UNIT-1 ⇒ Differential calculus

Review of succesive differentiation.

formula :-

i. $D^n(e^{ax+b}) = a^n e^{ax+b}$

ii. $D^n(a^{mx}) = a^{mx}(\log a)^n m^n$

iii. $D^n(ax+b)^m = m.(m-1).(m-2)\dots(m-n+1).a^n(ax+b)^{m-n}$; $m>n$

↳ if $n>m$, then $D^n(ax+b)^m = 0$

↳ if $n=m$, then $D^n(ax+b)^n = n!a^n$

iv. $D^n \sin(ax+b) = a^n \sin(ax+b+n\pi/2)$

v. $D^n \cos(ax+b) = a^n \cos(ax+b+n\pi/2)$

vi. $D^n e^{ax}\sin(bx+c) = (\sqrt{a^2+b^2})^x e^{ax}\sin(bx+c+n\tan^{-1}b/a)$

vii. $D^n e^{ax}\cos(bx+c) = (\sqrt{a^2+b^2})^x e^{ax}\cos(bx+c+n\tan^{-1}b/a)$

ix. $D^n \log(ax+b) = \dfrac{a^n(-1)^{n-1}(n-1)!}{(ax+b)^n}$

Ex $y = \sin^2 x$ n^{th} derivative.

Sol $y = (\sin x)^2$

$y' = 2\sin x.\cos x$

$y' = 2\sin 2x$

$y'' = \cos 2x.2$

$y''' = -2\sin 2x.2 = -4\sin 2x$

$y_n = 2^{n-1}\cancel{\cos}(\cancel{\sin 2x}.n\frac{\pi}{2}+2x)$

$1-2\sin^2 x = \cos 2x$

$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$

diff n times

$= 0 - \frac{1}{2}2^n\cos(2x+n\pi/2)$

$= -2^{n-1}\cos(2x+n\pi/2)$

Leibnitz theorem.

$$D^n(u.v) = D^n(u).V + {}^nC_1.D^{n-1}(u).D(V) + {}^nC_2.D^{n-2}(u).D^2(V) \dots + {}^nC_n.u.D^n(V).$$

${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

Ex- find the n^{th} differential coefficiante of $x^3\cos x$.

Sol Given, $y = x^3\cos x$

$\Rightarrow D^n(x^3\cos x) = D^n(\cos x).x^3 + {}^nC_1.D^{n-1}(\cos x).D^1(x^3) + {}^nC_2.D^{n-2}(\cos x).D^2(x^3) + {}^nC_3.D^{n-3}(\cos x).D^3(x^3)$

$\cos(n\pi/2 - \pi/2 + x)$

$\cos(\pi/2 - (n\pi/2 + x))$

$\sin(n\pi/2 + x)$

$\Rightarrow D^n(x^3\cos x) = \cos(n\pi/2+x).x^3 + n.\cos((n-1)\pi/2+x).3x^2 + \frac{n(n-1)}{2}\cos((n-2)\pi/2+x).6x + \frac{n(n-1)(n-2)}{6}\cos((n-3)\pi/2+x).6$

$\Rightarrow D^n(x^3\cos x) = x^3\cos(n\pi/2+x) + 3x^2 n\sin(n\pi/2+x)$

$- 3x.n(n-1)\cos(n\pi/2+x) - n(n-1)(n-2)\sin(n\pi/2+x)$

DATE____ PAGE____

$$D^n(x^3\cos x) = x^3[x^2 - 3x(n-1)]\cos\left(x+\frac{n\pi}{2}\right) + n[6x^2-(n-1)(n-2)]\sin\left(x+\frac{n\pi}{2}\right)$$ Ans.

θ find the n^{th} derivative of $(1-x^2)y_2 + xy_1 + y = 0$

Sol Given $(1-x^2)y_2 + xy_1 + y = 0$.

$\because$ $D^n[(1-x^2)y_2] = D^n(y_2).(1-x^2) + {}^nC_1 D^{n-1}(y_2).D(1-x^2) + {}^nC_2 D^{n-2}(y_2).D^2(1-x^2) + 0$

$= y_{n+2}(1-x^2) + n.y_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2)$

$= (1-x^2)y_{n+2} - 2nx\,y_{n+1} - n(n-1)y_n$

$\because$ $D^n(x.y_1) = D^n(y_1)x + nD^{n-1}(y_1)D(x) + 0$

$= y_{n+1}.x + ny_n$

Thus, $(1-x^2)y_{n+2} - 2nx.y_{n+1} - n(n-1)y_n + xy_{n+1} + ny_n + y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n-1)x\,y_{n+1} - (n(n-1)-n-1)y_n = 0$

$\Rightarrow y_{n+2}(1-x^2) + y_{n+1}x(1-2n) + y_n(-n^2+2n+1) = 0$

→ it is called recurn. Ans

$$\# \quad D^n(ax+b)^{-1} = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$$

θ find y^n, $y = \frac{1}{1-5x+6x^2}$

Sol $y = \frac{1}{6x^2-5x+1} = \frac{1}{(3x-1)(2x-1)} = \frac{A}{(3x-1)} + \frac{B}{(2x-1)}$

$\frac{2-3}{3}$ $-\frac{1}{3}$

by P.f $A=-3$ & $= \frac{-3}{(3x-1)} + \frac{2}{(2x-1)}$

$\frac{3-7}{2}$ $\frac{1}{2}$

Thus, $y^n = -\frac{(-1)^n n!\,3^{n+1}}{(3x-1)^{n+1}} + \frac{(-1)^n n!\,2^{n+1}}{(2x-1)^{n+1}}$ Ans

{Trick} P.f

$$\frac{x-1}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$$

i) orignal $=0$ $(x+1)^2=0) \Rightarrow x=-1$

ii) for find B put $x=-1$ in main oution except orignal $B=-2$

iii) Put $x=0$ in every where

$\frac{-1}{1^2} = \frac{A}{1} + \frac{-2}{1^2}$

$A = 1$

Maclo
Am
& thi
{ f
θ y
Sol.
⇒
⇒
⇒
⇒
θ
Sol

DATE____ PAGE____

Maclourin's Series :-

An $f(x)$ can be expanded in ascending power of x & this expersion be differentiable any no: of times thro

$$\left\{ f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0) \right\}$$

Q $y = \frac{e^x}{1+e^x}$; Expand by maclourin's theorem for the term containing x^3.

