Unit -1
Matrix And Determinants

Matrix And Determinants

The system of no. arranged in rectangular array (r & c) and bounded by a bracket is called matrix.

Various types of matrix :-

(i) Row Matrix :- If a matrix has only one row and any no. of column then it's called row m.

Ex :-

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$A_{1\times n} = [a_{ij}]_{i,j=1 \text{ to } n}$

(ii) Column M :- A mat. having one colmn and any no. of row is called a colmn matrix. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(iii) Null M :- Any Mat. in which all the elements are zero is called zero mat.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(iv) Sq. Mat :- {row no. = no. of column}

Ex :- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\{3\times3, 4\times4\}$

(v) Diagonal Mat :- A sq. mat. is called a diagonal Mat. if all non-diagonal Mat. are zero.

$$A_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(v) **Scalar:-** A diagonal Mat. in which all the elements are equal to some scalar

$$\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

(vi) **Unit Mat:-** A sq. Mat. is called a unit matrix if all the diagonal elements are unity and non-diagonal elements are zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(vii) **Symmetric Mat:-** A sq. Mat. is called a symm. Mat if are all values :- ~~i and j~~

$a_{ij} = a_{ji}$ then it is sy. mat

$A = A'$.

(ix) **Skew sym.** :- A sq. Mat. is called skew sym. matrix if writing 1st

(i) $A_{ij} = -a_{ji}$ $i, j = A'$

(ii) All diagonal elements are zero.

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

(x) Triangular Mat:- A sq. Mat. all of whose elements below the leading diago. are zero is called a upper triangular Mat. :-

$$A = \begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

low. Trig. Mat:-

$$\begin{bmatrix} a & 0 & 0 \\ h & b & 0 \\ g & f & c \end{bmatrix}$$

(xi) Orthogonal Mat:- A sq. mat. 'A' is called an ortho. mat. if $A \times A' = I$

$$\boxed{AA' = I}$$

(xii) Idempotent Mat:-

A mat. A is called an idempo. mat. if $\{A^2 = A\}$

(xiii) Periodic Mat:- A mat. A is called a periodic Mat, if

$A^{k+1} = A$ where k is the least positive integer for which $\{A^{k+1} = A\}$, then k is called the period of 'A'.

(xiv) Nilpotent Mat:-

$\{A^k = 0\}$ k is pos. integer

if A is the least pos. for $A^k = 0$ then k is the

index of nilpot. mat.

Involuntary Mat :-

A mat. A is called Invo. Mat if

(xv) $\{A^2 = I\}$

Elementary transformation :-

There are 3 types of elem. transformation.

(i) Interchanging any two rows or columns, this transf. is indicated as $\{R_{ij} \Leftrightarrow R_{ji}\}$ $C_i \rightarrow C_i \Leftrightarrow C_j$

$R_i \longrightarrow R_i \Leftrightarrow R_j$

(ii) Multiplication of the elements R_{ij} by any k

kR_i / kC_i

(iii) Add^n of the const. multiplication other row R_i to the another row kR_j

$$\boxed{R_i \rightarrow R_i + kR_j}$$

(iv) If a matrix 'B' is obtained from a matrix 'A' by one or more elementary operations or transformation then 'B' is said to be equavalent to 'A'

$A \sim B$

The rank of a matrix is said to be 'r' it has atleast one zero minor of order

Every minor 'A' higher than R is

Non zero row is that row in which all the elements are non-zero.

The rank of prod. matrix A.B of 2 mat. A & B is less than the rank of A & B matrix.

(ii) corresponding to every non-sing. matr. 'A' of rank 'R' their exist non-sing. mat. 'P' and 'Q' such that P and Q

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$ r is order of unit matrix.

Rank of a matrix is equals to no. of non-zero row in upp. triag. matrix.

(iii) Normal form :- By performing elementary transformation any non-zero matr. 'A' can be reduced to one of the following 4 forms

- $[I_r]$
- $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- $[I_r \ \ 0]$
- $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The no. r is rank of 'A'

$\rho(A) = r$

$A^{-1} = QP$

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Que: Find the rank of given matrix $\begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

$\rho(A) = 3$
$\rho(A) = 3$

In this method we will use only elementary row transformation.

Que: Find the rank of ~~trans~~ given matrix converting it into the normal form.

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 3 & 4 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix} \qquad \rho(A) = 3$$

on solve

Q: $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ Find two sq. matrix P and Q such that PAQ is the normal form and find A^{-1} if it exist otherwise give reason.

$A_{3\times3} = I_{3\times3} A I_{3\times3}$

opr. $R_1 \to R_1 - R_2$

opt at $R_2 \to R_2 - 2R_1$

opt. at $C_2 \to C_2 + C_3$

opt. at $C_3 \to C_3 - 4C_1$

$I_3 = PAQ$

where $P = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$ $|P| \neq 0$

$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$

$|Q| \neq 0$

$A^{-1} = QP$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n = b_1$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & & & & \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Since we have to apply rank theory so we construct a new matrix with the help of A, and B which is known as obedient matrix

$\boxed{\rho(A) = \rho(C)}$ then system is consistent.

If

- $\rho(A) = \rho(C) = (r) \leftarrow$ some no. $=$ total no. of unknowns then system is consistent with unique sol^n.

If

- $\rho(A) = \rho(C) = (r < n)$, then sys. is consistent but ∞ many sol^{n}s.

- $\rho(A) \neq \rho(C)$ then system is inconsistent and their is no sol^n.

Que:- Test consistency and hence solve the given system of equation

$$x_1 + 2x_2 + x_3 = 2$$
$$3x_1 + x_2 - 2x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 3$$
$$2x_1 + 4x_2 + 2x_3 = 4$$

Sol^n:- The given set of eq^n in the matrix form

can be written as

$$AX = B \quad \text{----} (1)$$

where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$B = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ since we have to apply rank so we construct a new matrix with the help of the elements of A and B which is known as

$$C = [A, B]$$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$

$\rho(A) = 3 = \rho(C)$ which is equal to unknown so system is consistent with unique sol^n, for the sol^n we will replace

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ in eq. 1st

Linear dependence and independence of vectors

$X_1, X_2, X_3 \cdots X_n$ are said to be dependent "If"

(1) all the vectors are of the same order.

(2) n scalars $\lambda_1, \lambda_2, \lambda_3$ towards λ_n not all zero exist such that

$$\lambda_1 x_1 + \lambda_2 X_2 + \lambda_3 X_3 + \cdots + \lambda_n X_n = 0$$

otherwise they are said to be linearly independent.

Qu:- Examine the following vectors for linear dependence and find the relation if it exist.

$$X_1 = (1, 2, 4), \; X_2 = (2, -1, 3), \; X_3 = (0, 1, 2),$$
$$X_4 = (-3, 7, 2)$$

Sol^n:-

Consider the matrix eq^n

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0.$$

$$\lambda_1 (1, 2, 4) + \cdots + \lambda_4 (-3, 7, 2) = 0$$

$$(\lambda_1) + 2\lambda_2 + \lambda_4 (-3) = 0 \quad \cdots (1)$$

$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0 \quad \cdots (2)$$

$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0 \quad \cdots (3)$$

$$\Rightarrow A\lambda = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

operation at $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}$$

operation at $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} \lambda_3 + \lambda_4 = 0 \Rightarrow \\ -5\lambda_2 + \lambda_3 + 13\lambda_4 = 0 \\ \lambda_1 + 2\lambda_2 - 3\lambda_4 = 0 \end{cases}$$

$t \neq 0$ $\quad \lambda_4 = t$

$\lambda_3 = -t$

$\lambda_2 = \frac{12}{5} t$

$\lambda_1 = -\frac{9}{5} t$

$$\frac{-9}{5} t + \lambda_1 + \frac{12t}{5} \lambda_2 - t\lambda_3 + t\lambda_4 = 0$$

$$-\frac{t}{5}(9\lambda_1 - 12\lambda_2 + 5\lambda_3 - 5\lambda_4) = 0 \quad \because t \neq 0$$

$$\Rightarrow 9\lambda_1 - 12\lambda_2 + 5\lambda_3 - 5\lambda_4 = 0$$

even values and even vectors

Let

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$AX = Y$ ....①

where A is ~~matrix~~ ~~clm~~ vector and Y is also column matrix.

c^{lm} matrix is transformed into col^{mn} vector Y, by means of eq. matrix A.

~~Let X be a such vector which transforms~~

Let X be a such vector which transforms into $\lambda(X)$ ~~the~~ by means the transformations I, $\lambda_2 X$

suppose the linear transformation $Y = AX$ transform X into a scalar multiple of itself, $\lambda_2 X$

$$AX = Y = \lambda X$$

$$AX - \lambda I X = 0$$

$$[A - \lambda I]X = 0 \quad \text{....②}$$

Thus the unknown scalar (λ) is known as the hygen value of the matrix A and corresponding non-zero sol^n and co-respond non-zero vector (X) is called hygen vector.

The matrix $[A - \lambda I]$ is known as characteristic matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

Characteristic polynomial :-

The determinant $|A - \lambda I|$ when expanded will give a polynomial which we call as characteristic polynomial of matrix 'A'

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$\Downarrow$$

$$-\lambda^3 - 7\lambda^2 + 11\lambda - 5$$

Characteristic eq^n :-

The eq^n $|A - \lambda I| = 0$ is called characteristic eq^n and root of this eq^n is known as characteristic root or hygen values.

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & \end{bmatrix} \qquad +\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\lambda = 1, 1, 5$$

Properties of Hy. values

(1) Any sq. matrix A and its transpose A' have the same hygen value.

(2) The sum of the hygen values of a matrix is equal to trace of the matrix.

(2) The product of the hygen values of a matrix 'A' is equal to determinant of the matrix.

(4) If $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_3$ ~~towards~~ are hygen values 'A' then the hygen values of :-

(i) KA are $K\lambda_1, K\lambda_2 \cdots K\lambda_n$

(ii) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \cdots \frac{1}{\lambda_n}$

(iii) A^m are $\lambda_1^m, \lambda_2^m \cdots \lambda_n^m$.

(1) Two ideal vectors of 'X' of a matrix 'A' is not unique.

(2) If '$\lambda_1, \lambda_2 \cdots \lambda_n$' be desctint hygen values of an matrix then corresponding digonal vectors, $X_1, X_2 \cdots X_n$, form a linearly independent set.

(3) If two or more hygen values are equal it may or may not be possible to get linearly independent hygen vectors corresponding to the equal roots.

(4) Two hygen vectors X_1, X_2 are called orthogonal vectors if

$$\boxed{X_1' X_2 = 0}$$

(5) Hygen vectors of a symmetric matrix corresponding to diff. hygen values are paired wise orthogonal.

ii) Con symm. Matrice with non-repeated Eigen values-

find the Eigen value and Eigen vectors of the matrix

Given that

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

let the characteristic equation for given matrix is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0 \qquad \lambda = 2, 3, 5$$

$\boxed{\text{Eigen vectors :-}}$

The Eigen vectors of matrix 'A' corresponding to the Eigen value 'λ' is given by the non-zero sol^n of the eq^n

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (a)$$

first Eigen vector 'x_1' corresponding to Eigen value $\lambda x = 2$ in eqn 'a'

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 1.x + 1.y + 4z = 0 \\ 0.x + 0.y + 6z = 0 \\ 0.x + 6.y + 3z = 0 \end{cases}$$

$$x_1 = k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad k \neq 0$$

$$\boxed{\frac{x}{6} = \frac{y}{6} = \frac{z}{0} = \frac{k}{6}}$$

~~Find~~ Eigen vector $\lambda = 3$ is obtained by replacing $\lambda = 3$ in eq^n (a).

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\frac{x}{-2} = \frac{y}{6} = \frac{z}{0} = \frac{k}{-2}}$$

$\boxed{\lambda = 5}$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x+y+4z \\ 0x-3y+6z \\ 0 \quad 0 \quad 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{x}{6+12} = \frac{y}{+12-0} = \frac{z}{6-0} = \frac{k}{}$$

Find the Eigen value and vector

$$A = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \quad \text{at } (1, 1, 3)$$

$\lambda = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{vmatrix}$$

$$\boxed{x + y + z = 0}$$

Symm. matrix with non-repeated Eigen values

Qu:- Find the Eig. values and vectors of

$$A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -(2+\lambda) & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -(2+\lambda) \end{bmatrix} = 0$$

$$-(2+\lambda)$$
$$-\{(7-\lambda)(2+\lambda)-25\}$$
$$-5(-(2+\lambda)5-16)$$
$$+4(25-28-4\lambda)$$

$$-(2+\lambda)\{-(7-\lambda)(2+\lambda)$$

$$-(2+\lambda)\left[-(14+7\lambda-2\lambda-\lambda^2)-25\right]$$

$$-5(-10-5\lambda-16)+4(25-3-4\lambda)$$

$$-(2+\lambda)\left[-(14+5\lambda-\lambda^2)\right]-5(-10-5\lambda-16)\ 100-(4\lambda)$$

$$-(2+\lambda)\left[\right.$$

$\lambda = -3, 5, 12$

$\lambda = -3, 5, 12$

$$\Rightarrow \begin{bmatrix} (2+3) & 5 & 4 \\ 5 & 7+6 & 5 \\ 4 & 5 & -14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 4 \\ 5 & 13 & 5 \\ 4 & 5 & -14 \end{bmatrix}$$

$\Rightarrow$

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Sym. matrix with repeated Eigen values :-

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} \Rightarrow 2-\lambda\left[(2-\lambda)^2 - 1\right] + 1\left((\lambda-2)+1\right) + 1\left(1-2+\lambda\right)$$

$$\Rightarrow (2-\lambda)\left[4+\lambda^2-4\lambda-1\right] + 1(\lambda-1) + (\lambda-1)$$

$$\Rightarrow 8-2\lambda^2-8\lambda-2-4\lambda-\lambda^3+4\lambda^2+\lambda+2\lambda-2$$

$$\Rightarrow -\lambda^3+2\lambda^2-9\lambda-12=0$$

$$\Rightarrow \lambda^3-2\lambda^2+9\lambda+12=0 \qquad (\lambda = 4, 1, 1)$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}_{3\times3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = 1$$

$-2x - y + z$

Since given matrix is symm. and Eigen values are repeated, so we take 3rd Eigen vector

$$x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$ which is pair wise orthogonal to x_1 and x_2.

$x_1' x_3 = 0$

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$x_2' x_3 = 0 \Rightarrow \begin{bmatrix} 1, & 1 & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$

$$x_2 = \begin{bmatrix} k_1 \\ k_2 \\ k_2 - k_1 \end{bmatrix}$$

Q1- Cayley Hamilton theorem :-

Every sq. matrix satisfies its own characteristic equation

$$|A - \lambda I| = (-1^n)\left(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n\right)$$

$$|A - \lambda I| = 0,$$

$$\left(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0\right) \cdots (1)$$

Q: find the characteristic eq^n of the matrix 'A' find A^{-1}.

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$$

$(1-\lambda)\left[(1-\lambda)^2 + (4) - 2(3(1-\lambda)+2) + 1(-6+\lambda-1)\right]$

$(1-\lambda)\left[1+\lambda^2-2\lambda+4-6+6\lambda-1-7+\lambda\right.$

$= (1-\lambda)\left[\right.$

$\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$ ---- (1)

Acc. to cayley melton theorem eq. is satisfied

A :-

$A^3 - 6A^2 + 6A - 11I = 0$ ---- (2)

$A^{-1}(A^3 - 6A^2 + 6A - 11I) = A^{-1} \times 0$

$A^2 - 6A + 6I - 11A^{-1} = 0$

$\boxed{11A^{-1} = A^2 - 6A + 6I}$

Unit:- 2

Ordinary Differential Equations

1) $\frac{dy}{dx} = \tan x$ $O = 1, D = 1$ 1, 2, 3 → O.D.E.

2) $\frac{d^2y}{dx^2} + y = x$ $O = 2, D = 1$ 4 → PDE.

(3) $\frac{d^2y}{dx^2} + \sqrt{1 + \frac{dy}{dx}} = 0$ $O = 2, D = 2$

(4) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x$. $O = 1, D = 1$

An equation containing dependent, independent ~~dept~~ variables with their derivatives.

Solⁿ:- An expression which satisfies the differential equation containing constants.

