Linear DE of n^{th} order

Linear DE are those eq^n in which the dependent variable & ~~its~~ derivatives offer only to the 1^{st} degree & not multiplied together.

Eg ① $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 5y = x^3 + x^2 + 1$

② $\frac{d^2x}{dt^2} + \frac{2dx}{dt} + 5x = \phi(t)$

Non-Linear DE -

A DE which is not linear is non-linear DE.

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 5y^2 = x^3 + x^2 + 1$

2 Questions

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Linear DE of n^{th} order with constant coefficients-

A DE of the form $a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = dx$ → Eq ①

Above DE in differential operation

i.e. $\frac{d}{dx} = D$, can be written as,

$\frac{dy}{dx} = Dy$

$\frac{d^2y}{dx^2} = D^2y$

$\frac{d^ny}{dx^n} = D^ny$

$\frac{1}{D} = \int dx$

$\frac{1}{D^2} = \iint dx\, dx$

$\frac{1}{D^n} = \iiint \dots dx\, dx \dots$ (n times)

$a_0 D^n y + a_1 D^{n-1} y + \dots + a_n y = \phi(x)$ → Eq ②

$(a_0 D^n + a_1 D^{n-1} + \dots + a_n) y = \phi(x)$

$f(D) = \phi(x)$

complete solution → CS = CF + PI

CF → complementary function

PI → Particularly integer.

Methods for finding Complementary Function [CF]

(1) Firstly we replace RHS by zero.

RHS → 0

(2) Let $y = C_1 e^{mx}$; be the comp. funct. of given eq^n.

$a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n y = 0$

$\boxed{y = C_1 e^{mx}}$

$\frac{dy}{dx} = C_1 m e^{mx} \dots \frac{d^ny}{dx^n} = C_1 m^n e^{mx}$

$$a_0 C_1 m^n e^{mx} + a_1 C_1 m^{n-1} e^{mx} + \dots + a_n C_1 e^{mx} = 0$$

$$(a_0 m^n + a_1 m^{n-1} + \dots a_n) C_1 e^{mx} = 0$$

$$(a_0 m^n + a_1 m^{n-1} + \dots a_n) y = 0$$

$$a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$$

Note :-

In the process of complementary funct

(1) We find oscillary eqn by replacing RHS by zero. $D \to m$ and $y \to 1$ in Eq②

Imp Points-

(1) Roots of Osc. eqn are real and distinct.

Let roots be- $m = m_1, m_2, m_3 \dots m_n$

$$CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$$

(2) Roots are equal-

$m_1 = m_2 = m_3 = \dots = m_n$

$$CF = (C_1 + xC_2 + x^2 C_3 + \dots + x^{n-1} C_n) e^{mx}$$

(3) Roots of OE are imaginary-

$m = \alpha \pm i\beta$ -

$$CF = e^{\alpha x} [C_1 \cos\beta x + C_2 \sin\beta x]$$

Method to find PI

$f(D) y = \underline{Q(x)}$ RHS

$$PI = \frac{1}{f(D)} Q(x)$$

(I) $Q(x) = e^{ax}$

$$PI = \frac{1}{f(D)} e^{ax}$$

$D \to a$

$$= \frac{1}{F(a)} e^{ax} \; [F(a) \neq 0]$$

If $F(a) = 0$ then,

In case of failure.

i.e. $F(a) = 0$ then

$$= x \frac{1}{F'(a)} e^{ax} \; [F'(a) \neq 0]$$

(II) $Qx = \sin ax$ OR $\cos ax$

$$PI = \frac{1}{F(D^2)} \sin ax \text{ or } \cos ax$$

$D^2 \to -a^2$

#12 marks question

$$= \frac{1}{F(-a^2)} \sin ax \text{ OR } \cos ax \; [f(-a^2) \neq 0]$$

(III) $Q(x) = x^n$

$$PI = \frac{1}{F(D)} x^n$$

$$= [F(D)]^{-1} x^n$$

{→ Expand $[F(D)]^{-1}$ in ascending powers of 'D' as far as the operation on x^n becomes zero}

(IV) $Q(x) = e^{ax} f_1(x)$

$$PI = \frac{1}{F(D)} e^{ax} f_1(x)$$

$D \to D + a$

$$= e^{ax} \frac{1}{f(D+a)} f_1(x)$$

(v) $\phi(x)$ = Any function in x.

