Beta and Gamma function.

① BETA FUNCTION:

$(m,n) > 0$, then $\beta(m,n)$ is denoted by and given by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Properties of Beta function:

⇒ ① $\beta(m,n) = \beta(n,m)$

② $\beta(m,n) = \int_0^\infty \frac{x \, dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}$

③

→ P I N P

③ P

Gamma

If $n > 0$,

• Its Prop

1) $\Gamma 1 = 1$

4) Γ

5 = Γ

en by

③ $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1} d\theta$

Gamma function

If $n > 0$, then Γn (gamma of n) $\boxed{= \int_0^\infty e^{-t} t^{n-1} dt = \Gamma n}$

- Its Properties :

1) $\Gamma 1 = 1$, 2) $\Gamma n+1 = n\Gamma n$, 3) $\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx$

4) $\Gamma n = \int_0^1 (\log \frac{1}{y})^{n-1} dy$

5 = $\Gamma n+1 = \int_0^\infty e^{-y^{1/n}} dy$

⇒ Relationship b/w Beta and Gamma function.

or

⇒ Prove that $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma (m+n)}$ where $(m, n > 0)$

→ Sol^n ∵ we know that

$$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx \quad -(1)$$

⇒ $\Gamma n = z^n \int_0^\infty e^{-zx} x^{n-1} dx$

⇒ multiply both sides by $e^{-z} z^{m-1}$

unction.

$(m,n>0)$

Prove that

Q.1 Gamma $\Gamma\frac{1}{2} = \sqrt{\pi}$

$\Rightarrow$ We know that : $\beta(m,n) = \frac{\Gamma n \, \Gamma m}{\Gamma m+n}$ where $m,n>0$

$\Rightarrow$ put $m = n = \frac{1}{2}$

$\Rightarrow \beta(\frac{1}{2},\frac{1}{2}) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1} = \overline{\Gamma 1} = 1$

$\Rightarrow \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \Gamma(\frac{1}{2})^2$

$\Rightarrow$

put $x = s^2\theta$

$\Rightarrow = \int_0^{\pi/2} \sin^{-1/2}(1-s^2\theta)^{-1/2} \sin 2\theta \, d\theta \Rightarrow \Gamma(\frac{1}{2})^2$

$\Rightarrow 2\left[\frac{\pi}{2} - 0\right) = \Gamma\left(\frac{1}{2}\right)^2$

$\Rightarrow \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$

$\Gamma n \, e^{-z} z^{m-1} = \int_0^\infty e^{-zx} x^{n-1} z^n e^{-z} z^{m-1} dx$

$\Rightarrow \Rightarrow \int_0^\infty e^{-z(1+x)} x^{n-1} z^{m+n-1} = \Gamma n \, e^{-z} z^{m-1}$

$\Rightarrow \int$ both the sides from limits $0 \to \infty$

$\Rightarrow \Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty \int_0^\infty e^{-z(1+x)} x^{n-1} z^{m+n-1} dx \, dz.$

$\Rightarrow \Gamma n \, \Gamma m = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} x^{n-1} z^{m+n-1} dx\right] dz$

$e^{-z(1+x)} x^{n-1} dx \, . \, z^{m+n-1} dx$

$x^{n-1} \int \left(e^{-z(1+x)} z^{m+n-1} dz\right) dx = \frac{\Gamma m+n}{(1+x)^{m+n}}$

$\Rightarrow \Gamma n \, \Gamma m = \quad \llcorner m+n \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$\Rightarrow \quad 1+x = t$

$\Rightarrow \left[\text{from prop. } (2) \text{ of beta f}^{n}\right] \Rightarrow \int_0^{\infty} \frac{x}{(1+x)^{m+n}} = \beta(m,n) = \Gamma n \cdot \Gamma m.$

$\Rightarrow$ statement, to prove Legendre's Duplication formula :

1) to prove this or Gamma $\Gamma_{m+1/2} \, \Gamma m = \frac{\sqrt{\pi}}{2^{2m-2}} \, \Gamma 2m$ where $m > 0$,

Sol^n $2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$ — (1)

put $\left[m = \frac{1}{2}\right]$ in eq^n 1st

putting $n = 1/2$ in

$\Rightarrow \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m}$

$\Rightarrow$ Putting $m = n \Rightarrow$

$\rightarrow \int_0^{\pi/2} (\sin$

2^{2m}

ut

$(m,n) = \Gamma n \cdot \Gamma m$.

