Integration

formula

i) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

ii) $\int dx = x + c$

iii) $\int \frac{1}{x}\,dx = \log|x| + c$

iv) $\int a^x\,dx = \frac{a^x}{\log a} + c$

v) $\int e^x dx = e^x + c$

vi) $\int \sin x\,dx = -\cos x + c$

vii) $\int \cos x\,dx = \sin x + c$

viii) $\int \sec^2 x\,dx = \tan x + c$

ix) $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

x) $\int \sec x.\tan x = \sec x + c$

xi) $\int \operatorname{cosec} x.\cot x\,dx = -\operatorname{cosec} x + c$

xi) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$

xii) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$

xiii) $\int \frac{1}{\sqrt{1-x^2}} = -\cos^{-1}x + c$

xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c$

xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1}x + c$

- $\int (u.v)\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx + c$
- $\int dx/x^2+a^2 = \frac{1}{a}\tan^{-1}\left[\frac{x}{a}\right] + c$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left[\frac{x}{a}\right] + c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x+\sqrt{x^2+a^2}| + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + c$
- $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + c$
- $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + c$
- $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^3}{2}\sin^{-1}\frac{x}{a} + c$

UNIT - 4 Multiple Integral

→ Double integral
→ Triple integral.

Q $\int_0^1\int_0^{x^2} e^{y/x}.\,dy\,dx$

Sol. $= \int_0^1 \left[\int_{y=0}^{x^2} e^{y/x}.\,dy\right] dx$

$= \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} . dx$

$= \int_0^1 [x.\,e^{y/x}]_0^{x^2}$

$= \int_0^1 [xe^x - xe^0]\, dx$

$= \int_0^1 (xe^x - x)\,dx$

$= \int_0^1 \overset{I}{x}\,(\overset{II}{e^x - 1})\,dx$

$= x\int(e^x-1)dx - \int_0^1 1\,(e^x - x)\,dx.$

$\int_{I}\int II - \int \frac{d(I)}{dx}\int II$

$= x\,[e^x - x]_0^1 - \left[e^x - \frac{x^2}{2}\right]_0^1$

$= e - 1 - [e - 1/2 - 1] = 1/2$ Ans

Q $\int_0^1\int_0^1 \frac{dy.\,dx}{\sqrt{(1-x^2)(1-y^2)}}$

Sol $I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy\right] . dx$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\sin^{-1} y\right]_0^1 dx$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left(\frac{\pi}{2}\right) . dx$

$= \frac{\pi}{2}\left[\sin^{-1} x\right]_0^1$

$= \frac{\pi}{2} \times \frac{\pi}{2} = \frac{\pi^2}{4}$ Ans

Q $\int_2^4 \int_{y=0}^{x} \int_{z=0}^{x+y} z \cdot dz\, dx\, dy$

Sol $\int_2^4 \int_{y=0}^{x} \left[\int_0^{x+y} z \cdot dz \right] dx \cdot dy.$

$= \int_2^4 \int_0^x \left[\frac{z^2}{2} \right]_0^{x+y} dx \cdot dy$

$= \int_2^4 \int_0^x \left[\frac{(x+y)^2}{2} \right] \cdot dx \cdot dy.$

$= \int_2^4 \frac{1}{2} \left[\int_0^x x^2 + y^2 + 2xy \cdot dy \right] dx$

$= \int_2^4 \frac{1}{2} \left[y x^2 + \frac{y^3}{3} + \frac{2xy^2}{2} \right]_0^x dx$

$\cancel{= \frac{1}{2} \int_2^4 \left[x^2 y + \frac{y^3}{3} + x y^2 \right] dx} = \int_2^4 \frac{1}{2} \left[x^3 + \frac{x^3}{3} + x^3 \right] dx$

$\cancel{= \frac{1}{2} \left[y \frac{x^3}{3} + \frac{y^3 x}{3} + \frac{y^2 x^2}{2} \right]_2^4}$

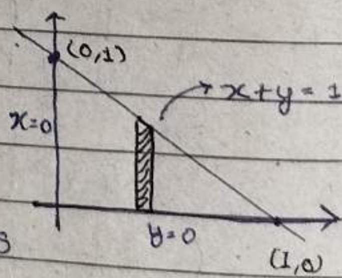
$= \frac{1}{2} \left[\frac{x^4}{4} + \frac{x^4}{3 \times 4} + \frac{x^4}{4} \right]_2^4 = \frac{x^4}{8} \left[2 + \frac{1}{3} \right]$

$= \frac{7}{3 \times 8} \left[x^4 \right]_2^4$

$= \frac{7}{3 \times 8} \left[4^4 - 2^4 \right] = 70$ Ans

Q $\iint_R x \cdot y\, dx \cdot dy$ over the region in positive quadrat for which $x + y \leq 1$.