Sol. $y = \frac{e^x}{1+e^x}$; $y]_0 = \frac{1}{1+1} = \frac{1}{2}$

$\Rightarrow$ $y']_0 = \frac{e^x(1+e^x) - e^x . e^x}{(1+e^x)^2} = \frac{2-1}{(1+1)^2} = \frac{1}{4}$

$\frac{e^x}{1+e^x} - \left(\frac{e^x}{1+e^x}\right)^2 \Rightarrow y - y^2 = y'$

$\Rightarrow$ $y'']_0 = y_1 - 2yy_1 = \frac{1}{4} - \frac{2 \times 1 \times 1}{2 \, 4} = 0$

$\Rightarrow$ $y''']_0 = \cancel{y'' - 2y'y''} = y'' - 2[y'y' + yy'']$

$= 0 - 2[\frac{1}{4} \times \frac{1}{4} + \frac{1}{2} \times 0] = -\frac{1}{8}$

$\Rightarrow$ $y = \frac{1}{2} + \frac{x}{4} + 0 - \frac{x^3}{6} \times \frac{1}{8} + \dots$ Ans.

Q $y = \log(\sec(x))$, Expand by maclorines theorem

Sol $y = \log(\sec x)$ $\qquad \log 1 = 0$

$\Rightarrow$ $y_0 = \log(\sec 0) = 0$

$\Rightarrow$ $y_1 = \frac{1}{\sec x} . \sec x . \tan x = 0$

$2\sec^2 x . \tan x$

$2[(2\sec x . \sec x . \tan x . \tan x)$

$+ 2\sec^2 x \sec^2 x$

$\Rightarrow$ $y_2 = \sec^2 x = 1$

$\Rightarrow$ $y_3 = 2 \sec x (\sec x . \tan x) = 0$

$\Rightarrow$ $y_4 = 2\sec^2 x \sec^2 x + 2\tan(2\sec x)(\sec x . \tan x) = 2$

Thus, $\log(\sec(x)) = \frac{x^2}{2} + \frac{x^4 \times 2}{24} + \dots$

Taylor's theorem :-

an $f(a+h)$ is a funᶜ of varible h (a=constant) then $f(a+h)$ can be expanding in ascending power of h such that

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f'(a) + \dots \frac{h^n}{n!} f^n(a).$$

$\because$ $x = a+h \Rightarrow h = x-a$

now, $f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \frac{(x-a)^n}{n!} f^n(a).$

Q Expand $\tan x$ in assending power of $x - \frac{\pi}{4}$

Sol Given $f(x) = \tan x$ & $h = x - \frac{\pi}{4}$, then $a = \frac{\pi}{4}$

$\Rightarrow f(a) = f(\pi/4) = \tan \pi/4 = 1$

$\Rightarrow f'(a) = \sec^2 x = 1 + \tan^2 x = 2$

$1 + y^2$

$\Rightarrow f''(a) = 2 y y_1 = 2 \times 1 \times 2 = 4$

$\Rightarrow f'''(a) = 2(y_1 y_1 + y y_2) = 2(2\times 2 + 1 \times 4) = 8 + 8 = 16$

$\Rightarrow f''''(a) = 4 y_1 y_2 + 2(y_1 y_2 + y y_3) = 80.$

By taylor series.

$$\tan x = 1 + x - \pi/4\,(2) + \frac{(x-\pi/4)^2 (4)}{2} + \frac{(x-\pi/4)^3 (16)}{6} + \frac{(x-\pi/4)^4 (80)}{24}$$

$$\tan x = 1 + 2(x - \pi/4) + 2(x-\pi/4)^2 + \frac{8}{3}(x-\pi/4)^3 + \dots$$

Ans.

Q Expand $\log x$ in power $(x-1)$ & find $\log(1.1)$ in for decimal place

Sol Given, $f(x) = \log x$ & $h = x - 1$

then, $a = 1$

$\Rightarrow f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$\Rightarrow f''(x) = -1 x^{-2} \Rightarrow f''(1) = -1$

$\Rightarrow f'''(x) = -1 \times -2 \times x^{-3} = \frac{2}{x^3} \Rightarrow f'''(1) = 2$

$\Rightarrow f^4(x) = 2 \times -3\, x^{-4} \Rightarrow f^4(1) = -6$

$\Rightarrow f^5(x) = -6 \times -4\, x^{-5} \Rightarrow f^5(1) = 24$

By taylor series.

$\log x = 0 + \frac{x-1}{1}(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6) + \frac{(x-1)^5}{24\times 5}(24)$

$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}$

Put $x = 1.1$, then $x-1 \Rightarrow 1.1-1 = 0.1$

$\log 1.1 = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} - \frac{(0.1)^5}{5}$

$\log 1.1 = 0.09531$ Ans.

Q Expand $\tan(x+\pi/4)$ as far as the term x^4 and find $\tan(46.5)$ upto four significant figure.

Sol $f(x) = \tan(x+\pi/4)$ & $h = x$

By previcious ques $a = \pi/4$

$y = 1,\ y_1 = 2,\ y_2 = 4,\ y_3 = 16,\ y_4 = 80$

By Taylor's series.