Solution
- General
- Particular

Exact Differential Equations

$Mdx + Ndy = 0$ is said to be Exact if $\frac{\partial M}{\partial dy} = \frac{\partial N}{\partial x}$

Exact Differential Equations are those which are obtained from their solutions by direct differentiation and are not reduced or simplified further.

Condition for differential equation of first order and first degree to be Exact is that if the differential equation can be rearranged as $Mdx + Ndy = 0$, $M, N \to x, y$ is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution of Exact Differential Equation

$$\int_{y=\text{constant}} M\,dx + \int \left(\text{those terms of } N \text{ not containing } x\right) dy = c.$$

(i) Solve the following differential Equation

$(1+4xy+2y^2)\,dx + (1+4xy+2x^2)\,dy = 0$

$M = 1+4xy+2y^2$

$N = (1+4xy+2x^2)$

$\frac{\partial M}{\partial y} = 4x+4y$, $\frac{\partial N}{\partial x} = 4y+4x$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So, equation is exact.

Solution: $\int M\,dx +$

$\int (1+4xy+2y^2)\,dx + \int 1\,dy = c$

$x + 2x^2y + 2y^2x + y = c$

$$(x+y)(2xy+1) = c$$

② $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

$M = 1+e^{x/y}$

$N = e^{x/y}\left(1-\frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = -e^{x/y}\cdot\frac{x}{y^2}$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(0-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\cdot\frac{1}{y}$$

$$= \frac{-e^{x/y}}{y} + \frac{e^{x/y}}{y} - \frac{x}{y^2}e^{x/y}$$

$$= -e^{x/y}\frac{x}{y^2}$$

Here,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution = $\int\left(1+e^{x/y}\right) + 0 = c$

$$x + \frac{e^{x/y}}{y} = c$$

③ $(e^y+1)\cos x\,dx + e^y\sin x\,dy = 0$

$M = (e^y+1)\cos x, \quad N = e^y\sin x$

$$\frac{\partial M}{\partial y} = e^y \cos x \quad , \quad \frac{\partial N}{\partial x} = e^y \cos x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\cancel{\sin x\, e^y} \int (e^y + 1) \cos x \, dx$$

$$(e^y + 1) \sin x = c$$

(4) $(x^2y - 2xy^2)dx - (x^3 + 3x^2y)dy = 0 \rightarrow (1)$

$M = x^2y - 2xy^2$

$N = 3x^2y - x^3$

$$\frac{\partial M}{\partial y} = x^2 - 4xy$$

$$\frac{\partial N}{\partial x} = 6xy - 3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Rules for converting non-Exact Differential Equation into Exact

(1) If the Equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$ then the integrating factor (I.F) is $\frac{1}{Mx + Ny}$

$$Mx + Ny \Rightarrow (x^2y - 2xy^2)x + (3x^2y - x^3)y$$

$$\Rightarrow x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$$

$$\Rightarrow x^2y^2$$

$\therefore$ IF is $\dfrac{1}{x^2y^2}$

Multiplying 1 by I.F

$$\left(\frac{x^2y - 2xy^2}{x^2y^2}\right) dx - \left(\frac{x^3 - 3x^2y}{x^2y^2}\right) dy = 0$$

$$\left[\frac{1}{y} - \frac{2}{x}\right] dx - \left[\frac{x}{y^2} - \frac{3}{y}\right] dy = 0$$

$$M = \frac{1}{y} - \frac{2}{x}, \quad N = \frac{3}{y} - \frac{x}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2}, \quad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

So, Eqn is Exact now

General solution is

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y} dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$Mx + Ny \Rightarrow (x^2y - 2xy^2)x + (3x^2y - x^3)y$$

$$\Rightarrow x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$$

$$\Rightarrow x^2y^2$$

$\therefore$ If is $\dfrac{1}{x^2y^2}$

Multiplying 1 by I.F

$$\left(\frac{x^2y - 2xy^2}{x^2y^2}\right) dx - \left(\frac{x^3 - 3x^2y}{x^2y^2}\right) dy = 0$$

$$\left[\frac{1}{y} - \frac{2}{x}\right] dx - \left[\frac{x}{y^2} - \frac{3}{y}\right] dy = 0$$

$$M = \frac{1}{y} - \frac{2}{x}, \quad N = \frac{3}{y} - \frac{x}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2}, \quad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

So, Eq^n is Exact now

General solution is

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y} dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = c$$

$$x^2y\,dx - (x^3+y^3)\,dy = 0.$$

$$M = x^2y$$

$$N = -x^3-y^3$$

$$\frac{\partial M}{\partial y} = x^2 \quad , \quad \frac{\partial N}{\partial x} = -3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$Mx + Ny = 0$$

$$x^3y - x^3y - y^4 = 0$$

$$-y^4$$

$$I.f = -\frac{1}{y^4}$$

Multiplying (1) by I.f

$$-\frac{x^2y}{y^4}\,dx + \left(\frac{x^3+y^3}{y^4}\right)dy = 0$$

$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

$$M = \frac{-x^2}{y^3}, \quad N = \frac{x^3}{y^4} + \frac{1}{y}$$

$$\frac{\partial M}{\partial y} = \cancel{\frac{-2x}{y^3}} \; \frac{3x^2}{y^4}, \quad \frac{\partial N}{\partial x} = \frac{-3x^2}{y^4}.$$

So, the equation is exact now

Solution $\Rightarrow$ $\int M dx + \int \frac{1}{y} dy$

$$\int \frac{-x^2}{y^3} dx + \int \frac{1}{y} dy$$

$$\cancel{\frac{-2x}{y^2}} + \frac{-x^3}{3y^3} + \log y = c.$$

(2) If the equation $Mdx + Ndy = 0$ is of the form

$$f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$$

$$Mx - Ny \neq 0$$

$$I.F. = \frac{1}{Mx - Ny}.$$

Q. Solve the equation

$$(xy \cdot \sin(xy) + \cos(xy))\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0.$$

$$M = [xy \cdot \sin(xy) + \cos(xy)]\, y$$
$$N = [xy \sin(xy) - \cos xy]\, x$$

$$\frac{\partial N}{\partial y} = [xy\sin(xy) + \cos(xy)] + \sin xy\,(x) + \cos(xy)\cdot x$$

$$= 2xy\sin xy$$

$$Mx - Ny$$

$$x^2y^2\sin xy + xy\cos y - x^2y^2\sin xy + xy\cos xy$$

$$2xy\cos xy$$

$$\frac{1}{Mx-Ny} = \frac{1}{2xy\cos xy}$$

Multiplying by $\frac{1}{2xy\cos xy}$

$$\left(\frac{y}{2}\tan xy + \frac{1}{2xy}\right)dx + \left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = 0$$

$$M = \frac{y}{2}\tan xy + \frac{1}{2x}$$

$$N = \frac{x}{2}\tan xy - \frac{1}{2y}$$

$$\frac{\partial M}{\partial y} = \frac{1}{2}\tan xy + \frac{y}{2}\cdot \sec^2 xy \cdot x$$

$$\Rightarrow \frac{1}{2}\tan xy + \frac{xy}{2}\sec^2 xy$$

$$\frac{\partial N}{\partial x} = \frac{1}{2}\tan xy + \frac{x}{2}\sec^2 xy \cdot y$$

$$\Rightarrow \frac{1}{2}\tan xy + \frac{xy}{2}\sec^2 xy.$$

Here,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

so,

$$\int\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \int\left(\frac{-1}{2y}\right)dy = c$$

$$\frac{y}{2}\cdot\frac{\log(\cos(xy))}{y} + \frac{1}{2}\log x - \frac{1}{2}\log y = \log c$$

$$\frac{\cancel{y}}{2}\cdot\frac{\log(\cos(xy))}{\cancel{y}} + \frac{1}{2}\log\left(\frac{x}{y}\right) = \log c$$

$$-\frac{1}{2}\log(\cos(xy)) + \frac{1}{2}\log\left(\frac{x}{y}\right) = \log c$$

$$\frac{1}{2}\log\left(\frac{x}{y\cos(xy)}\right) = \log c$$

$$\log\left(\frac{x}{y\cos(xy)}\right) = \log c^2$$

$$\frac{x}{y\cos(xy)} = c^2$$

$$x = y\cos(xy)\,c^2.$$

Rule:- (3) $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ only

$I.F = e^{\int f(x)dx}$

Q $(x^2+y^2+2x)dx + 2y\,dy = 0$

Here, $M = x^2+y^2+2x$

$N = 2y$

$\frac{\partial M}{\partial y} = 2y$

$\frac{\partial N}{\partial x} = 0.$

Here,

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

$\frac{1}{2y}\left(2y\right)$

$\Rightarrow 1 = f(x)$

$I.f = e^{\int 1\,dx} \Rightarrow e^{x}$

Multiply (1) by I.f

$(x^2+y^2+2x)e^x\,dx + 2ye^x\,dy = 0.$

Solving for exactness with new M and N,

$M = (x^2+y^2+2x)e^x$

$N = 2ye^x$

$\frac{\partial M}{\partial y} = 2ye^x$

$\frac{\partial N}{\partial x} = 2ye^x$

Here,

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The solution is Exact

So,

General solution is

$\Rightarrow \int (x^2+y^2+2x)e^x\,dx + \int 0\,dy$

$\Rightarrow e^x(x^2+y^2+2x) - \int e^x(2x+2)$

$\Rightarrow e^x(x^2+y^2+2x) - 2\left[e^x(x+1) - e^x\right]$

$\Rightarrow e^x(x^2+y^2+2x) - 2e^x(x+1) + 2e^x + c$

$\Rightarrow e^x x^2 + e^x y^2 + \cancel{2xe^x} - \cancel{2xe^x} - \cancel{2e^x} + \cancel{2e^x} + c$

$\Rightarrow e^x x^2 + e^x y^2 + c$

Rule 4:- $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ only

$I.F = e^{\int f(y)dy}$

Q. Solve :-

$$(xy^3+y)dx + 2(x^2y^2+x+y^4)\,dy = 0$$

$M = xy^3+y$

$N = 2(x^2y^2+x+y^4)$

$\frac{\partial M}{\partial y} = 3xy^2+1$

$\frac{\partial N}{\partial x} = 2(2xy^2+1)$

$\frac{\partial N}{\partial x} = 4xy^2+2$

Here,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So,

Equation is not Exact

So,

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\frac{1}{(xy^3+y)\cancel{2(x^2y^2+x+y^4)}}\left[(4xy^2+2)-(3xy^2+1)\right]$$

$$\frac{1}{xy^3+y}\left[xy^2+1\right]$$

$$\Rightarrow \frac{1}{y(xy^2+1)}(xy^2+1)$$

$$\Rightarrow \frac{1}{y}$$

$$I.F = e^{\int \frac{1}{y}dy}$$

$$e^{\log y}$$

$$y$$

Multiplying by y

$$(xy^4+y^2)\,dx + 2(x^2y^3+xy+y^5)\,dy = 0$$

Now,

checking the exactness for new M and N

$$M = xy^4+y^2 \quad , \quad N = 2(x^2y^3+xy+y^5)$$

$$\frac{\partial M}{\partial y} = 4xy^3+2y$$

$$\frac{\partial N}{\partial x} = 2\left[2x^2y^2+y\right]$$

$$\Rightarrow 4x^2y^2+2y$$

Here,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So,

the equation is Exact now

So,

particular solution will be

$$\int M dx + \int$$

$$\Rightarrow \int (xy^4 + y^2)\, dx + 2\int y^5\, dy$$

$$\Rightarrow \frac{x^2}{2}y^4 + xy^2 + \frac{y^6}{3} = c$$

$$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c.$$

Q. $$\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}\left(\cancel{x}\, 1 + y^2\right) x\, dy = 0$$

$$M = y + \frac{y^3}{3} + \frac{x^2}{2}$$

$$N = \frac{x}{4} + \frac{xy^2}{4}$$

$$\frac{\partial M}{\partial y} = 1 + y^2 + \cancel{x}$$

$$\frac{\partial N}{\partial x} = \frac{1}{4} + \frac{y^2}{4}$$

Here

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So,

Equation is not exact

So,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$\frac{4}{x(1+y^2)}\left[1+y^2 - \frac{1}{4} - \frac{y^2}{4}\right]$$

$$\frac{4}{x(1+y^2)}\left[\frac{3}{4} + \frac{3y^2}{4}\right]$$

$$\Rightarrow 3/x$$

Now,

$$I.F = e^{\int 3/x\, dx}$$

$$\Rightarrow e^{3\log x}$$

$$\Rightarrow e^{\log x^3}$$

$$\Rightarrow x^3$$

Multiply by x^3 in Eqn (1)

$$\Rightarrow \int\left(yx^3 + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \cancel{\frac{1}{4}\left(x^4 + x^4y^2\right)dy}$$

$$\frac{yx^4}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12} = c$$

$$3yx^4 + x^4y^3 + x^6 = c.$$

Q. $(3x^2y^4+2xy)dx + (2x^3y^3 - x^2)dy = 0$

$M = 3x^2y^4 + 2xy$

$N = 2x^3y^3 - x^2$

$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$

$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$

Here

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So,

Equation is not exact.

3rd Rule

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$\frac{1}{2x^3y^3 - x^2}\left(12x^2y^3 + 2x - 6x^2y^3 + 2x\right)$$

$$\Rightarrow \frac{1}{2x^3y^3 - x^2}\left(6x^2y^3 + 4x\right)$$

4th Rule

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\frac{1}{3x^2y^4+2xy}\left(6x^2y^3 - 2x - 12x^2y^3 - 2x\right)$$

$$\Rightarrow \frac{1}{3x^2y^4+2xy}\left(-6x^2y^3 - 4x\right)$$

$$\Rightarrow \frac{-2}{y}\left[\frac{3x^2y^3+2x}{3x^2y^3+2x}\right]$$

$$\Rightarrow \frac{-2}{y}$$

$$I.F. = e^{-2\int \frac{1}{y}dy}$$

$$\Rightarrow e^{-2\log y}$$

$$\Rightarrow e^{\log y^{-2}}$$

$$\Rightarrow \frac{1}{y^2}$$

$$\int\left(\frac{3x^2y^2+2x}{y}\right)dx$$

$$\frac{x^3y^2}{y} + \frac{2x^2}{2} = c$$

$$\Rightarrow \boxed{\frac{x^3y^2}{y} + \frac{x^2}{y} = c}$$

Q $(1+xy)y\,dx + (1-xy)x\,dy = 0.$

$M = (1+xy)y$

$= y + xy^2$

$\frac{\partial M}{\partial y} = 1 + 2xy.$

$N = (1-xy)x$

$= x - x^2y$

$\frac{\partial N}{\partial x} = 1 - 2xy$

Here,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So,

Rule no. 2

$$\cancel{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$$

$$\cancel{\frac{1}{1-2xy}\left(1+2xy - (1-2\right.}$$

$Mx - Ny = (1+2xy)x - (1-2xy)y^2$

$= \cancel{x + 2x^2y - y + 2xy^2}$

$\Rightarrow (y+xy^2)x - (x-x^2y)y$

$\Rightarrow \cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2$

$\Rightarrow 2x^2y^2$

$$\frac{1}{\cancel{I.F}} = \frac{1}{2x^2y^2}$$

Multiplying Eqⁿ by ~~Ø~~ I.F.

$$\frac{(1+xy)}{2x^2y^2}\, y\, dx + \frac{(1-xy)}{2x^2y^2}\, x\, dy = 0$$

$$\left(\frac{1+xy}{2x^2y}\right)dx + \left(\frac{1-xy}{2xy^2}\right)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$

$$M = \frac{1}{2x^2y} + \frac{1}{2x}$$

$$N = \frac{1}{2xy^2} - \frac{1}{2y}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{2x^2y^2}$$

$$\frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$$

Here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So,

the Equation is Exact now.

$$\int M\,dx + \int N \text{ not containing the term of } x = C$$

$$\int\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int\left(\frac{-1}{2y}\right)dy = c$$

$$\frac{1}{2y}\int \frac{1}{x^2}dx + \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int y\,dy = c$$

$$\frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\frac{-1}{2xy} + \frac{1}{2}\log\frac{x}{y} = c$$

Given this Equation if we can arrange in the form

$$x^a y^b (my\,dx + nx\,dy) + x^r y^s (py\,dx + qx\,dy) = 0$$

Then the Integrating factor is

$$I.F = x^h . y^k \text{ where}$$

h, k are to be found as by using condition of Exactness.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Q. Solve :-

$$(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0.$$

$$M = 2y + 6xy^2$$

$$N = 3x + 8x^2y$$

$$\frac{\partial M}{\partial y} = 2 + 12xy$$

$$\frac{\partial N}{\partial x} = 3 + 16xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Let, $x^h y^k$ be the Integrating factor

