right]_0^1$

$= e - 1 - [e - 1/2 - 1] = 1/2$ Ans

Q $\int_0^1\int_0^1 \dfrac{dy.dx}{\sqrt{(1-x^2)(1-y^2)}}$

Sol $I = \int_0^1 \dfrac{1}{\sqrt{1-x^2}}\left[\int_0^1 \dfrac{1}{\sqrt{1-y^2}}\,dy\right]\cdot dx$

$= \int_0^1 \dfrac{1}{\sqrt{1-x^2}}\left[\sin^{-1} y\right]_0^1 dx$

$= \int_0^1 \dfrac{1}{\sqrt{1-x^2}}\left(\dfrac{\pi}{2}\right)\cdot dx$

$= \dfrac{\pi}{2}\left[\sin^{-1} x\right]_0^1$

$= \dfrac{\pi}{2}\times\dfrac{\pi}{2} = \dfrac{\pi^2}{4}$ Ans

Q $\int_2^4 \int_{y=0}^{x} \int_{z=0}^{x+y} z \, dz\, dx\, dy$

Sol $\int_2^4 \int_{y=0}^{x} \left[\int_0^{x+y} z \, dz \right] dx \, dy$

$= \int_2^4 \int_0^x \left[\frac{z^2}{2} \right]_0^{x+y} dx \, dy$

$= \int_2^4 \int_0^x \left[\frac{(x+y)^2}{2} \right] dx \, dy$

$= \int_2^4 \frac{1}{2} \left[\int_0^x x^2 + y^2 + 2xy \, dy \right] dx$

$= \int_2^4 \frac{1}{2} \left[yx^2 + \frac{y^3}{3} + \frac{2xy^2}{2} \right]_0^x dx$

$\cancel{= \frac{1}{2} \int_2^4 \left[x^2 y + \frac{y^3}{3} + xy^2 \right] dx} = \int_2^4 \frac{1}{2} \left[x^3 + \frac{x^3}{3} + x^3 \right] dx$

$\cancel{= \frac{1}{2} \left[y\frac{x^3}{3} + \frac{y^3 x}{3} + \frac{y^2 x^2}{2} \right]_2^4}$

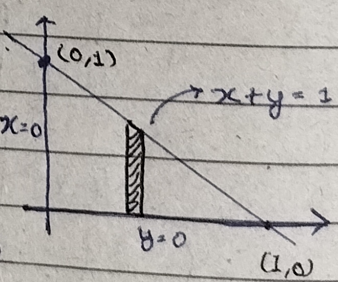
$= \frac{1}{2} \left[\frac{x^4}{4} + \frac{x^4}{3\times4} + \frac{x^4}{4} \right]_2^4 = \frac{x^4}{8} \left[2 + \frac{1}{3} \right]$

$= \frac{7}{3\times8} [x^4]_2^4$

$= \frac{7}{3\times8} [4^4 - 2^4] = 70$ Ans

Q $\iint_R x \cdot y \, dx \, dy$ over the region in positive quadrant for which $x + y \le 1$.

Sol consider a strip which is parallel to y axis & limit of y is 0 to $(1-x)$ then,



move the strip along the x-axis therefore limit of x is 0 to $(1-y)$

now,

$\iint_R x \cdot y \, dx \, dy = \int_0^1 \int_0^{1-x} xy \, dy \, dx$

$= \int_0^1 \left[\int_0^{1-x} xy \, dy \right] dx$

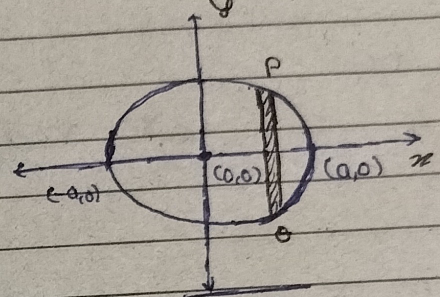
$=$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_0^{1-x} . dx$$

$$= \int_0^1 x (1-x)^2 \, dx$$

$$= 1/24 \quad \text{Ans.}$$

Q find $\iint_R (x+y)^2 \, dx \, dy$ over the region bounded by ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right)$



Sol. consider a strip which is parallel to y-axis and the limit of y is $+\frac{b}{a}\sqrt{a^2-x^2}$ to $-\frac{b}{a}\sqrt{a^2-x^2}$ then

$x^3 \Big] dx$

move the strip along x-axis, therefore limit of x is a to -a.

