Unit :- 3

Partial Differential Equation

$Z = f(x, y)$ ☆ formation of P.D.E.

☆ Elimination of Arbitrary Constants

$\frac{\partial z}{\partial x} = p$

$\frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial x^2} = r$

$\frac{\partial^2 z}{\partial x \partial y} = s$

$\frac{\partial^2 z}{\partial y^2} = t$

Q1 Find a P.D.E. by eliminating arbitrary constants h and k from the relation

$(x-h)^2 + (y-k)^2 + z^2 = c^2$

Differentiating partially w.r.t x

$2(x-h) + 2z\frac{\partial z}{\partial x} = 0$

$\cancel{\frac{\partial z}{\partial x}} = 2(x-h) = -2zp$

$x-h = -zp$

Differentiating w.r.t y partially

$2(y-k) + 2z\frac{\partial z}{\partial y} = 0$

$y-k = -zq$.

$\therefore$ (1) gives

$(-zp)^2 + (zq)^2 + z^2 = c^2$

$z^2p^2 + z^2q^2 + z^2 = c^2$

$z^2(1+p^2+q^2) = c^2$

Q.2 $z = ax + by + ab$

$\frac{\partial z}{\partial x} = a$

$p = a$

$\frac{\partial z}{\partial y} = b$

$b = q$

$\boxed{z = px + qy + pq}$

Q. 3. $z = A e^{-p^2 t} \cos px$

$\frac{\partial z}{\partial x} = -pA e^{-p^2 t} \sin px$

$\frac{\partial^2 z}{\partial x^2} = -p^2 A e^{-p^2 t} \cos px$

$\frac{\partial^2 z}{\partial x^2} = -p^2 z$

$p^2 = -\frac{1}{z} \cdot \frac{\partial^2 z}{\partial x^2}$

$\frac{\partial z}{\partial t} = -p^2 A \cos px \, e^{-p^2 t}$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}}$$

Q. $z = f(x+iy) + F(x-iy)$

$\frac{\partial z}{\partial x} = f'(x+iy) + F'(x-iy)$

$\frac{\partial z}{\partial y} = f'(x+iy)\, i - F'(x-iy)\, i$

$\frac{\partial z}{\partial y} = i\left(f'(x+iy) - F'(x-iy)\right)$

$\frac{\partial^2 z}{\partial y^2} = f''(x+iy) + F''(x-iy)$

$\frac{\partial^2 z}{\partial y^2} = -f''(x+iy) - F''(x-iy)$

$\cancel{r + t = 2F''(x-iy)}$

$\cancel{\frac{r+t}{2} = f''(x+iy)}$

$\boxed{r + t = 0}$

Q. ✭ <u>Elimination</u> of <u>Arbitrary function</u>

form the P.D.E. by eliminating arbitrary function 'f' from the relation

$z = f\left(\frac{y}{x}\right)$

$p = \frac{\partial z}{\partial x} = -f'\left(\frac{y}{x}\right) \cdot \frac{y}{x^2}$

$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$

$-\frac{px^2}{y} = qx$

$-px = qy$

$px + qy = 0$

Q. Construct a P.D.E.

$f(x^2+y^2+z^2, z^2-2xy) = 0$

let, $x^2+y^2+z^2 = u$

$z^2 - 2xy = v$

$\phi(u, v) = 0$

Differentiating w.r.t u and v

$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}\right)$

$+ \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x}\right)$

$$\frac{\partial f}{\partial u}\left(2x\cdot 1+2y\cdot 0+2z\cdot p\right)+\frac{\partial f}{\partial v}\left(-2y+(-2x)0+2z\cdot p\right]=0$$

$$\frac{\partial f}{\partial u}\left(2x+2zp\right)+\frac{\partial f}{\partial v}\left(-2y+2zp\right)=0$$

$$\frac{\partial f}{\partial u}\{2x+2zp)=\frac{\partial f}{\partial v}(2y-2zp)$$

$$\therefore\ \frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}}=\frac{y-zp}{x+zp}$$

