$x^{m-1}(1-x)^{n-1}$

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Unit-3

Beta and Gamma Functions

Beta Function :-

Let $(m,n) > 0$ then beta of (m,n) is denoted by $B(m,n)$ n is given by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Properties of Beta Function

① $B(m,n) = B(n,m)$

② $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

③ $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Gamma Function

If $n > 0$ then

$$\Gamma n = \int_0^\infty e^{-t} t^{n-1}\,dt$$

$\boxed{x^{m-1}(1-x)}$

$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx$

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$\Gamma n = \int_0^1 (\log 1/y)^{n-1}$

PROPERTIES of GAMMA fun

① $\Gamma 1 = 1$

$\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin n\pi}$

② $\Gamma n+1 = n \, \Gamma n$

③ $\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx$

④ $\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$

⑤ $\Gamma n-1 = \int_0^\infty e^{-y^{1/n}} dy$

$\Gamma n+1 = n!$

Establish relationship b/w Beta and gamma fun

Or

Proove that $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

where $m, n > 0$

$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx$

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since we know that

$$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx$$

$$\Gamma n = \int_0^\infty e^{-zx} x^{n-1} z^n dx$$

Multiply both the sides by $e^{-z} z^{m-1}$

$$\Gamma n \, e^{-z} z^{m-1} = \int_0^\infty e^{-zx} x^{n-1} z^n e^{-z} z^{m-1} dx$$

$$= \int_0^\infty e^{-z(x+1)} x^{n-1} z^{m+n-1} dx$$

Now integrating both the sides wrt z in limit 0 to ∞

$$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1} \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz \right] dx$$

$$\Gamma n \, \Gamma m = \int_0^\infty x^{n-1} \frac{\Gamma m+n}{(1+x)^{m+n}} dx$$

$$\int_0^\infty e^{-xz} x^{n-1} dx = \frac{\Gamma n}{z^n}$$

$e^{-zx} x^{n-1} dx = \frac{\Gamma n}{x^n}$

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$$\Gamma n \, \Gamma m = \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} = B(m,n)$$

Q Proof that $\Gamma\frac{1}{2} = \sqrt{\pi}$

Hence we know that

$$\frac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)} = B(m,n) \quad -(1)$$

Put $m = n = \frac{1}{2}$ in eq (1)

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1}$$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \left(\Gamma\frac{1}{2}\right)^2$$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n)$$

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$$\int_0^1 x^{1/2-1}(1-x)^{1/2-1}\,dx = \left(\Gamma\tfrac{1}{2}\right)^2$$

$$\int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx = \left(\Gamma\tfrac{1}{2}\right)^2$$

$x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$.

L.L	U.L
$x=0$	$x=1$
$\theta=0$	$\theta=90$

$$2\int_0^{\pi/2} \frac{1}{\sqrt{\sin^2\theta}\sqrt{1-\sin^2\theta}}\cdot\sin\theta\cos\theta\,d\theta = \left(\Gamma\tfrac{1}{2}\right)^2$$

$$2\int_0^{\pi/2} \frac{1}{\cancel{\sin\theta}\,\cancel{\cos\theta}}\cdot\cancel{\sin\theta}\,\cancel{\cos\theta}\,d\theta = \left(\Gamma\tfrac{1}{2}\right)^2$$

$$2\int_0^{\pi/2}\theta = \left(\Gamma\tfrac{1}{2}\right)^2$$

$$\cancel{2}\times\frac{\pi}{\cancel{2}} = \left(\Gamma\tfrac{1}{2}\right)^2$$

$$\sqrt{\pi} = \Gamma\tfrac{1}{2}$$

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State and proof Legendre's Duplication formula

OR

$$\Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m) \quad \text{where } m > 0$$

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \rightarrow (1)$$

Putting $n = \frac{1}{2}$ in eq (1)

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(\tfrac{1}{2})}{2\,\Gamma(m+\tfrac{1}{2})}$$

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cancel{\cos} \, d\theta = \frac{\Gamma(m)\,\sqrt{\pi}}{2\,\Gamma(m+\tfrac{1}{2})} \quad (2)$$

Putting $n = m$ in eq (1)

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(m)}{2\,\Gamma(2m)}$$

Definite.

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$$\int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\int_0^{\pi/2} \left(\frac{\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta = \frac{(\Gamma m)^2}{2\,\Gamma 2m}$$

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1}\,d\theta = (\Gamma m)^2$$

Put $2\theta = \phi$

$2\,d\theta = d\phi$

LoL — $\theta = 0$, $\phi = 0$

UL — $\theta = \pi/2$, $\phi = \pi$

$$\frac{1}{2}\int_0^{\pi} (\sin\phi)^{2m-1}\,d\phi = \frac{2^{2m-1}(\Gamma m)^2}{2\,\Gamma 2m}$$

$$\frac{1}{2}\cdot 2\int_0^{\pi/2} (\sin\phi)^{2m-1}\,d\phi = \frac{2^{2m-1}(\Gamma m)^2}{2\,\Gamma 2m}$$

Replacing ϕ by θ.

$$\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta = \frac{2^{2m-1}(\Gamma m)^2}{2\,\Gamma 2m} \longrightarrow (3)$$

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On comparing eq^n 2nd & 3rd.

$$\frac{\cancel{\Gamma m}\,\Gamma \tfrac{1}{2}}{\cancel{2}\,\Gamma m+\tfrac{1}{2}} = \frac{2^{2m-1}(\Gamma m)^2}{\cancel{2}\,\Gamma 2m}$$

$$\frac{\Gamma 2m\,\Gamma \pi}{2^{2m-1}} = \Gamma m\,\Gamma m+\tfrac{1}{2}$$

Q Proof that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$ where $m > 0$.

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

Taking ~~RHS~~ ~~$\beta(m+n)$~~

$\beta(m+1,n) + \beta(m,n+1)$

$$\frac{\Gamma m+1\,\Gamma n}{\Gamma m+n+1} + \frac{\Gamma m\,\Gamma n+1}{\Gamma m+n+1} = \frac{\Gamma m+1\,\Gamma n + \Gamma m\,\Gamma n+1}{\Gamma m+n+1}$$

$$\Gamma n+1 = n\Gamma n$$

$$= \frac{m\Gamma m\Gamma n + \Gamma m\, n\Gamma n}{m+n\ \Gamma m+n}$$

$$\frac{x^8(1-x)^{14}}{x^{m-1}} = \frac{x^{14}(1-x)^8}{(1-x)^{n-1}}$$

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$$\frac{\Gamma m \, \Gamma n \, (m+n)}{(m+n) \, \Gamma m+n} = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

→ Answer

Q find $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1-x)^{24}} dx$

$$= \frac{x^8 - x^{14}}{(1-x)^{24}} = \frac{x^8 \, dx}{(1-x)^{24}} - \frac{x^{14}}{(1-x)^{24}} dx.$$