Sol consider a strip which is parallel to y axis & limit of y is 0 to $(1-x)$ then,



move the strip along the x-axis therefore limit of x is 0 to $(1-y)$

now,
$\iint_R x \cdot y\, dx\, dy = \int_0^1 \int_0^{1-x} xy\, dy \cdot dx$

$= \int_0^1 \left[\int_0^{1-x} xy\, dy \right] \cdot dx$

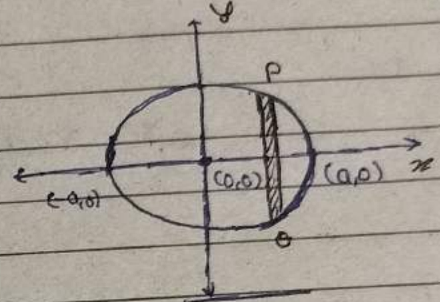
$=$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_0^{1-x} . dx$$

$$= \int_0^1 x (1-x)^2 dx$$

$$= 1/24 \quad Ans.$$

Q find $\iint_R (x+y)^2 dx dy$ over the region bounded by ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right)$



Sol. consider a strip which is parallel to y-axis and the limit of y is $+\frac{b}{a}\sqrt{a^2-x^2}$ to $-\frac{b}{a}\sqrt{a^2-x^2}$ then

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

move the strip along x-axis, therefore limit of x is a to -a.

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

$x^3 \Big] dx$

now, $$\iint_R (x+y)^2 dx \cdot dy = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x+y)^2 dx \cdot dy$$

$$= \int_{-a}^{a} \int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2 + y^2 + 2xy) dy \cdot dx$$

$\frac{\pi ab}{4}(x^2+y^2)$

$$= \int_{-a}^{a} \left[x^2 y + \frac{y^3}{3} + \frac{2xy^2}{2} \right]_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} dx$$

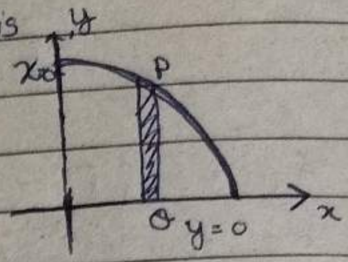
quadrat

$$= \int_{-a}^{a} x^2 \left(\frac{2b}{a}\sqrt{a^2-x^2}\right) + \left(\frac{2b}{a}\sqrt{a^2-x^2}\right)^3 / 3 + 2x \left(\frac{2b}{a}\sqrt{a^2-x^2}\right)^2 dx$$

$$= \frac{3b}{a}\int_{-a}^{a} x^2 \sqrt{a^2-x^2} + \left(\frac{2b}{a}\right)^3 \frac{1}{3} \int_{-a}^{a} (\sqrt{a^2-x^2})^3 dx + \left(\frac{2b}{a}\right)^2 \int_{-a}^{a} x(a^2-x^2) dx$$

Q find $\iint_R x \cdot y \, dx \cdot dy$ where R is the quadrant of a circle $x^2+y^2=a^2$, $x, y \geq 0$

Sol. consider a strip parallel to y axis & limit of y is 0 to $\sqrt{a^2-x^2}$ then, move strip along x-axis therefore limit of x is 0 to a



$y \rightarrow 0$ to $\sqrt{a^2-x^2}$ & $x \rightarrow 0$ to a

now

$$\iint_R x \cdot y \, dx dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} x \cdot y \, dx dy$$

$$= \int_0^a x \left[\int_0^{\sqrt{a^2-x^2}} y \, dy \right] \cdot dx$$

$$= \int_0^a x \left[\frac{y^2}{2} \right]_0^{\sqrt{a^2-x^2}} \cdot dx$$

$$= \frac{1}{2} \int_0^a x (a^2-x^2) \cdot dx$$

$$= \frac{1}{2} \int_0^a xa^2 - x^3 \, dx$$

$$= \frac{1}{2} \left(a^2 \left[\frac{x^2}{2} \right]_0^a - \left[\frac{x^4}{4} \right]_0^a \right)$$

$$= \frac{1}{2} \left(\frac{a^2}{2} (a^2-0) - \frac{1}{4} (a^4-0) \right)$$

$$= \frac{1}{2} \left(\frac{a^4}{2} - \frac{a^4}{4} \right)$$

$$= \frac{a^4}{8} \quad \underline{\text{Ans}}$$

Q

Change of order of integration.

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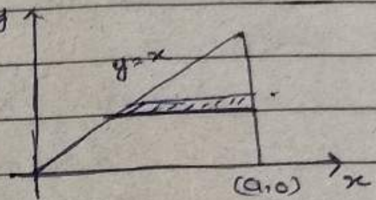
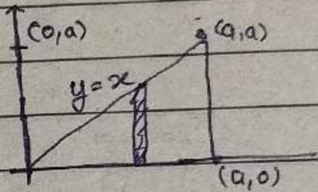
$$\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy = \int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dy\,dx.$$

Q. $\int_0^a \int_y^a \frac{x}{(x^2+y^2)}\,dx \cdot dy$. chang the order of integration.

sol.