Differentiating w.r.t. y

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial y}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right.=0$$

$$\frac{\partial f}{\partial u}\left(0+2y\cdot+2z\cdot q\right)+\frac{\partial f}{\partial v}\left(-2y(0)+(-2x)+2z\cdot q\right)=0$$

$$\frac{\partial f}{\partial u}\left(2y+2zq\right)=\frac{\partial f}{\partial v}\left(2x-2zq\right)$$

$$\frac{\partial f}{\partial u}(y+zq)=\frac{\partial f}{\partial v}(x-zq)$$

$$\frac{\partial f/\partial u}{\partial f/\partial v}=\frac{x-zq}{y+zq}$$

$$\frac{y-zp}{x+zp}=\frac{x-zq}{y+zq}$$

$$(y-zp)(y+zq)=(x-zq)(x+zp)$$

$$\cancel{y^2-z^2p^2=x^2-z^2p^2}$$

$$(y-zp)(y+zq) = (x-zq)(x+zp)$$

$$y^2 + yzq - zpy - \cancel{z^2pq} = x^2 + zxp - xzq - \cancel{z^2pq}$$

$$x^2 - y^2 + zp(x+y) - zq(x+y) = 0.$$

$$(x-y)(x+y) + zp(x+y) - zq(x+y) = 0$$

$$x - y + zp - zq = 0$$

Q find partial differentiation Equation $\phi(x^2 - z^2, x^3 - y^3) = 0$

$u = x^2 - z^2$

$v = x^3 - y^3$

$\phi(u, v) = 0 \rightarrow (1)$

Diff (1) w.r.t. x

$$\frac{\partial \phi}{\partial u}\left[2x - 2z\frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[3x^2\right] = 0$$

$$\therefore \frac{\partial \phi / \partial u}{\partial \phi / \partial v} = \frac{-3x^2}{2x - 2zp} \rightarrow (2)$$

Partially Diff (1) w.r.t y

$$\frac{\partial \phi}{\partial u}\left[-2z \cdot \frac{\partial z}{\partial y}\right] + \frac{\partial \phi}{\partial v}\left[-3y^2\right] = 0$$

$$\therefore \frac{\partial \phi / \partial u}{\partial \phi / \partial v} = \frac{3y^2}{-2zq} \rightarrow (3)$$

from eq^n (2) & (3)

$$\frac{-3x^2}{2(x-zp)} = \frac{3y^2}{-2zq}$$

$$\frac{x^2}{x-zp} = \frac{y^2}{zq}$$

$$x^2zq = y^2x - y^2zp$$

or

$$z(x^2q - y^2p) = y^2x$$

Lagrange's Equation :-

$$Pp + Qq = R$$

where, P, Q, R is $f(x,y)$

The subsidiary Equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$u(x,y,z) = a$$
$$v(x,y,z) = b$$

General solution is

$$\phi\ F(u,v) = 0$$

Q Solve, $xp + yq = z$

The subsidiary eq^n are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking :-