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \qquad \begin{matrix} m = 9 \\ n = 15 \end{matrix} \qquad \begin{matrix} m = 15 \\ n = 9 \end{matrix}$$

$$\cancel{\frac{x^8}{(1-x)^{24}} = \frac{x^{9-1}}{(1+(-x))^{9+15}}}$$

$$\int_0^\infty \frac{x^8}{(1-x)^{24}} dx - \int_0^\infty \frac{x^{14}}{(1-x)^{24}} dx$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \beta(m,n)$$

$$\int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}} dx$$

$$\beta(9,15) - \beta(15,9) \qquad \beta(m,n) = \beta(n,m)$$

0

Ans: $4/27$

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Q. Find $\int_0^\infty \sqrt{x}\, e^{-3\sqrt{x}}\, dx$

$\sqrt{x} = t$ $\quad$ $\cancel{x = t^2}$ $\quad$ $3\sqrt{x} = t$ $\quad$ $\sqrt{x} = \frac{t}{3}$

$x^{1/2} = t$ $\quad$ $\frac{3}{2\sqrt{x}} = dt$

$\frac{1}{2}\, \frac{1}{2\sqrt{x}} = dt$

$\cancel{\int_0^\infty \sqrt{x}\, e^{-3\sqrt{x}}\, dx = \int_0^\infty e^{-3t} \cdot t^{3-1} \cdot \frac{1}{2\sqrt{x}}} = \boxed{3}$ $\quad n = 3$

$e^{-t}\, t^{n-1}\, dt$

Q. $\int_0^\infty \sqrt{x}\, e^{-3\sqrt{x}}\, dx$ $\quad$ $3\sqrt{x} = t$ $\quad$ $\sqrt{x} = \frac{t}{3}$

$\int_0^\infty \frac{t}{3} e^{-t} \cdot \frac{2}{3}\sqrt{x}\, dt$ $\quad$ $\frac{3}{2\sqrt{x}} dx = dt$

$dx = \frac{2\sqrt{x}}{3} dt$

$n-1 = 2$

$n = 3$

$\frac{2}{27}\int_0^\infty t^2 e^{-t}\, dt$

$\frac{2}{27}\int_0^\infty t^{3-1} e^{-t}\, dt$

$\frac{2}{27}\boxed{3} = \frac{2 \times 2 \times 1}{27} = \frac{4}{27}$

~~$\frac{1}{3}e^{-t}dt$~~ t^{-3}

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Q. Solve $\int_0^\infty e^{-4x} x^{5/2} dx$ $\qquad \Gamma n = \int_0^\infty e^{-t} t^{n-1}$

$4x = t$
$4\,dx = dt$
$dx = \frac{1}{4} dt$
$x = \frac{t}{4}$

$$= \frac{1}{4}\int_0^\infty e^{-t} \cdot \left(\frac{t}{4}\right)^{5/2} \cdot dt$$

$$\frac{1}{4 \cdot (4)^{5/2}} \int_0^\infty e^{-t} (t)^{5/2} dt$$

$n - 1 = \frac{5}{2}$

$$\frac{1}{(4)^{7/2}} \int_0^\infty e^{-t} (t)^{7/2 - 1} dt$$

$n = \frac{5}{2} + 1 = \frac{7}{2}$

$$\frac{1}{4^{7/2}} \cdot \Gamma 7/2$$

$\frac{5}{2} + 1$

$\frac{5}{2} + 1 = \frac{5}{2} \Gamma 5/2$

$\frac{5}{2}!$

$$\frac{1}{4^{7/2}} \frac{5}{2} \Gamma 5/2 = \frac{1}{4^{7/2}} \cdot \frac{5}{2} \cdot \frac{3}{2} \Gamma 3/2$$

$\frac{5}{2} - 1 = 3/2$

$$= \frac{1}{4^{7/2}} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma 1/2$$

$\frac{7}{2} + 1$

$9/2$

$$= \frac{1}{4^{7/2}} \cdot \frac{15}{4 \cdot 2} \cdot \sqrt{\pi} = \frac{15}{4^{9/2}} \cdot \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \frac{15}{4^{7/2}} \cdot \frac{\sqrt{\pi}}{8}$$

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Q Proof that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

$x^2 = t \quad x = \sqrt{t}$

$2x\,dx = dt$

$dx = \frac{1}{2x} dt$

$$\int_0^\infty e^{-t} \cdot \frac{1}{2\sqrt{t}} dt$$

$$\frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

$n-1 = -\frac{1}{2}$

$n = -\frac{1}{2}+1$

$$\frac{1}{2}\int_0^\infty e^{-t}\, t^{1/2 - 1}\, dt$$

$$\frac{1}{2}\sqrt{1/2} = \frac{1}{2}\sqrt{\pi}$$

Q Express the integral $\int_0^1 x^m (1-x)^n$

$\int_0^1 x^m (1-x^n)^p \, dx$ in terms of Gamma fun. and hence find

$x \cdot x^2 \quad x^2 = t$

$t^{3/2}$

(1) $\int_0^1 x^3 (1-x^2)^4 dx$

(2) $\int_0^1 x^5 (1-x^3)^{10} dx$

$\frac{m}{n} - \frac{1}{n}$

$\frac{m}{n} + \frac{1}{n} - 1 \quad \frac{m+1}{n}$

$B(m,n) = \int_0^1 x^m (1-x)^{n-1}$

$\cancel{x}^{-1-1/n}$

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$B(\cancel{4},5)$ put $x^n = y$ $\quad x = y^{1/n}$ $\quad n(y)^{\frac{1}{n}(n-1)}$

$n x^{n-1}\, dx = dy$ $\quad y^{1-1/n}$

$dx = \frac{1}{n x^{n-1}} dy$

$= \frac{1}{n} y^{1/n - 1}\, dy$

$\int_0^1 (y^{1/n})^m (1-(y^{1/n})^n)^p \cdot \frac{1}{n} y^{1/n-1}$

$\frac{1}{n}\int_0^1 y^{m/n} (1-y)^p\, y^{1/n-1}\, dy$

$\frac{1}{n}\int_0^1 y^{m/n}(1-y)^p \cdot y^{1/n-1}$

$\frac{1}{n}\int_0^1 y^{\frac{m-n+1}{n}} (1-y)^p\, dy$

$\frac{1}{n}\int_0^1 y^{\frac{m+1}{n}-1} (1-y)^{(p+1)-1}$

$m = \frac{m+1}{n}$ $\quad n = p+1$

$m = \frac{3+1}{2} = \frac{4}{2} = 2$ $\quad n = 4+1$, $n = 5$

$\int_0^1 x^3 (1-x^2)^4$

$x^2 = t$

$2x\,dx = dt$

$dx = \frac{1}{2x} dt$

$x = \sqrt{t}$

$t^{3/2}$

$\int_0^1 t^{3/2} (1-t)^{5-1}$

$\int_0^1 t^{5/2-1} (1-t)^{5-1}$

$m = \frac{5}{2}$ $\quad n = 5$

$\frac{3}{2}+1$, $\frac{5}{2}$

$B(\frac{5}{2},5) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}$

$= \frac{\Gamma 5/2\, \Gamma 5}{\Gamma 15/2}$

$\frac{5}{2}+5$, $\frac{15}{2}$

$= \frac{\frac{3}{2}\Gamma 3/2 \cdot 4\times3\times2\times1}{\frac{13}{2}\Gamma 13/2}$