formula :

here $m>0$,

— (1)

putting $n = \frac{1}{2}$ in eq^n (1)

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma \frac{1}{2}}{2\,\Gamma m+\frac{1}{2}} \quad \text{— (2) eq}^n$$

$$= \frac{1}{2} \frac{\Gamma m \, \Gamma \pi}{\Gamma m+\frac{1}{2}}$$

$$\Rightarrow \text{Putting } \underline{\underline{m=n}} \Rightarrow 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{\Gamma 2m}$$

$$\rightarrow \int_0^{\pi/2} \frac{(\sin 2\theta)^{2m-1}}{2^{2m-1}} d\theta = \frac{\Gamma m \, \Gamma m}{2\,\Gamma 2m} \qquad \left[\div \; \& \times \text{ with } 2^{2m-1} \right]$$

lets put $2\theta = \varphi$

$$\Rightarrow \begin{matrix} 2\theta = \varphi \\ d\theta = \frac{d\varphi}{2} \end{matrix} \Rightarrow \int_0^{\pi/2} \frac{(\sin\varphi)^{2m-1}}{2^{2m-1}} \frac{d\varphi}{2} = \frac{\Gamma m \, \Gamma m}{2\,\Gamma 2m}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{(\sin \phi)^{2m-1}}{2^{2m-1}} \left(\frac{d\phi}{2}\right) = \frac{\Gamma m \, \Gamma m}{2 \, \Gamma 2m} \quad -(3)$$

$$\Rightarrow \frac{1}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \frac{(\sin \phi)^{2m-1}}{2^{2m-1}} \, d\phi = \quad \because \int_0^a f(x) = 2\int_0^{a/2} f(x)$$

[using propeorties of definite integral

$\Rightarrow$ Replacing ϕ by θ

$$\Rightarrow \int_0^{\pi/2} (\sin\theta)^{2m-1} \, d\theta = \frac{2^{2m-1} \, (\Gamma m)^2}{2 \, \Gamma 2m} \quad -(4)$$

on c

$\Rightarrow \frac{\sqrt{\pi}}{2^{2m-1}}$

Q. Prove

$\Rightarrow \beta(m$

$\Rightarrow \beta(m$

$\frac{\Gamma m}{2m}$ - (3)

$f(x)$ [using properties of definite integral

$)^2$ - (4)

On comparing eq. (4) with eq.^{n} (2)

$$\frac{\Gamma m \, \sqrt{\pi}}{2\,\Gamma\left(m+\frac{1}{2}\right)} = \frac{2^{2m-1}\left(\Gamma m\right)^2}{2\,\Gamma 2m}$$

$$\Rightarrow \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m = \Gamma m \, \Gamma\left(m+\frac{1}{2}\right)$$

Q. Prove that

$$\Rightarrow \beta(m,n) = \beta(m+1,n) + \beta(m,n+1), \quad \underline{\underline{m,n>0}}$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+n+1)} + \frac{\Gamma m \, \Gamma(n+1)}{\Gamma(m+n+1)} \Rightarrow \frac{\Gamma m\,\Gamma(n+1) + \Gamma(m+1)\,\Gamma n}{\Gamma(m+n+1)}$$

$\because \left\{ \Gamma(n+1) = n\,\Gamma(n) \right\}$

$$\Rightarrow \frac{\Gamma(m)\,\Gamma(n+1) + \Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)}$$

due this

$$\Rightarrow \frac{n\,\Gamma(m)\,\Gamma(n) + m\,\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1) \text{ or } (m+n)\,\Gamma(m+n)}$$

$\cancel{n\,\Gamma(m)}$ -

$\Rightarrow$ Que. find $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

⇒1) Traci

2) Pola

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Sym

curve

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⇒ 1) Tracing curve in Cartesian

2) Polar curve.

⇒ 1) Tracing of Cartesian curve -
helpful points for tracing it are -

① Symmetry :

[1 A curve is symmetrical about x axis, in the equation are given, for ex - $y^2 = 4ax$

[2 " about y axis, if all power of y in the equation are given, for ex - $x^2 = 4ay$

[3 A symmetrical about line $y = x$, if the equation of given curve remains unchanged on interchanging x & y, for ex $x^2 + y^2 = a^2$

If the eq^n of given curve, remain unchanged when y is changed to $-y$, vice versa for x, then curve is symmetrical in opp. quadrant,

$x^3+y^3 = 3a x x^2 y$.