$$\frac{dx}{x} = \frac{dy}{y}$$

On integration

$$\log x = \log y + \log a$$

$$\log \frac{x}{y} = \log a$$

$$\frac{x}{y} = a.$$

Taking :-

$$\frac{dy}{y} = \frac{dz}{z}$$

On integration

$$\log y = \log z + \log b$$

$$\log \frac{y}{z} = \log b$$

$$\frac{y}{z} = b$$

$\therefore$ General solⁿ is

$$f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

Q. $\left(\frac{y^2 z}{x}\right) p + xzq = y^2$

The subsidiary Eqⁿ is

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Dividing by x throughout

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

(i) $$\frac{dx}{y^2 z} = \frac{dy}{x^2 z}$$

$$\frac{dx}{y^2} = \frac{dy}{x^2}$$

$$x^2 dx = y^2 dy$$

$$\frac{x^3}{3} - \frac{y^3}{3} = a$$

or

$$x^3 - y^3 = a$$

(ii) $$\frac{dx}{y^2 z} = \frac{dz}{y^2 x}$$

$$x\,dx = z\,dz$$

On integrating

$$\frac{x^2}{2} - \frac{z^2}{2} = b$$

or

$$x^2 - z^2 = b$$

General solⁿ is

$$f(x^3 - y^3, x^2 - z^2) = 0$$

Q. $zp = -x$

The subsidiary Eqⁿ is

$$\therefore \frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$$

(i) $$\frac{dx}{z} = \frac{dz}{-x}$$

$$x\,dx = -z\,dz$$

On integrating

$$\frac{x^2}{2} + \frac{z^2}{2} = a$$

or $(x^2 + z^2) = a$

(ii) $\frac{dx}{z} = \frac{dy}{0}$

$dy = 0$

$y = b$

General solⁿ is

$f(x^2+z^2, y) = 0$

Q. $(mz - ny)p + (nx - lz)q = ly - mx$

The subsidiary Eqⁿ is

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

(i) l, m, n as multipliers

$$= \frac{l\,dx + m\,dy + n\,dz}{lmz - lny + mnx - mlz + lny - nmx}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

Taking,

$$\frac{dx}{mz-ny} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$l\,dx + m\,dy + n\,dz = 0$$

On integrating

$$lx + my + nz = a$$

(ii) x, y, z as multipliers

$$= \frac{xdx + ydy + zdz}{mxz - nxy + nxy - lzy + lyz - mxz}$$

$$= \frac{xdx + ydy + zdz}{0}$$

Taking this with anyone of the ratios

$$\therefore \quad xdx + ydy + zdz = 0$$

$$x^2 + y^2 + z^2 = b$$

General sol^n is

$$f(lx + my + nz, x^2 + y^2 + z^2) = 0$$

Q. $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$

Multiplying by xyz throughout,

$$x(y-z)p + y(z-x)q = (x-y)z$$

The subsidiary Eq^{n}s is

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{(x-y)z}$$

(i) 1, 1, 1 as multipliers

$$\frac{dx + dy + dz}{xy - xz + yz - yx + xz - yz} = \frac{dx + dy + dz}{0}$$

Taking this with anyone of the ratios

$$dx + dz + dy = 0$$

On integrating

$$x + y + z = a$$

(ii) $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integrating

$$\log x + \log y + \log z = \log b$$

$$xyz = b$$

general solution is

$$f(x+y+z, xyz) = 0$$

Q $xzp + yzq = xy$

The subsidiary Eq^n's is

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

(i) $$\frac{dx}{xz} = \frac{dy}{yz}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

$$\log x - \log y = \log a$$

$$\log \frac{x}{y} = \log a$$

$$\frac{x}{y} = a$$

(ii) $$\frac{dy}{yz} = \frac{dz}{xy}$$

$$\frac{dy}{z} = \frac{dz}{x}$$

$$x\,dy = z\,dz$$

$$a y\, dy = z\, dz$$

$$\frac{a y^2}{2} = \frac{z^2}{2} + b$$

$$\frac{a y^2}{2} - \frac{z^2}{2} = b$$

$$\frac{x}{y}\frac{y^2}{2} - \frac{z^2}{2} = b$$

$$xy - z^2 = b$$

So, General solution is

$$f\left(\frac{x}{y}, xy - z^2\right) = 0$$

Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

Subsidiary Equation is

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

$1/x, 1/y, 1/z$ as multiplier

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$$

$$\frac{1/x\, dx + 1/y\, dy + 1/z\, dz}{0}$$

Taking this with any one of the ratio:

