DIFFERENTIAL EQUATION -

Equations which involve differential coefficient is called a diff. equation.

Eg. ① $\frac{dy}{dx} = \frac{1+x^2}{1-y^2}$ $O \to 1$ $D \to 1$ ② $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$ $O \to 2$ $D \to 1$

③ $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$ $O \to 2$ $D \to 2$ after sq both side ④ $x\frac{dy}{dx} + y\frac{dy}{dy} = ny$ $O \to 1$ $D \to 1$

⑤ $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial y}$ $O \to 2$ $D \to 1$

Two types of DE -

① Ordinary DE - A differential eqn involving derivaties wrt a single independent variable is called ordinary DE.

② Partial DE - A DE involving partial derivatives wrt more than one independent variable is called a partial DE.

Order & Degree of a D.E.

The order of a DE is the order of the highest differential coefficient present in the eqn.

The degree of a DE is the degree of the highest derivative after removing the radical sign and fraction.

Formation of DE-

[order = No. of arbitary constants]

The DE can be formed by differentiating the ordinary eq^n and eleminating the arbitary const.

(Q.) Form the DE. from $y = ax + a^2$ by eliminating arbitary constant 'a'.

It is given that,

$$y = ax + a^2 \quad \rightarrow eq ①$$

Diff eq ① wrt x.

we get,

$$\frac{dy}{dx} = \frac{d(ax + a^2)}{dx}$$

$$\frac{dy}{dx} = a\frac{dx}{dx} + x\frac{da}{dx} + \frac{da^2}{dx}$$

$$\frac{dy}{dx} = a.1 + x.0 + 0.$$

$$\frac{dy}{dx} = a \quad \rightarrow eq ②$$

Substitute the value of 'a' in eq ①

$$y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2$$

$$x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 - y = 0$$

Order = 1

Degree = 2

(Q) Find DE by eleminating arb. const. A and B from $y = A\cos x + B\sin x$.

It is given that,

$$y = A\cos x + B\sin x \quad \rightarrow eq ①$$

Diff both side wrt x,

we will get,

$$\frac{dy}{dx} = \frac{d(A\cos x + B\sin x)}{dx}$$

$$\frac{dy}{dx} = \frac{A\,d\cos x}{dx} + \cos x\frac{dA}{dx} + \frac{B\,d\sin x}{dx} + \sin x\frac{dB}{dx}$$

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

DE. of the 1^{st} order and 1^{st} Degree -
(Std Methods)

1. Variable seperable.
2. Homogeneous
3. Reducible to homogeneous
4. ~~Redu~~ Linear
5. Reducable to linear

Exact DE

An eq^n of the type $\boxed{Mdx + Ndy = 0}$
where, $M = f(x, y)$
$N = f(x, y)$.

is said to be Exact DE if $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

and its sol^n is given by intg of -

$$\boxed{\underset{\substack{\text{keeping}\\ y \text{ as constant}}}{\int M dx} + \int (\text{terms of } N \text{ free from } x)\, dy = C}$$

Que. Solve-

$(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$

Let $\hookrightarrow$ eq ①

Comparing ① with $Mdx + Ndy = 0$

So, $M = 5x^4 + 3x^2y^2 - 2xy^3 \rightarrow$ eq ②

$N = 2x^3y - 3x^2y^2 - 5y^4 \rightarrow$ eq ③

Differentiate eq ② partially wrt. y

$$\frac{\partial M}{\partial y} = \frac{\partial (5x^4 + 3x^2y^2 - 2xy^3)}{\partial y}$$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$$

Now,

Diff. eq ③ partially wrt. x.

$$\frac{\partial N}{\partial x} = \frac{\partial (2x^3y - 3x^2y^2 - 5y^4)}{\partial x}$$

$$\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} = 6x^2y - 6xy^2$$

$\therefore$ Given eq ① is Exact DE.

Sol^n is given by -

$$\underset{\text{keeping y as const}}{\int M dx} + \underset{\text{Those terms of N do not contain x}}{\int N dy} = C$$

$$\int (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int -5y^4\,dy = C$$

$$x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

DE reduciable to the exact eqⁿ

If a DE which is not exact may becomes so on multiplication by a suitable function known as integrating factor.

Rules -

① If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of 'x' only; '$f(x)$'

Then, $IF = e^{\int f(x)\,dx}$.