⇒ (3) Origin & Tangents :

1) If the points (0,0) satisfy eq^n of the given curve, then curve passes through the origin.

2) If the curve passes through origin then nature of origin (node, cusp) etc, for this we find that the tangent at origin by equating the lowest degree terms to zero. for ex $x^2+y^2+x+y=0$

$\Rightarrow x+y=0 \Rightarrow x=-y$

1st. ab

⇒ If we get two tangent at origin then we find its nature,

a If both tangents are real and distinct then origin is called node.

b If both tangent are real & coincident then origin is called cusp

⇒ c If both tangents are imaginary then origin is isolated point.

2nd. (3) POI with coordinate axes.

Put $x=0$, $y=0$

⇒ In the eq-n of curve respectively, it gives points of the curve, on y axis & x axis. respectively
of I

(4) ⇒ Asymptotes:

[1] from origin to which the given curve doesnot intersecting and touches it at '∞', It means that a asymptote is limiting / bound line for the curve.
ary.

[2] Asymptotes || to x axis equating coefficient of highest power 'x' in the given eq-n of curve,

[3] " || to y axis equating coefficient of highest power 'y'

[4] " of curve.

★ Remark - It is not necessary every curve has asymptotes
for ex ⇒ $x^2 + y^2 = a^2$.

[5] To find Region of curve, We solve the equation of the curve, for 'y' or x which ever is convenient.

• Suppose we solve the eq-n of curve Y,

1st then Examine following points:

a] we find those value x for which y tends to $\rightarrow \infty$

b] we find " the interval for x in which the values of y become imaginary.

c] we find the interval for x, for which the value of it ↑ & ↓.

2nd [b] find $\frac{dy}{dx}$ by the eq-n of curve;

a] $\frac{dy}{dx} = 0$, for some value of x then tangent ∥ to x-axis.

[b] $\frac{dy}{dx} \Rightarrow \infty$ for some value of x, then Tangent is ∥ to y-axis

c] $\frac{dy}{dx} > 0$

interv

[d] sam

Q. trace

y=a

(7) y = +

$x \rightarrow 0 \Rightarrow$

$\rightarrow \infty$

values

the

[c] $\frac{dy}{dx} > 0$, for $x\ (a,b)$ the it $f(x)\uparrow$ in this (increasing) interval (a,b).

[d] same for decreasing for $\frac{dy}{dx} < 0$

Q. trace the curve.

$y^2(a-x) = x^2(a+x)$

$\sqrt{\frac{a+x}{a-x}} > 0$

$\therefore$ Sy w.r.t x

the shape curve is

$\Rightarrow a^2y^2 = x^2(a^2-x^2)$ Draw it, trace it.

*10/NOV

-Trancing of curve, to find shape of curve, $r = f(\theta)$, the following procedure is adopted.

1) Symmetry -(A) When θ is replaced by $(-\theta)$ then curve equation remain unchanged, then curve is symmetrical about 'initial line' i.e. 'x axis'.

(B) If θ is replaced by $(\pi-\theta)$ then equation curve is unchanged then it is symmetrical about line $\boxed{\theta = \frac{\pi}{2}}$ i.e. y axis.

B.2
(•) If θ is " $-\theta$ and r is replaced '$-r$' then equation curve is unchanged then, the curve is symmetrical about line $\theta = \frac{\pi}{2}$ i.e. 'y-axis'.

(C) If r is replaced by $-r$, equation of curve is unchanged the curve is symmetrical about pole i.e. Origin$(0,0)$.

1st. a C. 2 **Pole and tangent**, if $r = 0$, in given equation curve
2pt. then θ is equal to α, (i.e. finite value), is the tangent of curve at pole. $= \alpha$

D. Solve the equation curve for 'r', prepare table of corr. value of θ and r. This table gives a no. of pts on the curve. The shape of curve is obtained by plotting these points

θ	0	.					2π
r							

2nd

(θ, r) = points on the curve

If the value of r comes out to be imaginary or -ve in the region given by, ______

then the

* NOTE :

Q.

then the curve will not exist in region.

* NOTE: sometimes, it is convenient to Cartisan curve to polar curve equation or VICE VERSA.

Q.

Que. Trace the curve

$r = a(1 + \cos\theta)$

⇒Q. $r = 2a\cos\theta$.