AT Q, given limit strip is parallel to x-axis which bounded by the line $x=y$ to 0 & a



by change in order of integratia strip is parallel to y & limit of y is 0 to x & limit of x is 0 to a.

now,

$$\int_0^a \int_y^a \frac{x}{(x^2+y^2)}\,dx \cdot dy = \int_0^a\int_0^x \frac{x}{x^2+y^2}\cdot dy\,dx.$$

$$= \int_0^a \left[\int_0^x \frac{x}{x^2+y^2}\cdot dy \right] dx$$

$\frac{\tan^{-1} y}{x}$

$\left(\frac{a\pi}{4}\right)$

$$= \int_0^a \left[\frac{1}{x} \int_0^x \frac{1}{1+(y/x)^2}\cdot dy \right]_0^x dx$$

$$= \int_0^a \frac{1}{x} \tan^{-1}\frac{y}{x}\,dx$$

$$= \int_0^a \frac{\cancel{x}}{x}\frac{\pi}{4}\,dx$$

$$= \frac{\pi}{4}\int_0^a 1\,dx$$

$$= \frac{a\pi}{4} \quad \text{Ans}$$

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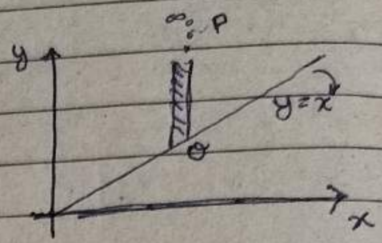
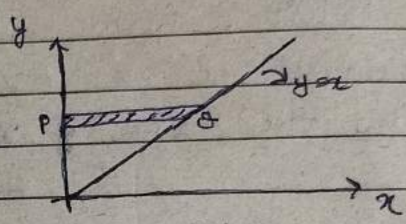
θ The change in order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$.

Sol. AT θ, Given limit of y-axis is x to ∞ then, strip is ∥ to y axis.



by change in order of integration strip is ∥ to x-axis with limit 0 to y & strip is moving along y-axis then limit of y is 0 to ∞



$x = 0$ to y

$y = 0$ to ∞

now, $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \cdot dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y} \cdot dx\, dy$

(1)

$$= \int_0^\infty \left[\int_0^y \frac{e^{-y}}{y} \cdot dx \right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} [x]_0^y \, dy$$

$$= \int_0^\infty e^{-y} \cdot dy$$

$$= -[e^{-y}]_0^\infty$$

$$= -\left[\frac{1}{e^\infty} - \frac{1}{e^0} \right]$$

$$= 1 \text{ Ans}$$

θ find $\iint_A r \sin\theta \cdot dr \cdot d\theta$ over the area of cardiod $r = a(1+\cos\theta)$ above the intial line. $0 \le \theta \le \pi$

Sol. Symmetry : put $\theta = -\theta$, value r remain same then, curve is symmetric about intial line

pole & tangent : put $r = 0$

$0 = a(1+\cos\theta) \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

curve is passing through pole & π is tangent at pole.

Table : θ

r

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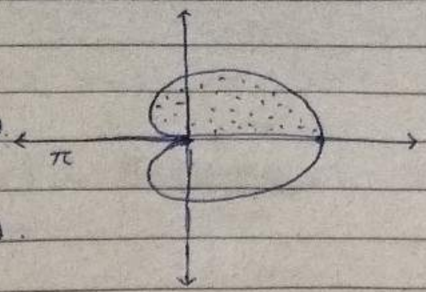
now, $\iint_R r\sin\theta \, dr\, d\theta = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta \, dr \, d\theta$



$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$

$= \frac{1}{2}\int_0^{\pi} \sin\theta \,(a(1+\cos\theta))^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi} \sin\theta \,.\, a^2(1+\cos\theta)^2 \, d\theta$

put $1+\cos\theta = t$ $\qquad \theta = \pi \rightarrow t = 0$

$-\sin\theta = \frac{dt}{d\theta}$ $\qquad \theta = 0 \rightarrow t = 2$

$= \frac{1}{2}\int_2^0 \cancel{\sin\theta} \,.\, a^2 t^2 . \frac{dt}{-\cancel{\sin\theta}}$

$= \frac{a^2}{2}\int_0^2 t^2 . dt$

$= \frac{a^2}{2}\left[\frac{t^3}{3}\right]_0^2$

$= \frac{a^2}{6}[8] = \frac{4a^2}{3}$ sq^n unit.

Q $\int_0^{\pi}\int_0^{a(1+\cos\theta)} \sin\theta \, r^2 \, dr \, d\theta$.

Sol. $= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} . d\theta$

$= \frac{1}{3}\int_0^{\pi} \sin\theta \, a^3 (1+\cos\theta)^3 \, d\theta$

$= \frac{a^3}{3}\int_2^0 \cancel{\sin\theta} \, t^3 . \frac{dt}{-\cancel{\sin\theta}}$

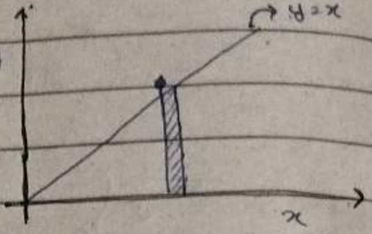
$= \frac{a^3}{3}\int_0^2 t^3 dt$

$= \frac{a^3}{3}\left[\frac{t^4}{4}\right]_0^2$

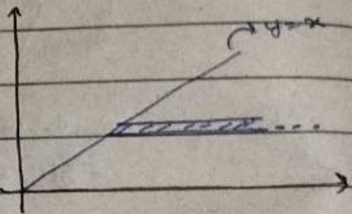
$= \frac{a^3}{3\times\cancel{4}}[\cancel{16}\,4] = \frac{4a^3}{3}$ Ans

Q find the value of integral $\int_0^\infty \int_0^x x e^{-x^2/y} \, dy \, dx$ by changing the order of integral.