$f(D) \to$ factorize.

$$\frac{1}{(D+a)} Q(x) = e^{-ax}\int e^{ax} Q(x)\,dx$$

$$\frac{1}{(D-a)} \phi(x) = e^{ax}\int e^{-ax} \phi(x)\,dx.$$

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(Q) Solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$.

It is given, $\to$ Eq ①

Above DE in diff operator form i.e. $\frac{d}{dx} = D$, can be written as -

$$D^2 - 6D + 9y = 6e^{3x} + 7e^{-2x} - \log 2. \to \text{Eq ②}$$

In the process of CF, first we find AE by replacing RHS by 0, D by m and $y \to 1$ in Eq ②. So, AE is as follows

$$m^2 - 6m + 9 = 0$$

Solve for 'm'

$$m = \frac{6 \pm \sqrt{36-36}}{2} = 3, 3$$

So, $CF = (C_1 + x C_2) e^{3x} \to$ Eq ③

Now, $PI = \frac{1}{D^2-6D+9}(6e^{3x} + 7e^{-2x} - \log 2)$

(OR) $PI = \frac{1}{(D^2-6D+9)} 6e^{3x} + \frac{1}{(D^2-6D+9)}(7e^{-2x}) + \frac{1}{(D^2-6D+9)}(\log 2) \to$ Eq ④

$\downarrow PI_1 \quad \downarrow PI_2 \quad \downarrow PI_3$

$PI = PI_1 + PI_2 - PI_3$

(v) $\phi(x)$ = Any function in x.

$f(D) \to$ factorize.

$$\frac{1}{(D+a)} Q(x) = e^{-ax}\int e^{ax} Q(x)\,dx$$

$$\frac{1}{(D-a)} \phi(x) = e^{ax}\int e^{-ax} \phi(x)\,dx.$$

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Solve for 'm'

$$m = \frac{6 \pm \sqrt{36-36}}{2} = 3, 3$$

So, $CF = (C_1 + x C_2) e^{3x} \to$ Eq ③

Now, $PI = \frac{1}{D^2-6D+9}(6e^{3x} + 7e^{-2x} - \log 2)$

(OR) $PI = \frac{1}{(D^2-6D+9)} 6e^{3x} + \frac{1}{(D^2-6D+9)}(7e^{-2x}) + \frac{1}{(D^2-6D+9)}(\log 2) \to$ Eq ④

$\downarrow PI_1 \quad \downarrow PI_2 \quad \downarrow PI_3$

$PI = PI_1 + PI_2 - PI_3$

$PI_1 = \frac{1}{D^2-6D+9} 6e^{3x}$

$D \to$ (x ka coeff.) 3

$PI_1 = \frac{1}{3^2-6(3)+9} 6e^{3x}$

$= 6 \frac{1}{9-18+9} e^{3x}$ (Rule failed)

$PI_1 = 6x \frac{1}{(2D-6)} e^{3x}$

$D \to 3$

$= 6x \frac{1}{6-6} e^{3x}$ (Rule failed)

$PI_1 = 6x^2 \frac{1}{2} e^{3x}$

$\boxed{PI_1 = 3x^2 e^{3x}}$

Now,

For $PI_2 = \frac{1}{D^2-6D+9} (7e^{-2x})$

$D \to -2$

$= 7 \times \frac{1}{(-2)^2-6(-2)+9} e^{-2x}$ $4+12+9$

$= \frac{7e^{-2x}}{25}$

$\boxed{PI_2 = \frac{7e^{-2x}}{25}}$ (No case of failuer)

Now,

$PI_3 = \frac{1}{D^2-6D+9} (\log 2) \boxed{e^{0x}}$ → For const quantity.

$PI_3 = \frac{1}{0^2-6(0)+9} (\log 2)$

$\boxed{PI_3 = \frac{\log 2}{9}}$

By Eq ⓑ

So, $PI = 3x^2e^{3x} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$ → Eq ⓒ

So, complete solution is -

$CS = CF + PI$

$y = (c_1 + x c_2) e^{3x} + 3x^2 e^{3x} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$

(Que) Solve: $\frac{d^4y}{dx^4} - \frac{3d^2y}{dx^2} - 4y = 5\sin 2x$

↳ Eq ①

$\frac{dy}{dx} = D$ in Eq ①

$D^4 - 3D^2 - 4y = 5\sin 2x - e^{-2x}$ → Eq ②

RHS → 0, D → m, y → 1 For CF

$m^4 - 3m^2 - 4 = 0$

$m^2 = \frac{+3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = 4, -1$

$m = \pm 2, \pm i$.