~~1~~ 1st . a S-1 ⇒ the curve sym. about the line i.e. x=axis $\frac{1}{\sqrt{2}}$

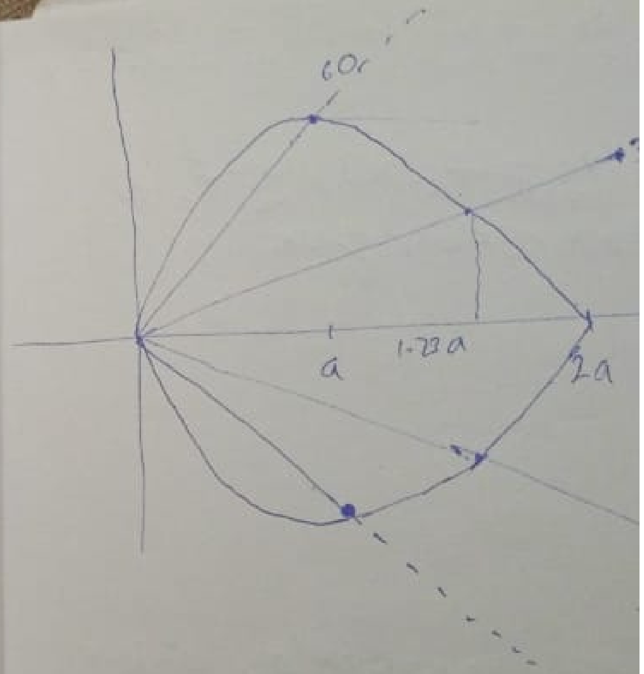
⇒ S-2 ⇒ $\boxed{r=0} \rightarrow 0 = 2a\cos\theta \Rightarrow \theta = \pi/2$, curve passes through the pole, and $\theta = \pi/2$ is tangent of curve at pole,

⇒ S-3, Prep. a table corresponding value r & θ,

θ	0	30	60	90	120	150	180	210
$r = 2a\cos\theta$	2a	1.73a	a	0	–	–	–	–

2nd

When $\theta \cancel{>} 90$, the value of r is negative so curve does not exist, using above points the approximate shape of curve