$n+1 = 5$, $n = 4$

$n+1 = \frac{15}{2}$

$n = \frac{15}{2} - 1$

$\frac{13}{2}$

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$$\frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right) = \frac{1}{n}\frac{\Gamma\frac{m+1}{n}\ \Gamma p+1}{\Gamma\frac{m+1}{n}+p+1}$$

$$\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot 4\cdot 3\cdot 2\cdot 1}{\frac{13}{2}\times\frac{11}{2}\times\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}$$

$$\frac{3\times 4\times 3\times 2\times 1\ \sqrt{\pi}\ \times 2^7}{13\times 11\times 9\times 7\times 5\times 3\times 1\times\sqrt{\pi}\times 2^2}$$

(1) $\int_0^\infty x^3 (1-x^2)^4 \, dx$

$m = \frac{m+1}{n}$ $n = p+1$

$m = \frac{4}{2} = 2$ $n = 5$

$$\frac{\Gamma 2\ \Gamma 5}{\Gamma 7} = \frac{\cancel{5}\times\cancel{5}\times 4\times 3\times 2}{\cancel{7}\times\frac{6}{3}\times 5\times 4\times 3\times 2} = \frac{1}{30}\times\frac{1}{2}$$

$$= \frac{1}{60}$$

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Q12 $\int_0^1 x^5 (1-x^3)^{10} dx$

$m = \frac{m+1}{n} = \frac{5+1}{3} = 2$

$n = 11$

$\frac{1}{3} \frac{\Gamma 2 \; \Gamma 11}{\Gamma 13}$

$\frac{1}{3} \times \frac{1 \times \cancel{10!}}{12 \times 11 \times \cancel{10!}}$

$= \frac{1}{396}$

Q Prove that

$\int_0^\infty \frac{x^c}{c^x} dx = \frac{\Gamma c+1}{(\log c)^{c+1}}$

$c^x = e^t$

$c^x \log c \, dx = e^t \, dt$

Q $\int_0^\infty \frac{x^c}{c^x} dx = \frac{\Gamma c+1}{(\log c)^{c+1}}, \; c > 1$

$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$I = \int_0^\infty \frac{x^c}{c^x} dx$

$c^x = e^t$

$\cancel{x \log c = t}$

$\log c^x = t$

$x = 0 \quad t = 0$

$x = \infty \quad t = \infty$

$x \log c = t$

$x = \frac{t}{\log c}$

$dx = \frac{dt}{\log c}$

$I = \int_0^\infty \left(\frac{t}{\log c}\right)^c e^t \cdot \frac{dt}{\log c}$

$\frac{1}{(\log c)^{c+1}} \int_0^\infty t^c \cdot e^{-t} \cdot$

$\frac{1}{(\log c)^{c+1}} \Gamma c+1 =$

$\frac{1}{2}$

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Q $\int_0^1 y^{q-1}\left(\log \frac{1}{y}\right)^{p-1} dy = \frac{\Gamma p}{q^p}$

$\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt$

$\Gamma p = \int_0^\infty e^{-x} x^{p-1} dx$

Put $x = qt$

$x \to 0 \; t \to 0 \quad dx = q\,dt$

$x \to \infty \; t \to \infty$

$\Gamma p = \int_0^\infty e^{-qt} (qt)^{p-1} dx$

$\Gamma p = \int_0^\infty e^{-qt} q^{p-1} t^{p-1} q\,dt$

$\Gamma p = \int_0^\infty e^{-qt} q^p t^{p-1} dt$

$\frac{\Gamma p}{q^p} = \int_0^\infty e^{-qt} t^{p-1} dt$

$t = \log \frac{1}{y}$

$e^t = \frac{1}{y}$

$y = e^{-t}$

$dy = -e^{-t} dt$

$\cancel{\frac{\Gamma p}{q^p}} = \int_0^\infty$

$e^t dy = -dt$

$\frac{1}{y} dy = -dt$

$t \to 0 \quad y \to 1$

$t \to \infty \quad y \to 0$

$\frac{\Gamma p}{q^p} \quad -\int_1^0 (e^{-t})^q \cdot \left(\log \frac{1}{y}\right)^{p-1} \frac{1}{y}$

$\int_0^1 y^{q-1} \log\left(\frac{1}{y}\right)^{p-1}$

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TRACING OF CURVE

① Cartesian Curve

② Polar Curve.

TRACING OF CARTESIAN CURVE,,

① WORK

Following Points are helpful while tracing of cartesian curve

$y = f(x)$

① Symmetry :- 1) A curve is symmetrical about the x-axis if all powers of y in the equation of curve are even $y^2 = 4ax$

b) A curve is symmetry about the y axis if all powers of ~~the~~ x in equation of curve is even $x^2 = 4ay$

c) A curve is symmetrical about the line $y = x$ if the equation of the given curve remains unchanged on the interchanging x & y.

For example - Circle $x^2 + y^2 = a^2$

d) If the eq^n of the curve remains unchanged when x is changed by $(-x)$ and y is changed by $(-y)$ then curve is symmetrical about the opposite quadrant

2) Origin & Tangent - i) If the point $(0,0)$ Satisfy the eq^n of the given curve then the curve passes through the Origin

If the curve passes through the Origin then check the nature of the Origin (node, cusp etc) for this we find the tangent at Origin by equating the lowest degree term to zero

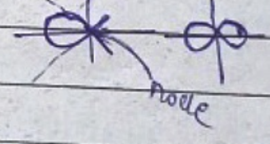
For example $x^2+y^2+x+y=0$

$x+y=0$

$\{y=-x \;,\; x=-y\}$ tangent

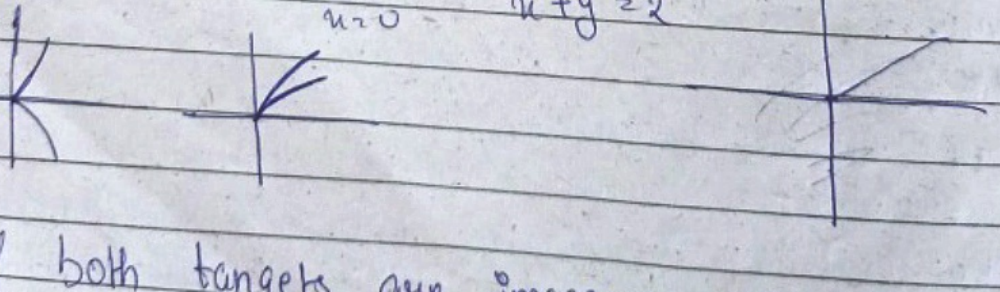
If we get two tangent at origin then we find its nature

a) if both the tangent are real & distinct the origin is called node



b) if both tangets are real & coincident then origin is called cust

for ex:- $x^2=0$ $x=0$ $x=0$ $x+y=2$



c) if both tangets are imaginery then Origin is called isolated point

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3> Point of intersection of curve with co-ordinate axis
Put $x=0$ & $y=0$ in the eq^n of curve, it gives point of intersection of the curve on y axis and x axis respectively.