Sol. A.T.Q, the given limit of y is 0 to x, then strip is ∥ to y-axis



By the chane in order of integration we draw strip ∥ to x-axis, then the limit of x is y to ∞ & strip is move along y-axis then, the limit of y is 0 to ∞.



now, $$\int_0^\infty \int_0^x x e^{-x^2/y} dy \cdot dx = \int_0^\infty \int_y^\infty x e^{-x^2/y} dx \cdot dy$$

put $x^2 = t$

$2x = \frac{dt}{dx}$, then

limit $x = y \rightarrow t = y^2$

$x = \infty \rightarrow t = \infty$

$$= \int_0^\infty \left[\int_{y^2}^\infty x e^{-t/y} \cdot \frac{dt}{2x} \right] dy$$

$$= \frac{1}{2} \int_0^\infty \left[\int_{y^2}^\infty e^{-t/y} \cdot dt \right] dy$$

$$= \frac{1}{2} \int_0^\infty \left[-y e^{-t/y} \right]_{y^2}^\infty dy$$

$$= \frac{1}{2} \int_0^\infty \underset{\mathrm{I}}{y} \, \underset{\mathrm{II}}{e^{-y}} \, dy$$

$$= \frac{1}{2} \left[y \cdot -e^{-y} - \int e^{-y} dy \right]_0^\infty$$

$$= \frac{1}{2} \left[-y e^{-y} + e^{-y} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{1}{e^y} - \frac{y}{e^y} \right]_0^\infty$$

$$= -\frac{1}{2} \left[\frac{1}{1} \right]$$

$$= -\frac{1}{2}$$ Ans

PAGE

dy dx by

$y=x$

x

$y=x$

$x \cdot dy$

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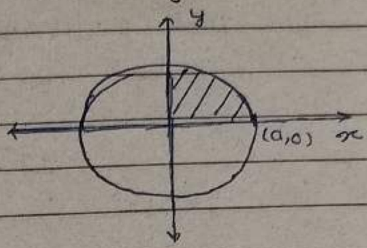
Volume using double & triple integration.

$\rightarrow \quad V = \iint_D Z \, dx \, dy \qquad \rightarrow \quad V = \iiint_D dx \, dy \, dz.$

Q prove that area of circle of $\pi r^2 / \pi a^2$

Sol.



$$A = \iint_D dx\,dy$$

$$A = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx$$

$$A = \int_0^a \left[y \right]_0^{\sqrt{a^2-x}} d\cancel{y}$$

$$A = \int_0^a \sqrt{a^2-x^2}\, dx$$

$x \rightarrow 0$ to a

$y \rightarrow 0$ to $\sqrt{a^2-x^2}$

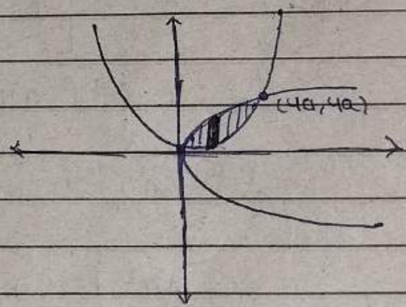
$$A = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$$

$$A = 4\left[\left(0 + \frac{a^2}{2}\sin^{-1} 1\right) - 0 \right] = 4\left[\frac{a^2}{2} \times \frac{\pi}{2} \right]$$

$$A = \pi a^2 \quad \text{Ans}$$

Q find the area of region by $\iint y^2 = 4ax$ & $x^2 = 4ay$.

Sol.



$$A = \iint_D dx\,dy$$

$$A = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy \cdot dx$$

$$A = \int_0^{4a} \left[y \right]_{x^2/4a}^{2\sqrt{ax}} \cdot dx$$

$$A = \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] \cdot dx$$

$x \rightarrow 0$ to $4a$

$y \rightarrow x^2/4a$ to $2\sqrt{ax}$

$$A = \frac{1}{4a} \int_0^{4a} 8a^{3/2}x^{1/2} - x^2 \; dx$$

$$= \frac{1}{4a}\left[8a^{3/2}\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]_0^{4a} = \frac{1}{3\times 4a}\left[16\sqrt{a}\,x^{3/2} - x^3 \right]_0^{4a}$$

$$\cancel{\frac{1}{4a}\left[16a^{3/2}\, 4^{3/2} a^{3/2} - \frac{4 \times 16 a^3}{3} \right]} = \frac{1}{3 \times 4a}\left[16\sqrt{a}\, a\, 4^{3/2} a^{3/2} - 4^3 a^3 \right.$$

$$A = \frac{16}{3} a^2 \quad \text{Ans}$$

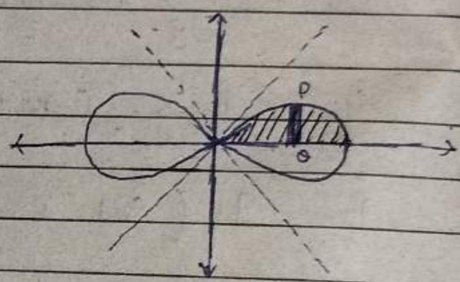
Q find the area bonded by lemniscate $r^2 = a^2\cos 2\theta$ using double integration.