So,

$CF = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos x + c_4 \sin x$

Now, $PI = \frac{1}{(D^4-3D^2-4)} (5\sin 2x - e^{-2x})$

$PI = \underbrace{\frac{1}{(D^4-3D^2-4)} (5\sin 2x)}_{PI_1} - \underbrace{\frac{1}{(D^4-3D^2-4)} e^{-2x}}_{PI_2}$

$PI = PI_1 - PI_2$ → ⓑ

$PI_1 = \frac{1}{D^4 - 3D^2 - 4} 5\sin 2x$

$D^2 \to -2^2$

$PI_1 = \frac{1}{(-2^2)^2 - 3(-2^2) - 4} 5\sin 2x$

$PI_1 = 5 \frac{1}{16+12-4} \sin 2x$

$PI_1 = \frac{5\sin 2x}{24}$

Ques Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = x + e^x \cos x$.

↳ Eq ①

Put D in the place of $\frac{d}{dx}$

$D^2 - 2D + 2y = x + e^x \cos x$

For CF,

$D \to m$, $y \to 1$, $RHS \to 0$

$m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$

$m = 1+i, 1-i$

$CF = e^{x}[C_1 \cos x + C_2 \sin x]$

Now, $PI = \frac{1}{D^2 - 2D + 2}(x + e^x \cos x)$

$$PI = \underbrace{\frac{1}{D^2-2D+2} x}_{PI_1} + \underbrace{\frac{1}{D^2-2D+2} e^x \cos x}_{PI_2}$$

$PI = PI_1 + PI_2$

$PI_1 = \frac{1}{D^2 - 2D + 2} x$

By Binomial theorem

$PI_1 = \frac{1}{2(2 - 2D + D^2)} x$

$= \frac{1}{2}\left(1 - D + \frac{D^2}{2}\right)^{-1} x$

$= \frac{1}{2}\left(1 - \left(D - \frac{D^2}{2}\right)\right)^{-1} x$

Expand

$= \frac{1}{2}\left(1 + D - \frac{D^2}{2} + \dots\right) x$

$= \frac{1}{2}\left(x + Dx - \frac{1}{2}D^2 x + \dots\right)$

$PI_1 = \frac{1}{2}(x+1)$

Now,

$PI_2 = \frac{1}{D^2 - 2D + 2} e^x \cos x$

$D \to (D+1)$

$PI_2 = e^x \frac{1}{[(D+1)^2 - 2(D+1) + 2]} \cos x$

$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 2} \cos x$

$= e^x \frac{1}{(D^2+1)} \cos x$.

$= e^x \frac{1}{}$

(Que) Solve: $\frac{d^2y}{dx^2} + 9y = \sec 3x$

$\hookrightarrow$ Eq ①

Put $\frac{d}{dx} = D$

$D^2 + 9y = \sec 3x$

For CF, $D \to m$, RHS $\to 0$, $y = 1$

$m^2 + 9 = 0$

$m = 0 \pm 3i$

$CF = e^{0x}[c_1 \cos 3x + c_2 \sin 3x]$

$CF = c_1 \cos 3x + c_2 \sin 3x$

Now,

$PI = \frac{1}{(D^2+3^2)} \sec 3x$

$= \frac{1}{(D+3i)(D-3i)} \sec 3x$

$= \frac{1}{6i}\left[\frac{1}{(D-3i)} - \frac{1}{(D+3i)}\right] \sec 3x$

$= \frac{1}{6i}\left(\frac{1}{(D-3i)} \sec 3x - \frac{1}{(D+3i)} \sec 3x\right) \rightarrow$ Eq ②

sign se vary karta toh ek solve karke

Now,

$\frac{1}{(D-3i)} \sec 3x = e^{3ix} \int e^{-3ix} \cdot \sec 3x \, dx$. sign change karke put kardege i wale terms cancel ho jayege. PI ki value aa jayegi

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Cauchy-Euler Homogeneous Linear DE.