=axis $\frac{1}{\sqrt{2}}$

$=\pi/2$, curve passes

is tangent of curve

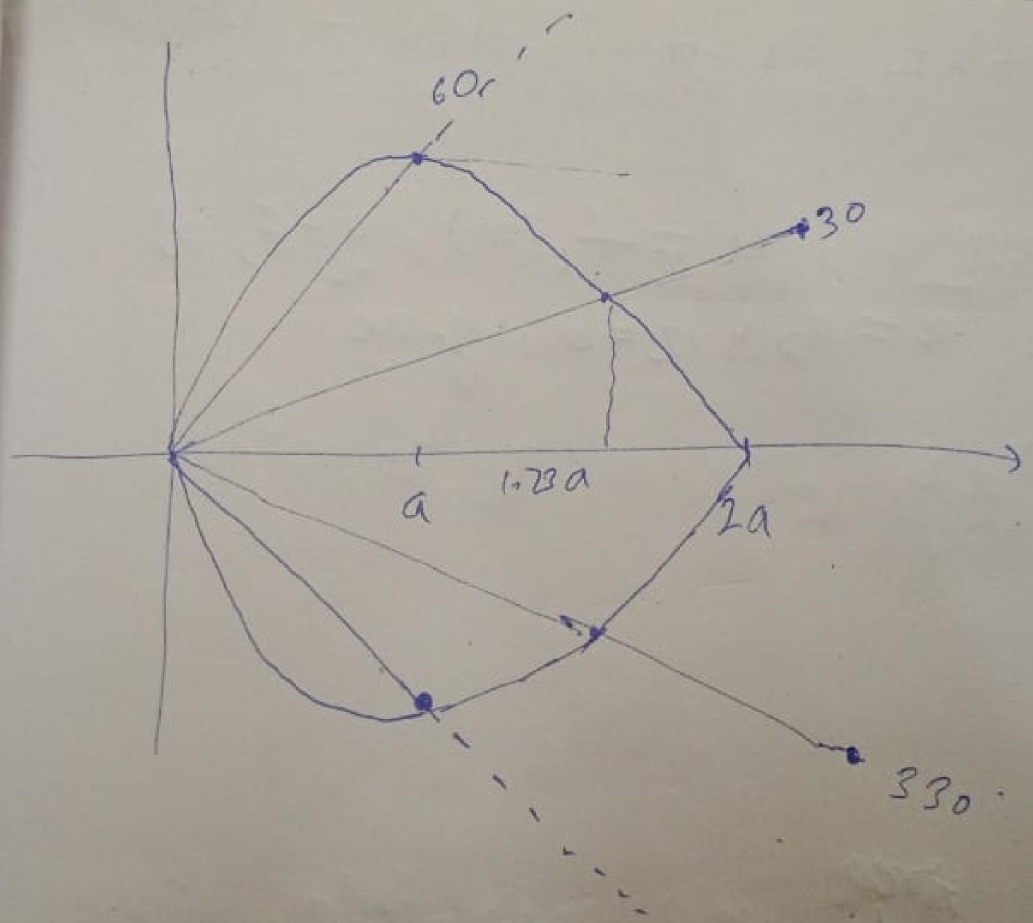
r & θ,

50	1	18	210
	-		-

is negative

g above

curve



AREA OF BOUNDED CURVE IN CARTISAN - 22 Nov

1st Q. a #

Area b/w two curve, $y_1 = f_1(x)$, $y_2 = f_2(x)$ and the ordinates $x = a$, & $x = b$, give

$$A = \int_a^b (y_2 - y_1)\,dx$$

$$\boxed{A = \int_a^b y\,dx}$$

2nd

Q.1 find the common Area to the parabola, $x^2 = 4ay$, $y^2 = 4ax$

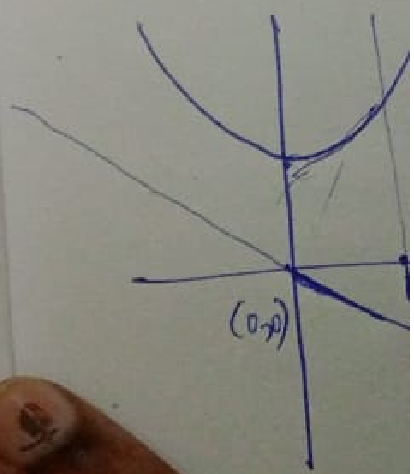
⇒ Result ⇒ $\frac{16}{3}|ab|$

⇒ Solve by aov

⇒ POI (0,0) &

=

Q. $y = x^2$+



$, y_2 = f_2(x)$
ive

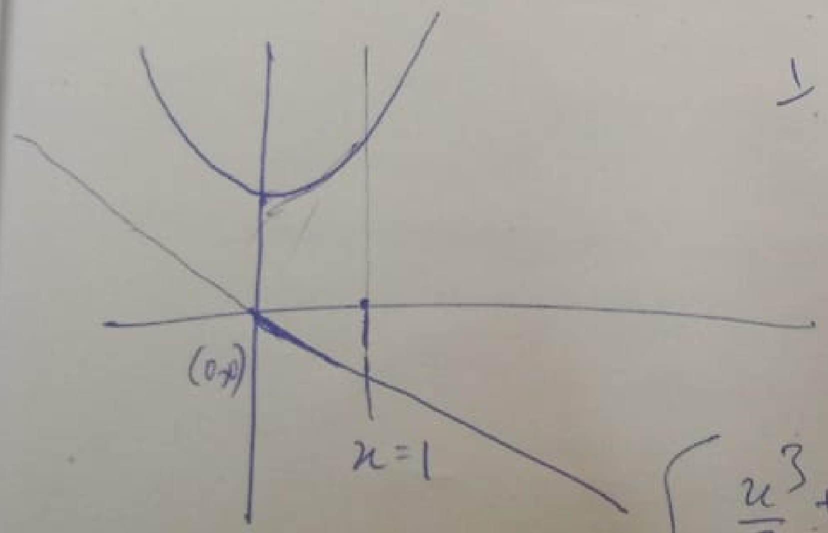
$\Rightarrow$ POI $(0,0)$ & $(4a, 4a)$

$A = \left(\frac{-1}{\lambda}\right)^2$

$=$

la, $x^2 = 4ay$
$y^2 = 4ax$

Q. $y = x^2 + 2$, $x = 0$, $x = 1$, $y = -x$



$y = x^2 + 1$

$y = -x$

$$\int_0^1 \left[(x^2+1) - (-x)\right] dx$$

$$\int_0^1 (x^2 + x + 1)$$

$$\left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]$$

Q. Find area of loop of curve $ay^2 = x^2(a-x)$

Ans. $\frac{18}{5}$

$\Rightarrow$