Sol i, Symetry - ~~geo~~ curve is symmetric about initial axis.

ii, pole & tangent - put $r=0$, then

$$0 = a^2\cos 2\theta$$
$$\cos 2\theta = 0$$
$$2\theta = \pi/2 \Rightarrow \theta = \pi/4$$

iii, table



now, $A = 4\iint r\,dr\,d\theta$

$\theta \to 0$ to $\pi/4$

$r \to 0$ to $a\sqrt{\cos 2\theta}$

$$= 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}}$$

$$= 4\int_0^{\pi/4}\frac{a^2\cos 2\theta}{2}\,d\theta$$

$$= 2a^2\int_0^{\pi/4}\cos 2\theta\,d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= 2a^2\left[\frac{1}{2} - 0\right] = a^2 \text{ Ans}$$

Q find $\iiint_R (x-2y+z)\,dx\,dy\,dz$ where R is Rigion bounded $0 \le x \le 1$, & $0 \le y \le x^2$, $0 \le z \le x+y$.

Sol. $= \int_0^1\int_0^{x^2}\int_0^{x+y} x - 2y + z \; dz\,dy\,dx$

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$$= \int_0^1 \int_0^{x^2} \left[xz - 2yz + \frac{z^2}{2} \right]_0^{x+y} dy \cdot dx$$

$$= \int_0^1 \int_0^{x^2} \left[x^2 + xy - 2yx - 2y^2 + \frac{x^2+y^2+2xy}{2} \right] dy\, dx$$

$$= \int_0^1 \int_0^{x^2} \frac{1}{2} \left[2x^2 + 2xy - 4yx - 4y^2 + x^2 + y^2 + 2xy \right] dy\, dx$$

$$= \int_0^1 \int_0^{x^2} \frac{1}{2} \left[3x^2 - 3y^2 \right] dy\, dx$$

$$= \int_0^1 \frac{1}{2} \left[3x^2 y - \frac{3y^3}{3} \right]_0^{x^2} dx$$

$$= \int_0^1 \frac{1}{2} \left[3x^4 - x^6 \right] \cdot dx$$

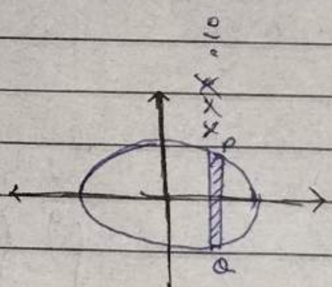
$$= \frac{1}{2} \left[\frac{3x^5}{5} - \frac{x^7}{7} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{3}{5} - \frac{1}{7} \right] = \frac{1}{2} \times \frac{21-5}{35} = \frac{16}{35 \times 2} = \frac{8}{35}$$

Q find the volume of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by using $\iint$ & $\iiint$.

Sol. Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b}}$$



now,

$$V = \iint_V z\, dx\, dy$$

$$V = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b}}\, dy \cdot dx$$

$x \to 0$ to a

$y \to 0$ to $b\sqrt{1-\frac{x^2}{a^2}}$

put $b\sqrt{1 - \frac{x^2}{a^2}} = t$

$$b^2\left(1 - \frac{x^2}{a^2}\right) = t^2$$

$$V = \frac{8c}{\quad} \int_0^a \int_0^t \sqrt{\frac{t^2}{b^2} - \frac{y^2}{b^2}}\, dy\, dx$$

eight quadrant.

$$= \frac{8c}{b}\int_0^a \int_0^t \sqrt{t^2-y^2}\, dy\, dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y\sqrt{t^2-y^2}}{2} + \frac{t^2}{2}\sin^{-1}\left(\frac{y}{t}\right)\right]_0^t dx$$

$$= \frac{8c}{b}\int_0^a \left(\frac{t^2}{2}\sin^{-1}(1)\right) dx$$

$$= \frac{8c\pi}{4b}\int_0^a t^2\, dx$$

$$= \frac{8c\pi}{4b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right) dx$$

$$= \frac{8c\pi}{4b} \times \frac{b^2}{a^2}\int_0^a (a^2-x^2)\, dx$$

$$= \frac{2bc\pi}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{2bc\pi}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{2bc\pi}{3\cancel{a^2}} \times 2\cancel{a^3}$$

$$= \frac{4}{3}\pi abc \quad \text{Ans}$$

II method -

Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — i,

$V = \iiint_R dx\, dy\, dz$ — ii

$$V = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-y^2/b^2}} dx\, dy\, dz$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \left[z \right]_0^{c\sqrt{1-\frac{x^2}{a^2}-y^2/b^2}} dy\, dx$$

$$V = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\; dy \cdot dx$$

multiply by 8 to find full volume

Centre of a gravity using $\iint$ integration & $\iiint$

Sol. Double integration.

- mass (M) $= \iint_D f(x,y)\,dx\,dy$

$= \iint_D \rho\,dx.dy$ where $\rho \to f(x,y)$.