Multiplying (1) by I.F

$$(2y^{k+1}x^h dx + 3x^{h+1}y^k dy) + (6x^{h+1}y^{k+2}dx + 8x^{h+2}y^{k+1}dy$$

As, 2 is exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M = 2y^{k+1}x^h + 6x^{h+1}y^{k+2}$$

$$\frac{\partial M}{\partial y} = 2(k+1)y^k x^h + 6x^{h+1}(k+2)y^{k+1}$$

$$N = 3x^{h+1}y^k + 8x^{h+2}y^{k+1}$$

$$\frac{\partial N}{\partial x} = 3(h+1)x^h y^k + 8(h+2)x^{h+1}y^{k+1}$$

$$2(k+1) = 3(h+1)$$

$$2k+2 = 3h+3$$

$$3h-2k+1=0 \rightarrow (1)$$

$$6(k+2) = 8(h+2)$$

$$6k+12 = 8h+16$$

$$\cancel{8h}\ 3k+6 = 4h+8$$

$$4h-3k+2=0$$

$$12h-8k+4=0$$

$$12h-9k+6=0$$

$$- \quad + \quad -$$

$$k-2=0$$

$$\boxed{k=2}$$

$$\boxed{h=1}$$

$$2y^3x\,dx + 3x^2y^2dy + 6x^2y^4dx + 8x^3y^3dy$$

$$(2y^3x + 6x^2y^4)\,dx + (3x^2y^2 + 8x^3y^3)\,dy = 0$$

$$\int (2y^3x + 6x^2y^4)\,dx$$

$$\cancel{2}y^3 \cdot \frac{x^2}{\cancel{2}} + \frac{\cancel{6}x^3y^4}{\cancel{3}_1} = c$$

$$y^3x^2 + 2x^3y^4 = c$$

$$\boxed{y^3x^2 + 2x^3y^4 = c}$$

. Solve

$$(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$$

$$M = y^3 - 2x^2y \quad , \quad N = 2xy^2 - x^3$$

$$\frac{\partial N}{\partial y} = 3y^2 - 2x^2$$

$$\frac{\partial N}{\partial x} = 2y^2 - 3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Rearranging,

$$y^3dx + 2xy^2dy - (2x^2y\,dx + x^3dy) = 0$$

$$y^2(y\,dx + 2x\,dy) - x^2(2y\,dx + x\,dy) = 0$$

Let,

$x^h \cdot y^k$ be the Integrating factor:

$\cancel{x^h y^1}$

$$(x^h y^{k+3} - 2x^{h+2} y^{k+1})\,dx + (2x^{h+1} y^{k+2} - x^{h+3} y^k)\,dy = 0$$

$$\frac{\partial M}{\partial y} = (k+3) y^{k+2} x^h - 2(k+1) y^k x^{h+2}$$

$$\frac{\partial N}{\partial x} = 2(h+1) x^h y^{k+2} - (h+3) x^{h+2} y^k$$

$$(k+3) = 2(h+1)$$

$$k+3 = 2h+2$$

$$2h - k = 1 \rightarrow (1)$$

$$2k+2 = h+3$$

$$\cancel{2k-1}\ h - 2k = -1 \rightarrow (2)$$

$$\begin{aligned} 2h - k &= 1 \\ 2h - 4k &= -2 \\ - \quad + \quad & + \\ \hline 3k &= 3 \end{aligned}$$

$$\boxed{k = 1}$$

$$\boxed{h = 1}$$

$$(xy^4 - 2x^3y^2)\,dx + (2x^2y^3 - x^4y^1)\,dy = 0$$

$$\int (xy^4 - 2x^3y^2)\,dx = 0$$

$$\int xy^4 dx - 2\int x^3y^2 dx = c$$

$$\frac{x^2y^4}{2} - \frac{x^4y^2}{2} = c.$$

Linear Differential Equation of Higher Order with constant coefficient

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_ny = f(x) \rightarrow (1)$$

$$D = \frac{d}{dx}$$

$$D^ny + a_1D^{n-1}y + a_2D^{n-2}y + \cdots + a_ny = f(x)$$

$$D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_ny = f(x) \rightarrow (2)$$

$$f(D)y = f(x)$$

Let $y = e^{mx}$ be trial sol^n.

(1) gives,

$$\Rightarrow m^ne^{mx} + a_1m^{n-1}e^{mx} + \cdots a_ne^{mx} = f(x)$$

$$\Rightarrow e^{mx}(m^n + a_1m^{n-1} + \cdots + a_n) = f(x) \rightarrow (3)$$

Auxiliary

Case 1:- $f(x) = 0$, (1) homogenous.

$\therefore$ (3) gives:-

$$e^{mx}(m^n + a_1m^{n-1} + \cdots + a_n) = 0$$

$\Rightarrow m^n + a_1 m^{n-1} + ___ + a_n = 0$

n roots,

(1) roots are real solution :

$m_1, m_2 \ldots m_n$

$\therefore$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + ___$

(2) equal

2 roots are equal

$y = (c_1 + c_2 x) e^{m_1 x} + ___$

3 roots are equal

$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + ___$

(3) when roots are imaginary

$\alpha \pm i\beta$

$m_1 = \alpha + i\beta$

$m_2 = \alpha - i\beta$

$\therefore$ General solution is

$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$

$y = e^{\alpha x} \left[c_1 e^{i\beta x} + c_2 e^{-i\beta x} \right]$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[\cos\beta x(c_1 + c_2) + \sin\beta x(ic_1 - ic_2)\right]$$

$$\Rightarrow e^{\alpha x}\left[A_1\cos\beta x + A_2\sin\beta x\right].$$

(1) solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$

$$D^2y - 3Dy - 4y = 0$$

$$y(D^2 - 3D - 4) = 0$$

Auxiliary Eqn. $(D \to m)$

$\cancel{D^2 - 3D - 4 = 0}$ $\quad m^2 - 3m - 4 = 0$

$m^2 - 4m + m - 4 = 0$

$\cancel{D^2 - 4D + D - 4 = 0}$ $\quad (m-4)(m+1) = 0$

$m = -1, 4.$

General solution is.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y = c_1 e^{-x} + c_2 e^{4x}$$

(2) $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$

$$D^3y - 9D^2y + 23Dy - 15y = 0$$

$$y(D^3 - 9D^2 + 23D - 15) = 0$$

Auxiliary Eqⁿ is

$$m^3 - 9m^2 + 23m - 15 = 0$$

$$m^2(m-1) - 8m(m-1) + 15(m-1) = 0$$

$$(m-1)(m^2 - 8m + 15) = 0$$

$$(m-1)(m^2 - 5m - 3m + 15) = 0$$

$$(m-1)(m-5)(m-3) = 0$$

$$m = 1, 3, 5$$

So,

General solution will be

$$\boxed{y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x}.}$$

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$$

$$D^3y - 2D^2y - 4Dy + 8y = 0$$

$$y(D^3 - 2D^2 - 4D + 8) = 0$$

Auxiliary Eqⁿ is

$$m^3 - 2m^2 - 4m + 8 = 0$$

$$m^2(m-2) + 0(m-2) - 4(m-2) = 0$$

$$(m-2)(m^2-4) = 0$$

$$m = 2, \pm 2$$

$$m = 2, 2, -2$$

$$y = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x}.$$

(3) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

$D^2y + 2Dy + 5y = 0$

$y(D^2+2D+5) = 0$

$m^2+2m+5 = 0$

$m = \frac{-2 \pm \sqrt{4-20}}{2}$

$\Rightarrow \frac{-2 \pm 4i}{2}$

$\Rightarrow -1 \pm 2i$ $\quad \alpha = -1-2i,\ \beta = -1+2i.$

General solⁿ:-

$y = c_1 e^{(-1-2i)x} + c_1 e^{(-1+2i)x}$

or $y = e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$

(4) $(D^2+D+1)^2(D-2)y = 0$

$(m^2+m+1)^2(m-2) = 0$

2, $\quad m = \frac{-1 \pm \sqrt{1-4}}{2}$

$\Rightarrow \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$

$\alpha = \frac{-1}{2} \quad \beta = \frac{\sqrt{3}}{2}$

$y = c_1 e^{2x} + c_2 e^{-x/2}\left[(c_2 + c_3 x)\cos\left(\frac{\sqrt{3}x}{2}\right) + (c_4 + c_5 x)\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$

Non Homogenous Linear Equations with constant coefficients :-

$$f(D)y = f(x) \quad (\neq 0)$$

General solution

C.f. + P.I →

C.f. → Complementary function

P.I → Particular Integral

In C.f. → R.H.S. should be 0.

-: Special methods to find Particular Integral :-

(i) $f(x) = e^{ax}$

$$P.I = \frac{1}{f(D)} \cdot f(x)$$

$$P.I = \frac{1}{f(a)\ (\neq 0)} \cdot e^{ax} \quad (D \to a)$$

$$G.S. = C.f. + P.I.$$

Q. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = e^{2x}$

$$4D^2y + 4Dx - 3y = e^{2x}$$

$$y(4D^2 + 4D - 3) = e^{2x}$$

$f(D) = 4D^2+4D-3$

Auxiliary eqn

$4m^2+4m-3=0$

$4m^2+6m-2m-3=0$

$2m(2m+3)-1(2m+3)=0$

$m = \frac{1}{2}, \frac{-3}{2}$

$C.F. = c_1 e^{x/2} + c_2 e^{-3x/2}$

$P.I = \frac{1}{f(a)} \cdot e^{ax}$

$\frac{1}{4D^2+4D-3} \cdot e^{2x}$

$\frac{1}{4\times 2^2+4\times 2-3} \cdot e^{2x}$

$\frac{e^{2x}}{16+8-3}$ or $\frac{e^{2x}}{21}$

General solution is

$y = c_1 e^{x/2} + c_2 e^{-3x/2} + \frac{e^{2x}}{21}$

Q. $(D^3+1)y = (e^x+1)^2$

$D^3+1=0$

$m^3+1=0$

$m^3+1=0$

$(m+1)(m^2-m+1)=0$

$$m^2 - m + 1 = 0$$

$m = -1,$ $\quad \dfrac{1 \pm \sqrt{1-4}}{2}$

$$\Rightarrow \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

$$C.F = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

For $P.I = \quad e^{2x} + 2e^{x} + 1$

$$\frac{e^{2x}}{D^3+1}, \quad \frac{2e^{x}}{D^3+1}, \quad \frac{e^{0x}}{1}$$

$$\frac{e^{2x}}{2^3+1} \Rightarrow \frac{e^{2x}}{9} \qquad \Rightarrow \frac{2e^{x}}{2} \Rightarrow e^{x} \qquad \Rightarrow 1$$

$$y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}x}{2}\right)\right] + \frac{e^{2x}}{9} + e^{x} + 1.$$

② $f(x) = \sin ax / \cos ax$

$$P.I = \frac{1}{f(D^2)} \sin ax$$

$$= \frac{1}{f(-a^2)} \cdot \sin ax \quad (\neq 0)$$

Q. 1 Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin 3x$.

$$D^2y - 5Dy + 6y = \sin 3x$$

Auxiliary eqn is

$$\cancel{CF} = m^2 - 5m + 6 = 0$$

$$m^2 - 3m - 2m + 6 = 0$$

$$(m-3)(m-2)$$

$$m = 2, 3.$$

$$C.F = c_1 e^{2x} + c_2 e^{3x}.$$

$$P.I = \frac{1}{D^2 - 5D + 6} \cdot \sin 3x$$

$$\frac{1}{-3^2 - 5D + 6} \cdot \sin 3x$$

$$\frac{\sin 3x}{-3 - 5D} = \frac{-\sin 3x}{(3+5D)} \times \frac{3-5D}{3-5D}$$

$$= \frac{-\sin 3x\,(3-5D)}{9 - 25D^2}$$

$$= \frac{-\sin 3x\,(3-5D)}{9+225}$$

$$\frac{-3\sin 3x}{234} + \frac{15\cos 3x}{234}$$

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{15}{234}\cos 3x - \frac{\sin 3x}{78}.$$

Q. $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$

$$D^2y - 4y = e^x + \sin 2x$$

Auxiliary Eqn is

$$m^2 - 4 = 0$$
$$m = 2, -2$$

$$C.F. = c_1 e^{2x} + c_2 e^{-2x}$$

$$P.I = \frac{e^x}{D^2-4} + \frac{\sin 2x}{D^2-4}$$
$$= \frac{-e^x}{3} + \frac{\sin 2x}{(-8)}$$
$$= \frac{-e^x}{3} - \frac{\sin 2x}{8}$$

General solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{e^x}{3} - \frac{\sin 2x}{8}$$

Q $(D^3 - 2D^2 + 3)y = \cos x$

$$m^3 - 2m^2 + 3 = 0$$

$$m^2(m+1) - 3m(m+1) + 3(m+1) = 0$$
$$(m^2 - 3m + 3)(m+1) = 0$$
$$m = \frac{3 \pm \sqrt{9-12}}{2} \Rightarrow \frac{3 \pm \sqrt{3}i}{2} \Rightarrow \frac{3}{2} \pm \frac{\sqrt{3}}{2}i \quad \alpha = \frac{3}{2}, \beta = \frac{\sqrt{3}}{2}$$

$$C.F = c_1 e^{-x} + e^{3x/2}\left[c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right]$$

$$P.I = \frac{\cos x}{D^3 - 2D^2 + 3}$$

$$\frac{\cos x}{D^3 + 2D + 3}$$

$$\frac{\cos x}{D^3 + 5}$$

$$\frac{\cos x}{-D + 5} \times$$

$$\frac{\cos x}{5 - D} \times \frac{5 + D}{5 + D}$$

$$\frac{5\cos x - \sin x}{25 - D^2}$$

$$\frac{5\cos x - \sin x}{26}$$

$$\frac{5}{26}\cos x - \frac{\sin x}{26}$$

$$y = c_1 e^{-x} + e^{3x/2}\left[c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right] + \frac{5}{26}\cos x - \frac{1}{26}\sin x.$$