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

now,

$$\iint_R (x+y)^2 dx \cdot dy = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x+y)^2 \, dx \cdot dy$$

$\frac{\pi ab}{4}(x^2+y^2)$

$$= \int_{-a}^{a} \int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2 + y^2 + 2xy) \, dy \cdot dx$$

$$= \int_{-a}^{a} \left[x^2 y + \frac{y^3}{3} + \frac{2xy^2}{2} \right]_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} dx$$

quadrat

$$= \int_{-a}^{a} x^2 \left(\frac{2b\sqrt{a^2-x^2}}{a} \right) + \left(\frac{2b\sqrt{a^2-x^2}}{a} \right)^3 \Big/ 3 + 2x \left(\frac{2b}{a}\sqrt{a^2-x^2} \right)^2 dx$$

$$= \frac{2b}{a} \int_{-a}^{a} x^2 \sqrt{a^2-x^2} + \left(\frac{2b}{a} \right)^3 \Big/ 3 \int_{-a}^{a} (a^2-x^2)^3 \, dx + \left(\frac{2b}{a} \right)^2 \int_{-a}^{a} x(a^2-x^2) \, dx$$

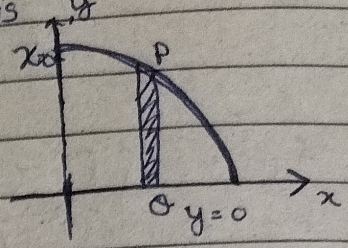
Q find $\iint_R x.y \, dx.dy$ where R is the quadrant of a circle $x^2+y^2=a^2$, $x, y \geq 0$

Sol. consider a strip parallel to y axis

& limit of y is 0 to $\sqrt{a^2-x^2}$

then, move strip along x-axis

therefore limit of x is 0 to a



$y \to 0$ to $\sqrt{a^2-x^2}$ & $x \to 0$ to a

now

$$\iint_R x.y \, dx dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} x.y \, dx dy$$

$$= \int_0^a x \left[\int_0^{\sqrt{a^2-x^2}} y \, dy \right] . dx$$

$$= \int_0^a x \left[\frac{y^2}{2} \right]_0^{\sqrt{a^2-x^2}} . dx$$

$$= \frac{1}{2} \int_0^a x (a^2-x^2) . dx$$

$$= \frac{1}{2} \int_0^a xa^2 - x^3 \, dx$$

$$= \frac{1}{2} \left(a^2 \left[\frac{x^2}{2} \right]_0^a - \left[\frac{x^4}{4} \right]_0^a \right)$$

$$= \frac{1}{2} \left(\frac{a^2}{2} (a^2-0) - \frac{1}{4} (a^4-0) \right)$$

$$= \frac{1}{2} \left(\frac{a^4}{2} - \frac{a^4}{4} \right)$$

$$= \frac{a^4}{8}$$ Ans

Q

Change of order of integration.

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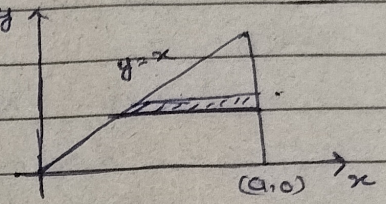
- $$\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy = \int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dy\,dx.$$

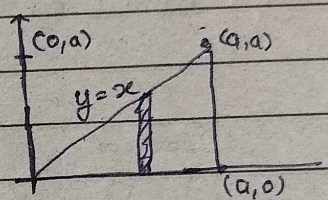
Q. $\int_0^a \int_y^a \frac{x}{(x^2+y^2)}\,dx\cdot dy$. chang the order of integration.

sol.