- centre of gravity

Let (x_c, y_c) be the coordinate of centre of gravity.

$$x_c = \frac{\iint_D x\rho\,dx\,dy}{\iint_D \rho\,dx\,dy}, \quad y_c = \frac{\iint_D y\rho\,dx\,dy}{\iint_D \rho\,dx\,dy}$$

triple integration

- mass (M) $= \iiint_V \rho\,dx\,dy\,dz$

- centre of gravity

Let (x_c, y_c, z_c) be the coordinate of centroid.

$$x_c = \frac{\iiint_V x\rho\,dx.dy\,dz}{\iiint_V \rho\,dx\,dy\,dz} \qquad y_c = \frac{\iiint_V y\rho\,dx\,dy\,dz}{\iiint_V \rho\,dx\,dy\,dz}$$

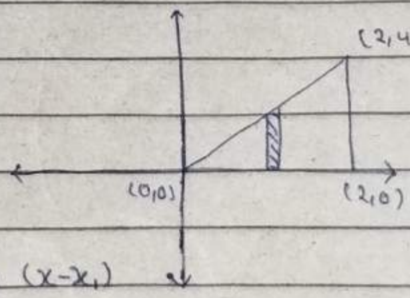
$$z_c = \frac{\iiint_V z\rho\,dx\,dy\,dz}{\iiint_V \rho\,dx\,dy\,dz}$$

Q A triangular thin plate with vertices (0,0), (2,0) & (2,4) has density $\rho = 1+x+y$ then find the mass of the plate & coordianate of centre of gravity.

Sol

mass (M) $= \iint_D \rho\,dx\,dy$

$= \int_0^2\int_0^{2x} 1+x+y\,dx\,dy$



$= \int_0^2 \left[y+xy+\frac{y^2}{2}\right]_0^{2x} dx$

$y - y_1 = \frac{y_1-y_2}{x_1-x_2}(x-x_1)$

$= \int_0^2 \left[2x+2x^2+\frac{\cancel{4}^{2}x^2}{\cancel{2}}\right] dx$

$x \to 0$ to 2

$y = 2x$ $y \to 0$ to $2x$

$= \int^2 2x + 4x^2\,dx$

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$$= \left[\frac{2x^2}{2} + \frac{5x^3}{2\times 3} \right]_0^2 =$$

$$= (4-0) + \frac{5}{6}(8-0)$$

$$= 4 + \frac{20}{3} = \cancel{\frac{32}{3}} \quad \cancel{\text{Ans}} \qquad \frac{44}{3} \text{ Ans}$$

Let (x_c, y_c) be the coordinate of centroid.

$$x_c = \frac{\int_0^2\int_0^{2x} x(1+x+y)\,dx\,dy}{\int_0^2\int_0^{2x} (1+x+y)\,dx\,dy}$$

$$x_c = \frac{3}{44}\int_0^2\int_0^{2x} x(1+x+y)\,dx\,dy.$$

$$= \frac{3}{44}\int_0^2\int_0^{2x} x + x^2 + xy \;dy\,dx$$

$$= \frac{3}{44}\int_0^2 \left[xy + x^2y + \frac{xy^2}{2} \right]_0^{2x} dx$$

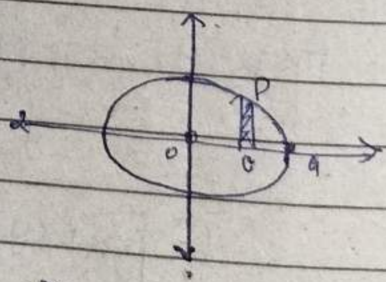
$$= \frac{3}{44}\int_0^2 2x^2 + 2x^3 + \frac{4x^3}{2}\;dx$$

$$= \frac{3}{44}\int_0^2 2x^2 + 4x^3\,dx$$

$$= \frac{3}{44}\left[\frac{2x^3}{3} + \frac{4x^4}{4} \right]_0^2$$

$$= \frac{3}{44}\left[\frac{2\times 8}{3} + 16 \right] = \frac{16}{44} = \frac{4}{11} \text{ Ans}$$

Q find the mass & centre of gravity of elliptic plate $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $\rho = 44xy$.



$x \to 0$ to a

$y \to 0$ to $b\sqrt{1 - \frac{x^2}{a^2}}$

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Q. if $r = (t^3 + t^2 + t)\,i + (t^2 + t)\,j + (t+1)\,k$ find $\frac{dr}{dt}$ & $\frac{d^2r}{dt^2}$

Sol. $\frac{dr}{dt} = (3t^2 + 2t)\,\hat{i} + (2t)\,\hat{j} + \hat{k}$

$\frac{d^2r}{dt^2} = (6t + 2)\,\hat{i} + 2\hat{j}$ Ans

Q A partical moving along. $x = 4\cos t$, $y = 4\sin t$, $z = 6t$ find valocity & acc at time $= 0$ & $t = \pi/2$.