A DE of the form

$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_{n-2} x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_0 y = \phi(x)$

$a_0, a_1, a_2 \dots$ are constants

To solve,

$x = e^z$ or $z = \log x$, $\frac{d}{dz} = D$

Now, $\frac{dy}{dx} = \frac{dy}{dz}\left(\frac{dz}{dx}\right) = \frac{1}{x}\frac{dy}{dz}$

$$\boxed{x\frac{dy}{dx} = Dy}$$

Now,

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$

$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dz}{dx}\right)\left(\frac{dy}{dz}\right)$

$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$

$$\boxed{x^2\frac{d^2y}{dx^2} = D(D-1)y}$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$\vdots$$

$$x^n\frac{d^ny}{dx^n} = D(D-1)(D-2)\cdots(D-(n-1))y$$

(Que) Solve: $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$

↳ Eq ①

Put $x = e^z$ or $z = \log x$ & $\dfrac{d}{dz} = D$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$D(D-1)y - 2Dy - 4y = e^{4z}$$

$$(D^2 - D - 2D - 4)y = e^{4z}$$

$$(D^2 - 3D - 4)y = e^{4z} \quad \rightarrow ⓐ$$

$D = \dfrac{3 \pm \sqrt{9+16}}{2}$

$D = \dfrac{3 \pm 5}{2}$

$D = 4, -1$

$(D-4)(D+1)y = e^{4z}$

roots: 4 & −1

$$CF = C_2e^{4z} + C_1e^{-z}$$

$$\boxed{CF = \frac{C_1}{x} + C_2x^4}$$

Now, for PI.

$$PI = \frac{1}{(D^2-3D-4)}e^{4z}$$

$D \to 4$!

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$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$\vdots$$

$$x^n\frac{d^ny}{dx^n} = D(D-1)(D-2)\cdots(D-(n-1))y$$

(Que) Solve: $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$

↳ Eq ①

Put $x = e^z$ or $z = \log x$ & $\dfrac{d}{dz} = D$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$D(D-1)y - 2Dy - 4y = e^{4z}$$

$$(D^2 - D - 2D - 4)y = e^{4z}$$

$$(D^2 - 3D - 4)y = e^{4z} \quad \rightarrow ⓐ$$

$D = \dfrac{3 \pm \sqrt{9+16}}{2}$

$D = \dfrac{3 \pm 5}{2}$

$D = 4, -1$

$(D-4)(D+1)y = e^{4z}$

roots: 4 & −1

$$CF = C_2e^{4z} + C_1e^{-z}$$

$$\boxed{CF = \frac{C_1}{x} + C_2x^4}$$

Now, for PI.

$$PI = \frac{1}{(D^2-3D-4)}e^{4z}$$

$D \to 4$!

Methods of variation of Parameters:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = \phi(x) \rightarrow (1)$$

$CF : Ay_1 + By_2$

$\left\{ \begin{array}{l} A, B : \text{const} \\ y_1, y_2 : \text{funct in } x \end{array} \right\}$

Assume $PI = uy_1 + vy_2$

where, $u = \int \frac{-y_2 \phi(x)}{y_1 y_2' - y_1' y_2} dx$

$v = \int \frac{y_1 Q(x)}{y_1 y_2' - y_1' y_2} dx$

Que: Solve: $\frac{d^2y}{dx^2} + y = \text{cosec } x$.

For CF,

$\frac{dy}{dx} \rightarrow D$; RHS $\rightarrow 0$; $y \rightarrow 1$

$D^2 + 1 = 0$

$D = \pm i$

$CF = C_1 \cos x + C_2 \sin x$

Now, for $PI = uy_1 + vy_2$

$(y_1 = \cos x, \ y_2 = \sin x) \ (\phi(x) = \text{cosec } x)$

$u = \int \frac{-\sin x \ \text{cosec } x}{\cos x(\cos x) - (-\sin x)\cos x} dx$

$= -x$

$v = \int \frac{\cos x \ \text{cosec } x \ dx}{\cos x(\cos x) + \sin x(\sin x)}$

$v = \int \cot x \ dx$

$v = \ln$

Simaltaneous linear DE.

If two or more dependent variables are functions of a single independent variable the eq^n. involving the derivative are simaltaneous linear DE.