3rd Rule:- $f(x)$ = polynomial :-

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + ___$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + ___$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \ldots$$

$$P.I = \frac{1}{f(D)} \text{ polynomial}$$

$$[f(D)]^{-1} \text{ polynomial}$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x$$

$$D^2y - 5Dy + 6y = x$$

$$(D^2 - 5D + 6)y = x$$

$$m^2 - 5m + 6 = 0$$

$$m = 2, 3$$

$$CF = c_1e^{2x} + c_2e^{3x}$$

$$P.I = \frac{1}{D^2 - 5D + 6}x$$

$$\Rightarrow (D^2 - 5D + 6)^{-1}x$$

$$\Rightarrow 6^{-1}\left(1 + \frac{D^2 - 5D}{6}\right)^{-1}x$$

$$\Rightarrow \frac{1}{6}\left[1 + \frac{D^2 - 5D}{6}\right]^{-1}x$$

$$\Rightarrow \frac{1}{6}\left[1 - \left(\frac{D^2 - 5D}{6}\right) + \left(\frac{D^2 - 5D}{6}\right)^2\right]x$$

$$\Rightarrow \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{D^4}{6} + \frac{25D^2}{36} - \frac{10D^3}{6}\right]x$$

$$\frac{1}{6}\left[1+\frac{5D}{6}\right]x$$

$$\frac{1}{6}x+\frac{5}{36}Dx$$

P.I. $\Rightarrow$ $\frac{1}{6}x+\frac{5}{36}$

General solution

$$y = c_1e^{2x}+c_2e^{3x}+\frac{x}{6}+\frac{5}{36}$$

Q. $\frac{d^2y}{dx^2}-4y = x^2$

$$D^2y-4y = x^2$$

$$(D^2-4)y = x^2$$

$$m = \pm 2$$

$$C.F = c_1e^{2x}+c_2e^{-2x}$$

P.I =

$$\frac{1}{(D^2-4)}x^2$$

$$\frac{1}{-4\left(\frac{D^2}{4}-1\right)}x^2$$

$$\frac{-1}{4\left(1-\frac{D^2}{4}\right)}x^2$$

$$\frac{-1}{4}\left(1-\frac{D^2}{4}\right)^{-1}x^2$$

$$\frac{-1}{4}\left[1+\frac{D^2}{4}\right]x^2$$

$$-\frac{x^2}{4}-\frac{2}{16}$$

$$-\frac{x^2}{4}-\frac{1}{8}$$

$$y = c_1e^{2x}+c_2e^{-2x}-\frac{x^2}{4}-\frac{1}{8}$$

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$$

$$D^3y + 3D^2y + 2Dy = x^2$$

$$y(D^3 + 3D^2 + 2D) = x^2$$

$$m^3 + 3m^2 + 2m = 0$$

$$m(m^2 + 3m + 2) = 0$$

$$m(m+1)(m+2) = 0$$

$$m = 0, 1, -2,$$

$$C.F = c_1e^{0x} + c_2e^{x} + c_3e^{-2x}$$

$$c_1 + c_2e^{x} + c_3e^{-2x}$$

$$P.I = \frac{1}{D^3 + 3D^2 + 2D} x^2$$

$$\frac{1}{D(D^2 + 3D + 2)} x^2$$

$$\cancel{\frac{x^3}{3(D^2+3D+2)}}$$

$$\frac{1}{D}\left[\frac{1}{D^2 + 3D + 2} \cdot x^2\right]$$

$$\frac{1}{2D}\left[\frac{1}{1 + \frac{D^2 + 3D}{2}}\right] x^2$$

$$\frac{1}{2D}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}x^2$$

$$\frac{1}{2D}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{D^4}{4}+\frac{9D^2}{4}+\right]x^2$$

$$\frac{1}{2D}\left[1-\frac{D^2}{2}+\frac{9D^2}{4}\right]x^2$$

$$\frac{1}{2D}\left[1+\frac{7D^2}{4}\right]x^2$$

$$\frac{1}{2}\left[\frac{1}{D}+\frac{7D^2}{4D}\right]x^2$$

$$\frac{1}{2}\left[\frac{1}{D}+\frac{7}{4}D\right]x^2$$

$$\frac{1}{2}\left[\frac{1}{D}x^2+\frac{7}{4}Dx^2\right]$$

$$\frac{x^3}{6}+\frac{7x}{4}$$

$$\frac{1}{2D}\left[1-\frac{3D}{2}+\frac{7D^2}{4}\right]x^2$$

$$\left[\frac{1}{2D}-\frac{3}{4}+\frac{7}{8}D\right]x^2$$

$$\frac{1}{2D}x^2-\frac{3}{4}x^2+\frac{7}{8}Dx^2$$

$$\frac{x^3}{6}-\frac{3}{4}x^2+\frac{7}{4}x$$

$$y = c_1+c_2e^{x}+c_3e^{-2x}+\frac{x^3}{6}-\frac{3}{4}x^2+\frac{7}{4}x$$

(4) $f(x)=e^{ax}\cdot V$ where V is some function of x

$$P.I = \frac{1}{f(D)}e^{ax}\cdot V$$

$$e^{ax}\frac{1}{f(D+a)}\cdot V$$

Q. $(D^2-2D+5)y = e^{2x}\cdot\sin x$

$m^2-2m+5=0$

$m = \dfrac{2\pm\sqrt{4-20}}{2}$

$\Rightarrow \dfrac{2\pm 4i}{2}$

$\Rightarrow 1\pm 2i$

$\alpha = 1, \quad \beta = 2$

$C.F = e^{x}\left[c_1\cos 2x + c_2\sin 2x\right]$

$P.I = \dfrac{1}{f(D)}\cdot e^{2x}\sin x$

$e^{2x}\cdot\dfrac{1}{f(D+2)}\sin x$

$e^{2x}\cdot\dfrac{1}{(D+2)^2-2(D+2)+5}\sin x$

$e^{2x}\cdot\dfrac{1}{D^2+4+4D-2D-4+5}\sin x$

$e^{2x}\cdot\dfrac{1}{D^2+5+2D}\sin x$

$e^{2x}\cdot\dfrac{\sin x}{D^2+2D+5}$

$e^{2x}\cdot\dfrac{\sin x}{-1+2D+5}$

$e^{2x}\cdot\dfrac{\sin x}{2D+4}$

$\dfrac{e^{2x}}{2}\dfrac{\sin x}{D+2}\cdot\dfrac{D-2}{D-2} \Rightarrow \dfrac{e^{2x}}{2}\cdot\dfrac{(\cos x-2\sin x)}{(-5)}$

$$-\frac{e^{2x}}{10}\left[\cos x - 2\sin x\right]$$

$$y = e^{x}\left[c_1\cos 2x + c_2\sin 2x\right] - \frac{e^{2x}}{10}\left[\cos x - 2\sin x\right].$$

$$(D^3+3D^2+3D+1)y = x^2e^{-x}$$

$$m^3+3m^2+3m+1$$

$$m^2(m+1) + 2m(m+1) + 1(m+1) = 0$$

$$(m^2+2m+1)(m+1) = 0$$

$$m = -1, -1, -1$$

$$CF = (c_1 + c_2x + c_3x^2)e^{-x}$$

$$P.I = \frac{1}{\cancel{f(D)}}\frac{1}{D^3+3D^2+3D+1}\cdot x^2e^{-x}$$

$$= e^{-x}\cdot\frac{1}{(D-1)^3+3(D-1)^2+3(D-1)+1}\,x^2$$

$$e^{-x}\cdot\frac{x^2}{(D+1-1)^3}$$

$$e^{-x}\cdot\frac{x^2}{D^3}$$

$$e^{-x}\cdot\iiint x^2$$

$$e^{-x}\cdot\iint\frac{x^3}{3}$$

$$e^{-x}\cdot\frac{1}{12}\iint x^4 = e^{-x}\cdot\frac{x^5}{60}$$

$$y = (c_1 + c_2 x + c_3 x^2) e^{-x} + e^{-x} \cdot \frac{x^5}{60}$$

Q. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

$$D^2 y - 2Dy + y = e^x$$

$$D^2 - 2D + 1 = 0$$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$C.F. = (c_1 + c_2 x) e^x$$

$$P.I = \frac{e^x \cdot 1}{(D-1)^2}$$

$$e^x \frac{1}{(D-1+1)^2}$$

$$e^x \cdot \frac{1}{D^2}$$

$$\frac{e^x \cdot x^2}{2!}$$

Rule (5) $f(x) = x \cdot V$ where V is the function of x then particular Integral is

$$P.I = \frac{1}{f(D)} \cdot xV$$

$$= x \cdot \frac{1}{f(D)} \cdot V + \frac{d}{dD}\left(\frac{1}{f(D)}\right) \cdot V$$

$(D^2-2D+1)\,y = x\cdot\sin x.$

$CF = (c_1+c_2x)\,e^{x}.$

$P.I = \cancel{\dfrac{1}{(D-1)^2}}\cdot\dfrac{x\sin x}{D^2-2D+1}$

$x\cdot\dfrac{1}{D^2-2D+1}\cdot\sin x + \left(\dfrac{d}{dD}\left[\dfrac{1}{D^2-2D+1}\right]\right)\sin x \rightarrow (1)$

(i) $\dfrac{x}{2}\cdot\dfrac{-1}{D}\sin x$

$\dfrac{x\cos x}{2}$

(ii) $\left(\dfrac{d}{dD}(D^2-2D+1)^{-1}\right)\sin x$

$-(D^2-2D+1)^{-2}\cdot(2D-2)\sin x$

$\dfrac{-2(D-1)}{(D^2-2D+1)^2}\sin x \Rightarrow \dfrac{-2(D-1)\sin x}{4D^2}$

$= \dfrac{1}{2}\cdot(D-1)\sin x$

$\cancel{-2}(D-1)\dfrac{\sin x}{\cancel{-2}D^2}$

$\dfrac{1}{2}[\cos x-\sin x]$

$\cancel{-(D-1)\cos x}$

$\cancel{(D-1)(-\cos x)}$

$\cancel{\sin x+\cos x}$

$y = C.F + P.I$

$= (c_1 + c_2 x)e^{x} + \frac{x}{2}\cos x + \frac{1}{2}\cos x - \frac{1}{2}\sin x$

Q. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x\cos x$

$m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

$C.F. = c_1 + c_2 e^{-x}$

$P.I : \frac{1}{D^2+D} x\cos x$

$x \frac{\cos x}{D^2+D} + \frac{d}{dD}\left[\frac{1}{D^2+D}\right] \cdot \cos x$

$x \frac{\cos x}{D-1} \times \frac{D+1}{D+1}$

$x \frac{(D+1)\cos x}{D^2-1}$

$x \frac{(D+1)\cos x}{-2}$

$-\frac{x}{2}[-\sin x + \cos x)$

$\frac{x}{2}[\sin x - \cos x]$

(ii) $\frac{d}{dD}\left[(D^2+D)^{-1}\right]\cos x$

$-1(D^2+D)^{-2}\cdot \cancel{\cos x}\,(2D+1)\cos x$

$-\frac{(2D+1)\cos x}{(D^2+D)^2}$

$-\frac{(2D+1)\cos x}{(D-1)^2}$

$+\frac{1}{2}(2D+1)(D+1)\cos x$

$\frac{1}{2}(2D^2+2D+D+1)\cos x$

$\frac{1}{2}(2D^2+3D+1)\cos x$

$-\cos x - \frac{3}{2}\sin x$

$-\frac{(2D+1)\cos x}{D^4+D^2+2D^3}$

$-\frac{(2D+1)\cos x}{1-1+2D^3}$

$-\frac{1}{2}(2D+1)\frac{1}{D^3}\cos x$

$+\frac{1}{2}(2D+1)\frac{1}{D}\cos x$

$\frac{1}{2}(2D+1)\sin x$

$\cos x + \frac{\sin x}{2}$

$y = c_1 + c_2e^{-x} + \frac{x\sin x}{2} - \frac{x\cos x}{2} + \cos x + \frac{\sin x}{2}$

Q. $\cancel{(D^2+2D+1)y}$ $(D^2+3D+2)y = x\cos 2x$

$m^2+3m+2=0$

$m^2+2m+m+2=0$

$m(m+2)+1(m+2)=0$

$m_2 -1, -2.$

$C.F = c_1e^{-x}+c_2e^{-2x}.$

$$P.I = \frac{1}{D^2+3D+2} x\cos 2x$$

$$\frac{x\cos 2x}{D^2+3D+2} + \frac{d}{dD}\left[\frac{1}{D^2+3D+2}\right]\cos 2x$$

(i) $\dfrac{x\cos 2x}{-4+2+3D}$

$\dfrac{x\cos 2x}{3D-2}$

(~~ii~~) $x\,\dfrac{\cos 2x\,(3D+2)}{9D^2-4}$

$\dfrac{x\,(3D+2)\cos 2x}{-36-4}$

$\Rightarrow \dfrac{-x}{40}(3D+2)\cos 2x$

$\dfrac{-x}{40}(-6\sin 2x+2\cos 2x)$

$\Rightarrow \dfrac{3x}{20}\sin 2x - \dfrac{\cos 2x}{20}.$

(iii) $\cancel{x}\,\dfrac{d}{dD}\left[\dfrac{1}{D^2+3D+2}\right]\cos 2x$

$$\frac{d}{dD}\left[\frac{1}{D^2+3D+2}\right]\cdot\cos 2x$$

$$\frac{(D^2+3D+2)\,0 - (2D+3)\cos 2x}{(D^2+3D+2)^2}$$

$$-(2D+3)\cdot\frac{\cos 2x}{(D^2+3D+2)^2}$$

$$-(2D+3)\frac{\cos 2x}{(-4+3D+2)^2}$$

$$-(2D+3)\cdot\frac{\cos 2x}{(3D-2)^2}$$

$$-(2D+3)\cdot\frac{\cos 2x}{9D^2+4-12D}$$

$$-(2D+3)\cdot\frac{\cos 2x}{-36+4-12D}$$

$$-(2D+3)\cdot\frac{\cos 2x}{-32-12D}$$

$$\frac{1}{4}(2D+3)\frac{\cos 2x}{(8+3D)}$$

$$\frac{1}{4}(2D+3)\cdot\frac{\cos 2x}{8+3D}\cdot\frac{8-3D}{8-3D}$$

$$\frac{1}{4}(2D+3)\frac{\cos 2x\,(8-2D)}{64-9D^2}$$

$$\frac{1}{4}(2D+3)\frac{\cos 2x\,(8-2D)}{64+36}$$

$$\frac{1}{4}(2D+3)(8-2D)\frac{1}{100}\cos 2x$$

$$\frac{1}{400}(16D-4D^2+24-6D)\cos 2x$$

$$\frac{-1}{400}\cdot 32\sin 2x + \frac{4\times 4}{400}\cos 2x + \frac{24\cos 2x}{400} + \frac{12\sin 2x}{400}$$

$$\frac{-2}{25}\sin 2x + \frac{1}{25}\cos 2x + \frac{3}{50}\cos 2x + \frac{3}{100}\sin 2x$$

$$-\frac{8}{100}\sin 2x + \frac{3}{100}\sin 2x + \frac{2}{50}\cos 2x + \frac{3}{50}\cos 2x$$

$$\Rightarrow -\frac{1}{20}\sin 2x + \frac{1}{10}\cos 2x$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{3x}{20}\sin 2x - \frac{1}{20}\cos 2x - \frac{1}{20}\sin 2x + \frac{1}{10}\cos 2x$$

$$= c_1 e^{-x} + c_2 e^{-2x} + \frac{3x}{20}\sin 2x + \frac{1}{20}\cos 2x - \frac{1}{20}\sin 2x.$$

Cases of failure :-

(1) $f(x) = e^{ax}$

$$= \frac{1}{f(D)} \cdot e^{ax}$$

$$= \frac{1}{0} e^{ax}$$

It is a case of failure

(i) $\frac{1}{D-a} e^{ax} = x e^{ax}$

(ii) $\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax}$

(iii) $\frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$

Q $\frac{d^2y}{dx^2} - a^2y = e^{ax}$

$m = \pm a$

$CF = c_1 e^{ax} + c_2 e^{-ax}$

$\frac{1}{(D-a)(D+a)} \cdot e^{ax}$

$\frac{1}{2a} \frac{1}{(D-a)} e^{ax}$

$\frac{x e^{ax}}{2a}$

Q. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 2e^{-3x}$

$m^2 + 6m + 9 = 0$

$(m+3)^2 = 0$

$m = -3, -3.$

$C.F = (c_1 + c_2 x) e^{-3x}$

$P.I. = \frac{1}{(D+3)^2} \cdot 2e^{-3x}$

$2 \frac{1}{(D+3)^2} e^{-3x}$

$2 \cdot \frac{x^2}{2!} e^{-3x}$

$= x^2 e^{-3x}.$

Q (2) $f(x) = \sin ax$

$P.I = \frac{1}{f(D^2)} \sin ax$

$\frac{1}{f(-a^2)} \sin ax$

$= 0$

case fails.

$P.I = \frac{-x \cos ax}{2a}$

$\frac{1}{f(D^2)} \cdot \cos ax$

$\frac{1}{f(-a^2)} \cos ax$

$= 0$

$P.I. = \frac{x \sin ax}{2a}$

Q $\frac{d^2y}{dx^2} + a^2y = \cos ax$

$c_1 \cos ax + c_2 \sin ax$

$C.F = \cancel{c_1 e^{-ax} + c_2 e^{ax}}$

$\frac{1}{D^2 + a^2} \cdot \cos ax$

$P.I = \frac{x}{2a} \sin ax$

$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{2a} \sin ax$

$$\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$

$$P.I = \frac{e^x}{5} - \frac{x}{4} \cos 2x$$

General Method:-

① $P.I = \frac{1}{D-a} f(x)$

$$e^{ax} \int f(x) \cdot e^{-ax} dx$$

② $P.I = \frac{1}{(D-a_1)(D-a_2)} f(x)$

$$= \frac{1}{(D-a_1)} e^{a_2 x} \int f(x) e^{-a_2 x} dx$$

$$= e^{a_1 x} \int g(x) \cdot e^{-a_1 x} dx$$

Apply General Method to solve.