② If $\dfrac{-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}}{M}$ is a function of 'y' only; '$f(y)$'

Then, $IF = e^{\int f(y)\,dy}$

③ If M is on the form - $M = y f(xy)$ and $N = x f(xy)$ then $IF = \dfrac{1}{Mx - Ny}$.

④ If a DE is of the form $x^m \times y^n (ay\,dx + bx\,dy) + x^{n'} * y^{n'}(a'y\,dx + b'x\,dy) = 0$

Then $IF = x^h y^k$ (where h & k is obtained by solving following eqⁿ -

$$\frac{m+h+1}{a} = \frac{n+k+1}{b} \quad \text{and} \quad \frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

⑤ If the given expression $Mdx + Ndy = 0$ is homogeneous eqⁿ and $Mx + Ny \neq 0$. Then, $IF = \dfrac{1}{Mx + Ny}$

(Que) Solve : $(2x\log x - xy)dy + 2y\,dx = 0$

It is given that, ↳ eq ①

Compare eq ① with $Mdx + Ndy = 0$

So, $M = 2y$
$N = 2x\log x - xy$

Now, Diff M partially with resp to. y.

$$\frac{\partial M}{\partial y} = \frac{\partial(2y)}{\partial y}$$

$$\frac{\partial M}{\partial y} = 2 \rightarrow ⓐ$$

Now, Diff N partially wrt x.

$$\frac{\partial N}{\partial x} = \frac{\partial(2x\log x - xy)}{\partial x}$$

$$\frac{\partial N}{\partial x} = 2 + 2\log x - y \rightarrow ⓑ$$

From ⓐ and ⓑ

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

→ Not Exact DE. But it become so by multiplying by IF.

where, IF can be find by :- $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2 - 2 - 2\log x + y}{x(2\log x - y)}$

$$IF = -\frac{1}{x}\frac{(2\log x - y)}{(2\log x - y)} = -\frac{1}{x}$$

Then $IF = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$

Now Multiply eq ① by IF we will get -

$$(2\log x - y)\,dy + \frac{2y}{x}\,dx = 0 \rightarrow Eq\ ②$$

Now checking Eq ② is wheather Exact DE or not -

Comparing Eq ② with $M_1 dx + N_1 dy = 0$

$$M_1 = \frac{2y}{x}$$

$$N_1 = \frac{2\log x - y}{1}$$

Diff. M_1 partially wrt y -

$$\frac{\partial M_1}{\partial y} = \frac{\partial\left(\frac{2y}{x}\right)}{\partial x}$$

$$\frac{\partial M_1}{\partial y} = \frac{2}{x} \rightarrow (a_1)$$

Diff. N_1 partially wrt x -

$$\frac{\partial N_1}{\partial x} = \frac{\partial(2\log x - y)}{\partial x}$$

$$\frac{\partial N_1}{\partial x} = \frac{2}{x} \rightarrow (b_1)$$

$\because$ Eq (a_1) and (b_1) are same

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$\therefore$ Eq ② is Exact DE -

② $(y^4 + 2y)\,dx + (\cancel{x}y^3 + 2y^4 - 4x)\,dy = 0$ ↳ Eq ①

Compare Eq ① with $M dx + N dy = 0$

So,

$$M = y^4 + 2y$$

$$N = \cancel{2}y^3 + 2y^4 - 4x$$

Diff M wrt y - partially -

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \rightarrow (a)$$

Diff- N wrt x partially-

$$\frac{\partial N}{\partial x} = y^3 - 4 \rightarrow (b)$$

$\because$ (a) $\neq$ (b)

$\therefore$ Eq (1) is not Exact DE.

Now, finding IF - by Rule #2

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y^3 - 4 - (4y^3 + 2)}{y^4 + 2y}$$

$$= \frac{-3y^3 - 6}{y^4 + 2y} = -\frac{3}{y}$$

$$IF = e^{\int -\frac{3y^3+6}{y^4+2y} dy} = 3/y$$

doubt

$$xy + \frac{2x}{y^2} + y^2 = c \longrightarrow \text{Final Ans-}$$

(Que) $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)\, dy = 0$ $\rightarrow$ Eq (1)

Comparing with $Mdx + Ndy = 0$

So, $M = xy^2 + 2x^2y^3$

$N = x^2y - x^3y^2$

Diff M wrt y partially.