For Eg,

$\left. \begin{array}{l} \frac{dx}{dt} + 4x + 3y = t \\ \frac{dy}{dt} + 2x + 5y = e^t \end{array} \right\}$ x, y dependent, t independent

Methods of solving these eq^n

is based on the process of elimination as we solve algebric simaltaneous equation.

Let $\frac{d}{dt} = D$

$(D+4)x + 3y = t \rightarrow Eq (1)$

$2x + (D+5)y = e^t \rightarrow Eq (2)$

(1) Elimination

Eq (1) $\times 2$ & Eq (2) $\times (D+4)$

$2(D+4)x + 6y = 2t \rightarrow (3)$

$2(D+4)x + (D+5)(D+4)y = (D+4)e^t$

$2(D+4)x + (D^2+9D+20)y = e^t + 4e^t \rightarrow (4)$

$De^t = \frac{d}{dt}$

(4) − (3)

$(D^2 + 9D + 20 - 6)y = 5e^t - 2t$

$(D^2 + 9D + 14)y = 5e^t - 2t \rightarrow (5)$

$D = \frac{-9 \pm \sqrt{81 - 56}}{2}$

By solving for CF & PI find y

Partial Differential Equations-

Those equations which contains partial derivatives, independent & dependent variables.

Independent variable will be denoted by 'x' and 'y' and the dependent variable by 'z'
The partial coefficients are denoted as follows:

$$\frac{\partial z}{\partial x} = p \quad , \quad \frac{\partial z}{\partial y} = q \rightarrow$$ First order, First degree. (Not r,

$$\downarrow$$

$$\frac{\partial^2 z}{\partial x^2} = r \quad , \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

and p, q are higher order DE.

Order and Degree of a partial DE.

It is same as the order and degree of an ordinary DE.

Methods of forming Partial DE.

(1) By Eliminating arbitary constants:

(Que) Form the Partial DE. by Eliminating arbitary const 'a' & 'b'.

$$2x + 0 + 2(z-a)\frac{\partial}{\partial x}(z-a) = 0$$

$$2x + 2(z-a)\left(\frac{\partial z}{\partial x} - \frac{\partial a}{\partial x}\right) = 0$$

$$2x + 2(z-a)\,p = 0$$

$$(z-a) = \frac{-x}{p} \rightarrow (a)$$

Diff both side in Eq① parially wrt 'y'

$$\frac{\partial}{\partial y}\left(x^2+y^2+(z-a)^2\right) = \frac{\partial (b^2)}{\partial y}$$

$$0 + 2y + 2(z-a)\left(\frac{\partial z}{\partial y} - \frac{\partial a}{\partial y}\right) = 0$$

$$2y + 2(z-a)\,q = 0$$

$$z-a = \frac{-y}{q} \rightarrow (b)$$

From (a) & (b)

$$\frac{-x}{p} = \frac{-y}{q}$$

$$\boxed{qx - py = 0}$$ required DE.

NOTE: If number of

(2) By Eliminating Arbitary functions

(Que) Form partial DE -

$z = f(x^2 - y^2)$ arbitary function

Given that,

$z = f(x^2 - y^2) \rightarrow$ Eq ①

Diff both side partially wrt 'x'

So we get,

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(f(x^2 - y^2))$

$\frac{\partial z}{\partial x} = f'(x^2 - y^2) \frac{\partial (x^2 - y^2)}{\partial x}$

$\frac{\partial z}{\partial x} = f'(x^2 - y^2)\, 2x.$

$f'(x^2 - y^2) = \frac{\partial z}{\partial x} \frac{1}{2x}$ ⓐ

$— " — = \frac{p}{2x}$

Now,

Diff Eq ① both side partially wrt 'y'

$\frac{\partial z}{\partial y} = f'(x^2 - y^2) \frac{\partial}{\partial y}(x^2 - y^2)$

$\frac{\partial z}{\partial y} = -f'(x^2 - y^2) \cdot 2y$

$f'(x^2 - y^2) = \frac{-q}{2y} \rightarrow$ ⓑ

From ⓐ & ⓑ

$\frac{-q}{2y} = \frac{p}{2x}$

$\boxed{qx + py = 0}$ Req. Patial DE.

1 func → 1st order 1st degree partial DE.

more than 1 arbitary funct → Higher order & degree partial DE.