ATQ, given limit strip is parallel to x-axis which bounded by the line $x=y$ to a & y



by change in order of integratia strip is parallel to y & limit of y is 0 to x & limit of x is 0 to a.



now,

$$\int_0^a \int_y^a \frac{x}{(x^2+y^2)}\,dx\cdot dy = \int_0^a\int_0^x \frac{x}{x^2+y^2}\cdot dy\,dx.$$

$$= \int_0^a \left[\int_0^x \frac{x}{x^2+y^2}\cdot dy\right] dx$$

$\tan^{-1}\frac{y}{x}$

$\left(\frac{a\pi}{4}\right)$

$$= \int_0^a \left[\frac{1}{x}\int_0^x \frac{1}{1+(y/x)^2}\cdot dy\right]_0^x dx$$

$$= \int_0^a \frac{1}{x}\tan^{-1}\frac{y}{x}\,dx$$

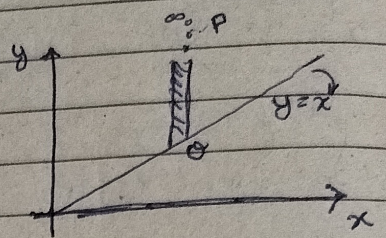
$$= \int_0^a \frac{\cancel{x}}{x}\frac{\pi}{4}\,dx$$

$$= \frac{\pi}{4}\int_0^a 1\,dx$$

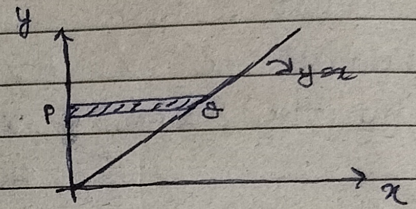
$$= \frac{a\pi}{4} \quad \text{Ans}$$

Q. The change in order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$.

Sol. AT Q, Given limit of y-axis is x to ∞ then, strip is $\parallel$ to y axis.



by change in order of integration strip is $\parallel$ to x-axis with limit 0 to y & strip is moving along y-axis then limit of y is 0 to ∞



$x = 0$ to y

$y = 0$ to ∞

now, $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \cdot dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y} \cdot \cancel{dy}\, dx\, \cancel{dy}$

(1)

$$= \int_0^\infty \left[\int_0^y \frac{e^{-y}}{y} \cdot dx \right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} [x]_0^y\, dy$$

$$= \int_0^\infty e^{-y} \cdot dy$$

$$= -[e^{-y}]_0^\infty$$

$$= -\left[\frac{1}{e^\infty} - \frac{1}{e^0} \right]$$

$$= 1 \quad \text{Ans}$$

Q find $\iint_A r \sin\theta \cdot dr \cdot d\theta$ over the area of cardiod $r = a(1+\cos\theta)$ above the intial line. $0 \le \theta \le \pi$

Sol. Symmetry : put $\theta = -\theta$, value r remain same then, curve is symmetric about intial line

pole & tangent : put $r = 0$

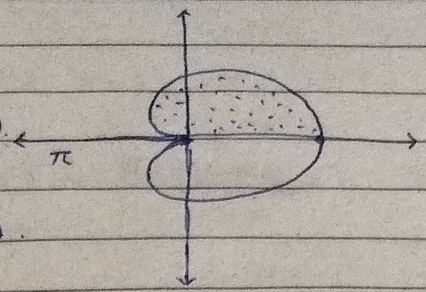
$0 = a(1+\cos\theta) \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

curve is passing through pole & π is tangent at pole.

Table : θ

r

now, $\iint_R r\sin\theta \, dr\, d\theta = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta \, dr \, d\theta$



$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$

$= \frac{1}{2}\int_0^{\pi} \sin\theta \, (a(1+\cos\theta))^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi} \sin\theta . a^2 (1+\cos\theta)^2 \, d\theta$

put $1+\cos\theta = t$ $\qquad \theta = \pi \rightarrow t = 0$

$-\sin\theta = \frac{dt}{d\theta}$ $\qquad \theta = 0 \rightarrow t = 2$

$= \frac{1}{2}\int_2^{0} \cancel{\sin\theta} . a^2 t^2 . \frac{dt}{-\cancel{\sin\theta}}$

$= \frac{a^2}{2}\int_0^{2} t^2 . dt$

$= \frac{a^2}{2}\left[\frac{t^3}{3}\right]_0^2$

$= \frac{a^2 [8]}{\cancel{6}} = \frac{4a^2}{3}$ sq^n unit.