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

On integrating

$$\log x + \log y + \log z = \log a$$

$$xyz = a$$

(ii) x, y, z as multiplier

$$\frac{x\,dx + y\,dy + z\,dz}{0}$$

Taking this with any one of the ratio

$$x\,dx + y\,dy + z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b$$

or

$$x^2 + y^2 + z^2 = b$$

So, General solution is

$$f(\qquad , x^2+y^2+z^2) = 0$$

Q. Solve

$$x(y^2+z)p - y(x^2+z)q = z(x^2-y^2) = 0$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

(i) Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

$$= \frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{0}$$

Taking this with one of the ratios

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

On integrating

$$\log x + \log y + \log z = \log a$$

$$\log xyz = \log a$$

$$a = xyz$$

(ii) Taking $x, y, -1$ as multiplier

$$\frac{x\,dx + y\,dy + dz}{x^2(y^2 + z) - y^2(x^2 + z) \mp z(x^2 - y^2)}$$

$$\frac{x\,dx + y\,dy \mp dz}{\cancel{x^2 y^2} + x^2 z - \cancel{y^2 x^2} - y^2 z - zx^2 + zy^2}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

Taking this with one of the ratios

$$x\,dx + y\,dy - dz = 0$$

$$\frac{x^2}{2}+\frac{y^2}{2}-z = b$$

$$x^2+y^2-2z = b'$$

General solution is

$$f\left(xyz,\ x^2+y^2-2z\right) = 0$$

Q $z(xp-yq) = y^2-x^2$

$$zxp-zyq = y^2-x^2$$

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2}$$

(i) $\dfrac{dx}{x} = \dfrac{dy}{-y}$

$$\log x = -\log y + \log a$$

$$\log xy = \log a$$

$$\boxed{a = xy}$$

(ii) Taking x,y,z as multipliers

$$\frac{x\,dx + y\,dy + z\,dz}{0}$$

Taking this with any one of the multiplier

$$x\,dx + y\,dy + z\,dz = 0$$

On integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=b$$

$$x^2+y^2+z^2=b$$

So,

General solution is :-

$$f(xy,\ x^2+y^2+z^2)=0$$

Homogenous linear Equation with constant coefficients.

$z \to x, y$

$$\frac{\partial^n z}{\partial x^n}+\frac{A_1\partial^n z}{\partial x^{n-1}\partial y}+\frac{A_2\partial^n z}{\partial x^{n-2}\partial y^2}+\ldots+A_n\frac{\partial^n z}{\partial y^n}=f(x,y) \quad (1)$$

$$D=\frac{\partial}{\partial x},\quad D'=\frac{\partial}{\partial y}$$

$$(D^n+A_1D^{n-1}D'+A_2D^{n-2}D'^2+A_3D^{n-3}D'^3+\ldots+A_nD'^n)z=f(x,y)$$

Working Rule.

for CF:-

Replace D by m, D' by 1, we get auxiliary equation as

$$m^n+A_1m^{n-1}+A_2m^{n-2}+A_3m^{n-3}+\ldots A_n=0$$

(i) Distinct

$$CF=f_1(y+m_1x)+f_2(y+m_2x)+\ldots$$

(2) Equal

$$C.F. = f_1(y+m_1x) + x f_2(y+m_1x)$$

Q. Find the general solution of following Equations

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$$

$$(D^2 - DD' - 6(D')^2)z = 0$$

Auxiliary Equation is

$$\cancel{D^2 - D} - m^2 - m - 6 = 0$$

$$m^2 - 3m + 2m - 6 = 0$$

$$m(m-3) + 2(m-3) = 0$$

$$(m-3)(m+2) = 0$$

$$m = 3, -2$$

General solution

$$z = f_1(y-2x) + f_2(y+3x)$$

Q. $(D+2D')(D-3D')^2 z = 0$

$$(m+2)(m-3)^2 = 0$$

$$m = -2, 3, 3$$

$$z = f_1(y-2x) + f_2(y+3x) + x f_3(y+3x)$$

Q. $4r - 12s + 9t = 0$

$4m^2 - 12m + 9 = 0$

$4m^2 - 6m - 6m + 9 = 0$

$2m(2m-3) - 3(2m-3).$

$(2m-3)(2m-3) = 0$

$m = \frac{3}{2}, \frac{3}{2}$

$z = f_1\left(y + \frac{3}{2}x\right) + xf_2\left(y + \frac{3}{2}x\right)$

Q $\frac{\partial^2 z}{\partial x^2} + a^2 \frac{\partial^2 z}{\partial y^2} = 0$