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \rightarrow (a)$$

Diff N wrt x partially-

$$\frac{\partial N}{\partial x} = 2xy - 3x^2y^2 \rightarrow (b)$$

$\because$ (a) $\neq$ (b)

Now finding IF by Rule #3

$$IF = \frac{1}{Mx - Ny} = \frac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3} = \frac{1}{3x^3y^3}$$

$$\frac{3x^2y^2}{x^2y - x^3y^2}$$

$$\frac{3y}{1 - xy}$$

Multiply eq ① by IF

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right) dy = 0$$

↳ Eq ②

$$M' = \left(\frac{1}{3x^2y} + \frac{2}{3x}\right) \qquad N' = \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)$$

$$\frac{\partial M'}{\partial y} = \frac{-\cancel{2}}{3x^3y} \cancel{+\frac{2}{3x^2}} = \frac{\partial N'}{\partial x} = \frac{-1}{3x^2y^2}$$

$$\frac{-1}{3x^2y^2}$$

Exact DE → eq ②

Ques $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$

↳ Eq ①

$$M = y^3 - 2x^2y$$
$$N = 2xy^2 - x^3$$

$$\frac{\partial M}{\partial y} = 3y^2 - 2x^2 \neq \frac{\partial N}{\partial x} = 2y^2 - 3x^2$$

Finding IF by Rule #5

\# Rule #4

$$IF = \frac{1}{Mx + Ny}$$

$$IF = \frac{1}{y^3x - 2x^3y + 2xy^3 - x^3y} = \frac{1}{3xy^3 - 3x^3y} = \frac{1}{3xy(y^2 - x^2)}$$

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$(xy^3)dy - (x^3+y^3)dx = 0$ → Eq ①

$M = -(x^3+y^3)$

$N = xy^3$

$\frac{\partial m}{\partial y} = -3y^2$

$\frac{\partial n}{\partial x} = y^3$

By Rule # 5

$IF = \frac{1}{Mx+Ny} = \frac{1}{xy^3 - x^4 - y^3x} = \frac{1}{x(y^4-x^3-y^3)}$

$= \frac{-1}{x^4}$

multiply Eq ① with IF

$\left(\frac{-y^3}{x^3}\right)dy + \left(\frac{1}{x}\right.$

11/10/2023

Linear DE of n^{th} order

linear DE are those eq^n in which the dependent variable & ~~its~~ derivatives offer only to the 1^{st} degree & not multiplied together.

Eg ① $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 5y = x^3 + x^2 + 1$

② $\frac{d^2x}{dt^2} + \frac{2dx}{dt} + 5x = \phi(t)$

Non-Linear DE -

A DE which is not linear is non-linear DE.

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 5y^2 = x^3 + x^2 + 1$

2 Questions

Linear DE of n^{th} order with constant coefficients -

A DE of the form $a_0 \frac{d^n y}{\partial x^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = dx$

↳ Eq (1)

Above DE in differential operation

i.e. $\frac{d}{dx} = D$, can be written as,

$\frac{dy}{dx} = Dy$

$\frac{d^2y}{dx^2} = D^2 y$

$\frac{d^n y}{dx^n} = D^n y$

$\frac{1}{D} = \int dx$

$\frac{1}{D^2} = \iint dx\, dx$

$\frac{1}{D^n} = \iiint \dots dx\, dx \dots$ (n times)

$a_0 D^n y + a_1 D^{n-1} y + \dots + a_n y = \phi(x) \rightarrow$ Eq (2)

$(a_0 D^n + a_1 D^{n-1} + \dots + a_n)\, y = \phi(x)$

$f(D) = \phi(x)$

complete solution → CS = CF + PI

CF → complementary function

PI → Particularly integer.

Methods for finding Complementary Function [CF]

(1) Firstly we replace RHS by zero.

RHS → 0

(2) Let $y = C_1 e^{mx}$; be the comp. funct. of given eq^n.