Solution of Partial DE

→ By Direct Integration

(Que) Solve: $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

We have, ↳ Eq ①

Int. Eq ① wrt 'x'

So we get,

$\frac{\partial^2 z}{\partial x \partial y} = \left[\int \cos(2x + 3y)\, \partial x\right] + \phi_1(y)$ arbitary const in terms of arbitary function

$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2} \sin(2x + 3y) + \phi_1(y).$ ↳ Eq ②

Now,

Int. Eq ② wrt 'x'

$\frac{\partial z}{\partial y} = \left[\int \left[\frac{1}{2} \sin(2x + 3y) + \phi_1(y)\right] \partial x\right] + \phi_2(y)$

$\frac{\partial z}{\partial y} = \frac{-1}{4} \cos(2x + 3y) + \phi_1(y) \int \partial x + \phi_2(y)$

$\frac{\partial z}{\partial y} = \frac{-1}{4} \cos(2x + 3y) + \phi_1(y)\, x + \phi_2(y)$ ↳ Eq ③

Now, Int Eq ③ wrt 'y'

So, we get,

$z = \int \frac{-1}{4} \cos(2x + 3y)\, dy + \int \phi_1(y)\, x\, dy + \int \phi_2(y)\, d$

$+ \phi_3(x).$

$z = \frac{-1}{12} \sin(2x + 3y) + x\underbrace{F_1(y)}_{\int \phi_1(y) dy} + \underbrace{F_2(y)}_{\int \phi_2(y) dy} + \phi_3(x)$

Lag Linear Partial DE

An Eqⁿ of the type.

$$Pp + Qq = R$$

where, P, Q, R are functions of x, y, z and $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ is called Lagrenge's linear Partial DE.

And its solⁿ is given by

$f(u, v) = 0$ (OR)
$u = f(v)$
(OR)
$v = f(u)$
] u, v functions of x, y, and z.

Working Rule to solve -

$$Pp + Qq = R$$

① Auxillary Eqⁿ $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

② Solve the above auxillary Eqⁿ (A E)
let the two solⁿ be -
$u = C_1$ and $v = C_2$

③ Then
$f(u, v) = 0$
$u = f(v)$
$v = f(u)$
] is the required solution.

Grouping Method (By Que).

Que: Solve : $yq - xp = z$ (small p or q → Lag.)
Given that ↳ Eq ①

compare partial DE ① with
$Pp + Qq = R$.

So, $P = -x$
$Q = y$
$R = z$

So, AE will be -
We know that,
for $Pp + Qq = R$
is given by -
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

So, AE for partial DE ① will be

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

For 1st solⁿ, we are taking 1st two fractio

i.e. $\frac{dx}{-x} = \frac{dy}{y} \Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$
which on int gives -

$\ln x + \ln y = \ln C_1$
$\ln(xy) = \ln(C_1)$
After taking anti Log -
$xy = C_1$ (constant).

Now for 2nd solⁿ - we are taking last 2 frac
i.e. $\frac{dy}{y} = \frac{dz}{z} \Rightarrow \frac{dy}{dy} - \frac{dz}{z} = 0$

After int

$\ln y - \ln z = \ln c_2$

After taking anti ln

$\frac{y}{z} = c_2$

So, Required sol^n is -

$f[(x,y), (y/z)] = 0$

$xy = f(y/z)$

OR

$y/z = f(xy)$

Que: Solve: $y^2p - xyq = x(z-2y)$ ↳ Eq ①

given that

Comparing eq ① with $Pp + Qq = R$

we get,

$P = y^2$

$Q = -xy$

$R = x(z-2y)$

We know that,

Integrating by both sides

$\boxed{x^2 + y^2 = c_1}$

Now, For 2^{nd} sol^n, taking last 2^n fractions

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{(z-2y)}{y}dy + dz = 0$$

$$z\,dy - 2y\,dy + y\,dz = 0$$

$$zy - 2y^2 + zy = c_2$$

$$\boxed{2(zy - y^2) = c_2}$$

Method of Multipliers -

Let the AE be -

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Let l, m, n may be constants or functions of x, y, z.