4> Asymptotes
A straight line at a finite distance from origin to which the curve does not intersect it and touches it at infinity. It means that the asymptotes is the boundary line for the curve.

a) Asymptotes $\parallel^{lel}$ to x-axis.
equating the coefficient of highest power of x to zero in the given eq^n of curve.

b) Asymptotes $\parallel^{lel}$ to y-axis
equating the coefficient of highest power of y to zero in the given eq^n of curve.

c) It is not necessary that every curve has asymptotes, the curve $x^2+y^2=a^2$ has no asymptotes.

5> To find the reason of the curve we solve the equation of the curve for y or x whichever is convenient. Suppose we solve the equation of curve for y then we examine foll points
a) we find those value of x for which y tends to infinity
(b) We find the interval for x in which value of y become imaginary.

(6) find $\frac{dy}{dx}$ (i) if $\frac{dy}{dx} = 0$ for some value of x then the tangent ||lel to x axis

(c) $\frac{dy}{dx}$ = +ve tangent → inc (ii) if $\frac{dy}{dx} \to \infty$ (iv) tangent ||lel y axis

c) if find the interval of x in which the value of y is either increasing or decreasing

f) Trace the curve

$$y^2(a-x) = x^2(a+x)$$

Sol^n. $y^2a - xy^2 = x^2a + x^3$ $x^2a - yx^2 = y^2a + y^3$

1) Symmetry

a) Symmetrical to x axis i.e., power of y is even

(b) Not symmetrical to y-axis as power of x is not even

(c)

(d)

The curve is symmetry about x-axis

2) Origin & Tangent

The curve is passing through origin & equating the lowest degree term to zero

$y^2a = x^2a$ $a(x^2 - y^2) = 0$

$x^2 = y^2$

$x = y, -y$ (Real & distinct

$y = +x, -x$

Two Real & distinct tangent at Origin
Therefore Origin is node.

3) Point of Intersection on Coordinate axis

Put $x=0$ in eq ①
$y^2 a = 0$ $y^2 = 0$ $\boxed{y=0}$

Put $y=0$ in eq ①
$x^2(a+x) = 0$
$\cancel{x^2}$ $x = -a, 0$
So the points are $(0,0)$ $(-a,0)$

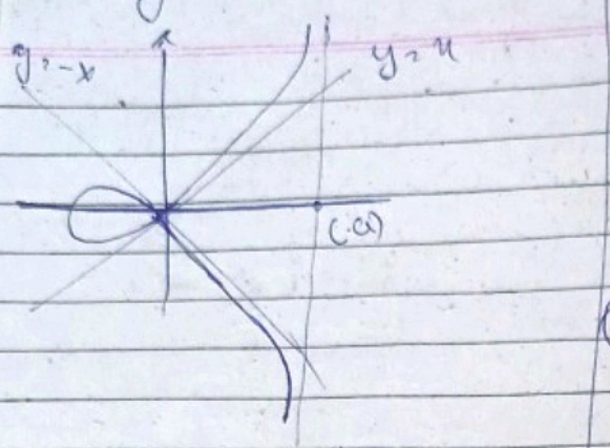
4) Asymptotes
asymptotes $\parallel^{el}$ to y axis
$y^2(a-x) = 0$
$a-x=0$
$\underline{\underline{x=a}}$

5) Region- Solving the given eq^n for y

$$y = \pm x\sqrt{\frac{a+x}{a-x}}$$

① y become imaginery when $x>a$
② y become imaginery when $x<-a$ for numerator
③ y exist between $-a<x<a$
④ For each x y has 2 equal & opp value
⑤ using about 5 points the approximate shape of the curve is

$a^2x^2 = y^2(a^2 - x^2)$



Q Trace the curve
$a^2y^2 = x^2(a^2 - x^2)$

① Symmetry
(1) along x-axis
(2) along y-axis
(3) opposi
The curve is symmetrical about x axis, y axis, opposite quadrant

② Origin & Tangent:
The curve passes through the origin & equating the lowest degree term to zero

$a^2y^2 = a^2x^2$

$a^2(y^2 - x^2) = 0$

$y^2 = x^2$

$y = \pm x$

Real & distinct

Here two real & distinct tangent ∴ origin is a node

③ Point of Intersection on Coordinate axis,

Put $x = 0$ in eq

$a^2y^2 = 0$

$y = 0$

Put $y = 0$ in eq

$0 = x^2a^2 - x^4$

$0 = x^2(a^2 - x^2)$

$x = 0 \quad a^2 - x^2 = 0$

$\boxed{\pm a = x}$

(0,0) (a,0) (-a,0)

hence point of intersection are
(0,0) (-a,0) (a,0)

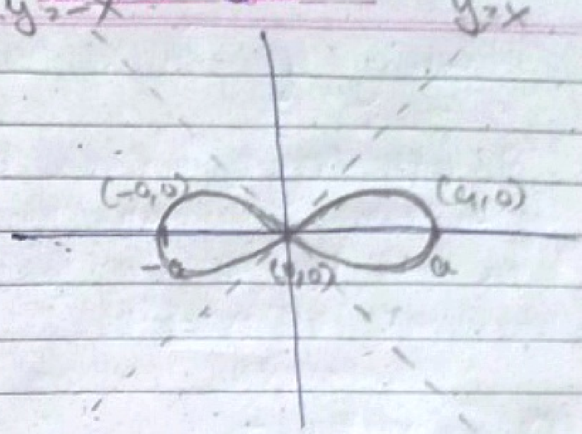
④ Asymptotes

highest degree term of x & y equate to zero

$0 = x^4 \quad a^2y^2 = 0$

$\boxed{x=0} \quad \boxed{y=0}$

here no asymptote exist



⑤ Reason

Solving eq ① for y

$$y = \pm\frac{x}{a}\sqrt{a^2-x^2} \longrightarrow ②$$

Observation

① y become imaginary when $x>a$, $x<-a$

for each x $(-a<x<a)$, y has two equal and opposite value.