① $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$m = 1, 2$

C.F $= c_1 e^x + c_2 e^{2x}$

$$\frac{1}{D^2 - 3D + 2} e^{5x}$$

$$\frac{1}{(D-1)(D-2)} e^{5x}$$

$$\frac{1}{(D-1)(D-2)} = \frac{A}{D-1} + \frac{B}{D-3}$$

$$\frac{1}{(D-1)(D-2)} = \frac{AD - 3A + BD - B}{(D-1)(D-3)}$$

$1 = D(A+B) - 3A - B$

$A = -B$

$-3A - B = 1$

$3B - B = 1$

$B = 1$

$A = -1$

$$\frac{-1}{(D-1)} e^{5x} + \frac{1}{D-2} e^{5x}$$

$$-\left[e^x \int e^{5x} \cdot e^{-x} dx\right] + \left[e^{2x} \int e^{5x} \cdot e^{-2x} dx\right]$$

$$-\left[e^x \cdot \int e^{4x} dx\right] + \left[e^{2x} \frac{e^{3x}}{3} dx\right]$$

$$-\left[\frac{e^x \cdot e^{4x}}{4}\right] \quad \frac{e^{5x}}{3}$$

$$= -\frac{e^{5x}}{4}$$

$$-\frac{e^{5x}}{4} + \frac{e^{5x}}{3}$$

(2) $\frac{d^2y}{dx^2} - y = 2 + 5x$

$(D^2 - 1)\, y = 2 + 5x$

$m = \pm 1$

$C.F = c_1 e^{-x} + c_2 e^{x}$

$P.I = \frac{1}{(D-1)(D+1)} (2+5x)$

$\frac{A}{D-1} + \frac{B}{D+1} = \frac{1}{(D-1)(D+1)}$

$AD + A + BD - B = 1$

$A = -B$

$A - B = 1$

$-2B = 1$

$B = -1/2$

$A = 1/2$

$\frac{1}{2}\left[\frac{1}{(D-1)} (2+5x) - \frac{1}{(D+1)} (2+5x)\right]$

$\frac{1}{2}\left[e^{x} \int (2+5x) \cdot e^{-x} dx - e^{-x} \int (2+5x) e^{x} dx\right]$

$\cancel{\frac{1}{2}\left[e\right.}$ $e^{x} \int (2+5x)\, e^{-x} dx$

$e^{x}\left[-e^{-x}(2+5x) + \int 5e^{-x} dx\right.$

$e^{x}\left[-e^{-x}(2+5x) - 5e^{-x}\right]$

$e^{x}\left[-e^{-x}(2+5x) - 5e^{-x}\right]$

$\Rightarrow -(2+5x)-5$

$\Rightarrow e^{-x}\int(2+5x)e^{x}dx$

$\Rightarrow e^{-x}\left[(2+5x)e^{x}-5\int e^{x}dx\right]$

$e^{-x}\left[(2+5x)e^{x}-5e^{x}\right]$

$(2+5x)-5$

$\frac{1}{2}\left[-2-5x-5-((2+5x)-5)\right]$

$\frac{1}{2}\left[-7-5x-2-5x+5\right]$

$\frac{1}{2}\left[-4-10x\right]$

$-2-5x$

☆ <u>Linear Differential Equations with Variable coefficients</u>

$$x^{n}\frac{d^{n}y}{dx^{n}}+a_{1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n}y=f(x)$$

variable coefficients should reduce in constant coeffi..

Let,

$x=e^{z}$

$z=\log x$

$\frac{dz}{dx}=\frac{1}{x}$

$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$; $x\frac{dy}{dx}=\frac{dy}{dz}=Dy$ where $D=\frac{d}{dz}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dz}\right) = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2} = (D^2 - D)\,y \quad , D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)\,y \quad , D = \frac{d}{dz}$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)$$

Q.Q. solve :- $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$

Let, $x = e^z$

$z = \log x$

~~dz~~ and $x\frac{dy}{dx} = Dy$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

where $D = \frac{d}{dz}$

$$D(D-1)y - 3Dy + 4y = 2x^2$$

$$y(D^2-D-3D+4) = 2x^2$$

$$y(D^2-4D+4) = 2e^{2z}$$

$$m^2-4m+4=0$$

$$(m-2)^2=0$$

$$m=2,2$$

$$C.F = (c_1+c_2 z)e^{2z}$$

$$P.I = \frac{2e^{2z}}{(D-2)^2}$$

$$2\frac{e^{2z}}{(D-2)^2}$$

$$\cancel{2}\cdot\frac{z^2}{\cancel{2}}e^{2z}$$

$$\therefore \quad P.I. = z^2\cdot e^{2z}.$$

General solution is

$$y = (c_1+c_2 z)e^{2z} + z^2\cdot e^{2z}$$

$$(c_1+c_2\log x)e^{2\log x} + (\log x)^2\cdot e^{2\log x}$$

$$(c_1+c_2\log x)\cdot x^2 + (\log x)^2\cdot x^2.$$

Q. $x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 13y = \log x$

Let :

$x = e^z$

$z = \log x$

$x\frac{dy}{dx} = Dy$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$(D(D-1))y + 7Dy + 13y = z$

$y(D^2 - D + 7D + 13) = z$

$(D^2 + 6D + 13)y = z$

$m^2 + 6m + 13 = 0$

$\frac{-6 \pm \sqrt{36-52}}{2}$

$\frac{-6 \pm \sqrt{-16}}{2}$

$\frac{-6 \pm 4i}{2}$

$= -3 \pm 2i$

$\alpha = -3 \quad \beta = 2$

C.F. $= e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$

$$\frac{}{D^2+6D+13} \cdot z$$

$$\frac{}{13\left(1+\left(\frac{D^2+6D}{13}\right)\right)} z$$

$$\frac{1}{13}\left(1+\left(\frac{D^2+6D}{13}\right)\right)^{-1} \cdot z$$

$$\frac{1}{13}\left(1-\left(\frac{D^2+6D}{13}\right)+\left(\frac{D^2+6D}{13}\right)^2\right) \cdot z$$

$$\frac{1}{13}\left(z-\frac{6D \cdot z}{13}\right)$$

$$\frac{1}{13}\left(z-\frac{6}{13}\right)$$

$$P.I = \frac{z}{13} - \frac{6}{169}$$

$$\frac{\log x}{13} - \frac{6}{169}$$

$$\frac{1}{13}\left(\log x - \frac{6}{13}\right)$$

☆ General solution :-

$$y = e^{-3\log x}\left(c_1\cos(2(\log x)) + c_2\sin(2\log x)\right) + \frac{1}{13}\left(\log x - \frac{6}{13}\right)$$

$$y = x^{-3}\left(c_1\cos(2\log x) + c_2\sin(2\log x)\right) + \frac{1}{13}\left(\log x - \frac{6}{13}\right)$$

Q. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$

let, $x = e^z$

$z = \log x$

$$D(D-1)y - Dy + 4y = \cos z + e^z \sin z$$

$$y(D^2 - D - D + 4) = \cos z + e^z \sin z$$

$$y(D^2 - 2D + 4) = \cos z + e^z \sin z$$

Auxiliary Equation :-

$$m^2 - 2m + 4 = 0$$

$$m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} \Rightarrow \frac{2 \pm 2i\sqrt{3}}{2} \Rightarrow 1 \pm i\sqrt{3}$$

$$m = 1 \pm i\sqrt{3}$$

$$\alpha = 1, \ \beta = \sqrt{3}$$

So,

$$CF = e^z [c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z]$$

Now,

$$P.I = \frac{1}{D^2 - 2D + 4} (\cos z + e^z \sin z)$$

$$= \underbrace{\frac{1}{D^2 - 2D + 4} \cos z}_{I} + \underbrace{\frac{1}{D^2 - 2D + 4} e^z \sin z}_{II}$$

Taking Part I

$$= \frac{1}{D^2 - 2D + 4} \cos z$$

$$= \frac{1}{-1 - 2D + 4} \cos z$$

$$= \frac{1}{3 - 2D} \cos z$$

$$= \frac{1}{3 - 2D} \times \frac{3 + 2D}{3 + 2D} \cos z$$

$$= \frac{(3 + 2D)\cos z}{9 - 4D^2}$$

$$\frac{3+2D}{9+4}\cos z = \frac{(3+2D)\cos z}{13} = \frac{1}{13}(3\cos z - 2\sin z)$$

Taking Part II

$$= \frac{1}{D^2-2D+4} e^z \sin z$$

$$= e^z \frac{1}{(D+1)^2-2(D+1)+4} \sin z$$

$$= e^z \frac{1}{D^2+1+2D-2D-2+4} \sin z$$

$$\Rightarrow e^z \frac{1}{D^2+3} \sin z$$

$$\Rightarrow e^z \frac{1}{-1+3} \sin z$$

$$= \frac{1}{2} e^z \sin z$$

So, $P.I. = \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2} e^z \sin z$

So,

General solution is :-

$$C.F + P.I$$

$$\Rightarrow e^z[c_1\cos(\sqrt{3}z) + c_2\sin(\sqrt{3}z)] + \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2}e^z \sin z.$$

$$\Rightarrow e^{\log x}[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)] + \frac{1}{13}(3\cos(\log x) - 2\sin(\log x))$$
$$+ \frac{1}{2}e^{\log x}\sin(\log x)$$

$$\Rightarrow x[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x) + \frac{1}{13}(3\cos x(\log x) - 2\sin(\log x))$$
$$+ \frac{1}{2}x\sin(\log x).$$

Method of Variation of Parameters

$$\cancel{P}\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = f(x)$$

Working Rule :-

P and Q are constants, given this

① To find C.F

$C.F = c_1 u + c_2 v$

② Let the complete solution be $y = Au + Bv$

where, $\frac{dA}{dx} = \frac{-v f(x)}{w}$, $\frac{dB}{dx} = \frac{u f(x)}{w}$

where, $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

Wronskian's Determinant.

$A = \int ___ \, dx + c_1$

$B = \int ___ \, dx + c_2$

$\therefore$

Q. Solve:- $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

$(D^2+1)y = \operatorname{cosec} x$

$m^2+1 = 0$

$m = \pm i$

$C.F. = c_1 \cos x + c_2 \sin x$

$u = \cos x \quad v = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$W = 1$

$$\frac{dA}{dx} = \int -\sin x . \operatorname{cosec} x \, dx$$

$$\Rightarrow -\int 1 \, dx$$

$$\Rightarrow -x + c_1$$

$$\frac{dB}{dx} = \int \cos x . \operatorname{cosec} x \, dx$$

$$\int \cot x \, dx$$

$$B \Rightarrow \log \sin x + c_2$$

General solution:-

$Au + Bv$

$(-x+c_1)u + (\log\sin x + c_2)v$

$(-x+c_1)\cos x + (\log\sin x + c_2)\sin x$

$y = c_1\cos x + c_2\sin x - x\cos x + \sin x\log\sin x$

2. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = \frac{e^x}{2x}$

$D^2y - 2Dy + y$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

xe^x

$e^x + xe^x$

$e^x(1+x)$

$C.F = (c_1 + c_2x)e^x$

$c_1e^x + c_2e^x \cdot x$

$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(1+x) \end{vmatrix}$

$u = e^x \quad v = xe^x. \quad e^{2x}(1+x) - e^{2x}x$

$W = e^{2x}$

$A = \int \frac{-v \cdot f(x)}{W}$

$= -\int \frac{xe^x \cdot e^x}{W \; 2x} = \frac{-1}{2}\int \frac{\cancel{x}\, e^{2x}}{e^{2x} \cdot \cancel{x}} = \frac{-1}{2}x + c_1$

$B = \int \frac{e^x e^x}{e^{2x}\, 2x} = \frac{1}{2}\int \frac{1}{x}dx = \frac{1}{2}\log x + c_2$

$$\left(-\frac{x}{2}+c_1\right)e^x + \left(\frac{1}{2}\log x + c_2\right)xe^x$$

$$y = c_1e^x + c_2xe^x - \frac{xe^x}{2} + \frac{xe^x}{2}\log x$$

$$y = (c_1+c_2x)e^x + \frac{1}{2}xe^x(\log x - 1)$$

Q. $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$

$m = \pm 2i$

$C.F. = c_1\cos 2x + c_2\sin 2x$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$2\cos^2 2x + 2\sin^2 2x$$

$$W = 2$$

$$A = -\int \frac{\sin 2x \cdot 4\tan 2x}{2}$$

$$\Rightarrow -2\int \sin 2x \tan 2x \, dx$$

$$\Rightarrow -2\int \frac{\sin 2x \cdot \sin 2x}{\cos 2x} dx$$

$$\Rightarrow -2\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$\Rightarrow -2\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$A = -2\int(\sec 2x - \cos 2x)\,dx$$

$$\Rightarrow -2\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$\Rightarrow -2\left[\frac{1}{2}\log|\sec 2x + \tan 2x| - \frac{1}{2}\sin 2x\right]$$

$$\Rightarrow -\log|\sec 2x + \tan 2x| + \sin 2x + c_1$$

$$B = \int \frac{u \cdot f(x)}{w}$$

$$4\int \frac{\cos 2x \cdot \tan 2x}{2}\,dx$$

$$\Rightarrow 2\int \cos 2x \cdot \frac{\sin 2x}{\cos 2x}\,dx$$

$$\Rightarrow 2\int \sin 2x\,dx$$

$$\Rightarrow \frac{-2}{2}\cos 2x + c_2$$

$$\Rightarrow -\cos 2x + c_2$$

So,
General solution is

$$Au + Bv$$

$$(-\log|\sec 2x + \tan 2x| + \sin 2x + c_1)\cos 2x + (-\cos 2x + c_2)\sin 2x$$

$$\Rightarrow c_1\cos 2x + c_2\sin 2x + \cancel{\sin 2x\cos 2x} - \cancel{\sin 2x\cos 2x} - \log|\sec 2x + \tan 2x|$$

$$\Rightarrow c_1\cos 2x + c_2\sin 2x - \log|\sec 2x + \tan 2x|$$

Simultaneous Diff. Equations

(1) $\frac{dx}{dt} - 7x + y = 0$, $\frac{dy}{dt} - 2x - 5y = 0$

$(D-7)x + y = 0$, $(D-5)y - 2x = 0$

$2(D-7)x + 2y = 0$, $y = -(D-7)x$

$-2(D-7)x + (D-7)(D-5)y = 0$ $-(D-5)(D-7)x - 2x = 0$

$x((D-5)(D-7)+2) = 0$

$(D-7)(D-5)y + 2y = 0$ $D^2(D^2 - 12D + 37) = 0$

$y(D^2 - 12D - 33) = 0$

$m^2 - 12m - 33 = 0$

$\frac{12 \pm \sqrt{144 + 132}}{2}$

$\Rightarrow \frac{12 \pm \sqrt{12}}{2}$

$\Rightarrow \frac{12 \pm \sqrt{276}}{2}$

let $\frac{d}{dx} = D$

$$\frac{dy}{dx} + y = z + e^x$$

$$\frac{dz}{dx} + z = y + e^x$$

$$Dy + y - z = e^x$$

$$y(D+1) - z = e^x$$

$$-y + (D+1)z = e^x$$

$$\cancel{y(D+1)} - z = e^x$$

$$-\cancel{y(D+1)} + (D+1)^2 z = e^x(D+1)$$

$$(D+1)^2 z - z = e^x(D+2)$$

$$z(D^2 + \cancel{1} + 2D - \cancel{1}) = e^x(D+2)$$

$$z(D^2 + 2D) = e^x(D+2)$$

$$z = \frac{e^x(D+2)}{D^2+2D}$$

$$z = \frac{(D+2)\ e^x}{1+2}$$

$$z = \frac{(D+2)\ e^x}{3}$$

$$z = \frac{1}{3}(e^x + 2e^x)$$

$$z = \cancel{e^x}\ \frac{3e^x}{3} = e^x$$

$$D(D+2) = 0$$

$$D = 0 \quad D = -2$$

$$C.F = c_1 + c_2 e^{-2x}$$

General solution for z =

$$z = c_1 + c_2 e^{-2x} + e^x$$

$$(D+1)y - z = e^x$$
$$-y + (D+1)z = e^x$$

$$(D+1)^2 y - \cancel{(D+1)z} = e^x(D+1)$$
$$-y + \cancel{(D+1)z} = e^x$$

$$((D+1)^2 - 1)y = 3e^x$$

$$y(D^2 + \cancel{1} + 2D - \cancel{1})y = 3e^x$$

$$D(D+2) = 3e^x$$
$$D = 0, \quad D = -2$$

C.F =

$$y = c_1 + c_2 e^{-2x}$$

P.I = $\dfrac{3e^x}{e^x D^2 + 2D}$

$$\Rightarrow \frac{3e^x}{3}$$

$$\Rightarrow e^x$$

$$y = c_1 + c_2 e^{-2x} + e^x.$$

$$\frac{dx}{dt} + 4x + 3y = t$$

$\frac{d}{dt} = D$

$$\frac{dy}{dt} + 2x + 5y = e^t$$

$$Dx + 4x + 3y = t$$
$$(D+4)x + 3y = t$$
$$2x + (D+5)y = e^t$$

$$2(D+4)x + 6y = 2t$$
$$2(D+4)x + (D+4)(D+5)y = (D+4)e^t$$
$$- \qquad - \qquad -$$