Then we have -

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$ → So l, m, n are selected in such a way that $\boxed{lP + mQ + nR = 0}$

So, $l\,dx + m\,dy + n\,dz = 0$

which after integration gives 1^{st} set of sol^n

i.e. $u = c_1$

Illy we select 2^{nd} set of multiplier l_1, m_1, n_1 for the 2^{nd} sol^n

i.e. $v = c_2$

$\therefore$ $f(u, v) = 0$ (OR) $u = f(v)$ (OR) $v = f(u)$

is the req. sol^n

Solve 1: $x(y^2+z)\frac{\partial z}{\partial x} - y(x^2+z)\frac{\partial z}{\partial y} = z.(x^2-y^2)$ ↳ Eq ①

given that,

Compare with $Pp + Qq = R$. with eq ①

$x(y^2+z)\,p - y(x^2+z)\,q = z(x^2-y^2)$

$P = x(y^2+z)$

$Q = -y(x^2+z)$

$R = z(x^2-y^2)$

now, AE for Eq ①

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

(Grouping Not Applicable as one variable can not be cancelled completely in any 2 fraction).

Suppose, It's 1st set of multiplier- $\boxed{(x, y, -1)}$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{xdx+ydy-1dz}{x^2y^2+x^2z-x^2y^2-y^2z-zx^2+zy^2}$$

Each fraction $= \frac{xdx+ydy-dz}{0}$

$xdx+ydy-dz=0$

After integrating,

$\boxed{x^2+y^2-2z=C_1}$

Now, taking 2nd set of multiplier- $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

Each fraction $= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

Int by both side

$\ln x+\ln y+\ln z = \ln C_2$

so, $\boxed{xyz = C_2}$

Que 2: Solve: $x(z^2-y^2)\frac{dz}{dx} + y(x^2-z^2)\frac{dz}{dy} = z(y^2-x^2)$ ↳ Eq ①

It can be written as-

$x(z^2-y^2)\,p + y(x^2-z^2)\,q = z(y^2-x^2)$ ↳ Eq ②

Comparing it with $Pp+Qq=R$

we get,

$P = x(z^2-y^2)$

$Q = y(x^2-z^2)$

$R = z(y^2-x^2)$

Now, AE for Eq ① will be.

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

Suppose, 1st set of multiplier (x, y, z)

2nd $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$(z^2y^2+x^2z^2-(y^2x^2+y^2z^2)-(z^2x^2-z^2y^2)$

$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

$xyz=C_2$

Linear Homogeneous PDE of n^{th} order with const coefficient -

An equation of the type -

$$a_0\frac{\partial^n z}{\partial x^n} + \frac{a_1 \partial^n z}{\partial x^{n-1}\partial y} + \frac{a_2 \partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + \frac{a_n \partial^n z}{\partial y^n} = \phi(x,y) \to Eq\ ①$$

is known as the linear homogeneous PDE of n^{th} order with cont coeff.

Put $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$

Then. It will be -

$$a_0 D^n + a_1 D^{n-1}D' + a_2 D^{n-2}D'^2 + \cdots + a_n D'^n = \phi(x,y)$$

$$\Rightarrow F(D,D')z = \phi(x,y) \to Eq\ ②$$

$$CS\ (\text{of } 1\ \&\ 2) = CF + PI \to Eq\ ③$$

Rules for finding the CF -

In the process of finding CF first we find AE by replacing RHS by 0, $D \to m$, $D' \to 1$ and $z = 1$ in Eq ②

$\therefore$ AE $\Rightarrow F(m,1) = 0$

ie. $a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots a_n = 0 \to Eq\ ④$

1. Roots of AE are distinct.

$m = m_1, m_2, m_3, \cdots, m_n$

$CF = F_1(y+m_1x) + F_2(y+m_2x) + \cdots + F_m(y+m_nx)$

2. Roots of AE are equal.

$m = m_1 = m_2 = \cdots = m_n$

$CF = F_1(y+mx) + xF_2(y+mx) + x^2F_3(y+mx) + \cdots + x^nF_n(y+mx).$

Methods of finding PI

$F(D,D')z = \phi(x,y)$

$PI = \frac{1}{f(D,D')}\phi(x,y)$

(1) $\phi(x,y) = e^{ax+by}$

$PI = \frac{1}{f(D,D')}e^{ax+by}$

$D \to a$; $D' \to b$

$\therefore PI = \frac{1}{f(a,b)}e^{ax+by}$ $[f(a,b) \neq 0]$

If $f(a,b) = 0$ then

$PI = x\frac{1}{\frac{\partial}{\partial D}[f(D,D')]}e^{ax+by}$ D' will be constant here.