$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} \dots a_n y = 0$

$\boxed{y = C_1 e^{mx}}$

$\frac{dy}{dx} = C_1 m e^{mx} \dots \quad \frac{d^n y}{dx} = C_1 m^n e^{mx}$

$$a_0 C_1 m^n e^{mx} + a_1 C_1 m^{n-1} e^{mx} + \dots + a_n C_1 e^{mx} = 0$$

$$(a_0 m^n + a_1 m^{n-1} + \dots a_n) C_1 e^{mx} = 0$$

$$(a_0 m^n + a_1 m^{n-1} + \dots a_n) y = 0$$

$$a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$$

Note :-

In the process of complementary funct
(1) We find oscillary eq^n by replacing RHS by zero. $D \to m$ and $y \to 1$ in Eq ②

Imp Points-

(1) Roots of Osc. eq^n are real and distinct.

Let roots be- $m = m_1, m_2, m_3 \dots m_n$

$$CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$$

(2) Roots are equal -

$m_1 = m_2 = m_3 = \dots = m_n$

$$CF = (C_1 + x C_2 + x^2 C_3 + \dots + x^{n-1} C_n) e^{mx}$$

(3) Roots of OE are imaginary -

$m = \alpha \pm i\beta$

$$CF = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$$

\# Methods to find PI

$f(D) y = Q(x)$ R.H.S

$$PI = \frac{1}{f(D)} Q(x)$$

(I) $Q(x) = e^{ax}$

$$PI = \frac{1}{f(D)} e^{ax}$$

$D \to a$

$$= \frac{1}{F(a)} e^{ax} \quad [F(a) \neq 0]$$

If $F(a) = 0$ then,

In case of failure,

i.e. $F(a) = 0$ then

$$= x \frac{1}{F'(a)} e^{ax} \quad [F'(a) \neq 0]$$

(II) $Q(x) = \sin ax$ OR $\cos ax$

$$PI = \frac{1}{F(D^2)} \sin ax \text{ or } \cos ax$$

$D^2 \to -a^2$

#12 marks question

$$= \frac{1}{F(-a^2)} \sin ax \text{ OR } \cos ax \quad [f(-a^2) \neq 0]$$

(III) $\phi(x) = x^n$

$$PI = \frac{1}{F(D)} x^n$$

$$= [F(D)]^{-1} x^n$$

{→ Expand $[F(D)]^{-1}$ in ascending powers of 'D' as far as the operation on x^n becomes zero.}

(IV) $\phi(x) = e^{ax} f_1(x)$

$$PI = \frac{1}{F(D)} e^{ax} f_1(x)$$

$D \to D + a$

$$= e^{ax} \frac{1}{f(D+a)} f_1(x)$$

(v) $Q(x) =$ Any function in x.

$f(D) \to$ factorize.

$$\frac{1}{(D+a)} Q(x) = e^{-ax} \int e^{ax} Q(x)\, dx$$

$$\frac{1}{(D-a)} Q(x) = e^{ax} \int e^{-ax} Q(x)\, dx.$$

27/10/23

(Q) Solve! $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2.$

It is given, $\to$ Eq ①

Above DE in diff operator form
i.e. $\frac{d}{dx} = D$, can be written as -

$D^2 - 6D + 9y = 6e^{3x} + 7e^{-2x} - \log 2. \to$ Eq②

In the process of CF, first we find. AE
by replacing RHS by 0, D by m
and $y \to 1$ in Eq ②. So, AE is as follows-

$m^2 - 6m + 9 = 0$

solve for 'm'

$$m = \frac{6 \pm \sqrt{36 - 36}}{2} = 3, 3.$$

So, $CF = (C_1 + x C_2) e^{3x} \to$ Eq ⓐ

Now, $PI = \frac{1}{D^2 - 6D + 9} (6e^{3x} + 7e^{-2x} - \log 2)$

(or) $PI = \underbrace{\frac{1}{(D^2 - 6D + 9)} 6e^{3x}}_{PI_1} + \underbrace{\frac{1}{(D^2 - 6D + 9)} (7e^{-2x})}_{PI_2} + \underbrace{\frac{1}{(D^2 - 6D + 9)} (\log 2)}_{PI_3} \to$ Eq ⓑ

$PI = PI_1 + PI_2 - PI_3$

(v) $\phi(x)$ = Any function in x.