$\tan(x+\pi/4) = y + \frac{h}{1!}y_1 + \frac{h^2}{2!}y_2 + \frac{h^3}{3!}y_3 + \frac{h^4}{4!}y_4$

$\tan(x+\pi/4) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4$

$180° = \pi$

$1° = \frac{\pi}{180}$ Put $x = 1.5° \Rightarrow 0.02618$

$\tan(46.5°) = 1 + 2(0.02618) + 2(0.02618)^2 + 8/3(0.02618)^3 + 10/3(0.02618)^4$

$1.5° = \frac{3}{2}\times 0.01745$

$\tan(46.5°) = 1.0538$ (approx) Ans

$1.5° = 0.02618$

MST-1 2021

Q Find the n^{th} differential coefficient of $\cos^4 x$

Sol. Let $y = \cos^4 x$

$y = \frac{1}{4}\times 4\cos^4 x$

$\cos 2x = 2\cos^2 x - 1$ $\quad y = \frac{1}{4}(2\cos^2 x)^2$

$y = 1/4(1+\cos 2x)^2$

$y = 1/4(1 + \frac{2\cos^2 2x}{2} + 2\cos 2x)$

$y = \frac{1}{4} + \frac{1}{8}(1+\cos 4x) + \frac{1}{2}(\cos 2x)$

diff n time

$y_n = 0 + \frac{1}{8}(0 + 4^n\cos(n\pi/2 + 4x)) + \frac{1}{2}2^n\cos(n\pi/2+2x)$

$y_n = \frac{4^{n-1}}{2}\cos(\frac{n\pi}{2}+4x) + 2^{n-1}\cos(\frac{n\pi}{2}+2x)$

$y_n = 2^{2n-3}\cos(\frac{n\pi}{2}+4x) + 2^{n-1}\cos(\frac{n\pi}{2}+2x)$

Ans.

DATE______ PAGE______

MST-1 2019

Θ Expand $y = (\sin^{-1}x)^2$ by Maclaurins theorem.

Sol Given $y = (\sin^{-1}x)^2$

$\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right) = \frac{-1(1-x^2)^{-3/2} \cdot -2x}{2}$

$\Rightarrow y_0 = 0$

$= x(1-x^2)^{-3/2}$

$\Rightarrow y' = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$ at $0 = 0$

$y' = \frac{2\sqrt{y}}{\sqrt{1-x^2}} = 2\sqrt{\frac{y}{1-x^2}}$

$\Rightarrow y'' = 2\left(\frac{\left(\frac{1}{\sqrt{1-x^2}}\right)^2 - \sin^{-1}x \cdot x(1-x^2)^{-3/2}}{(\sqrt{1-x^2})^2}\right) =$

$\sqrt{(1-x^2)}(1-x^2)^{\frac{1}{2}+1}$

$\Rightarrow y'' = \frac{2 \times 1}{2} \cdot \frac{y}{1-x^2} \cdot \left[y'\sqrt{1-x^2} \quad . \quad 2\left(\frac{\frac{1}{1-x^2} -}{1-x^2}\right)\right.$

x——x——x——x——x——x

Asymptotes :- it is a straight line which touches the curve at infinty.

n th degree → n asymptotes

working rule -

i. Let $f(x,y) = 0$ be the curve of n^{th} degree

ii. put $y = mx + c$ in $f(x,y) = 0$ & simplify.

iii. name the coefficient of x^n, x^{n-1}, x^{n-2} is $\phi_n(m), \phi_{n-1}(m), \phi_{n-2}(m)$

iv. put $\phi_n(m) = 0$ & solve it for m

v. put $\phi_{n-1}(m) = 0$ & solve it for c, $c = g(m)$ find c for each m such that $c_1 = g(m_1)$ & $c_2 = g(m_2)$ .... etc.

vi. write asymptotes as $y = m_1x + c_1$, $y = m_2x + c_2$ & $y = m_3x + c_3$

Shorcut method -

i. simplify $f(x,y) = 0$ & put $x = 1$ & $y = m$ into highest degree term & name $\phi_n(m)$ & simlarly find $\phi_{n-1}(m)$, $\phi_{n-2}(m)$.

ii. put $\phi_n(m) = 0$ & solve it for m.

iii. find $c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)}$ (non repeated root) / $\phi''_n(m)c^2 + \phi'_{n-1}(m)c + \phi_{n-2}(m) = 0$ (2 repeated root)

iv. write asymptotes as $y = mx + c$

Θ write asymptotes of $y^3 - 3xy^2 - x^2y + 3x^3 - 3x^2 + 10xy - 3y^2 - 10x - 10y + 7 = 0$

Sol $f(x,y) \Rightarrow y^3 - 3xy^2 - x^2y + 3x^3 - 3x^2 + 10xy - 3y^2 - 10x - 10y + 7 = 0$

put $x = 1$ & $y = m$

$\phi_3(m) = m^3 - 3m^2 - m + 3$

$\phi_2(m) = -3 + 10m - 3m^2$

$\phi(m) = -10 - 10m$

Put $\phi_3(m) = 0$

$m^3 - 3m^2 - m + 3 = 0$

$m^2(m-3) - 1(m-3) = 0$

$(m-3)(m^2-1) = 0$

$(m-3)(m+1)(m-1) = 0$

$m = 3, -1, 1$

now, $c = \frac{-(-3m^2+10m-3)}{(3m^2-6m-1)} = \frac{3m^2-10m+3}{3m^2-6m-1}$

Put $m_1 = 1$

$c_1 = \frac{3-10+3}{3-6-1} = \frac{-4}{-4} = 1$ ∴ $y = x+1$

Put $m_2 = -1$

$c_2 = \frac{3(-1)^2-10(-1)+3}{3(-1)^2-6(-1)-1} = \frac{16}{8} = 2$ ∴ $y = -x+2$

put $m = 3$

$c_3 = \frac{3(3)^2-10(3)+3}{3(3)^2-6(3)-1} = \frac{27-30+3}{27-18-1} = \frac{0}{8} = 0$

∴ $y = 3x$

Thus, asymptotes is $y-x-1=0$, $y+x-2=0$ & $y-3x=0$ Ans

2019.