$m^2 + a^2 = 0$

$m = \pm ai$

$z = f_1(y + aix) + f_2(y - aix)$

Q. $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x^2 \partial y} - 6\frac{\partial^3 z}{\partial y^3} = 0$

$m^3 - 7m^2 - 6 = 0$

~~$m^2(m-3) - 4m(m-3) - 12(m-3)$~~

$m^2(m-1) - 6m(m-1) - 6(m-1) = 0$

$(m^2 - 6m - 6)(m-1) = 0$

$m = 1, 3 \pm \sqrt{15}$

$m = 1,\ 3+\sqrt{15},\ 3-\sqrt{15}$

$$z = f_1(y+x) + f_2(y+(3+\sqrt{15})x) + f_3(y+(3-\sqrt{15})x)$$

$$\phi(D, D')\, z = f(x, y)$$

$$G.S = C.F + P.I$$

$$C.F \to D,\ E$$

Rules for P.I

① $f(x, y) = e^{ax+by}$

$$P.I = \frac{1}{\phi(D, D')} e^{ax+by}$$

$$P.I = \frac{1}{\phi(a, b)} e^{ax+by}$$

① $$\frac{\partial^3 z}{\partial x^3} - \frac{3\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$$

$$D^3 z - 3D^2 D' z + 4D'^3 = e^{x+2y}$$

$$m^3 - 3m^2 + 4 = 0$$

$$m^2(m+1) - 4m(m+1) + 4(m+1) = 0$$

$$(m^2 - 4m + 4)(m+1) = 0$$

$$m = \frac{4 \pm \sqrt{16-16}}{2} \Rightarrow 2, 2.$$

$m = -1, 2, 2$

$$C.F = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$$

$$P.I = \frac{1}{D^3 - 3D^2D' + 4D'^3} \cdot e^{x+2y}$$

$$\frac{1}{1 - 3\times 2 + 4\times 8} e^{x+2y}$$

$$\frac{1}{33-6} e^{x+2y}$$

$$\frac{1}{27} e^{x+2y}$$

General solution is $= z = C.F + P.I$

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

(2) $f(x,y) = \sin(ax+by)$

$$P.I = \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax+by)$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax+by)$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} \quad \sin x \cos 2y$$