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Tracing of Polar Curve

To find the approximate shape of the curve, represent by polar eqn $r = f(\theta)$ the foll procedure is adopted

① Symmetry :-

(i) when (θ) is replaced by $(-\theta)$ then equation of curve remain same, then the curve is symmetrical about the initial line i.e x-axis

(ii) when (θ) is replaced by $(\pi-\theta)$ then the equation of curve is unchanged therefore it is symmetrical about the line $\theta = \pi/2$ i.e y-axis

(iii) Again θ is replaced by $(-\theta)$ & (r) is replaced by $(-r)$, & eqn of curve is unchanged, therefore curve is symmetrical about the line $\theta = \pi/2$ i.e y axis

(iv) When (r) is replaced by $(-r)$, eqn is unchanged curve is symmetrical about the pole (origin)

② Pole & Tangent

(i) Put $r = 0$ in given eqn of curve then $\theta = \alpha$ (Alpha) is the tangent of the curve at pole

(3) Points on the Curve

Solve the eq^n of curve for r, prepare a table of corresponding value of θ and r then this table gives us the number of points on the curve. The shape of the curve is obtained by plotting the points.

θ	0	$\pi/2$	$\pi/6$	$\pi/4$	$\pi/3$	...	2π
$r = f(\theta)$							

(θ, r) = points on the curve.

(4) Region :-

The value of r comes out to be imaginary or negative in the region given by $\theta_1 < \theta < \theta_2$ then the curve will not exist in this region.

Remark - Sometimes it is convenient to change cartesian eq^n to polar eq^n or vise versa.

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Q) Trace the curve $r = a(1+\cos\theta)$

(1) Symmetry

(i) Put ~~θ~~ replace θ with $(-\theta)$ eq^n remains same
$r = a(1+\cos(-\theta)) = r = a(1+\cos\theta)$, therefor given curve
symmetrical along initial line i.e., x-axis

(ii)

$r = a(1+\cos(\pi-\theta))$
$= a(1-\cos\theta)$ eq^n changes

(iii) ~~r=~~ ~~θ~~

(2) Pole & Tangent

Put $r=0$ in eq (1)
$0 = a(1+\cos\theta)$
$\cos\theta = -1$
$\boxed{\theta = \pi}$

Curve passes through the pole & $\theta = \pi$ is the tangent of curve at pole

① (modcir)
press 9 times $\frac{1}{\sqrt{2}}$
press 1

$a\left(1+\frac{\sqrt{3}}{2}\right)$ $\frac{2+\sqrt{3}}{2}$ $\frac{a}{2}(2+\sqrt{3})$

 $1-0.866$ $1-\frac{1}{2}$ $\frac{1}{2}$

$1-0.866$

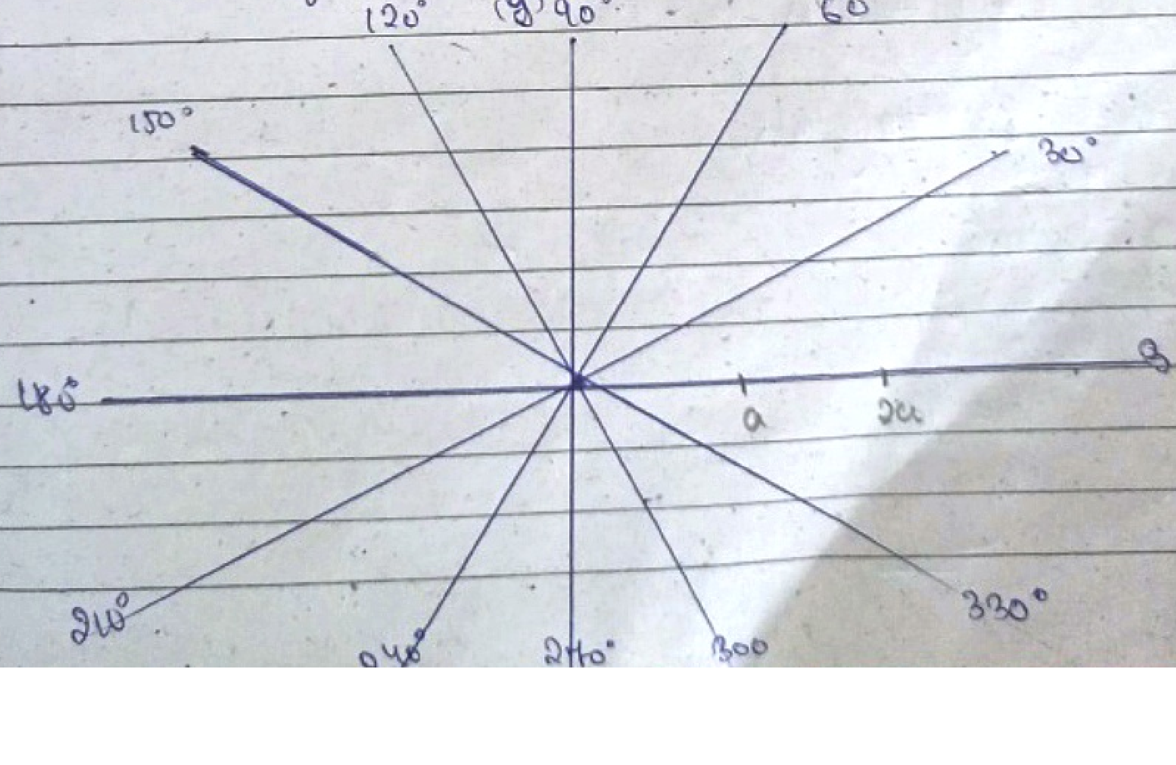
③ Point on the Curve

Preparing the table of corresponding value of θ in r

θ	0	30	~~45~~	60	90	120	150	180
$r=f(\theta)$ $r=a(1+\cos\theta)$	2a	1.86a		1.5a	a	0.5a	0.134a	0

210	240	270	300	330	360
0.134a	0.5a	a	1.5a	1.86a	2a

④ Curve exist for $0 \le \theta \le 2\pi$ so the approximate shape of the curve for



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Q) Trace the curve $r = 2a\cos\theta$

① Symmetry

(i) replace θ with $(-\theta)$ $r = 2a\cos(-\theta) = 2a\cos\theta$
eqn remains therefore it is symmetrical along x-axis

(ii) replace θ with $(\pi-\theta)$ $r = 2a\cos(\pi-\theta)$
$r = -2a\cos\theta$
eqn changes

(iii) replace $\theta = -\theta$ & $r = -r$
$-r = 2a\cos(-\theta)$ $= -r = 2a\cos\theta$
eqn changes

(iv) replace $r = -r$
eqn changes

② Poles & tangents,
Put $r = 0$
$0 = 2a\cos\theta$
$2a\cos\theta = 0$
$\cos\theta = 0$
$\theta = 90^\circ$
$\theta = \pi/2$