Q find the asymptote to the given curve:

$(x^2-y^2)(y^2-4x^2)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0$

Sol. Given, $x^2y^2-4x^4-y^4+4x^2y^2-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0$

$-4x^4-y^4+5x^2y^2-6x^3+5x^2y+3xy^2-2y^3$

put $y = m$ & $x = 1$ in 4th, 3rd, 2nd degree term.

$\phi_4(m) = -4-m^4+5m^2$

$\phi_3(m) = -6+5m+3m^2-2m^3$

$\phi_2(m) = -1+3m$

Put $\phi_4(m) = 0$

$m^4-5m^2+4 = 0$

$\cancel{m^2(m^2-5)+4=0}$

$m^4-4m^2-m^2+4$

$m^2(m^2-4)-1(m^2-4)=0$

$(m^2-1)(m^2-4)=0$

$m = \pm 1, \pm 4$

Thus, asymptotes is

$y = x$, $y = -x+1$

$y = 4x+\frac{33}{103}$, $y = -4x+\frac{75}{103}$

Ans.

now, $c = \frac{2m^3-3m^2-5m+6}{10m-4m^3}$

Put $m = +1$

$c_1 = \frac{2-3-5+6}{10-4} = 0$

put $m = -1$

$c_2 = \frac{-2-3+5+6}{-10+4} = 1$

put $m = +4$

$c_3 = \frac{2(64)-3(16)-5(4)+6}{10(4)-4(64)}$

$c_3 = \frac{128-48-20+6}{40-246} = \frac{66}{206} = \frac{33}{103}$

Put $m = -4$

$c_4 = \frac{-128-48+20+6}{-40+246} = \frac{150}{206} = \frac{75}{103}$

DATE ____ PAGE ____

Radius of curvature :-

$R.C \propto 1/\text{radius}$

$\kappa \propto 1/\rho$

i) Cartesian formula for Radius of curvature - if given curve is a cartesion plan in xr plane then ROC will be

$$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2}$$

ii) Polar formula for ROC - if $r = f(\theta)$ & $f(r,\theta) = 0$ then roc is

$$\rho = \frac{\{r^2 + (dr/d\theta)^2\}^{3/2}}{r^2 + 2(dr/d\theta)^2 - r\,d^2r/d\theta^2}$$

Q find R.o.C of a curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $(1/4, 1/4)$.

Sol $\sqrt{y} = \sqrt{a} - \sqrt{x}$

now, $\frac{1}{2} y^{-1/2} \frac{dy}{dx} = -\frac{1}{2} x^{-1/2}$

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}} \quad \text{at } (1/4, 1/4) \Rightarrow -1$$

again, $$\frac{d^2y}{dx^2} = -\left[\frac{\frac{1}{2} y^{-1/2} dy/dx \cdot \sqrt{x} - \frac{1}{2} x^{-1/2} y^{1/2}}{x}\right]$$

$$\frac{d^2y}{dx^2} = \left[-\frac{1}{2} \times \frac{1}{\sqrt{y}} \times \frac{-\sqrt{y}}{\sqrt{x}} \cdot \frac{\sqrt{x}}{x} + \frac{1}{2}\frac{\sqrt{y}}{\sqrt{x} \cdot x}\right]$$

$$\left.\frac{d^2y}{dx^2}\right]_{(1/4,1/4)} = \frac{1}{2\sqrt{x} \cdot x} + \frac{1}{2}\frac{\sqrt{y}}{\sqrt{x} \cdot x} = \cancel{\frac{1+\sqrt{y}}{2\sqrt{x}}} \quad 4 + 4 = 8$$

Thus, $$\rho = \frac{\{1 + (-1)^2\}^{3/2}}{8} = \frac{2^{3/2}}{8} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}} \text{ Ans}$$

Q Prove that Radius of circle is constant :- $x^2 + y^2 = a^2$

Sol $x = a\cos t$ & $y = a\sin t$

$\frac{dx}{dt} = -a\sin t$ & $\frac{dy}{dt} = a\cos t$

$\therefore \frac{dy}{dx} = -\cot t$ again, $\frac{d^2y}{dx^2} = \operatorname{cosec}^2 t \frac{dt}{dx}$

$$= \operatorname{cosec}^2 t \left(\frac{-1}{a\sin t}\right)$$

$$= -\frac{\operatorname{cosec}^3 t}{a}$$

DATE_____ PAGE_____

$$\rho = \frac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \frac{\{1+(-\cot t)^2\}^{3/2}}{-\operatorname{cosec}^3 t/a} = \frac{-a(\operatorname{cosec}^2 t)^{3/2}}{\operatorname{cosec}^3 t}$$

$$\rho = -a$$

Hence, R.O.C of circle is constant.

Q. ROC $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\rho = \frac{a^2b^2}{p^3}$ where p be the length of the of the ⊥ from the centre ~~the~~ ~~p~~ upon the tangent.