$$6y - (D+4)(D+5)y = 2t - (D+4)e^t$$

$$y(6 - (D^2+9D+20))y = 2t - e^t - 4e^t$$

$$y(-D^2-9D-14) = 2t - e^t - 4e^t$$

$$y(D^2+9D+14) = e^t + 4e^t - 2t$$

$$y(D^2+9D+14) = 5e^t - 2t$$

C.F = $m^2+9m+14=0$

$m^2+7m+2m+14=0$

$(m+7)(m+2)=0$

$m=-2, m=-7$

C.F = $c_1e^{-2t} + c_2e^{-7t}$

$$\frac{5e^t}{D^2+9D+14} - \frac{2t}{D^2+9D+14}$$

$$\frac{5e^t}{1+9+14} \qquad \frac{-2}{14}\left(1+\frac{D^2+9D}{14}\right)^{-1} t$$

$$\frac{5e^t}{24} \qquad \frac{-1}{7}\left(1-\left(\frac{D^2+9D}{14}\right)\right) t$$

$$\frac{-1}{7}\left(t - \frac{9}{14}Dt\right)$$

$$\frac{-1}{7}\left(t - \frac{9}{14}\right)$$

$$\frac{5}{128} - \frac{t}{7} + \frac{9}{14\times 7} - \frac{t}{7}$$

$$\boxed{y = c_1 e^{-2t} + c_2 e^{-7t} + \frac{5}{24} e^{t} + \frac{9}{128} - \frac{t}{7}}$$

$$(D+4)x + 3y = t$$
$$2x + (D+5)y = e^{t}$$

$$(D+4)(D+5)x + \cancel{3(D+5)y} = t(D+5)$$
$$6x + \cancel{3(D+5)y} = 3e^{t}$$

$$(D^2+9D+14)x = 1 + 5t - 3e^{t}$$

$$C.F = c_1 e^{-2x} + c_2 e^{-7x}$$

$$\frac{t^{0x}}{D^2+9D+14} + \frac{5}{D^2+9D+14} t - \frac{3}{D^2+9D+14} e^{t}$$

$$\frac{1}{14} + \frac{5}{14}\left(1 + \left(\frac{D^2+9D}{14}\right)\right)^{-1} t - \frac{3}{24} e^{t}$$

$$\frac{1}{14} - \frac{3}{24} e^{t} + \frac{5}{14}\left(t - \frac{9D}{14} t\right)$$

$$\frac{14}{196} - \frac{3}{24} e^{t} + \frac{5t}{14} - \frac{45}{196}$$

$$\frac{5t}{14} - \frac{3}{24} e^{t} + \frac{1}{14}\left(1 - \frac{45}{14}\right)$$

$$\frac{-31}{196}$$

$$\boxed{x = c_1 e^{-2x} + c_2 e^{-7x} + \frac{5t}{14} - \frac{3}{24} e^{t} - \frac{31}{196}}$$

(4) $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$

$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$

$Dx + (D-2)y = 2\cos t - 7\sin t \quad \times D$

$(D+2)x - Dy = 4\cos t - 3\sin t \quad \times (D-2)$

$D^2x + D(D-2)y = -2\sin t - 7\cos t$

$(D^2-4)x - D(D-2)y = -4\sin t - 3\cos t - 8\cos t + 6\sin t$

$(D^2+D^2-4)x = \cancel{-14\cos} -18\cos t$

$2m^2 - 4 = 0$

$m^2 = 2$

$m = \pm\sqrt{2}$

$c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x}$

$\frac{-18\cos t}{2D^2-4}$

$= \frac{-18\cos t}{-2-4}$

$= \frac{-18\cos t}{-6}$

$= 3\cos t$

$x = c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x} + 3\cos t.$

$(D+2)Dx + (D^2-4)y = -2\sin t - 7\cos t + 4\cos t - 14\sin t$

$D(D+2)x - D^2y = -4\sin t - 3\cos t$

$- \quad + \quad + \quad +$

$\cancel{D^2} - D(D^2-4+D^2)y = -12\sin t$

$2D^2 - 4 = 0$

$m^2 = 2$

$m = \pm\sqrt{2}$

$C.F = c_1e^{-\sqrt{2}x} + c_2e^{\sqrt{2}x}$

$$\frac{-12\sin t}{2D^2-4}$$

$$\Rightarrow \frac{-12\sin t}{-2-4}$$

$$\Rightarrow \frac{-12}{-6}\sin t$$

$$\Rightarrow 2\sin t$$

$$y = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} + 2\sin t$$

Differential Equations of 1st order and n^{th} degree

Let, $p = \frac{dy}{dx}$

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \text{———} + A_{n-1} p + A_n = 0$$

$$A_i \rightarrow f(x, y)$$

(1) Solvable for p.

Solve:- $p^2 - 5p + 6 = 0$

$p^2 - 3p - 2p + 6 = 0$

$p(p-3) - 2(p-3) = 0$

$(p-3)(p-2) = 0$

$p = 2, \; p = 3$

$p - 2 = 0$

$\frac{dy}{dx} = 2$

$\int dy = \int 2dx \Rightarrow y = 2x + c \Rightarrow (y - 2x - c) = 0$

$p - 3 = 0$

$\frac{dy}{dx} = 3$

$y = 3x + c$

$(y - 3x - c) = 0$

Gs. $(y - 2x - c)(y - 3x - c) = 0$

(2) $x^2p^2 + xyp - 6y^2 = 0$

$x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$

$xp(xp + 3y) - 2y(xp + 3y) = 0$

$(xp - 2y)(xp + 3y) = 0$

$p = \frac{2y}{x}, \frac{-3y}{x}$

$\frac{dy}{dx} = \frac{2y}{x}$

$\frac{1}{2}\int \frac{dy}{y} = \int \frac{dx}{x}$

$\frac{1}{2}\log y = \log x + c$

$\log\sqrt{y} = \log xc$

$\sqrt{y} = xc$

$y = x^2c^2$

$y - x^2c^2 = 0$

$\frac{dy}{dx} = \frac{-3y}{x}$

$-\frac{1}{3}\frac{dy}{y} = \frac{dx}{x}$

$y^{-1/3} = xc$

$y = x^{-3}c$

$yx^3 = c$

$(y - x^2c^2)(yx^3 - c) = 0$

(2) Solvable for y

$y = \sin p - p\cos p$

$\frac{dy}{dx} = \cos p \cdot \frac{dp}{dx} - \left(1\cos p\frac{dp}{dx} - p\sin p\frac{dp}{dx}\right)$

$p = \cancel{\cos p\frac{dp}{dx}} - \cancel{\cos p\frac{dp}{dx}} + p\sin p\frac{dp}{dx}$

$$\cancel{p} = \cancel{p}\sin p \frac{dp}{dx}$$

$$\int dx = \int dp \sin p$$

$$x = -\cos p + c$$

$$x + \cos p - c = 0$$

$$\cos p = c - x$$

$$p = \cos^{-1}(c-x)$$

Putting in 1:-

$$y = \sin(\cos^{-1}(c-x)) - \cos^{-1}(c-x) \cdot \cos(\cos^{-1}(c-x))$$

Q. $x^2 + p^2 x - yp = 0 \rightarrow (1)$

$$yp = x^2 + p^2 x$$

$$y = \frac{x^2}{p} + px \rightarrow \frac{dy}{dx} = \frac{p \cdot 2x - x^2 \frac{dp}{dx}}{p^2} + \frac{dp}{dx} \cdot x + p$$

$$y - px = \frac{x^2}{p} \qquad p = \frac{2xp}{p^2} - \frac{x^2}{p^2}\frac{dp}{dx} + x\frac{dp}{dx} + p$$

$$\frac{dp}{dx}\left(\frac{-x^2}{p^2} + x\right) + \frac{2x}{p} = 0 \qquad \cancel{p} = \frac{2x}{p} - \frac{x^2}{p^2}\frac{dp}{dx} + x\frac{dp}{dx} + \cancel{p}$$

$$-x + \frac{x^2}{p^2} = \frac{dx}{dp}\left(\frac{2x}{p}\right) \qquad -\frac{x^2}{p^2}\frac{dp}{dx} + x\frac{dp}{dx} + \frac{2x}{p} = 0$$

$$\left(\frac{x^2}{p^2} - x\right)\frac{p}{2x} = \frac{dx}{dp}$$

$$\left(\frac{x^2}{p^2} - x\right)\frac{p}{2x} = \frac{dx}{dp}$$

$$\frac{x}{2p} - \frac{p}{2} = \frac{dx}{dp}$$

$$\frac{dx}{dp} - \frac{x}{2p} = \frac{-p}{2}$$

$$I.F = e^{-\int \frac{1}{2p}}$$

$$e^{-\frac{1}{2}\int \frac{1}{p}}$$

$$e^{-\frac{1}{2}\log p}$$

$$p^{-1/2}$$

$$I.F = \frac{1}{\sqrt{p}}$$

$$x \cdot \frac{1}{\sqrt{p}} = -\int \frac{1}{\sqrt{p}} \cdot \frac{p}{2}\, dp$$

$$\frac{x}{\sqrt{p}} = -\frac{1}{2}\int \sqrt{p} + c$$

$$\frac{x}{\sqrt{p}} = -\frac{1}{2}\frac{p^{1/2+1}}{1/2+1} + c$$

$$-\frac{2}{2}\frac{p^{3/2}}{3}$$

$$\frac{x}{\sqrt{p}} = -\frac{1}{3}p^{3/2} + c$$

$$\frac{x}{\sqrt{p}} + \frac{p\sqrt{p}}{3} = c$$

$$x + \frac{p\sqrt{p}\sqrt{p}}{3} = c\sqrt{p}$$

$$x + \frac{p^2}{3} = c\sqrt{p} \rightarrow (2)$$

$$x = -\frac{p^2}{3} + c\sqrt{p} \rightarrow (2)$$

Eqn (1) and (2) collectively gives the required answer

(4) <u>Clairout's Equation</u>

The Equation $y = px + f(p)$ where p is $\frac{dy}{dx}$ is called Clairout's Equation.

Diff. w.r.t x

$$\cancel{\frac{dy}{dx}} = \cancel{p} + x\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}$$

$$\frac{dp}{dx}\left(x + f'(p)\right) = 0$$

Either $\frac{dp}{dx}=0$ or $x+f'(p)=0$, On solving ① one

$$\frac{dp}{dx}=0$$

$$dp=0$$

$$p=c$$

General solution is ~~$y=p$~~ $y=cx+f'(c)$

Q. find the general sol^n of $y=px+\frac{a}{p}$

$$y=cx+\frac{a}{c}$$

The Given Eq^n is in clairout's form. So, General solution is obtained by replacing p by c.

So,

G.S. is $y=cx+\frac{a}{c}$

$$y=px+p-p^2$$

$$y=cx+c-c^2$$

$$(y-px)(p-1)=p$$

$$y=px\ \cancel{p-1}+\frac{p}{p-1}$$

$$y=cx+\left(\frac{c}{c-1}\right)$$

(4) $\sin y \cdot \cos px - \cos y \cdot \sin px - p = 0$

$$\sin(y - px) = p$$

$$y - px = \sin^{-1} p$$

$$y = px + \sin^{-1} p$$

$$y = cx + \sin^{-1} c.$$

$$(x-a)p^2 + (x-y)p - y = 0$$

$$xp^2 - ap^2 + xp - yp - y = 0$$

$$x - y(p+1) + x(p+p^2) = ap^2$$

$$-y(p+1) + xp(1+p) = ap^2$$

$$-y + xp = \frac{ap^2}{1+p}$$

$$y - xp = \frac{-ap^2}{p+1}$$

$$y = xp - \frac{ap^2}{p+1}$$

$$y = xc - \frac{ac^2}{c+1}$$

(6) $axy\,p^2 + (x^2 - ay^2 - b)p - xy = 0$

Hint $x^2 = u$

$y^2 = v$

$2x\frac{dx}{dx} = \frac{du}{dx}$, $2y\frac{dy}{dx} = \frac{dv}{du}$

$$\frac{y\,dy}{x\,dx} = \frac{dv}{du} \Rightarrow \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du} \Rightarrow p = \frac{x}{y}\frac{dv}{du} \Rightarrow p = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$$

$$p = \frac{\sqrt{u}}{\sqrt{v}}\, P$$

$$a\sqrt{u}\sqrt{v}\,p^2 + (u - av - b)\,p - \sqrt{u}\sqrt{v} = 0$$

$$a\sqrt{u}\sqrt{v}\,p^2 + up - avp - bp - \sqrt{u}\sqrt{v} = 0$$

$$ap^2 + \frac{u}{\sqrt{u}\sqrt{v}}\,p - \frac{av}{\sqrt{u}\sqrt{v}}\,p - \frac{b}{\sqrt{u}\sqrt{v}}\,p - 1 = 0$$

Unit :- 3

Partial Differential Equation

$z = f(x, y)$ ☆ formation of P.D.E.

$\frac{\partial z}{\partial x} = p$ ☆ Elimination of Arbitrary Constants

$\frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial x^2} = r$

$\frac{\partial^2 z}{\partial x \partial y} = s$

$\frac{\partial^2 z}{\partial y^2} = t$

Q.1 find a P.D.E. by eliminating arbitrary constants h and k from the relation

$$(x-h)^2 + (y-k)^2 + z^2 = c^2$$

Differentiating partially w.r.t x

$$2(x-h) + 2z\frac{\partial z}{\partial x} = 0$$

$$\cancel{\frac{\partial z}{\partial x}} = 2(x-h) = -2zp$$

$$x - h = -zp$$

Differentiating w.r.t y partially

$$2(y-k) + 2z\frac{\partial z}{\partial y} = 0$$

$$y - k = -zq.$$

$\therefore$ (1) gives

$$(-zp)^2 + (zq)^2 + z^2 = c^2$$

$$z^2p^2 + z^2q^2 + z^2 = c^2$$

$$z^2(1 + p^2 + q^2) = c^2$$

Q.2 $z = ax + by + ab$

$\frac{\partial z}{\partial x} = a$

$p = a$

$\frac{\partial z}{\partial y} = b$

$b = q$

$$\boxed{z = px + qy + pq}$$

Q. 3. $z = A e^{-p^2 t} \cos px$

$$\frac{\partial z}{\partial x} = -pA e^{-p^2 t} \sin px$$

$$\frac{\partial^2 z}{\partial x^2} = -p^2 A e^{-p^2 t} \cos px$$

$$\frac{\partial^2 z}{\partial x^2} = -p^2 z$$

$$p^2 = -\frac{1}{z} \cdot \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial z}{\partial t} = -p^2 A \cos px \, e^{-p^2 t}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}}$$

Q. $z = f(x+iy) + F(x-iy)$

$$\frac{\partial z}{\partial x} = f'(x+iy) + F'(x-iy)$$

$$\frac{\partial z}{\partial y} = f'(x+iy)\, i - F'(x-iy)\, i$$

$$\frac{\partial z}{\partial y} = i\left(f'(x+iy) - F'(x-iy)\right)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+iy) + f''(x-iy)$$

$$\frac{\partial^2 z}{\partial y^2} = -f''(x+iy) - F''(x-iy)$$

$$r + t = 2F''(x-iy)$$

$$\frac{r+t}{2} = f''(x-iy)$$

$$\boxed{r + t = 0}$$

Q. Elimination of Arbitrary function

from the P.D.E. by eliminating arbitrary function 'f' from the relation

$$z = f\left(\frac{y}{x}\right)$$

$$p = \frac{\partial z}{\partial x} = -f'\left(\frac{y}{x}\right) \cdot \frac{y}{x^2}$$