$$(D^2 - DD')z = \frac{1}{2} 2\sin x \cos 2y$$

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0, 1$

$C.F = f_1(y) + f_2(y+x)$

$$P.I = \frac{1}{D^2 - DD'} \left[\sin(x+2y) + \sin(x-2y)\right]$$

$$\frac{1}{D^2 - DD'} \sin(x+2y) + \frac{1}{D^2 - DD'} \sin(x-2y)$$

$$\frac{1}{-1+2} \sin(x+2y) + \frac{1}{-1\cancel{+}-2} \sin(x-2y)$$

$$\frac{1}{2}\left(\sin(x+2y) - \frac{1}{3} \sin(x-2y)\right)$$

General solution =

$$f_1(y) + f_2(y+x) + \frac{1}{2}\sin(x+2y) - \frac{1}{6} \sin(x-2y)$$

Q. $(D^3 - 7D'D^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

$m^3 - 7m - 6 = 0$

$m^2(m+1) - m(m+1) - 6(m+1) = 0$

$(m^2 - m - 6)(m+1) = 0$

$(m-3)(m+2)(m+1) = 0$

$m = -1, -2, 3$

$m = -2, -1, 3$

$C.F = f_1(y-2x) + f_2(y-x) + f_2(y+3x)$

$P.I_2 = \frac{1}{D^3 - 7D'D^2 - 6D'^3} \sin(x+2y)$

$\frac{1}{-D + 7D' \times 4 + 24D'} \sin(x+2y)$

$\frac{1}{-D + 28D' + 24D'} \sin(x+2y)$

$\Rightarrow \frac{1}{27D + 24D'} \sin(x+2y)$

$\Rightarrow$ Multiplying dividing by D

$\frac{1}{D^3 - 7D'D^2 - 6D'^3} e^{2x+y}$

$\frac{D}{27D^2 + 24DD'} \sin(x+2y)$

$\frac{D\sin(x+2y)}{-27-48}$

$\frac{1}{8-14-6}$

$\frac{D\sin(x+2y)}{-75}$

$P.I_1 = -\frac{1}{12} e^{2x+y}$

$P.I_2 = -\frac{1}{75}\cos(x+2y)$

General solution is

$f_1(y-2x) + f_2(y-x) + f_3(y+3x) - \frac{1}{12} e^{2x+y} - \frac{1}{75}\cos(x+2y)$

General Method :-

$$P.I = \frac{1}{D - mD'} f(x,y)$$

$$= \int f(x, c - mx)\, dx$$

$$c = y + mx.$$

Q Solve :- $r - 4s + 4t = e^{2x+y}$

$$(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$C.F = f_1(y+2x) + x f_2(y+2x)$$

$$P.I. = \frac{1}{(D - 2D')^2} e^{2x+y}$$

Applying Rule no. 1

$$\frac{1}{(2 - 2\times 1)^2} e^{2x+y}$$

$$\frac{1}{0} e^{2x+y}$$

Case fails

Applying General Method:-

$$P.I. = \frac{1}{(D-2D')(D-2D')} e^{2x+y}$$

$$m = 2$$

$$\frac{1}{(D-2D')} \int e^{\cancel{2x}+c-\cancel{2x}}\, dx$$

$$\frac{1}{(D-2D')} \int e^{c}\, dx$$

$$\frac{1}{(D-2D')} e^{c} \cdot x\, dx$$

$$\frac{1}{(D-2D')} e^{y+2x} \cdot x\, dx$$

$$\int e^{c-\cancel{2x}+\cancel{2x}} \cdot x\, dx$$

$$\int e^{c} \cdot x\, dx$$

$$e^{c} \int x\, dx$$

$$\Rightarrow e^{c} \cdot \frac{x^2}{2}$$

$$P.I \Rightarrow e^{y+2x} \cdot \frac{x^2}{2}$$

Q2. $(D^2 - 2DD' - 15D'^2)z = 12xy$

$m^2 - 2m - 15 = 0$

$m^2 - 5m + 3m - 15 = 0$

$m(m-5) + 3(m-5) = 0$

$(m-5)(m+3) = 0$

$m = -3, 5$

$$CF = f_1(y-3x) + f_2(y+5x)$$

$$PI = \frac{1}{(D^2-2DD'-15D'^2)} \cdot 12xy$$

$$= \frac{12}{D^2}\left[\frac{1}{1-2\frac{D'}{D}-15\left(\frac{D'}{D}\right)^2}\right] \cdot xy$$

$$= \frac{12}{D^2}\left[1-\left(\frac{2D'}{D}+15\left(\frac{D'}{D}\right)^2\right)\right]^{-1} xy$$