$f(D) \rightarrow$ factorize.

$$\frac{1}{(D+a)} \phi(x) = e^{-ax} \int e^{ax} \phi(x)\, dx.$$

$$\frac{1}{(D-a)} \phi(x) = e^{ax} \int e^{-ax} \phi(x)\, dx.$$

27/10/23

(Q) Solve! $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2.$

It is given, $\rightarrow$ Eq ①

Above DE in diff operator form

i.e. $\frac{d}{dx} = D$, can be written as -

$$D^2 - 6D + 9y = 6e^{3x} + 7e^{-2x} - \log 2. \rightarrow \text{Eq ②}$$

In the process of CF, first we find. AE by replacing RHS by 0, D by m and $y \rightarrow 1$ in Eq ②. So, AE is as follows-

$$m^2 - 6m + 9 = 0$$

solve for 'm'

$$m = \frac{6 \pm \sqrt{36 - 36}}{2} = 3, 3.$$

So,
$$CF = (C_1 + x C_2) e^{3x} \rightarrow \text{Eq ⓐ}$$

Now, $PI = \frac{1}{D^2 - 6D + 9} (6e^{3x} + 7e^{-2x} - \log 2)$

(or) $PI = \underbrace{\frac{1}{(D^2-6D+9)} 6e^{3x}}_{PI_1} + \underbrace{\frac{1}{(D^2-6D+9)} (7e^{-2x})}_{PI_2} - \underbrace{\frac{1}{(D^2-6D+9)} (\log 2)}_{PI_3}$ $\rightarrow$ Eq ⓑ

$PI = PI_1 + PI_2 - PI_3$

$$PI_1 = \frac{1}{D^2-6D+9} 6e^{3x}$$

$D \to$ (x ka coeff.) 3

$$PI_1 = \frac{1}{3^2-6(3)+9} 6e^{3x}$$

$$= 6 \frac{1}{9-18+9} e^{3x} \quad \text{(Rule failed)}$$

$$PI_1 = 6x \frac{1}{(2D-6)} e^{3x}$$

$D \to 3$

$$= 6x \frac{1}{6-6} e^{3x} \quad \text{(Rule failed)}$$

$$PI_1 = 6x^2 \frac{1}{2} e^{3x}$$

$$\boxed{PI_1 = 3x^2 e^{3x}}$$

Now,
For $PI_2 = \frac{1}{D^2-6D+9} (7e^{-2x})$

$D \to -2$

$$= 7 \times \frac{1}{(-2)^2-6(-2)+9} e^{-2x}$$

$4+12+9$

$$= \frac{7e^{-2x}}{25}$$

$$\boxed{PI_2 = \frac{7e^{-2x}}{25}}$$ (No case of failuer)

Now,
$PI_3 = \frac{1}{D^2-6D+9} (\log 2) \boxed{e^{0x}} \rightarrow$ For const quantity.

$$PI_3 = \frac{1}{0^2-6(0)+9} (\log 2)$$

$$\boxed{PI_3 = \frac{\log 2}{9}}$$

By Eq ⓑ

So, $PI = 3x^2e^{3x} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$ → Eq ⓒ

So, complete solution is -

$CS = CF + PI$

$y = (c_1 + x c_2) e^{3x} + 3x^2 e^{3x} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$

(Que) Solve: $\frac{d^4y}{dx^4} - \frac{3d^2y}{dx^2} - 4y = 5\sin 2x - e^{-2x}$

↳ Eq ①

$\frac{dy}{dx} = D$ in Eq ①

$D^4 - 3D^2 - 4y = 5\sin 2x - e^{-2x}$ → Eq ②

$RHS \to 0$, $D \to m$, $y \to 1$. For CF

$m^4 - 3m^2 - 4 = 0$

$m^2 = \frac{+3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = 4, -1$

$m = \pm 2, \pm i.$

So,

$CF = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos x + C_4 \sin x.$

Now, $PI = \frac{1}{(D^4 - 3D^2 - 4)} (5\sin 2x - e^{-2x})$

$PI = \underbrace{\frac{1}{(D^4 - 3D^2 - 4)}(5\sin 2x)}_{PI_1} - \underbrace{\frac{1}{(D^4 - 3D^2 - 4)} e^{-2x}}_{PI_2}$

$PI = PI_1 - PI_2$ → ⓑ

$PI_1 = \frac{1}{D^4 - 3D^2 - 4} 5\sin 2x$

$D^2 \to -2^2$

$PI_1 = \frac{1}{(-2^2)^2 - 3(-2^2) - 4} 5\sin 2x$

$PI_1 = 5 \frac{1}{16+12-4} \sin 2x$

$PI_1 = \frac{5\sin 2x}{24}$

(Ques) Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = x + e^x \cos x$.