Sol^n :- $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —1,

now, $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

therefore, eq^n of tangent at poin (x,y).

$$Y-y_1 = \frac{dy}{dx}(X-x_1)$$

$$Y-y = \frac{-b^2x}{a^2y}(X-x)$$

$$(Y-y)a^2y = -b^2x(X-x)$$

$$\frac{Yy}{b^2}-\frac{y^2}{b^2} = -\frac{Xx}{a^2}+\frac{x^2}{a^2}$$

$$\frac{Xx}{a^2}+\frac{Yy}{b^2} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$$

$$\frac{Xx}{a^2}+\frac{Yy}{b^2} = 1$$

Put $x = a\cos t$ & $y = b\sin t$

$$\frac{\cancel{a}\cos t\, x}{a^{\cancel{2}}}+\frac{\cancel{b}\sin t\, y}{b^{\cancel{2}}} = 1$$

$$(b\cos t)x + (a\sin t)y = ab$$

if P is length of ⊥ ~~from~~ from the centre upon the tangent.

$$P = \frac{|ax_1+by_1+c_1|}{\sqrt{a^2+b^2}}$$

$$P = \frac{ab}{\sqrt{(a\sin t)^2+(b\cos t)^2}}$$

$$P^2 = \frac{a^2b^2}{a^2\sin^2 t+b^2\cos^2 t}$$

$$a^2\sin^2 t + b^2\cos^2 t = \frac{a^2b^2}{p^2}$$

from eq 1, $x^2/a^2+y^2/b^2=1$

$\because \frac{dy}{dx} = \frac{-b^2x}{a^2y}$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{y - \frac{dy}{dx}\cdot x}{y^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{y}{y^2} - \frac{b^2x^2}{a^2y\cdot y^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{b^2}{a^2}\left[\frac{b^2x^2}{a^2y^3}-\frac{1}{y}\right]$$

$$\frac{d^2y}{dx^2} = \frac{b^4}{a^4}\frac{x^2}{y^3} - \frac{b^2}{a^2}\frac{1}{y}$$

now, $\rho = \frac{\{1+b^4x^2/a^4y^2\}^{3/2}}{\frac{b^4x^2}{a^4y^3}-\frac{b^2}{a^2y}}$

Q find the ROC of $r^n = a^n \cos n\theta$

Sol. $r^n = a^n \cos n\theta$

taking log

$\log(r^n) = \log(a^n \cos n\theta)$

$n \log r = n \log a + \log \cos n\theta$

diff w r t x

$n \cdot \frac{1}{r} \cdot \frac{dr}{dx} = 0 + \frac{1}{\cos n\theta} \cdot -\sin n\theta \cdot n$

$$\frac{dr}{dx} = -r \frac{\sin n\theta}{\cos n\theta} = -r \tan n\theta.$$

again, diff w r t x

$$\frac{d^2r}{dx^2} = -\left[\frac{dr}{dx} \tan n\theta + r \sec^2 n\theta . n\right]$$

$$= r \tan^2 n\theta - nr \sec^2 n\theta$$

$$= \cancel{r}\,[$$

now,

$$\rho = \frac{\{r^2 + (dr/d\theta)^2\}^{3/2}}{r^2 + 2(dr/d\theta)^2 - r\, d^2r/d\theta^2}$$

$$\rho = \frac{\{r^2 + (-r \tan n\theta)^2\}^{3/2}}{r^2 + 2(-r \tan n\theta)^2 - r(r \tan^2 n\theta - nr \sec^2 n\theta)}$$

$$\rho = \frac{[r^2 + r^2 \tan^2 n\theta]^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r^2 \tan^2 n\theta - nr^2 \sec^2 n\theta}$$

$$\rho = \frac{r^3 [1 + \tan^2 n\theta]^{3/2}}{r^2(1 + \tan^2 n\theta - n \sec^2 n\theta)}$$

$$\rho = \frac{r \sec^3 n\theta}{\sec^2 n\theta - n \sec^2 n\theta}$$

$$\rho = \frac{r \sec n\theta}{(1-n)}$$

$$\rho = \frac{r}{(1-n)\cos n\theta} \qquad \left[\because \cos n\theta = \frac{r^n}{a^n}\right]$$

$$\rho = \frac{a^n}{(1-n)\, r^{n-1}} \qquad \text{Ans.}$$

DATE ____ PAGE ____

UNIT-II [6] Advanced Differential calculus

- Maxima & minima of function of two or more variable

working rule -

i, write a given func^n & find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ & $\frac{\partial u}{\partial z}$

ii, put $\frac{\partial u}{\partial x} = 0$ — i, & $\frac{\partial u}{\partial y} = 0$ — ii, solving eq.i, & eq.ii, suppos $x = a$ & $y = b$ is

sol^n of eq^n these point (a,b) is critical point.

iii, Find $r = \frac{\partial^2 u}{\partial x^2}$, $s = \frac{\partial^2 u}{\partial x . \partial y}$, $t = \frac{\partial^2 u}{\partial^2 y}$ at point (a,b)

iv, calculate $rt - s^2$ & examine following cases -

(a) $rt - s^2 > 0$, $r > 0$ then point (a,b) is minima
& $f(a,b)$ is minimum value of f^n.

(b) $rt - s^2 > 0$, $r < 0$ then point (a,b) is maxima
& $f(a,b)$ is maximam value of f^n

(c) $rt - s^2 < 0$, then point (a,b) is neither maxima nor minima
called saddle point.

d) $rt - s^2 = 0$, then further investigation is required.

v, Put the value on the given func^n then find u_{max} & u_{min}

Q Discuss the max & min value of the function $x^3y^2(1-x-y)$

sol $u = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

now, $\frac{\partial u}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$\frac{\partial u}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

Put $\frac{\partial u}{\partial x} = 0$ & $\frac{\partial u}{\partial y} = 0$

$x^2y^2(3-4x-3y) = 0$ & $x^3y(2-2x-3y) = 0$

$4x + 3y - 3 = 0$ — i, & $2x + 3y - 2 = 0$ — ii,

now solving eq i, & ii,

$4x + \cancel{3y} - 3 = 0$
$2x + \cancel{3y} - 2 = 0$
$- \quad - \quad +$
$2x - 1 = 0$
$x = 1/2$

put eq ii,

$\cancel{2} \times \frac{1}{\cancel{2}} + 3y - 2 = 0$

$y = \frac{1}{3}$

DATE ____ PAGE ____

The critical point is $\left(\frac{1}{2}, \frac{1}{3}\right)$, then.