Q $\int_0^{\pi}\int_0^{a(1+\cos\theta)} \sin\theta \, r^2 \, dr \, d\theta$.

Sol. $= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} . d\theta$

$= \frac{1}{3}\int_0^{\pi} \sin\theta \, a^3 (1+\cos\theta)^3 \, d\theta$

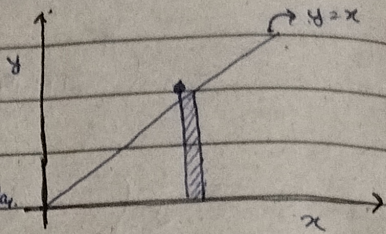
$= \frac{a^3}{3}\int_2^{0} \cancel{\sin\theta} \, t^3 . \frac{dt}{-\cancel{\sin\theta}}$

$= \frac{a^3}{\cancel{3}}\int_0^{2} t^3 dt$

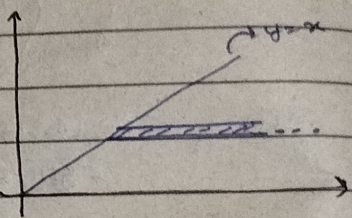
$= \frac{a^3}{\cancel{3}}\left[\frac{t^4}{4}\right]_0^2$

$= \frac{a^3}{3\times\cancel{4}}[\cancel{16}^{\,4}] = \frac{4a^3}{3}$ Ans θ find the value of integral $\int_0^\infty \int_0^x x e^{-x^2/y} \, dy \, dx$ by changing the order of integral.

Sol. A.T.θ, the given limit of y is 0 to x, then strip is ∥ to y-axis



By the chane in order of integration we draw strip ∥ to x-axis, then the limit of x is y to ∞ & strip is move along y-axis then, the limit of y is 0 to ∞.



now, $\int_0^\infty \int_0^x x e^{-x^2/y} dy \cdot dx = \int_0^\infty \int_y^\infty x e^{-x^2/y} dx \cdot dy$

put $x^2 = t$

$2x = \frac{dt}{dx}$, then

limit $x = y \to t = y^2$

$x = \infty \to t = \infty$

$= \int_0^\infty \left[\int_{y^2}^\infty x e^{-t/y} \cdot \frac{dt}{2x} \right] dy$

$= \frac{1}{2} \int_0^\infty \left[\int_{y^2}^\infty e^{-t/y} \cdot dt \right] dy$

$= \frac{1}{2} \int_0^\infty \left[-y e^{-t/y} \right]_{y^2}^\infty dy$

$= \frac{1}{2} \int_0^\infty \underset{\mathrm{I}}{y} \, \underset{\mathrm{II}}{e^{-y}} \, dy$

$= \frac{1}{2} \left[y \cdot -e^{-y} - \int e^{-y} dy \right]_0^\infty$

$= \frac{1}{2} \left[-y e^{-y} + e^{-y} \right]_0^\infty$

$= \frac{1}{2} \left[\frac{1}{e^y} - \frac{y}{e^y} \right]_0^\infty$

$= -\frac{1}{2} \left[\frac{1}{1} \right]$

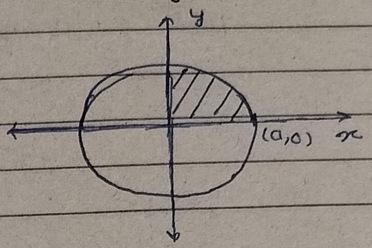
$= -\frac{1}{2}$ Ans

Volume using double & triple integration.

→ $V = \iint_D z \, dx\, dy$ → $V = \iiint_D dx\, dy\, dz$.