Sol we know that

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 6t\hat{k}$

$\therefore \frac{d\vec{r}}{dt} = -4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 6\hat{k}$

put $t = 0$

$\left.\frac{d\vec{r}}{dt}\right]_{t=0} = 4\hat{j} + 6\hat{k}$

$v = \sqrt{16+36} =$

$v = 2\sqrt{13}$

& put $t = \pi/2$

$\left.\frac{d\vec{r}}{dt}\right]_{t=\pi/2} = -4\hat{i} + 6\hat{k}$

$v = \sqrt{16+36}$

$v = 2\sqrt{3}$

$\therefore \frac{d^2r}{dt^2} = -4\cos t\,\hat{i} + 4\sin t\,\hat{j}$

put $t = 0$

$\frac{d^2r}{dt^2} = -4\hat{i}$

$a = \sqrt{16}$

$a = 4$

Put $t = \pi/2$

$\frac{d^2r}{dt^2} = 4\hat{j}$

$a = \sqrt{16}$

$a = 4$

Q $r = (2x^2y - x^4)\,\hat{i} + (e^{xy} - y\sin x)\,\hat{j} + x^2\cos y\,\hat{k}$, then prove $\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial^2 r}{\partial y \partial x}$.

Sol $\frac{dr}{\partial y} = 2x^2\,\hat{i} + (e^{xy}\cdot x - \sin x)\,\hat{j} - x^2\sin y\,\hat{k}$

$\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial y}\right) = 4x\,\hat{i} + (ye^{xy}\cdot x + e^{xy} - \cos x)\,\hat{j} - 2\sin y\,x\,\hat{k}$

$\frac{\partial r}{\partial x} = (4xy - 4x^3)\hat{i} + (e^{xy}.y - y(\cos x))\hat{j} + 2(\cos y x)\hat{k}$

$\frac{\partial r}{\partial x} = (4xy - 4x^3)\hat{i} + (e^{xy}.y - y(\cos x))\hat{j} - 2x\sin y\,\hat{k}$

$\frac{\partial}{\partial y}\left(\frac{\partial r}{\partial x}\right) = 4x\hat{i} + (y e^{xy}.x + e^{xy} - \cos x)\hat{j}$

Vector differentiation operator

The operator ∇ (delta or nevla) is known as vector differentiation operator is defined as.

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Gradient of scalar point func.

Let f be a scalar point func then gradient of f is denoted by ∇f & it is defined as

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)f$$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Q find gradient of scalar point func $\phi(x,y,z) = x^2+y^2-z$ at the point (1,2,5).

Sol. Given $\phi = x^2+y^2-z$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-z)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2-z) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2-z) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2-z)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

put $(x,y,z) = (1,2,5)$, then

$$\nabla\phi = 2\hat{i} + 4\hat{j} - \hat{k}$$ Ans.

Q If $r = x\hat{i} + y\hat{j} + z\hat{k}$, then show gradient $\vec{r} = \hat{r}$

Sol. Given, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r^2 = x^2+y^2+z^2$. —(i)

$\because 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

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now, Gradient of $\gamma = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\gamma$

$$\nabla\gamma = \left(\hat{i}\frac{\partial\gamma}{\partial x} + \hat{j}\frac{\partial\gamma}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= \left(\frac{x}{\gamma}\hat{i} + \frac{y}{\gamma}\hat{j} + \frac{z}{\gamma}\hat{k}\right)$$

$$= \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\gamma}\right)$$

$\boxed{\nabla\gamma = \hat{\gamma}}$ H.P.

Q if $\vec{\gamma} = x\hat{i} + y\hat{j} + z\hat{k}$ show that gradient $\gamma^n = n\gamma^{n-2}\gamma$

Sol. Given, $\vec{\gamma} = x\hat{i} + y\hat{j} + z\hat{k}$.

~~partial diff of γ^n w.r.t x, y, z, then~~

$$\cancel{n\gamma^{n-1}}\left(\frac{\partial}{\partial x}(\gamma^n)\hat{i} + \frac{\partial}{\partial y}(\gamma^n)\hat{j} + \frac{\partial}{\partial z}(\gamma^n)\hat{k}\right.$$

$$= n\gamma^{n-1}\frac{\partial\gamma}{\partial x}\hat{i} + n\gamma^{n-1}\frac{\partial}{\partial y}(\gamma)\hat{j} + n\gamma^{n-1}\frac{\partial(\gamma)}{\partial z}\hat{k}$$

$$= n\gamma^{n-1}\frac{x}{\gamma}\hat{i} + n\gamma^{n-1}\frac{y}{\gamma}\hat{j} + n\gamma^{n-1}\frac{z}{\gamma}$$

$$= n\gamma^{n-2}x\hat{i} + n\gamma^{n-1}y\hat{j} + n\gamma^{n-2}z\hat{j}.$$

$$= n\gamma^{n-2}(x\hat{i} + y\hat{j} + z\hat{j})$$

$\boxed{\nabla\gamma^n = n\gamma^{n-2}\vec{\gamma}}$ H.P

$$\nabla\gamma^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\gamma^n$$

$$\nabla\gamma^n = \left(\frac{\partial(\gamma^n)}{\partial x}\hat{i} + \frac{\partial}{\partial y}(\gamma^n)\hat{j} + \frac{\partial}{\partial z}(\gamma^n)\hat{k}\right)$$