$r = \frac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$.

$s = \frac{\partial^2 u}{\partial x . \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 6x^2y - 8x^3y - 9x^2y^2$.

$t = \frac{\partial^2 u}{\partial y^2} = 2x^3 - 2x^4 - 6yx^3$

at poin $(\frac{1}{2}, \frac{1}{3})$ $r = -\frac{1}{9}$, $s = \frac{-1}{12}$, $t = -\frac{1}{8}$

now, $rt - s^2 = \left(-\frac{1}{9}\right)\cdot\left(-\frac{1}{8}\right) - \left(\frac{-1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} = +ve$.

& $r < 0$

Then, U_{max} at $\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{8} \times \frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right)$

$= \frac{1}{72} \times \frac{1}{6} \Rightarrow \frac{1}{432}$ Ans

- Partial differential.

Q if $u = (x^2 + y^2 + z^2)^{-1/2}$ then prove that.

i, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. & ii, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$

Sol $u = (x^2 + y^2 + z^2)^{-1/2}$

on taking power (-2) in both side.

$u^{-2} = x^2 + y^2 + z^2$ — i,

Partially diff w r t x

$-\cancel{2}u^{-3}\frac{\partial u}{\partial x} = \cancel{2}x$

$\frac{\partial u}{\partial x} = -xu^3$.

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\left[u^3 + 3u^2x.\frac{\partial u}{\partial x}\right]$

$= -\left[u^3 + 3u^2x.(-xu^3)\right]$

$= 3x^2u^5 - u^3$

- Partial diff eq- i, w r t y.

$-2u^{-3}\frac{\partial u}{\partial y} = 2y$

$\frac{\partial u}{\partial y} = -yu^3$

similarly, $\frac{\partial^2 u}{\partial y^2} = 3y^2u^5 - u^3$

DATE____ PAGE____

Similarly $\frac{\partial u}{\partial z} = -zu^3$

$\frac{\partial^2 u}{\partial z^2} = 3z^2u^5 - u^3$

i, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

$= 3x^2u^5 - u^3 + 3y^2u^5 - u^3 + 3z^2u^5 - u^3$

$= 3u^5(x^2+y^2+z^2) - 3u^3$

$= 3u^5(u^{-2}) - 3u^3$

$= 0$. H.P

ii, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$x(-xu^3) + y(-yu^3) + z(-zu^3)$

$= -u^3[x^2+y^2+z^2]$

$= -u^3[u^{-2}]$

$= -u$ H.P.

Q if $u = f\left(\frac{y}{x}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Sol $u = f\left(\frac{y}{x}\right)$

$\Rightarrow \frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)$

$\Rightarrow \frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right)$

now, $x f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right) + y f'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right)$

$= 0$ H.p.

Q if $u = \log(\tan x + \tan y + \tan z)$ then, show that

$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$

Sol. $\left\{ \frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot (\sec^2 x + \sec^2 y + \sec^2 z) \right\}$

$u = \log(\tan x + \tan y + \tan z)$

$e^u = e^{\log\{\tan x + \tan y + \tan z\}}$

$e^u = \tan x + \tan y + \tan z$

now, $e^u \frac{\partial u}{\partial x} = \sec^2 x$ ~~+ $\sec^2 y$ + $\sec^2 z$~~

$e^u \frac{\partial u}{\partial y} = \sec^2 y$

$e^u \frac{\partial u}{\partial z} = \sec^2 z$

now, $\sin 2x \cdot \frac{\sec^2 x}{e^u} + \sin 2y \frac{\sec^2 y}{e^u} + \sin 2z \frac{\sec^2 z}{e^u}$

$\Rightarrow \frac{2\sin x \cos x}{\cos^2 x \cdot e^u} + \frac{2\sin y \cos y}{\cos^2 y\, e^u} + \frac{2\sin z \cos z}{\cos^2 z\, e^u}$

$\Rightarrow \frac{2\tan x}{e^u} + \frac{2\tan y}{e^u} + \frac{2\tan z}{e^u}$

$\Rightarrow 2\left(\frac{e^u}{e^u}\right) = 2$ Ans

Euler's theorem :- Applied when given fuc^n is homogenous.

State & prove Euler's theorem for homogenous function of degree 'n'.

Statement : if u is a homogenous function of degree 'n' in two variable then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof : u is homogenous function of degree 'n' in two variable (x, y). then

$$u = x^n f\left(\frac{y}{x}\right) \quad \text{—i)}$$

∴ Partially diff w r t x

$$\frac{\partial u}{\partial x} = nx^{n-1} \cdot f\left(\frac{y}{x}\right) + x^n \cdot f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

multiply by x.

$$x\frac{\partial u}{\partial x} = nx^n \cdot f\left(\frac{y}{x}\right) - x^{n-1} y . f'\left(\frac{y}{x}\right) \quad \text{—ii)}$$

∴ Partially diff w r t y.

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{1}{x}\right)$$

multiply by y

$$y\frac{\partial u}{\partial y} = x^n y\, f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad \text{—iii)}$$

now adding eq ii) & iii) , then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) - \cancel{x^{n-1} y\, f'\left(\frac{y}{x}\right)} + \cancel{x^{n-1} y\, f'\left(\frac{y}{x}\right)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right)$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$ Hence proved.