Q prove that area of circle of $\pi r^2 / \pi a^2$

Sol.



$A = \iint_D dx dy$

$A = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx$

$A = \int_0^a [\, y \,]_0^{\sqrt{a^2-x}} d\cancel{y}$

$A = \int_0^a \sqrt{a^2-x^2}\, dx$

$x \to 0$ to a

$y \to 0$ to $\sqrt{a^2-x^2}$

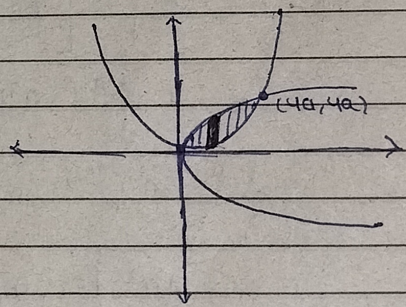
$A = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$

$A = 4\left[\left(0 + \frac{a^2}{2} \sin^{-1} 1 \right) - 0 \right] = 4\left[\frac{a^2}{2} \times \frac{\pi}{2} \right]$

$A = \pi a^2$ Ans

Q find the area of region by $\iint y^2 = 4ax$ & $x^2 = 4ay$.

Sol.



$A = \iint_D dx\, dy$

$A = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy \cdot dx$

$A = \int_0^{4a} [\, y \,]_{x^2/4a}^{2\sqrt{ax}} \cdot dx$

$A = \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] \cdot dx$

$x \to 0$ to $4a$

$y \to x^2/4a$ to $2\sqrt{ax}$

$A = \frac{1}{4a} \int_0^{4a} 8 a^{3/2} x^{1/2} - x^2 \, dx$

$= \frac{1}{4a} \left[8a^{3/2} \frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]_0^{4a} = \frac{1}{3 \times 4a} \left[16\sqrt{a}\, x^{3/2} - x^3 \right]_0^{4a}$

$\cancel{= \frac{1}{4a} \left[16 a^{3/2} 4^{3/2} a^{3/2} - \frac{4 \times 16\, a^{3}}{3} \right]} = \frac{1}{3 \times 4a} \left[16\sqrt{a}\, 4^{3/2} a^{3/2} - 4^3 a^3 \right.$

$A = \frac{16}{3} a^2$ Ans

Q find the area bonded by lemniscate $r^2 = a^2\cos 2\theta$ using double integration.

Sol i, symetry - ~~gra~~ curve is symmetric about initial axis.

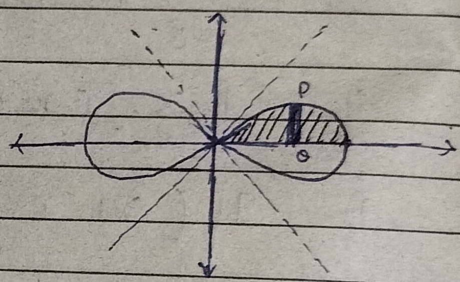
ii, pole & tangent - put $r=0$, then

$0 = a^2\cos 2\theta$

$\cos 2\theta = 0$

$2\theta = \pi/2 \Rightarrow \theta = \pi/4$

iii, table



now, $A = 4\iint r\,dr\,d\theta$

$\theta \to 0$ to $\pi/4$

$r \to 0$ to $a\sqrt{\cos 2\theta}$

$= 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r \, dr\, d\theta$

$= 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}}$

$= 4\int_0^{\pi/4} \frac{a^2\cos 2\theta}{2}\, d\theta$

$= 2a^2\int_0^{\pi/4}\cos 2\theta\, d\theta$

$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$

$= 2a^2\left[\frac{1}{2} - 0\right] = a^2$ Ans

Q find $\iiint_R (x-2y+z)\,dx\,dy\,dz$ where R is Rigion bounded $0 \le x \le 1$, & $0 \le y \le x^2$, $0 \le z \le x+y$.