$$= \frac{12}{D^2}\left[1+\frac{2D'}{D}\right] xy$$

$$= \frac{12}{D^2}\left[xy+\frac{2D'}{D}xy\right]$$

$$\frac{12}{D^2}\left[xy+2\frac{1}{D}\cdot x\right]$$

$$\frac{12}{D^2}\left[xy+2\int x\right]$$

$$\frac{12}{D^2}\left[xy+2\frac{x^2}{2}\right]$$

$$\frac{12}{D^2}\left[xy+x^2\right]$$

$$\frac{12}{D}\left[\frac{x^2}{2}y+\frac{x^3}{3}\right]$$

$$12\left[\frac{x^3}{6}y+\frac{x^4}{12}\right]$$

$$= 2x^3y + x^4$$

Q.8. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cdot \cos x$

$m^2 + m - 6 = 0$

$m^2 + 3m - 2m - 6 = 0$

$m(m+3) - 2(m+3) = 0$

$(m+3)(m-2) = 0$

$m = 2, -3.$

$\therefore$ C.F: $f(y+2x) + f(y-3x)$

P.I = $\cancel{\frac{1}{(m+3)(m-2)} \cdot y\cos x}$

$\frac{1}{D^2 + DD' - 6(D')^2} \cdot y\cos x$

$\frac{1}{D^2 + 3DD' - 2DD' - 6(D')^2} \cdot y\cos x$

$\frac{1}{D(D+3D') - 2D'(D+3D')} \cdot y\cos x$

$\frac{1}{(D+3D')(D-2D')} y\cos x$

$\frac{1}{(D+3D')} \int (c-2x)\cos x \, dx$

$\frac{1}{(D+3D')} \left[c\int \cos x\,dx \; 2\int x\cos x\,dx \right]$

$\frac{1}{(D+3D')} \left[c\sin x - 2\left[x\sin x + \cos x \right] \right]$

$\frac{1}{(D+3D')} \left[c\sin x - 2x\sin x - 2\cos x \right]$

$\frac{1}{(D+3D')} \left[(y+2x)\sin x - 2x\sin x - 2\cos x \right]$

$\frac{1}{(D+3D')} \left[y\sin x + \cancel{2x\sin x} - \cancel{2x\sin x} - 2\cos x \right]$

$\frac{1}{(D+3D')} \left[y\sin x - 2\cos x \right]$

$$\int((c+3x)\sin x - 2\cos x)\,dx$$

$$\int c\cdot\sin x\,dx + 3\int x\sin x\,dx - 2\int\cos x\,dx$$

$$c\int\sin x\,dx + 3\int x\sin x\,dx - 2\int\cos x\,dx$$

$$\Rightarrow -c\cos x + 3\left[-x\cos x + \sin x\right] - 2\sin x$$

$$\Rightarrow -c\cdot\cos x - 3x\cos x + 3\sin x - 2\sin x$$

$$\Rightarrow -c\cos x - 3x\cos x + \sin x$$

$$\Rightarrow -(y-3x)\cos x - 3x\cos x + \sin x$$

$$\Rightarrow -y\cos x + 3x\cancel{\cos x} - 3\cancel{x}\cos x + \sin x$$

$$\Rightarrow -y\cos x + \sin x.$$

(4) $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

$m^2 - m - 2 = 0$

$m^2 - 2m + m - 2 = 0$

$m(m-2) + (m-2) = 0$

$(m-2)(m+1) = 0$

$m = -1, 2$

C.F. $= f(y-x) + f(y+2x)$

$$P.I = \frac{1}{D^2 - DD' - 2D'^2}\cdot(y-1)e^x$$

$$= \frac{1}{D^2 - 2DD' + DD' - 2D'^2}\cdot(y-1)e^x$$

$$= \frac{1}{D(D-2D') + D'(D-2D')}\cdot(y-1)e^x$$

Here $m=-1$ let $y = c+x$

$$\Rightarrow \frac{1}{(D-2D')(D+D')} \cdot (y-1)e^x$$

$$\frac{1}{(D-2D')} \int (c+x-1)\, e^x$$

$$\frac{1}{(D-2D')} \left[c\int e^x dx + \int x e^x dx - \int e^x dx \right]$$

$$\frac{1}{(D-2D')} \left[c.e^x + xe^x - e^x - e^x \right]$$

$$\frac{1}{(D-2D')} \left[ce^x + xe^x - 2e^x \right]$$

$$\frac{1}{(D-2D')} \left[(y-x)\, e^x + xe^x - 2e^x \right]$$

$$\frac{1}{(D-2D')} \left[ye^x \cancel{-xe^x} + \cancel{xe^x} - 2e^x \right]$$

$$\frac{1}{(D-2D')} \left[ye^x - 2e^x \right]$$

Here $m=2$.