$\therefore PI = \frac{x}{f'(a,b)}e^{ax+by}$ $[f'(a,b) \neq 0]$

(2) $\phi(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$PI = \frac{1}{F(D^2, DD', D'^2)}\sin(ax+by)$ or $\cos(ax+by)$

$D^2 \to -a^2$; $DD' = -ab$, $D'^2 = -b^2$

$\therefore PI = \frac{1}{F(-a^2,-ab,-b^2)}\sin(ax+by)$ or $\cos(ax+by)$

$[F(-a^2, -ab, -b^2) \neq 0]$

If zero, then partially differentiate wrt D.

(3) $\phi(x,y) = x^m y^n$

$PI = \frac{1}{F(D,D')}x^m y^n$

(i) If $m > n$, then $\frac{1}{F(D,D')}$ is expanded in powers $\frac{D'}{D}$.

(ii) If $m < n$, then $\frac{1}{F(D,D')}$ is expanded in powers $\frac{D}{D'}$.

then we'll have choice.

(4) $\phi(x,y)$ = Any function in (x,y)

$$PI = \frac{1}{f(D,D')} \phi(x,y)$$

Factorize

$$PI = \frac{1}{(D-m_1D')(D-m_2D')\ldots(D-m_nD')} \phi(x,y)$$

$$\frac{1}{(D-m_1D')} \phi(x,y) = \int \phi(x, c+mx)\,dx \qquad \left[\begin{array}{l} y = c+mx \\ c \text{ is a constant} \end{array}\right]$$

After integrating, we replace c by $y-mx$. We repeat the same process for the remaining factors.

Questions:

(1) Solve : $\frac{\partial^2 z}{\partial x^2} - \frac{3\partial^2 z}{\partial x \partial y} + \frac{2\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos(x+2y)$

Given PDE can be written in the form of differential operation, i.e., $\frac{\partial}{\partial x} = D$, $\frac{\partial}{\partial y} = D'$ as follows:

$z(D^2 - 3DD' + 2D'^2) = e^{2x-y} + e^{x+y} + \cos(x+2y) \rightarrow$ Eq①

In the process of finding CF first we find AE by replacing RHS $\rightarrow 0$, $D \rightarrow m$, $D' \rightarrow 1$ and $z \rightarrow 1$

$\therefore$ AE $\Rightarrow (m^2 - 3m + 2) = 0$

$m = 1, 2.$

$\therefore$ CF = $F_1(y+x) + F_2(y+2x) \rightarrow$ Eq②

Now, $PI = \frac{1}{D^2 - 3DD' + 2D'^2} \cdot e^{2x-y} + e^{x+y} + \cos(x+2y)$

$PI = \frac{1}{D^2-3DD'+2D'^2} e^{2x-y} + \frac{1}{D^2-3DD'+2D'^2} e^{x+y} + \frac{1}{D^2-3DD'+2D'^2} \cos(x+2y)$

$PI = PI_1 + PI_2 + PI_3$

$PI_1 = \frac{1}{D^2-3DD'+2D'^2} e^{2x-y}$

$\Rightarrow D \rightarrow 2$, $D' - 1$

$PI_1 = \frac{1}{12} e^{2x-y}$

$PI_2 = \frac{1}{D^2-3DD'+2D'^2} e^{x+y}$

$\Rightarrow D \rightarrow 1$, $D' \rightarrow 1$

$PI_2 = \frac{1}{1-3+2=0} e^{x+y}$ (rule fails)

$= x \frac{1}{2D-3D'} e^{x+y}$

$= x \frac{1}{2-3} e^{x+y}$

$PI_2 = -x e^{x+y}$

$PI_3 = \frac{1}{D^2-3DD'+2D'^2} \cos(x+2y)$

$D^2 = -1$, $DD' = -2$, $D'^2 = -4$

$PI_3 = \frac{1}{-1+6-8} \cos(x+2y)$

$= \frac{-1}{3} \cos(x+2y)$

$PI = \frac{1}{12} e^{2x-y} + (-x e^{x+y}) + \left(-\frac{1}{3}\cos(x+2y)\right) \rightarrow$ Eq③

So, from ~~eq~~ ② and ③

$CS = F_1(y+x) + F_2(y+2x) + \frac{1}{12} e^{2x-y} - x e^{x+y} - \frac{1}{3}\cos(x+2y)$