Q find directional derivative of $\phi = xy + yz + zx$ on the direction of the vector $\vec{n} = \hat{i} + 2\hat{j} + 2\hat{k}$ at the point $(1,2,$

Sol. Directional derivative = (gradient of ϕ at point) $\hat{n}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = (y+z)\hat{i} + (z+x)\hat{j} + (y+x)\hat{k}$$

$$\nabla\phi = 2\hat{i} + \hat{j} + 3\hat{k}$$

$$DD = 2\hat{i} + \hat{j} + 3\hat{k} \cdot \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

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$$DD = \frac{1}{3}(2+2+6) = \frac{10}{3}$$ Ans

- Directional Derivative

for any scalar point function at any point P(x,y) that is $x\hat{i} + y\hat{j} + z\hat{k}$ is the direction of $\vec{P}$ is defined as

$$\text{Directional Derivative} = (\text{gradiet at point P}) \cdot \frac{\vec{N}}{|\vec{N}|}$$

Q find the directional derivatēve of function $\phi = x^2 - y^2 + 2z^2$ at the point (1,2,3) in the direction of line $\vec{pq}$ when q has coordinate (5,0,4).

Sol
$$\phi = x^2 - y^2 + 2z^2$$
$$\vec{P} = \hat{i} + 2\hat{j} + 3\hat{k} \qquad \vec{q} = 5\hat{i} + 4\hat{k}$$
$$\vec{PQ} = \cancel{\hat{i} + 2\hat{j} + 3\hat{k}} \quad 4\hat{i} - 2\hat{j} + \hat{k}$$

and
$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$
$$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$
$$= 2\hat{i} - 4\hat{j} + 12\hat{k}$$

then,
$$D.D = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

$$D.D = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

- Divergence -

Divergence of a vector point func.
let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and vector point function defined in a certain field then

$$\nabla \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$\text{Div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$ which is a scalar point function called Divergence of f

when divergence of f = 0 the f is said to be solenoidal vector.

- Curl of a vector point function.

Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any vector point function of defined as a certain field that.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

If $\nabla \times F = 0$, then vector is irrotational.

Q Show that the vector $\nabla(x^3 + y^3 + z^3 - 3xyz)$ is irrotational also find divergence of function.

Sol. $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

$\vec{F} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= (-3x + 3x)\hat{i} - (3y - 3y)\hat{j} + (-3z + 3z)\hat{k}$$

$\text{curl } \vec{F} = 0$, then

$\vec{F}$ is irrotational.

and $$\text{Div } \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 6x + 6y + 6z$$ Ans

Q Prove that the vector $\vec{V} = 8y^2z^2\hat{i} + 4x^2z^2\hat{j} - 3x^2y^2\hat{k}$ is a solenoid vector. & find curl of $\vec{V}$

Sol. $$\text{Div } \vec{V} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 0 + 0 + 0$$

then, $\vec{V}$ is solenoid vector.

$$\text{curl } \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(3y^2z^2\hat{i} + 4x^2z^2\hat{j} - 3x^2y^2\hat{k}\right)$$

and

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y^2z^2 & 4x^2z^2 & 3x^2y^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(3x^2y^2) - \frac{\partial}{\partial z}(4x^2z^2)\right)\hat{i} - \left(\frac{\partial}{\partial x}(3x^2y^2) - \frac{\partial}{\partial z}(3y^2z^2)\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}(4x^2z^2) - \frac{\partial}{\partial y}(3y^2z^2)\right)\hat{k}$$

$$= (-6x^2y - 8zx^2)\hat{i} - (6xy^2 - 6y^2z)\hat{j} + (8xz^2 - 6yz^2)\hat{k}$$

Q prove that divergence of $\hat{r} = \frac{2}{r}$ & curl of $\hat{r} = 0$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol. Given, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

now,

$$\text{Div } \hat{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \hat{r}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \left(\frac{\sqrt{x^2+y^2+z^2} - x \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x}{x^2+y^2+z^2}\right) + \left(\frac{\sqrt{x^2+y^2+z^2} - y\,\frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x}{x^2+y^2+z^2}\right)$$

$$+ \left(\frac{\sqrt{x^2+y^2+z^2} - z \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2z}{x^2+y^2+z^2}\right)$$

$$= \frac{x^2+y^2+z^2 - x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2 - y^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2 - z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{y^2+z^2 + x^2+z^2 + x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2}{|\vec{r}|}$$ Hence, proved.

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and $\text{curl } \hat{r} = \nabla \times \hat{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial}{\partial z} f_2\right)\hat{i} - \left(\frac{\partial}{\partial x} f_3 - \frac{\partial f_1}{\partial z}\right)\hat{j} + \left(\frac{\partial}{\partial x} f_2 - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

$$\cancel{= \left(\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}} - \left(\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right)\right)\hat{i} - \left(\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}} - \left(\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right)\right)\hat{j}}$$

$$\cancel{+ \left(\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} - \left(\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right)\right)\hat{k}}$$

$$= \left(z \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2y - y \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2z\right)\hat{i} + \ldots$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= 0$$

$\therefore$ $\text{curl } \hat{r} = 0$ Hence, proved.