Note : if u is a homogenous function then $\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$

Relation b/w 2nd order derivative of homogenous function.

where, n is degree of homogenous func.

i. $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \cdot \partial y} = (n-1)\frac{\partial u}{\partial x}$

ii. $y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial x \cdot \partial y} = (n-1)\frac{\partial u}{\partial y}$

iii. $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \cdot \partial y} = n(n-1)u.$

⇒ if u is not a homogenous func. but $f(u)$ is homogenous fun of degree n

i. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

ii. $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \cdot \partial y} = g(u)[g'(u) - 1]$

where, $g(u) = n\frac{f(u)}{f'(u)}$

Q if $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$

Sol. Here, u is non-homogenous function, then,

$$\sin u = \frac{x^2+y^2}{x+y}$$

$$\sin u = \frac{x^2(1+(y/x)^2)}{x(1+y/x)} = x f\left(\frac{y}{x}\right)$$

i.e $\sin u$ is homogenous func of degree '1'.

By euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u$$

$$x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = \sin u$$

$$x\cos u \cdot \frac{\partial u}{\partial x} + y\cos u \cdot \frac{\partial u}{\partial y} = \sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u \quad \text{H.P.}$$

Q if $u = \sin^{-1}\left(\frac{x-y}{\sqrt{x}+\sqrt{y}}\right)$ then prove that.

i. $x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = \frac{1}{2}\tan u$

ii. $x^2 \frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} = \frac{\sin u \cos 2u}{4\cos^3 u}$

Sol. $\because \sin u = \frac{x-y}{\sqrt{x}\ \sqrt{y}} = \frac{\overset{\sqrt{x}}{\cancel{x}}(1-y/x)}{\cancel{\sqrt{x}}(1-\sqrt{y/x})} = x^{1/2} f\left(\frac{y}{x}\right)$

Here, $f(u) = \sin u$ which is homogenous of x & y in degree $\frac{1}{2}$

By euler's theorem,

$$x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = \frac{1}{2}\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial x} = \frac{1}{2} \sin u.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2} \tan u \quad \text{H.P.}$$

now, $g(x) = n\, f(u)/f'(u) = \frac{1}{2} \frac{\sin u}{\cos u} = \frac{1}{2} \tan u \quad g'(x) = \frac{1}{2} \sec^2 u$

~~we~~ Know that

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x.\partial y} = g(x)[g'(x)-1]$$

$$= \frac{1}{4} \tan u[\sec^2 u - 1 - 1] = \frac{1}{2} \tan u \left[\frac{1}{2}\sec^2 u - 1\right]$$

$$= \frac{1}{4} \tan u[\tan^2 u - 1]$$

$\sec^2 x = 1+\tan^2 x.$

$$= \frac{1}{4} \frac{\sin^3 u}{\cos^3 u} - \frac{\sin u}{\cos u} \qquad = \frac{1}{4} \frac{\sin u}{\cos u}\times\frac{1}{\cos^2 u} - \frac{1}{2}\frac{\sin u}{\cos u}$$

$$= \frac{1}{4}\left(\frac{\sin^3 u - \sin u\cos^2 u}{\cos^3 u}\right) \qquad = \frac{1}{4} \frac{\sin u - 2\sin u\cos^2 u}{\cos^3 u}$$

$$= \frac{1}{4} \frac{\sin u(\sin^2 u - \cos^2 u)}{\cos^3 u} = \frac{1}{4}\frac{\sin u \cos 2u}{\cos^3 u} \quad \text{Ans}$$

Jacobian :-

Let u, v are func of variables x & y that is $u = u(x,y)$, $V = V(x,y)$ than the jacobian of u & v with respect to x & y is demotated by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J(u,v)$ is defined as

$$J(u,V) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial v/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

Similarly, $u = u(x,y,z)$ then

$V = V(x,y,z)$ $\quad j(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial v/\partial x & \partial w/\partial x \\ \partial u/\partial y & \partial v/\partial y & \partial w/\partial y \\ \partial u/\partial z & \partial v/\partial z & \partial w/\partial z \end{vmatrix}$

$w = w(x,y,z)$

and

$$J'(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}.$$

Q if $x = u(1+v)$, $y = v(1+u)$ find the jacobian of (x,y) with u & v.

Sol $J(x,y) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+v)(1+u) - vu$

Properties :-

i. $JJ' = 1 \Rightarrow \frac{\partial(u\ v)}{\partial(x\ y)} \times \frac{\partial(x\ y)}{\partial(u\ v)} = 1$

ii. if u & v are function of r and s where r and s are function of x & y, then

$$\frac{\partial(u.v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

iii. The variable x y u v are connected by implicit func that is $f_1(x,y,u,v) = 0$; where, u & v are implicit func $f_2(x,y,u,v) = 0$ of x & y then

$$\frac{\partial(u\ v)}{\partial(x,y)} = (-1)^n \frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}$$

where, n is no of independent variable

Q. If $u = \frac{2yz}{x}$, $v = \frac{3zx}{y}$ & $w = \frac{4xy}{z}$ calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol. $J(u,v,w) = \frac{\partial(u,v,w)}{\partial(u,v,w)} = \begin{vmatrix} -\frac{2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & -\frac{3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & -\frac{4xy}{z^2} \end{vmatrix} = \frac{2\times3\times4}{x^2y^2z^2}\begin{vmatrix} -yz & zx & yx \\ zy & -zx & xy \\ yz & xz & -xy \end{vmatrix}$

$\cancel{= -\frac{2yz}{x^2}\left[\left(-\frac{3zx}{y^2}\right)\left(-\frac{4xy}{z^2}\right) - \left(\frac{4x}{z}\right)\left(\frac{3x}{y}\right)\right] - \frac{2z}{x}\left[\left(\frac{3z}{y}\right)\left(-\frac{4xy}{z^2}\right) - \left(\frac{4y}{z}\right)\left(\frac{3x}{y}\right)\right] + \frac{2y}{x}\left[\left(\frac{3z}{y}\right)\left(\frac{4x}{z}\right) + \left(\frac{3zx}{y^2}\right)\left(\frac{4y}{z}\right)\right]}$

$$= \frac{24}{x^2y^2z}(x^2y^2z^2)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 24(4) = 96$$

Thus $J \cdot J' = 1$

$96 J' = 1 \Rightarrow J' = \frac{1}{96}$ Ans.