Sol. $= \int_0^1\int_0^{x^2}\int_0^{x+y} x-2y+z \; dz\, dy\, dx$

$$= \int_0^1 \int_0^{x^2} \left[xz - 2yz + \frac{z^2}{2} \right]_0^{x+y} dy \cdot dx$$

$$= \int_0^1 \int_0^{x^2} \left[x^2 + xy - 2yx - 2y^2 + \frac{x^2+y^2+2xy}{2} \right] dy \cdot dx$$

$$= \int_0^1 \int_0^{x^2} \frac{1}{2} \left[2x^2 + 2xy - 4yx - 4y^2 + x^2 + y^2 + 2xy \right] dy\,dx$$

$$= \int_0^1 \int_0^{x^2} \frac{1}{2} \left[3x^2 - 3y^2 \right] dy\,dx$$

$$= \int_0^1 \frac{1}{2} \left[3x^2 y - \frac{3y^3}{3} \right]_0^{x^2} dx$$

$$= \int_0^1 \frac{1}{2} \left[3x^4 - x^6 \right] \cdot dx$$

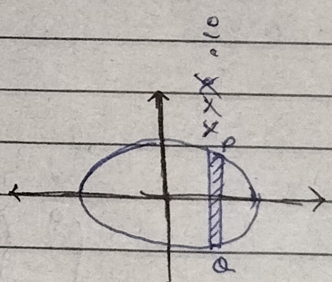
$$= \frac{1}{2} \left[\frac{3x^5}{5} - \frac{x^7}{7} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{3}{5} - \frac{1}{7} \right] = \frac{1}{2} \times \frac{21-5}{35} = \frac{16}{35 \times 2} = \frac{8}{35}$$ Ans

Q find the volume of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by using $\iint$ & $\iiint$.

Sol. Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b}}$$



now,

$$V = \iint_V z\,dx\,dy$$

$$V = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b}}\; dy \cdot dx$$

$x \to 0$ to a

$y \to 0$ to $b\sqrt{1-\frac{x^2}{a^2}}$

put $b\sqrt{1-\frac{x^2}{a^2}} = t$

$$b^2\left(1 - \frac{x^2}{a^2}\right) = t^2$$

$$V = 8c \int_0^a \int_0^t \sqrt{\frac{t^2}{b^2} - \frac{y^2}{b^2}}\; dy\,dx$$

eight quadrant.

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$$= \frac{8c}{b}\int_0^a \int_0^t \sqrt{t^2-y^2}\, dy\, dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y\sqrt{t^2-y^2}}{2} + \frac{t^2}{2}\sin^{-1}\left(\frac{y}{t}\right)\right]_0^t dx$$

$$= \frac{8c}{b}\int_0^a \left(\frac{t^2}{2}\sin^{-1}(1)\right) dx$$

$$= \frac{8c\pi}{4b}\int_0^a t^2\, dx$$

$$= \frac{8c\pi}{4b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right) dx$$

$$= \frac{8c\pi}{4b} \times \frac{b^2}{a^2}\int_0^a (a^2-x^2)\, dx$$

$$= \frac{2bc\pi}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{2bc\pi}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{2bc\pi}{3a^2} \times 2a^3$$

$$= \frac{4}{3}\pi abc \quad \text{Ans}$$

II method -

Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — i,

$$V = \iiint_R dx\, dy\, dz \quad \text{— ii}$$

$$V = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-y^2/b^2}} dx\, dy\, dz$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \left[z \right]_0^{c\sqrt{1-\frac{x^2}{a^2}-y^2/b^2}} dy\, dx$$

$$V = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\; dy \cdot dx$$

multiply by 8 to find full volume.

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Centre of a gravity using $\iint$ integration & $\iiint$

~~Sol~~. Double integration.