$\frac{1}{2}\times 2a$ $\frac{\sqrt{3}}{2}$ $1/2$ $-\frac{1}{2}$

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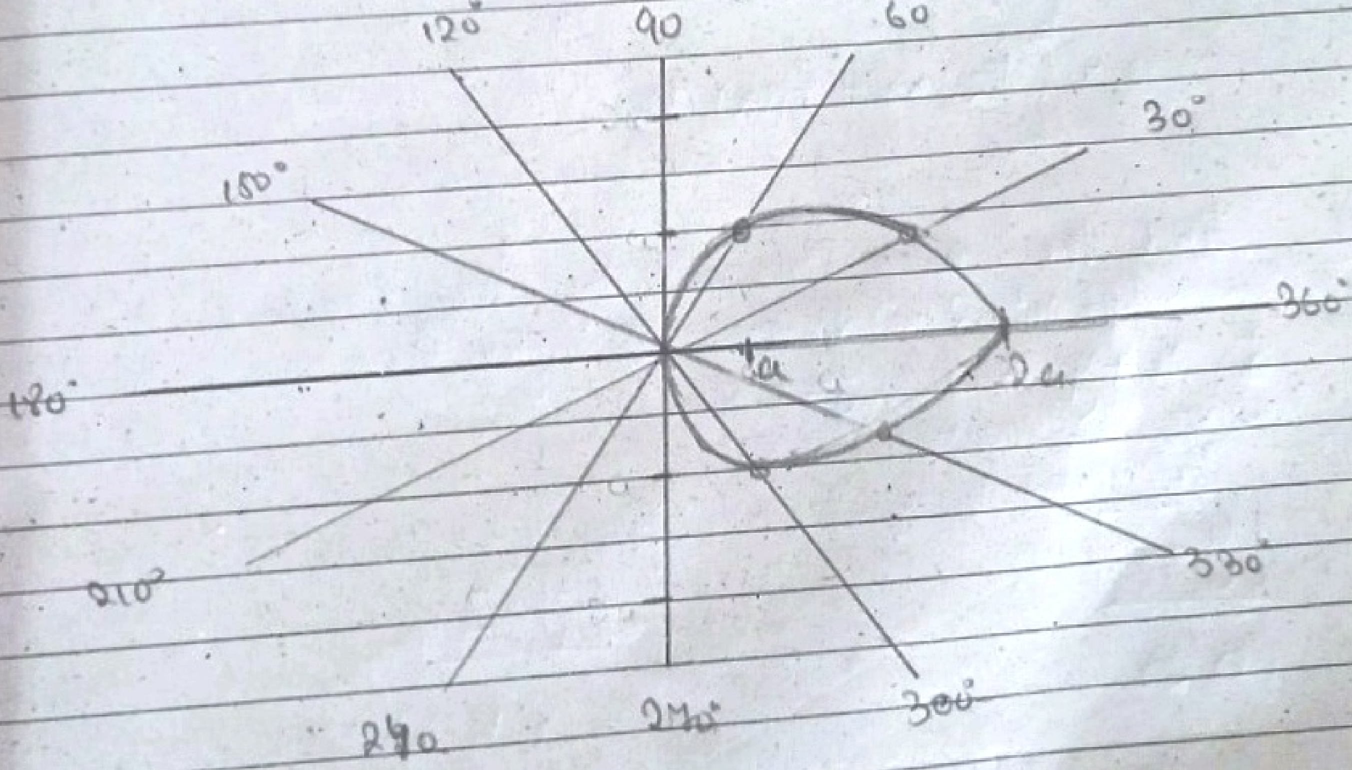
(3) Point on Curve.

θ	0	30	60	90	120	150	180
$r = 2a\cos\theta$	2a	$\sqrt{3}a$ 1.73a	a	0			

210	240	270	300	330	360
		0	a	1.73a	2a



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Area of Bounded Curve in cartesian Coordinates

$$A = \int_a^b f(x)\,dx$$

Area b/w two curves

Area enclosed b/w two curve $y_1 = f_1(x)$ & $y_2 = f_2(x)$ and the ordinates $x = a$ & $x = b$ is given by

$$A = \int_a^b (y_2 - y_1)\,dx$$

$$= \int_a^b f_2(x) - f_1(x)\,dx$$

Q Find the area bounded by the parabola $y = x^2 + 2$ — (1) & the straight line $x = 0$, $x =$ & $x + y = 0$ $y = -x$

POINT OF INTERSECTION,

Put $x = 0$ in eq (1)

$\boxed{y = 2}$

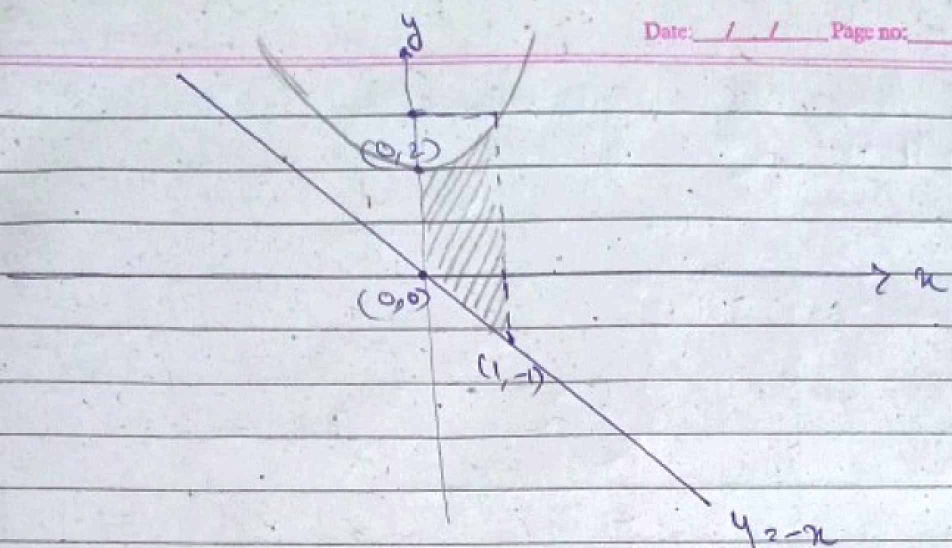
(0, 2)

Put $x = 1$ in eq (1)

$\boxed{y = 3}$

(1, 3)

Date: / / Page no: $y = x^2 + 2$



The parabola $x^2 = y - 2$ have vertex $(0,2)$ \$ axis along the y axis. The line $x = 1$ meet the parabola at $(1,3)$ \$ the line $x + y = 0$ at $(1,-1)$. The required area is

$$A = \int_a^b \{f_2(x) - f_1(x)\}\,dx$$

$$= \int_a^b \{x^2 + 2 - (-x)\}\,dx.$$

$$\int_0^1 (x^2 + x + 2)\,dx$$

$$\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 + \left[2x\right]_0^1$$

$$\frac{1}{3} - 0 + \frac{1}{2} + 2 = \frac{17}{6}$$

$\frac{1}{3} + \frac{1}{2} = \frac{5}{6} + \frac{2\times6}{1\times6}$

$an^2 - (n^3)$

Q Find the area of the loop of the curve
$ay^2 = x^2(a-x)$

By Curve tracing

① Symmetry
(i) even power of y symmetrical along x-axis

②

$ay^2 = x^2 a$
$a(y^2 - x^2) = 0$ $\boxed{y = \pm x.}$ $x = \pm y$

Real & distinct roots

$\therefore$ it is a node.