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$-\frac{px^2}{y} = qx$$

$$-px = qy$$

$$px + qy = 0$$

Q. Construct a P.D.E.

$$f(x^2+y^2+z^2,\ z^2 - 2xy) = 0$$

let $x^2+y^2+z^2 = u$

$z^2 - 2xy = v$

$$\phi(u, v) = 0$$

Differentiating w.r.t u and v

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}\right)$$

$$+ \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x}\right)$$

$$\frac{\partial f}{\partial u}\left(2x\cdot 1 + 2y\cdot 0 + 2z\cdot p\right) + \frac{\partial f}{\partial v}\left(-2y + (-2x)0 + 2z\cdot p\right) = 0$$

$$\frac{\partial f}{\partial u}\left(2x + 2zp\right) + \frac{\partial f}{\partial v}\left(-2y + 2zp\right) = 0$$

$$\frac{\partial f}{\partial u}\left(2x + 2zp\right) = \frac{\partial f}{\partial v}\left(2y - 2zp\right)$$

$$\therefore \frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = \frac{y - zp}{x + zp}$$

Differentiating w.r.t. y

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial f}{\partial u}\left(0 + 2y + 2z\cdot q\right) + \frac{\partial f}{\partial v}\left(-2y(0) + (-2x) + 2z\cdot q\right) = 0$$

$$\frac{\partial f}{\partial u}\left(2y + 2zq\right) = \frac{\partial f}{\partial v}\left(2x - 2zq\right)$$

$$\frac{\partial f}{\partial u}\left(y + zq\right) = \frac{\partial f}{\partial v}\left(x - zq\right)$$

$$\frac{\partial f/\partial u}{\partial f/\partial v} = \frac{x - zq}{y + zq}$$

$$\frac{y - zp}{x + zp} = \frac{x - zq}{y + zq}$$

$$(y - zp)(y + zq) = (x - zq)(x + zp)$$

$$\cancel{y^2 - z^2p^2 = x^2 - z^2p^2}$$

$$(y-zp)(y+zq) = (x-zq)(x+zp)$$

$$y^2 + yzq - zpy - \cancel{z^2pq} = x^2 + zxp - xzq - \cancel{z^2pq}$$

$$x^2 - y^2 + zp(x+y) - zq(x+y) = 0.$$

$$(x-y)(x+y) + zp(x+y) - zq(x+y) = 0$$

$$x - y + zp - zq = 0$$

Q find partial differentiation Equation $\phi(x^2-z^2, x^3-y^3) = 0$

$u = x^2 - z^2$

$v = x^3 - y^3$

$\phi(u,v) = 0 \rightarrow (1)$

Diff (1) w.r.t. x

$$\frac{\partial \phi}{\partial u}\left[2x - 2z\frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[3x^2\right] = 0$$

$$\therefore \frac{\partial \phi/\partial u}{\partial \phi/\partial v} = \frac{-3x^2}{2x - 2zp} \rightarrow (2)$$

Partially Diff (2) w.r.t y

$$\frac{\partial \phi}{\partial u}\left[-2z\cdot\frac{\partial z}{\partial y}\right] + \frac{\partial \phi}{\partial v}\left[-3y^2\right] = 0$$

$$\therefore \frac{\partial \phi/\partial u}{\partial \phi/\partial v} = \frac{3y^2}{-2zq} \rightarrow (3)$$

from Eq^n (2) & (3)

$$\frac{-3x^2}{2(x-zp)} = \frac{3y^2}{-2zq}$$

$$\frac{x^2}{x-zp} = \frac{y^2}{zq}$$

$$x^2zq = y^2x - y^2zp$$

or

$$z(x^2q - y^2p) = y^2x$$

Lagrange's Equation:-

$$Pp + Qq = R$$

where, P, Q, R is $f(x,y)$

The subsidiary Equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$u(x,y,z) = a$$
$$v(x,y,z) = b$$

General solution is

$$\phi\, F(u,v) = 0$$

Q Solve, $xp + yq = z$

The subsidiary eq^n are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking:-

$$\frac{dx}{x} = \frac{dy}{y}$$

On integration

$$\log x = \log y + \log a$$
$$\log \frac{x}{y} = \log a$$
$$\frac{x}{y} = a.$$

Taking :-

$$\frac{dy}{y} = \frac{dz}{z}$$

On integration

$$\log y = \log z + \log b$$
$$\log \frac{y}{z} = \log b$$
$$\frac{y}{z} = b$$

$\therefore$ General sol^n is

$$f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

Q. $\left(\frac{y^2 z}{x}\right)p + xzq = y^2$

The subsidiary Eq^n is

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Dividing by x throughout

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

(i) $$\frac{dx}{y^2 z} = \frac{dy}{x^2 z}$$

$$\frac{dx}{y^2} = \frac{dy}{x^2}$$
$$x^2 dx = y^2 dy$$
$$\frac{x^3}{3} - \frac{y^3}{3} = a$$

or

$$x^3 - y^3 = a$$

(ii) $$\frac{dx}{y^2 z} = \frac{dz}{y^2 x}$$

$$x\,dx = z\,dz$$

On integrating

$$\frac{x^2}{2} - \frac{z^2}{2} = b$$

or

$$x^2 - z^2 = b$$

General sol^n is

$$f(x^3 - y^3, x^2 - z^2) = 0$$

Q. $zp = -x$

The subsidiary Eq^n is

$$\therefore \frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$$

(i) $$\frac{dx}{z} = \frac{dz}{-x}$$

$$x\,dx = -z\,dz$$

On integrating

$$\frac{x^2}{2} + \frac{z^2}{2} = a$$

or $(x^2 + z^2) = a$

(iii) $\frac{dx}{z} = \frac{dy}{0}$

$dy = 0$

$y = b$

General solⁿ is

$f(x^2+z^2, y) = 0$

Q. $(mz - ny)p + (nx - lz)q = ly - mx$

The subsidiary Eqⁿ is

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

(i) l, m, n as multipliers

$$= \frac{ldx + mdy + ndz}{lmz - lny + mnx - mlz + lny - nmx}$$

$$= \frac{ldx + mdy + ndz}{0}$$

Taking,

$$\frac{dx}{mz-ny} = \frac{ldx + mdy + ndz}{0}$$

$$ldx + mdy + ndz = 0$$

On integrating

$$lx + my + nz = a$$

(ii) x, y, z as multipliers

$$= \frac{x\,dx + y\,dy + z\,dz}{mxz - nxy + nxy - lzy + lyz - mxz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

Taking this with anyone of the ratios

$$\therefore \quad x\,dx + y\,dy + z\,dz = 0$$

$$x^2 + y^2 + z^2 = b$$

General solⁿ is

$$f(lx + my + nz,\ x^2 + y^2 + z^2) = 0$$

Q. $$\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$$

Multiplying by xyz throughout,

$$x(y-z)p + y(z-x)q = (x-y)z$$

The subsidiary Eqⁿs is

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{(x-y)z}$$

(i) 1, 1, 1 as multipliers

$$\frac{dx + dy + dz}{xy - xz + yz - yx + xz - yz} = \frac{dx + dy + dz}{0}$$

Taking this with anyone of the ratios

$$dx + dz + dy = 0$$

On integrating

$$x + y + z = a$$

(iii) $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integrating

$$\log x + \log y + \log z = \log b$$

$$xyz = b$$

general solution is

$$f(x+y+z, xyz) = 0$$

Q $xzp + yzq = xy$

The subsidiary Eq^n's is

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

(i) $\frac{dx}{xz} = \frac{dy}{yz}$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

$$\log x - \log y = \log a$$

$$\log \frac{x}{y} = \log a$$

$$\frac{x}{y} = a$$

(ii) $\frac{dy}{yz} = \frac{dz}{xy}$

$$\frac{dy}{z} = \frac{dz}{x}$$

$$xdy = zdz$$

$$ay\,dy = z\,dz$$

$$\frac{ay^2}{2} = \frac{z^2}{2} + b$$

$$\frac{ay^2}{2} - \frac{z^2}{2} = b$$

$$\frac{x}{y}\frac{y^2}{2} - \frac{z^2}{2} = b$$

$$xy - z^2 = b$$

So, General solution is

$$f\left(\frac{x}{y}, xy - z^2\right) = 0$$

Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

Subsidiary Equation is

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

$1/x$, $1/y$, $1/z$ as multiplier

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$$

$$\frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{0}$$

Taking this with any one of the ratio.

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

On integrating

$$\log x + \log y + \log z = \log a$$

$$xyz = a$$

(ii) x, y, z as multiplier

$$\frac{x\,dx + y\,dy + z\,dz}{0}$$

Taking this with any one of the ratio

$$x\,dx + y\,dy + z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b$$

or

$$x^2 + y^2 + z^2 = b$$

So, General solution is

$$f(\qquad\qquad, x^2+y^2+z^2) = 0$$

Q Solve

$$x(y^2+z)p - y(x^2+z)q = z(x^2-y^2) = 0$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

(i) Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

$$= \frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{0}$$

Taking this neither one of the ratios

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

On integrating

$$\log x + \log y + \log z = \log a$$

$$\log xyz = \log a$$

$$a = xyz$$

(ii) Taking $x, y, -1$ as multiplier

$$\frac{x\,dx + y\,dy + dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$\frac{x\,dx + y\,dy - dz}{\cancel{x^2y^2} + x^2z - \cancel{y^2x^2} - y^2z - zx^2 + zy^2}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

Taking this neither one of the ratios

$$x\,dx + y\,dy - dz = 0$$

$$\frac{x^2}{2}+\frac{y^2}{2}-z = b$$

$$x^2+y^2-2z = b'$$

General solution is

$$f\left(xyz,\ x^2+y^2-2z\right) = 0$$

Q $z(xp-yq) = y^2-x^2$

$$zxp - zyq = y^2-x^2$$

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2}$$

(i) $\frac{dx}{x} = \frac{dy}{-y}$

$$\log x = -\log y + \log a$$

$$\log xy = \log a$$

$$\boxed{a = xy}$$

(ii) Taking xyz as multipliers

$$\cancel{xdx + ydy + zdz}$$

$$0$$

Taking this with any one of the multiplier

$$\therefore$$

$$xdx + ydy + zdz = 0$$

On integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=b$$

$$x^2+y^2+z^2=b$$

So,

General solution is :-

$$f(xy,\ x^2+y^2+z^2)=0.$$

Homogenous linear Equation with constant coefficients

$z \to x, y$

$$\frac{\partial^n z}{\partial x^n}+\frac{A_1\partial^n z}{\partial z^{n-1}\partial y}+\frac{A_2\partial^n z}{\partial z^{n-2}\partial y^2}+\cdots+A_n\frac{\partial^n z}{\partial y^n}=f(x,y) \quad (1)$$

$$D=\frac{\partial}{\partial x},\quad D'=\frac{\partial}{\partial y}$$

$$(D^n+A_1D^{n-1}D'+A_2D^{n-2}D'^2+A_3D^{n-3}D'^3+\cdots+A_nD'^n)z=f(x,y)$$

Working Rule.

for CF :-

Replace D by m, D' by 1, we get auxiliary Equation as

$$m^n+A_1m^{n-1}+A_2m^{n-2}+A_3m^{n-3}+\cdots \quad A_n=f(x,y)=0$$

(i) Distinct

$$CF=f_1(y+m_1x)+f_2(y+m_2x)+\cdots$$

② Equal

$$CF. = f_1(y+m_1x) + x f_2(y+m_1x)$$

Q Find the General solution of following Equations

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$$

$$(D^2 - DD' - 6(D')^2)z = 0$$

Auxiliary Equation is

$$\cancel{D^2 - D} - m^2 - m - 6 = 0$$

$$m^2 - 3m + 2m - 6 = 0$$

$$m(m-3) + 2(m-3) = 0$$

$$(m-3)(m+2) = 0$$

$$m = 3, -2$$

General solution,

$$z = f_1(y-2x) + f_2(y+3x)$$

Q $(D+2D')(D-3D')^2 z = 0$

$$(m+2)(m-3)^2 = 0$$

$$m = -2, 3, 3$$

$$z = f_1(y-2x) + f_2(y+3x) + x f_3(y+3x)$$

Q. $4r - 12s + 9t = 0$

$4m^2 - 12m + 9 = 0$

$4m^2 - 6m - 6m + 9 = 0$

$2m(2m-3) - 3(2m-3)$

$(2m-3)(2m-3) = 0$

$m = \frac{3}{2}, \frac{3}{2}$

$z = f_1\left(y + \frac{3}{2}x\right) + x f_2\left(y + \frac{3}{2}x\right)$

Q $\frac{\partial^2 z}{\partial x^2} + a^2 \frac{\partial^2 z}{\partial y^2} = 0$

$m^2 + a^2 = 0$

$m = \pm ai$

$z = f_1(y + aix) + f_2(y - aix)$

Q. $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x^2 \partial y} - 6\frac{\partial^3 z}{\partial y^3} = 0$

$m^3 - 7m^2 - 6 = 0$

~~$m^2(m-3) - 4m(m-3) - 12(m-3)$~~

$m^2(m-1) - 6m(m-1) - 6(m-1) = 0$

$(m^2 - 6m - 1)(m-1) = 0$

$m = 1, 3 \pm \sqrt{15}$

$m = 1,\ 3+\sqrt{15},\ 3-\sqrt{15}$

$z = f_1(y+x) + f_2(y+(3+\sqrt{15})x) + f_3(y+(3-\sqrt{15})x)$

$\phi(D, D')\ z = f(x, y)$

$G.S = C.F + P.I$

$C.F \to D,\ E$

Rules for P.I

(1) $f(x, y) = e^{ax+by}$

$$P.I = \frac{1}{\phi(D, D')} e^{ax+by}$$

$$P.I = \frac{1}{\phi(a, b)} e^{ax+by}$$

(1) $\dfrac{\partial^3 z}{\partial x^3} + 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$

$D^3 z - 3D^2 D' z + 4D'^3 = e^{x+2y}$

$m^3 - 3m^2 + 4 = 0$

$m^2(m+1) - 4m(m+1) + 4(m+1) = 0$

$(m^2 - 4m + 4)(m+1) = 0$

$m = \dfrac{4 \pm \sqrt{16-16}}{2} \Rightarrow 2, 2$

$m = -1, 2, 2$

$$C.F = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$$

$$P.I = \frac{1}{D^3 - 3D^2D' + 4D'^3} \cdot e^{x+2y}$$

$$\frac{1}{1 - 3\times 2 + 4\times 8} e^{x+2y}$$

$$\frac{1}{33-6} e^{x+2y}$$

$$\frac{1}{27} e^{x+2y}$$

General solution is $= z = C.F. + P.I$

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

(2) $f(x,y) = \sin(ax+by)$

$$P.I = \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax+by)$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax+by)$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$$

$$(D^2 - DD')z = \sin x \cos 2y$$

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0, 1$

$C.F = f_1(y) + f_2(y+x)$

$$P.I = \frac{1}{D^2 - DD'}\left[\sin(x+2y) + \sin(x-2y)\right]$$

$$\frac{1}{D^2 - DD'}\sin(x+2y) + \frac{1}{D^2 - DD'}\sin(x-2y)$$

$$\frac{1}{-1+2}\sin(x+2y) + \frac{1}{-1-2}\sin(x-2y)$$

$$\frac{1}{2}\left(\sin(x+2y) - \frac{1}{3}\sin(x-2y)\right)$$

General solution =

$$f_1(y) + f_2(y+x) + \frac{1}{2}\sin(x-2y) - \frac{1}{6}\sin(x-2y)$$

Q. $(D^3 - 7D'D^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

$m^3 - 7m - 6 = 0$

$m^2(m+1) - m(m+1) - 6(m+1) = 0$

$(m^2 - m - 6)(m+1) = 0$

$(m-3)(m+2)(m+1) = 0$

$m = -1, -2, 3$

$m = -2, -1, 3$

$C.F = f_1(y-2x) + f_2(y-x) + f_2(y+3x)$

$P.I_2 = \dfrac{1}{D^3 - 7D^2D' - 6D'^3} \sin(x+2y)$

$\dfrac{1}{-D + 7D^2 \times 4 + 24D'} \sin(x+2y)$

$\dfrac{1}{-D + 28D^2 + 24D'} \sin(x+2y)$

$\Rightarrow \dfrac{1}{27D + 24D'} \sin(x+2y)$

$\Rightarrow$ Multiplying dividing by D

$\dfrac{1}{D^3 - 7D^2D' - 6D'^3} e^{2x+y}$

$\dfrac{1}{8 - 14 - 6}$

$P.I_1 = -\dfrac{1}{12} e^{2x+y}$

$\dfrac{D}{27D^2 + 24DD'} \sin(x+2y)$

$\dfrac{D \sin(x+2y)}{-27 - 48}$

$\dfrac{D \sin(x+2y)}{-75}$

$P.I_2 = -\dfrac{1}{75} \cos(x+2y)$

General solution is

$f_1(y-2x) + f_2(y-x) + f_3(y+3x) - \dfrac{1}{12} e^{2x+y} - \dfrac{1}{75} \cos(x+2y)$