- mass (M) $= \iint_D f(x,y)\,dx\,dy$

$$= \iint_D \rho\,dx.dy. \quad \text{where } \rho \to f(x,y).$$

- centre of gravity

Let (x_c, y_c) be the coordinate of centre of gravity.

$$x_c = \frac{\iint_D x\rho\,dx\,dy}{\iint_D \rho\,dx\,dy}, \quad y_c = \frac{\iint_D y\rho\,dx\,dy}{\iint_D \rho\,dx\,dy}.$$

triple integration

- mass (M) $= \iiint_V \rho\,dx\,dy\,dz$

- centre of gravity

Let (x_c, y_c, z_c) be the coordinate of centroid.

$$x_c = \frac{\iiint_V x\rho\,dx.dy\,dz}{\iiint_V \rho\,dx\,dy\,dz} \qquad y_c = \frac{\iiint_V y\rho\,dx\,dy\,dz}{\iiint_V \rho\,dx\,dy\,dz}$$

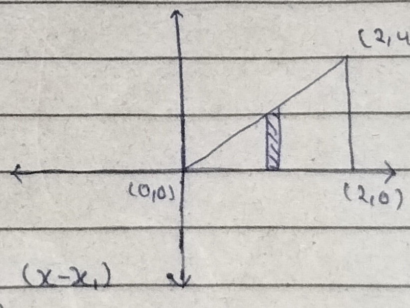
$$z_c = \frac{\iiint_V z\rho\,dx\,dy\,dz}{\iiint_V \rho\,dx\,dy\,dz}$$

Q A triangular thin plate with vertices (0,0), (2,0) & (2,4) has density $\rho = 1+x+y$ then find the mass of the plate & coordianate of centre of gravity.

Sol

mass (M) $= \iint_D \rho\,dx\,dy$

$$= \int_0^2\int_0^{2x} 1+x+y\,dx\,dy.$$



$$= \int_0^2 \left[y+xy+\frac{y^2}{2}\right]_0^{2x} dx$$

$$y-y_1 = \frac{y_1-y_2}{x_1-x_2}(x-x_1)$$

$$y = 2x$$

$x \to 0$ to 2

$y \to 0$ to $2x$

$$= \int_0^2 \left[2x+2x^2+\frac{4x^2}{2}\right]dx$$

$$= \int_0^2 2x + 4x^2\,dx$$

$$= \cancel{\left[\frac{2x^2}{2} + \frac{5x^3}{2\times 3} \right]_0^2} =$$

$$= \cancel{(4-0) + \frac{5}{6}(8-0)}$$

$$= \cancel{4 + \frac{20}{3} = \frac{32}{3}} \quad \cancel{\text{Ans}} \qquad \frac{44}{3} \text{ Ans}$$

Let (x_c, y_c) be the coordinate of centroid

$$x_c = \frac{\int_0^2\int_0^{2x} x(1+x+y)\,dx\,dy}{\int_0^2\int_0^{2x} (1+x+y)\,dx\,dy}$$

$$x_c = \frac{3}{44}\int_0^2\int_0^{2x} x(1+x+y)\,dx\,dy.$$

$$= \frac{3}{44}\int_0^2\int_0^{2x} x + x^2 + xy \;dy\,dx$$

$$= \frac{3}{44}\int_0^2 \left[xy + x^2y + \frac{xy^2}{2} \right]_0^{2x} dx$$

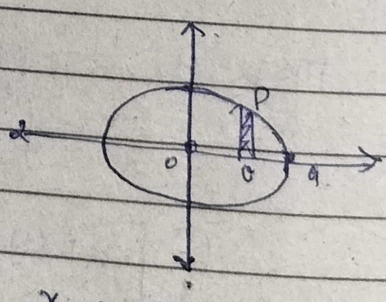
$$= \frac{3}{44}\int_0^2 2x^2 + 2x^3 + \frac{4x^3}{2}\,dx$$

$$= \frac{3}{44}\int_0^2 2x^2 + 4x^3\,dx$$

$$= \frac{3}{44}\left[\frac{2x^3}{3} + \frac{4x^4}{4} \right]_0^2$$

$$= \frac{3}{44}\left[\frac{2\times 8}{3} + 16 \right] = \frac{16}{44} = \frac{4}{11} \text{ Ans}$$

Q find the mass & centre of gravity of elliptic plate $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $\rho = 44xy$.



$x \to 0$ to a

$y \to 0$ to $b\sqrt{1 - \frac{x^2}{a^2}}$