$\hookrightarrow$ Eq (1)

Put D in the place of $\frac{d}{dx}$

$D^2 - 2D + 2y = x + e^x \cos x$

For CF,

$D \to m$, $y \Rightarrow 1$, RHS $\to 0$

$m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$

$m = 1+i, 1-i$

$CF = e^{x}[C_1 \cos x + C_2 \sin x]$

Now, $PI = \frac{1}{D^2 - 2D + 2}(x + e^x \cos x)$

$PI = \underbrace{\frac{1}{D^2-2D+2} x}_{PI_1} + \underbrace{\frac{1}{D^2-2D+2} e^x \cos x}_{PI_2}$

$PI = PI_1 + PI_2$

$$PI_1 = \frac{1}{D^2-2D+2}\, x$$

By binomial theorem

$$PI_1 = \frac{1}{(2-2D+D^2)}\, x$$

$$= \frac{1}{2}\left(1-D+\frac{D^2}{2}\right)^{-1} x$$

$$= \frac{1}{2}\left(1-\underbrace{\left(D-\frac{D^2}{2}\right)}\right)^{-1} x$$

Expand

$$= \frac{1}{2}\left(1+D-\frac{D^2}{2}+\cdots\right)x.$$

$$= \frac{1}{2}\left(x+Dx-\frac{1}{2}D^2x+\cdots\right)$$

$$PI_1 = \frac{1}{2}(x+1)$$

Now,

$$PI_2 = \frac{1}{D^2-2D+2}\, e^x \cos x$$

$$D \to (D+1)$$

$$PI_2 = e^x \frac{1}{[(D+1)^2-2(D+1)+2]} \cos x.$$

$$= e^x \frac{1}{D^2+\cancel{2D}+1-\cancel{2D}-\cancel{2}+\cancel{2}} \cos x$$

$$= e^x \frac{1}{(D^2+1)} \cos x.$$

$$= e \frac{1}{}$$

(Que) Solve: $\frac{d^2y}{dx^2} + 9y = \sec 3x$

↳ Eq ①

Put $\frac{d}{dx} = D$

$D^2 + 9y = \sec 3x$

For CF, $D \to m$, RHS → 0, y = 1

$m^2 + 9 = 0$

$m = 0 \pm 3i$

$CF = e^{0x} [C_1 \cos 3x + C_2 \sin 3x]$

$CF = C_1 \cos 3x + C_2 \sin 3x$

Now,

$PI = \frac{1}{(D^2+3^2)} \sec 3x$

$= \frac{1}{(D+3i)(D-3i)} \sec 3x$

$= \frac{1}{6i} \left[\frac{1}{(D-3i)} - \frac{1}{(D+3i)} \right] \sec 3x$

$= \frac{1}{6i} \left(\frac{1}{(D-3i)} \sec 3x - \frac{1}{(D+3i)} \sec 3x \right)$ → Eq ②

sign se vary karta toh ek solve karke

Now,

$\frac{1}{(D-3i)} \sec 3x = e^{3ix} \int e^{-3ix} \cdot \sec 3x \, dx$. sign change karke put kardege i wale terms cancel ho jayege.

PI ki value aa jayegi

30/10/23

Cauchy-Euler Homogeneous Linear DE.

A DE of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_0 y = \phi(x)$$

$a_0, a_1, a_2 \dots$ are constants.

To solve,

$x = e^z$ or $z = \log x$, $\frac{d}{dz} = D$

Now, $\frac{dy}{dx} = \frac{dy}{dz}\left(\frac{dz}{dx}\right) = \frac{1}{x}\frac{dy}{dz}$

$$\boxed{x\frac{dy}{dx} = Dy}$$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dz}{dx}\right)\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\boxed{x^2\frac{d^2y}{dx^2} = D(D-1)y}$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

|

|

$$x^n \frac{d^ny}{dx^n} = D(D-1)(D-2) \cdots (D-(n-1))y$$

(Que) Solve: $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

↳ Eq (1)

Put $x = e^z$ or $z = \log x$ & $\frac{d}{dz} = D$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$D(D-1)y - 2Dy - 4y = e^{4z}$$

$$(D^2 - D - 2D - 4)y = e^{4z}$$

$$(D^2 - 3D - 4)y = e^{4z} \rightarrow (a)$$

$D = \frac{3 \pm \sqrt{9+16}}{2}$

$D = \frac{3 \pm 5}{2}$

$D = 4, -1$

$(D-4)(D+1)y = e^{4z}$

roots: 4 & -1

$$CF = C_2 e^{4z} + C_1 e^{-z}$$

$$\boxed{CF = \frac{C_1}{x} + C_2 x^4}$$

Now, for PI.

$$PI = \frac{1}{(D^2 - 3D - 4)} e^{4z}$$

$D \rightarrow 4$!