③ Point of intersection.
Put $x=0$ in eq ①
$ay^2 = 0$
$y = 0$
$(0, 0)$

② Put $y=0$ in eq ①
$x^2(a-x) = 0$
$x=0$ $x=a$
$(0,0)$, $(a, 0)$

④ Asymptotes
$ay^2 = x^3$
$ay^2 = (-1)^3$ No asymptotes
$ay^2 = -1$
$y =$

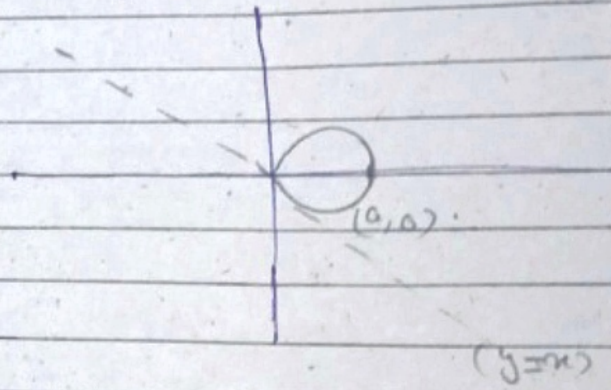
$$\boxed{y = \frac{x}{\sqrt{a}}\sqrt{a-x}.}$$

$a - x \geq 0$ $x > a$ Full area = half × 2

Date / / $\sqrt{a-x}$ $\frac{15}{2}$

$$y = \frac{x}{\sqrt{a}}\left(\sqrt{a-x}\right)$$

Imaginery value if $x > a$



$$\text{Area} = 2 \times \text{upper half of loop}$$

$$= 2 \times \int_0^a \frac{x}{\sqrt{a}}\left(\sqrt{a-x}\right) dx$$

$$= 2 \times \frac{1}{\sqrt{a}} \int_0^a x\sqrt{a-x}\, dx$$

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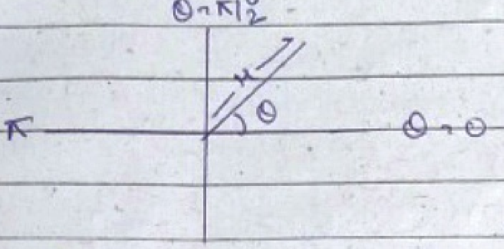
Area of Bounded Curve in Polar Coordinate



r = revolving line

x-axis → initial line.

$$\text{Area of Polar curve} = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta$$

Q Find the area of coordinate cardioide

$r = a(1+\cos\theta)$

using curve tracing

① $r = a(1+\cos(-\theta)) = a(1+\cos\theta)$

eqn remains same.

symmetrical about initial line

② Pole & Tangent

$a(1+\cos\theta) = 0$

$1+\cos\theta = 0$

$\theta = \pi$

The curve passes through pole at $\theta = \pi$

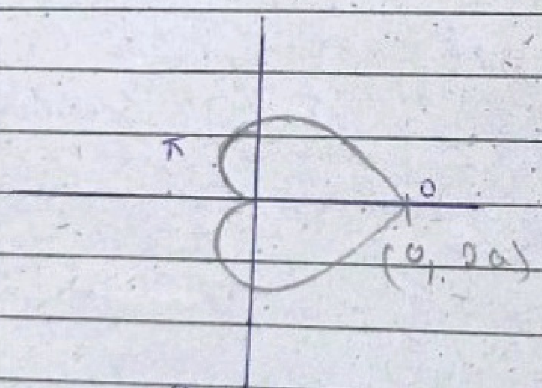
then tangent at the pole $\theta = \pi$

③ POINT OF INTERSECTION

θ	0	30	60	90	120	1
$r = a(1+\cos\theta)$ $a(1+1)$	$2a$			a		
	180	210	240	270	300	
	330	360				

Region

$0 \leq \theta \leq 2\pi$



$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$$

$\cancel{2a^2 \cos}$

$$\text{req } A = 2 \times \frac{1}{2} \int_0^{\pi} a^2(1+\cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} (1+\cos\theta)^2 \, d\theta$$

$$a^2 \int_0^{\pi} \left(2\cos^2 \theta/_2\right)^2 d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$1 + \cos 2\theta = 2\cos^2\theta$

$1 + \cos\theta = 2\cos^2 \theta/_2$

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$$4a^2 \int_0^{\pi} \cos^4 \frac{\theta}{2}\, d\theta$$

$\frac{\theta}{2} = \phi \qquad \theta = 2\phi$

$d\theta = 2\,d\phi$

$\theta = 0 \quad \phi = 0$
$\theta = \pi \quad \phi = \pi/2$

$$8a^2 \int_0^{\pi/2} \cos^4 \phi \, d\phi$$

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\ \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+1}{2}}$$

$$8a^2 \frac{\Gamma\frac{1}{2}\ \Gamma\frac{5}{2}}{2\,\Gamma 3}$$

$$= \frac{3}{2} a^2 \pi$$

$\pi \frac{a^2}{4}$

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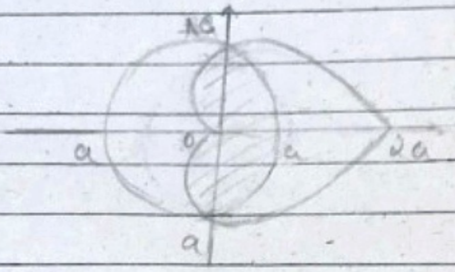
Q Find the area to the circle $r = a$ — ①

cos

$r = a(1+\cos\theta)$ — ②



$a = a(1+\cos\theta)$

$1 = 1+\cos\theta$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\text{Area} = \frac{1}{2}\int r^2 \, d\theta$

$A_1 = \frac{1}{2}\int_0^{\pi/2} a^2 \, d\theta$ —

$A_2 = \frac{a^2}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2 \, d\theta$

$\frac{a^2}{2}\int_{\pi/2}^{\pi} (1+\cos^2\theta + 2\cos\theta) \, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\frac{\cos 2\theta + 1}{2}$

$\frac{a^2}{2}\int_{\pi/2}^{\pi} \left(1 + \frac{\cos 2\theta + 1}{2} + 2\cos\theta\right) d\theta$

$\frac{a^2}{2}\int_{\pi/2}^{\pi} \left(\cancel{\frac{3}{2}}\, 1 + \frac{\cos 2\theta}{2} + \frac{1}{2} + 2\cos\theta\right) d\theta$

$\frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{4} + \frac{\theta}{2} + 2\sin\theta\right]_{\pi/2}^{\pi}$

$3\pi +$

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$$\frac{a^2}{2}\,\frac{a^2}{4}\left[3\theta + 4\sin\theta + \frac{\sin\theta}{2}\right]_{\pi/2}^{\pi}$$

$$\frac{a^3}{8}(2\pi - 8)$$

$$A = 2(A_1 + A_2)$$

$$\left(\frac{5}{4}\pi - 2\right)a^2$$

Length of Arc of curve.