let $y = c-2x$

$$\int \left[(c-2x)\, e^x - 2e^x \right] dx$$

$$\int c.\, e^x dx - 2xe^x dx - 2e^x dx$$

$$c\int e^x dx - 2\int xe^x dx - 2\int e^x dx$$

$$\Rightarrow c.e^x - 2\left[xe^x - e^x \right] - 2e^x$$

$$ce^x - 2xe^x + \cancel{2e^x} - \cancel{2e^x}$$

$$\Rightarrow (y+2x)\, e^x - 2xe^x$$

$$ye^x + \cancel{2xe^x} - \cancel{2xe^x}$$

$$\Rightarrow ye^x$$

① $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, $u(0,y) = 8e^{-3y}$ → ①

Let, $u = XY$ → ② be solution of ①
where X is a function of ~~x~~ x only
and Y is a function of y.
Only ② in ① gives :-

$X'Y = 4XY'$

$= \frac{X'}{X} = \frac{4Y'}{Y}$

$\frac{X'}{X} = \frac{4Y'}{Y} = k$ (say)

If $\frac{X'}{X} = k \Rightarrow X' - kX = 0$

auxiliary Eqⁿ $m - k = 0$

$m = k$

Solution is $X = c_1 e^{kx}$

If $\frac{4Y'}{Y} = k \Rightarrow 4Y' - kY = 0$

auxiliary Eqⁿ

$4m - k = 0$

$m = k/4$

Solution is $Y = c_2 e^{\frac{ky}{4}}$

General solution is

$u = c_1 c_2 e^{kx} \cdot e^{\frac{ky}{4}}$ → ③

$u(x,y) = u = c_1 c_2 e^{kx}$

Eqⁿ ③ put $x = 0$

$8e^{-3y} = c_1 c_2 e^{ky/4}$

$\boxed{c_1 c_2 = 8,}$

$\frac{k}{4} = -3$, $\boxed{k = -12}$

② $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

Gives, $u = 3e^{-y} - e^{-5y}$
when $= 0$

Let $u = XY$
be solution of ①, ① gives:-

$\Rightarrow 4X'Y + XY' = 3XY$

$\Rightarrow 4X'Y = 3XY - XY'$

$\Rightarrow 4X'Y = X(-Y' + 3Y)$

$= \frac{4X'}{X} = \frac{-Y' + 3Y}{Y} = k$

If $\frac{4X'}{X} = k$

$\Rightarrow 4X' = kX$

$\Rightarrow 4X' - kX = 0$

$4m - k = 0$

$m = \frac{k}{4}$

Solⁿ is

$X = c_1 e^{kx/4}$

If $\frac{-Y' + 3Y}{Y} = k$

$-Y' + 3Y = kY$

$-m + 3 = k$

$\therefore m = 3 - k$

Solⁿ is $Y = c_2 e^{(3-k)y}$

G.S is

$u = c_1 c_2 e^{kx/4} \cdot e^{(3-k)y}$

when $X = 0$, then on
comparing with 1st term

$c_1c_2e^{(3-k)y} = 3e^{-y}$

$\boxed{c_1c_2 = 3}$

$3-k = -1$

$\boxed{4 = k}$

$\boxed{u = 3e^{x-y}}$

→ when $x=0$, then comparing with 2^{nd} term

$c_1c_2e^{(3-k)y} = -e^{-5y}$

$c_1c_2 = -1$

$3-k = -5$

$k = 8$

$\boxed{u = e^{2x-5y}}$

(2) $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$

Q. $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u$

$u(x, 0) = 10e^{-x} - 6e^{-4x}$

$u = XT$

~~Ux~~ :