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

|

|

$$x^n\frac{d^ny}{dx^n} = D(D-1)(D-2)\cdots(D-(n-1))y$$

(Que) Solve: $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

↳ Eq ①

Put $x = e^z$ or $z = \log x$ & $\frac{d}{dz} = D$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$D(D-1)y - 2Dy - 4y = e^{4z}$$

$$(D^2 - D - 2D - 4)y = e^{4z}$$

$$(D^2 - 3D - 4)y = e^{4z} \quad \rightarrow ⓐ$$

roots: 4 & -1

$D = \frac{3 \pm \sqrt{9+16}}{2}$

$D = \frac{3 \pm 5}{2}$

$D = 4, -1$

$(D-4)(D+1)y = e^{4z}$

$$CF = C_2e^{4z} + C_1e^{-z}$$

$$\boxed{CF = \frac{C_1}{x} + C_2x^4}$$

Now, for PI.

$$PI = \frac{1}{(D^2-3D-4)}e^{4z}$$

$D \to 4$!

Methods of variation of Parameters:

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = \phi(x) \rightarrow ①$$

$CF = Ay_1 + By_2$

$\left\{ \begin{array}{l} A, B : \text{const} \\ y_1, y_2 : \text{funct in } x \end{array} \right\}$

Assume $PI = uy_1 + vy_2$

where, $u = \int \frac{-y_2 \phi(x)}{y_1 y_2' - y_1' y_2} dx$

$$v = \int \frac{y_1 Q(x)}{y_1 y_2' - y_1' y_2} dx.$$

Que: Solve: $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$

For CF,

$\frac{dy}{dx} \rightarrow D$; RHS $\rightarrow 0$; $y \rightarrow 1$

$D^2 + 1 = 0$

$D = \pm i.$

$CF = C_1 \cos x + C_2 \sin x.$

Now, for $PI = uy_1 + vy_2$

$(y_1 = \cos x, \; y_2 = \sin x) \; (\phi(x) = \operatorname{cosec} x)$

$u = \int \frac{-\sin x \operatorname{cosec} x}{\cos x(\cos x) - (-\sin x)\cos x} dx$; $v = \int \frac{\cos x \operatorname{cosec} x \, dx}{\cos x(\cos x) + \sin x(\sin x)}$

$= -x$; $v = \int \cot x \, dx$

$v = \ln$

Simaltaneous linear DE.

If two or more dependent variables are functions of a single independent variable the eq^n. Involving the derivative are simaltaneous linear DE.

For Eg,
$$\left.\begin{aligned}\frac{dx}{dt} + 4x + 3y &= t\\ \frac{dy}{dt} + 2x + 5y &= e^t\end{aligned}\right\}$$
x, y dependent
& independent

Methods of solving these eq^n
is based on the process of elimination as we solve algebric simaltaneous equation.

Let $\frac{d}{dt} = D$

$(D+4)x + 3y = t \rightarrow$ Eq (1)

$2x + (D+5)y = e^t \rightarrow$ Eq (2)

(1) Elimination

Eq (1) $\times 2$ & Eq (2) $\times (D+4)$.

$2(D+4)x + 6y = 2t \rightarrow$ (3)

$2(D+4)x + (D+5)(D+4)y = (D+4)e^t$

$2(D+4)x + (D^2 + 9D + 20)y = e^t + 4e^t \rightarrow$ (4)

$De^t = \frac{de^t}{dt} = e^t$

(4) − (3)

$(D^2 + 9D + 20 - 6)y = 5e^t - 2t$

$D = \frac{-9 \pm \sqrt{81-56}}{2}$

$(D^2 + 9D + 14)y = 5e^t - 2t \rightarrow$ (5)

By solving for CF & PI find y