① For Cartesian Coordinate

$$S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{where } x = g(y)$$

For Polar.

2) $r = f(\theta)$

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

$$S = \int_{r_1}^{r_2} \sqrt{1 + \left(\frac{r\,d\theta}{dr}\right)^2}\, dr \quad [\theta = f(r)]$$

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Q Find the length of arc of the curve (Semi cubical ~~pro~~ parabola) $ay^2 = x^3$ from the vertex $(0,0)$ to the point (a,a)

$$ay^2 = x^3 \quad \text{--- (1)}$$

$(0,0)$ (a,a)

$$ay^2 = x^3$$

$$y = \sqrt{\frac{x^3}{a}} = x\sqrt{\frac{x}{a}}$$

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$2\frac{dy}{dx} = \frac{3x^2}{a}$$

$$\frac{dy}{dx} = \frac{3}{2}\frac{x^2}{ay}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}\frac{x^4}{a^2y^2} = \frac{9}{4}\frac{x^{\cancel{4}}}{a x^{\cancel{3}}} = \frac{9}{4}\frac{x}{a}$$

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$\frac{1}{2}+1$
3

$$S = \int_0^a \sqrt{1+\frac{9x}{4a}}\, dx = \frac{1}{2\sqrt{a}}\int_0^a \sqrt{4a+9x}\, dx$$

$$\frac{1}{2\sqrt{a}}$$

$4a + 9a = t$

$\boxed{19\,dt}$

$\sqrt{13} - 9 \quad a\sqrt{a}$

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$\frac{3}{2} - \frac{1}{2} \quad \frac{2}{2} = 1$

$$\frac{1}{9 \times 2\sqrt{a}} \int_{4a}^{13a} t^{1/2}\,dt = \frac{1}{18\sqrt{a}} \left[\frac{t^{3/2}}{3} \times 2 \right]_{4a}^{13a}$$

$$\frac{1}{18\sqrt{a}} \quad \frac{1}{27\sqrt{a}} \quad \cancel{a^{3/2}}$$

$$\frac{(13)^{3/2}}{13\sqrt{13}} - 8 \qquad = \frac{1}{18} \cdot \frac{1}{27\sqrt{a}} \left[(13a)^{3/2} - (4a)^{3/2} \right]$$

$$= \frac{1}{27\sqrt{a}}\, a\sqrt{a}\,(13\sqrt{13} - 8)$$

$$S = \frac{a}{27}(13\sqrt{13} - 8)$$

Q Find the entire length of $r = a(1+\cos\theta)$ Show that the arc of the upper half is bisected by the line $\theta = \frac{\pi}{3}$

$\frac{1}{2}+1 \quad \frac{3}{2}$

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$r = a(1+\cos\theta) \qquad r^2 = a^2(1+\cos\theta)^2$

$$\frac{dr}{d\theta} = -a\sin\theta$$

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$$S = \int_{\theta_1}^{\theta_2} \sqrt{a^2(1+\cos\theta)^2 + (a\sin\theta)^2}\, d\theta$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta}\, d\theta$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{a^2 + 2a^2\cos\theta + a^2}\, d\theta$$

$$= \sqrt{2}a \int_{0}^{\pi} \sqrt{1+\cos\theta}$$

$\cos 2\theta = 2\cos^2\theta - 1$
$\cos 2\theta + 1 = 2\cos^2\theta$
$\cos\theta + 1 = 2\cos^2\theta/2$

$$= 2a \int_{0}^{\pi} \sqrt{\cos^2\theta/2}\, d\theta$$

$$= 2a \int_{0}^{\pi} \cos\theta/2\, d\theta$$

$$S = 2a\left[2\sin\theta/2\right]_0^{\pi} = \cancel{2a}\ \underline{\underline{4a}}$$

length of upper arc = $4a$

length of full arc = $2 \times 4a = \underline{\underline{8a}}$

$\frac{e^x+1-1-1}{e^x+1}$ $\frac{e^x+1}{}$ ~~$\frac{\pi}{3}$ $\frac{\pi}{2}$~~

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$1-\frac{2}{e^x+1}$

The arc of the length

The length of upper arc is 4a

Therefore

$$S_1 = \cancel{4}\cdot 2a\int_0^{\pi/3} \cos\theta/2 \, d\theta$$

$$2a\left[2\sin\theta/2\right]_0^{\pi/3} = \cancel{2a}\,\cancel{4}^{2}a\left[\frac{1}{\cancel{2}}\right] = \underline{\underline{2a}}$$

1

θ

$\theta/2$

Find the length of Curve

Q $y = \log\left[\frac{e^x-1}{e^x+1}\right]$ from $x=1$ to $x=2$

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{e^x+1}{e^x-1}\Big($$

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$$\frac{e^x-1}{e^x+1} = \frac{e^x+1-1-1}{e^x+1} = 1-\frac{2}{e^x+1} = 1-2(e^x+1)$$

$$2(e^x+1)^{-2}\cdot e^x = \frac{2e^x}{(e^x+1)^2}$$

$$\frac{dy}{dx} = \left(\frac{\cancel{e^x+1}}{e^x-1}\right)\frac{2e^x}{(e^x+1)^{\cancel{2}}} = \frac{2e^x}{(e^x)^2-(1)^2}$$

$$\frac{dy}{dx} = \frac{2e^x}{e^{2x}-1}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{4e^{2x}}{(e^{2x}-1)^2}$$

$$S = \int_1^2 \sqrt{1+\frac{4e^{2x}}{(e^{2x}-1)^2}}\,dx$$

$(e^{2x})^2$
$+(1)^2$
$+2e^{2x}$

$$S = \frac{1}{e^{2x}-1}\int_1^2 \sqrt{e^{4x}+1-2e^{2x}+4e^{2x}}\,dx$$

$$\frac{1}{e^{2x}-1}\int_1^2 \sqrt{e^{4x}+2e^{2x}+1}\,dx$$