Q. Find the value of the jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$ where $u = x^2 - y^2$, $v = 2xy$, $x = r\cos\theta$, $y = r\sin\theta$.

Sol. $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 + 4y^2 = 4u$

$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$

now, $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(r,\theta)}$

$4u \times r = \frac{\partial(u,v)}{\partial(r,\theta)}$ Ans.

Q. If $f_1 = x^2 + y^2 + u^2 - v^2$ & $f_2 = uv + xy$ then show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2 - y^2}{u^2 + v^2}$

Sol.

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}} = \frac{\begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix}}{\begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}}$$

$$= \frac{2x^2 - 2y^2}{2u^2 + 2v^2} = \frac{x^2 - y^2}{u^2 + v^2} \quad \text{H.p.}$$

function depamdamce :-

if $J(u,v) \neq 0 \Rightarrow$ Independent

if $J(u,v) = 0 \Rightarrow$ dependent

Q if $u = x^2 + y^2 + z^2$, $v = x + y + z$ & $w = xy + yz + zx$ $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. ~~u~~ vanishes identicaly also find the relationship b/w $(u, w \,\&\, v)$

Sol
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ y & z & x \end{vmatrix}$$

$$= 2x(x-z) - 2y(x-y) + 2z(z-y)$$
$$= 2x^2 - 2xz + 2y^2 - 2xy + 2z^2 - 2zy$$
$$= 2(x^2+y^2+z^2) - 2(xy+yz+zx)$$

Lagrange's method of undetermined multiplier.

Working rule : i) let $f(x,y,z)$ be the fun^c & $\phi(x,y,z)=0$ is any condition.

if $\begin{cases} f(x,y,z,\lambda) = f(x+y+z) + \lambda\,\phi(x,y,z) \\ f(x,y,z,\lambda_1,\lambda_2) = f(x,y,z) + \lambda_1\phi_1(x,y,z) + \lambda_2\phi_2(x,y,z) \end{cases}$

where, λ is Lagrang's multiplier.

eq^n is Know as lagrange's auxilary eq^n.

ii) find $\frac{\partial F}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$ & $\frac{\partial f}{\partial z}=0$ and solving them for find x,y,z in terms of λ

iii) By condition $\phi(x,y,z)=0$ put x,y,z & find λ with the help of λ find critical point (x,y,z)

iv) Examine the value of $f(x,y,z)$ at critical point. maxima / minima ??

Q find the maximum & minimum distance of a point (3,4,12) from the sphere $x^2+y^2+z^2=4$.

Sol Let $P(x,y,z)$ be a any point in the sphere $(x^2+y^2+z^2=4)$ and point A (3,4,12)

& $\phi(x^2+y^2+z^2-4)=0$ —— i)

now, The distance b/w point P & A

Let $f = AP^2 = (x-3)^2 + (y-4)^2 + (z-12)^2$ —— ii)

~~now~~, $\cancel{\frac{\partial f}{\partial x} = 2(x-3) = 0}$ making Lagrang's auxilary eq^n

$\cancel{x = \frac{3}{2}}$ $f = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2+y^2+z^2-4)$

now, $\frac{\partial f}{\partial x}=0$	$\frac{\partial f}{\partial y}=0$	$\frac{\partial f}{\partial z}=0$
$2(x-3)+2\lambda x=0$	$2(y-4)+2\lambda y=0$	$2(z-12)+2\lambda z=0$
$2x-6+2\lambda x=0$	$2y-8+2\lambda y=0$	$2z-24+2\lambda z=0$
$\cancel{2}x(1+\lambda)=\cancel{6}\,3$	$\cancel{2}y(1+\lambda)=\cancel{8}\,4$	$z(1+\lambda)=\cancel{24}\,12$
$x=\frac{3}{1+\lambda}$ —— iii)	$y=\frac{4}{1+\lambda}$ —— iv)	$z=\frac{12}{1+\lambda}$ —— v)

put value of x, y & z in eq i), then

$$\left(\frac{3}{1+\lambda}\right)^2 + \left(\frac{4}{1+\lambda}\right)^2 + \left(\frac{12}{1+\lambda}\right)^2 = 4$$

DATE____ PAGE____

$$\frac{9+16+144}{(1+\lambda)^2} = 4$$

$169 = 4(1+\lambda)^2$

$1+\lambda = \pm \frac{13}{2}$

(+ve) $\lambda = 11/2$

(-ve) $\lambda = -15/2$

now, put value of λ in eq iii, iv, v

+ve $(x, y, z) = \frac{6}{13}, \frac{8}{13}, \frac{24}{13}$

-ve $(x, y, z) = -\frac{6}{13}, \frac{-8}{13}, \frac{-24}{13}$

$\because$ There are two critical point $\left(\frac{6}{13}, \frac{8}{13}, \frac{24}{13}\right)$ & $\left(\frac{-6}{13}, \frac{-8}{13}, \frac{-24}{13}\right)$

$$f\left(\frac{6}{13}, \frac{8}{13}, \frac{24}{13}\right) = \left(\frac{6}{13}-3\right)^2 + \left(\frac{8}{13}-4\right)^2 + \left(\frac{24}{13}-12\right)^2$$

$AP^2 = 121$

$AP = 11$

and $f\left(\frac{-6}{13}, \frac{-8}{13}, \frac{-24}{13}\right) = \left(\frac{-6}{13}-3\right)^2 + \left(\frac{-8}{13}-4\right)^2 + \left(\frac{-24}{13}-12\right)^2$

$AP^2 = 225$

$AP = 15$

Thus, the maximum distance is 15

& minimum is 11 . Ans