General Method :-

$$P.I = \frac{1}{D - mD'} f(x, y)$$

$$= \int f(x, c - mx)\, dx$$

$$c = y + mx.$$

Q Solve :- $r - 4s + 4t = e^{2x+y}$

$$(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$C.F = f_1(y+2x) + x f_2(y+2x)$$

$$P.I. = \frac{1}{(D^2 - 2D')^2} e^{2x+y}$$

Applying Rule no. 1

$$\frac{1}{(2 - 2\times 1)^2} e^{2x+y}$$

$$\frac{1}{0} e^{2x+y}$$

Case fails

Applying General Method:-

$$P.I = \frac{1}{(D-2D')(D-2D')} e^{2x+y}$$

$m = 2$

$$\frac{1}{(D-2D')} \int e^{2x+c-2x} dx$$

$$\frac{1}{(D-2D')} \int e^{c} dx$$

$$\frac{1}{(D-2D')} e^{c} \cdot x \, dx$$

$$\frac{1}{(D-2D')} e^{y+2x} \cdot x \, dx$$

$$\int e^{c-2x+2x} \cdot x \, dx$$

$$\int e^{c} \cdot x \, dx$$

$$e^{c} \int x \, dx$$

$$\Rightarrow e^{c} \cdot \frac{x^2}{2}$$

$$P.I \Rightarrow e^{y+2x} \cdot \frac{x^2}{2}$$

Q2. $(D^2 - 2DD' - 15D'^2) z = 12xy$

$m^2 - 2m - 15 = 0$

$m^2 - 5m + 3m - 15 = 0$

$m(m-5) + 3(m-5) = 0$

$(m-5)(m+3) = 0$

$m = -3, 5$

$$CF = f_1(y-3x) + f_2(y+5x)$$

$$PI = \frac{1}{(D^2-2DD'-15D'^2)} \cdot 12xy$$

$$= \frac{12}{D^2}\left[\frac{1}{1-2\frac{D'}{D}-15\left(\frac{D'}{D}\right)^2}\right] \cdot xy$$

$$= \frac{12}{D^2}\left[1-\left(\frac{2D'}{D}+15\left(\frac{D'}{D}\right)^2\right)\right]^{-1} xy$$

$$= \frac{12}{D^2}\left[1+\frac{2D'}{D}\right] xy$$

$$= \frac{12}{D^2}\left[xy + \frac{2D'}{D}xy\right]$$

$$\frac{12}{D^2}\left[xy + 2\frac{1}{D}\cdot x\right]$$

$$\frac{12}{D^2}\left[xy + 2\int x\right]$$

$$\frac{12}{D^2}\left[xy + 2\frac{x^2}{2}\right]$$

$$\frac{12}{D^2}\left[xy + x^2\right]$$

$$\frac{12}{D}\left[\frac{x^2}{2}y + \frac{x^3}{3}\right]$$

$$12\left[\frac{x^3}{6}y + \frac{x^4}{12}\right]$$

$$= 2x^3y + x^4$$

Q.8. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cdot \cos x$

$m^2 + m - 6 = 0$

$m^2 + 3m - 2m - 6 = 0$

$m(m+3) - 2(m+3) = 0$

$(m+3)(m-2) = 0$

$m = 2, -3.$

$\therefore$ C.F : $f(y+2x) + f(y-3x)$

P.I : $\cancel{\dfrac{1}{(m+3)(m-2)} \cdot y\cos x}$

$$\frac{1}{D^2 + DD' - 6(D')^2} \cdot y\cos x$$

$$\frac{1}{D^2 + 3DD' - 2DD' - 6(D')^2} \cdot y\cos x$$

$$\frac{1}{D(D+3D') - 2D'(D+3D')} \cdot y\cos x$$

$$\frac{1}{(D+3D')(D-2D')} y\cos x$$

$$\frac{1}{(D+3D')} \int (c - 2x)\cos x \, dx$$

$$\frac{1}{(D+3D')} \left[c\int \cos x \, dx - 2\int x\cos x \, dx \right]$$

$$\frac{1}{(D+3D')} \left[c\sin x - 2\left[x\sin x + \cos x \right] \right]$$

$$\frac{1}{(D+3D')} \left[c\sin x - 2x\sin x - 2\cos x \right]$$

$$\frac{1}{(D+3D')} \left[(y+2x)\sin x - 2x\sin x - 2\cos x \right]$$

$$\frac{1}{(D+3D')} \left[y\sin x + \cancel{2x\sin x} - \cancel{2x\sin x} - 2\cos x \right]$$

$$\frac{1}{(D+3D')} \left[y\sin x - 2\cos x \right]$$

$$\int ((c+3x)\sin x - 2\cos x)\,dx$$

$$\int c\sin x\,dx + 3\int x\sin x\,dx - 2\int \cos x\,dx$$

$$c\int \sin x\,dx + 3\int x\sin x\,dx - 2\int \cos x\,dx$$

$$\Rightarrow -c\cos x + 3\left[-x\cos x + \sin x\right] - 2\sin x$$

$$\Rightarrow -c\cdot\cos x - 3x\cos x + 3\sin x - 2\sin x$$

$$\Rightarrow -c\cos x - 3x\cos x + \sin x$$

$$\Rightarrow -(y-3x)\cos x - 3x\cos x + \sin x$$

$$\Rightarrow -y\cos x + \cancel{3x\cos x} - \cancel{3x\cos x} + \sin x$$

$$\Rightarrow -y\cos x + \sin x$$

(4) $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

$m^2 - m - 2 = 0$

$m^2 - 2m + m - 2 = 0$

$m(m-2) + (m-2) = 0$

$(m-2)(m+1) = 0$

$m = -1, 2$

$C.F. = f(y-x) + f(y+2x)$

$$P.I = \frac{1}{D^2 - DD' - 2D'^2}\cdot(y-1)e^x$$

$$= \frac{1}{D^2 - 2DD' + DD' - 2D'^2}\cdot(y-1)e^x$$

$$= \frac{1}{D(D-2D') + D'(D-2D')}\cdot(y-1)e^x$$

Here m=1 let $y = c+x$

$$\Rightarrow \frac{1}{(D-2D')(D+D')} \cdot (y-1)e^x$$

$$\frac{1}{(D-2D')} \int (c+x-1)\, e^x$$

$$\frac{1}{(D-2D')} \cdot \left[c\int e^x dx + \int x e^x dx - \int e^x dx \right]$$

$$\frac{1}{(D-2D')} \left[c \cdot e^x + x e^x - e^x - e^x \right]$$

$$\frac{1}{(D-2D')} \left[c e^x + x e^x - 2e^x \right]$$

$$\frac{1}{(D-2D')} \left[(y-x)\, e^x + x e^x - 2e^x \right]$$

$$\frac{1}{(D-2D')} \left[y e^x - \cancel{x e^x} + \cancel{x e^x} - 2e^x \right]$$

$$\frac{1}{(D-2D')} \left[y e^x - 2e^x \right]$$

Here $m=2$

let $y = c-2x$

$$\int \left[(c-2x)\, e^x - 2e^x \right] dx$$

$$\int c \cdot e^x dx - 2x e^x dx - 2 e^x dx$$

$$c\int e^x dx - 2\int x e^x dx - 2\int e^x dx$$

$$\Rightarrow c \cdot e^x - 2\left[x e^x - e^x \right] - 2e^x$$

$$c e^x - 2x e^x + \cancel{2e^x} - \cancel{2e^x}$$

$$\Rightarrow (y+2x)\, e^x - 2x e^x$$

$$y e^x + \cancel{2x e^x} - \cancel{2x e^x}$$

$$\Rightarrow y e^x$$

(1) $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, $u(0,y) = 8e^{-3y} \to (1)$

Let, $u = XY \to (2)$ be solution of (1)
where X is a function of x only
and Y is a function of y.

Only (2) in (1) gives :-

$X'Y = 4XY'$

$= \frac{X'}{X} = \frac{4Y'}{Y}$

$\frac{X'}{X} = \frac{4Y'}{Y} = K$ (say)

If $\frac{X'}{X} = K \Rightarrow X' - KX = 0$

auxiliary Eqn $m - K = 0$

$m = K$

Solution is $X = c_1 e^{Kx}$

If $\frac{4Y'}{Y} = K \Rightarrow 4Y' - KY = 0$

auxiliary Eqn

$4m - K = 0$

$m = K/4$

Solution is $Y = c_2 e^{\frac{Ky}{4}}$

General solution is

$u = c_1 c_2 e^{Kx} \cdot e^{\frac{Ky}{4}} \to (3)$

$u(x,y) = u = c_1 c_2 e^{Kx}$

Eqn (3) put $x = 0$

$8e^{-3y} = c_1 c_2 e^{Ky/4}$

$\boxed{c_1 c_2 = 8,}$

$\frac{Ky}{4} = -3$, $\boxed{K = -12}$

(2) $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

Gives, $u = 3e^{-y} - e^{-5y}$
when $= 0$

Let $u = XY$
be solution of (1), (1) gives-

$\Rightarrow 4X'Y + XY' = 3XY$

$\Rightarrow 4X'Y = 3XY - XY'$

$\Rightarrow 4X'Y = X(-Y' + 3Y)$

$= \frac{4X'}{X} = \frac{-Y' + 3Y}{Y} = K$

If $\frac{4X'}{X} = K$

$\Rightarrow 4X' = KX$

$\Rightarrow 4X' - KX = 0$

$4m - K = 0$

$m = \frac{K}{4}$

Sol^n is

$X = c_1 e^{Kx/4}$

If $\frac{-Y' + 3Y}{Y} = K$

$-Y' + 3Y = KY$

$-m + 3 = K$

$\therefore \quad m = 3 - K$

Sol^n is $Y = c_2 e^{(3-K)y}$

G.S is

$u = c_1 c_2 e^{Kx/4} \cdot e^{(3-K)y}$

when $X = 0$, then on comparing with 1st term

$c_1c_2 e^{(3-k)y} = 3e^{-y}$

$\boxed{c_1c_2 = 3}$

$3-k = -1$

$\boxed{4 = k}$

$\boxed{u = 3e^{x-y}}$

→ when $x = 0$, then comparing with 2nd term

$c_1c_2 e^{(3-k)y} = -e^{-5y}$

$c_1c_2 = -1$

$3-k = -5$

$k = 8$

$\boxed{u = e^{2x-5y}}$

(2) $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$

Q. $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u$

$u(x, 0) = 10e^{-x} - 6e^{-4x}$

$u = XT$

~~Hint~~ -

UNIT-4

Probability and statistics.

⇒ **Probability** ⇐

If an event 'A' can happen in m ways and fail in n ways all their ways are equally likely to occur then the probability of the happening of 'A' is,

$$P(A) = \frac{\text{no. of fav. cases}}{\text{total no. of mutually exclusive \& equally likely cases}}$$

~~P(A)~~ = ____

Probability is measure of success or failure of an event is occur.

Q. From a pack of 52 cards one is drawn at random. Find the probability of getting a king.

Sol⇒ Given, A = taking king from 52 cards.

no. of fav. outcomes = 4

total no. of mutually exclusive & equally likely outcomes = 52.

So,

$$P(A) = \frac{\text{no. of fav. cases}}{\text{total no. of mutually exclusive \& equally likely cases}}$$

$$= \frac{4}{52} = \frac{1}{13}$$

⇒ **Addition law of probability** ⇐

If A & B are two events associated with an experiment then,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

prepare its ~~perform~~ proof by your own.

Multiplication law of probability :

If there are two independent events the respective probability of which are known then the probability that both will happen is the product of probabilities of their happening respectively.

indep. $\boxed{P(A \cap B) = P(A)\,P(B)}$

Conditional probability :

Let 'A' and 'B' be two events of a sample space S and Probability of $B \neq 0$ the conditional probability of event 'A' given 'B' denoted by, $P(A/B)$

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

Byes theorem :

If $B_1, B_2, \ldots B_n$ are mutually exclusive events with $P(B_i) \neq 0$, $(i = 1, 2, 3, \ldots n)$ of random experiment then for any arbitrary event of A of sample space of the above experiment, where $P(A) > 0$

$$P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A/B_i)}$$

prepare its proof by your own.

Q A bag contains 4 white & 2 black balls. and a second bag contains 3 of each colour. A bag is selected at random and a ball is drawn random from choosen bag. What is probability that the ball drawn is white.

Sol:) Given,

event A = bag 1 is choosen

event B = bag 2 is choosen.

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$

Now, let E_1 = white ball choosen from bag 1 & G_1 = white ball choosen from bag 2.

So,

E = white ball choosen.

So,

$P(E_1) = \frac{4}{6} = \frac{2}{3}$ = $\frac{\text{white ball in bag 1}}{\text{total balls}}$

$P(G_1) = \frac{3}{6} = \frac{1}{2}$ = $\frac{\text{white ball in bag 2}}{\text{total balls}}$.

$P(E) = \frac{2}{3} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{7}{12}$.

Q An article manufactured by a company consists of two parts 'A' and 'B'. In the process of manufacture of part A 9 out of 100 are likely to be defective similarly 5 out of 100 are likely to be defective in manufacture of part B. Calculate the probability that the assembled article not be defective. (Events of finding the part A, non-defective and that of B independent).

Sol:) Given,

let E_1 = non-defective part of A.

G_1 = non-defective part of B.

$P(E_1) = 1 - P(\text{Defective of A})$

$P(G_1) = 1 - P(\text{Defective of B})$

$P(E_1) = 1 - \frac{9}{100}$ $\left(\frac{\text{Defective of A}}{\text{total part}}\right)$

$P(E_1) = \frac{91}{100}$

$P(G_1) = 1 - \frac{5}{100} = \frac{95}{100}$

$P(\text{not Defective}) = \frac{91 \times 95}{100 \times 100}$

$= 8.6 \times 10^{3} \times 10^{-4}$

$= 8.64 \times 10^{-1} = 0.8645$

Q3. An urn contains 9 balls, 2 of which are red, 3 blue & 4 black. ~~Three~~ 3 balls are drawn from the urn at random. What is the probability that:

(1) The three balls are of different colour.

(2) The three balls are of same colour.

Sol:

urn

9 balls $\boxed{\begin{matrix}2 \text{ red} \\ 3 \text{ blue} \\ 4 \text{ black}\end{matrix}}$

Case I =>

$$= \frac{2}{9} \times \frac{3}{8} \times \frac{4}{7} =$$

$$= \frac{12}{72 \times 7} = \frac{2}{42} = \frac{1}{21}$$

$$= 0.48.$$

Case II =>

$$= \frac{3}{9} \times \frac{2}{8} \times \frac{1}{7} + \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7}$$

$$= \frac{6 + 24}{9 \times 8 \times 7} = \frac{30}{9 \times 8 \times 7} = \frac{5}{24 \times 7}$$

$$= \frac{5}{84} = 0.06.$$

Q An urn first contains three white & 4 red balls & an urn 2nd contains 5 white & 6 red balls. One ball is drawn at random from one of the urns and is found to be white. Find the probability that it was drawn from urn 1st.

Sol: Given.

$\frac{1}{2}$ = urn 1

$\frac{3}{7} \leftarrow \begin{cases} 3 \text{ white} \\ 4 \text{ red} \end{cases}$

Urn 2 → $\frac{1}{2}$

$\left.\begin{matrix} 5 \text{ white} \\ 6 \text{ red} \end{matrix}\right\} \rightarrow \frac{5}{11}$

$$\frac{\frac{1}{2} \times \frac{3}{7}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{5}{11}} = \frac{33 + 35}{\frac{3}{7}}$$

$$= \frac{\frac{3}{7}}{\frac{68}{77}} = \frac{33}{68} \quad 33 \quad 68 = 0.49$$

Q An insurance company insured 2000 scooter drivers, 4000 car drivers & 6000 truck drivers. The probability of accidents are 0.01, 0.03 & 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver.

Sol:

total no. of insured = 2000 + 4000 + 6000

= 12000

$$P(\text{scooter ins}) = \frac{2}{12} = \frac{1}{6} = 0.17$$

$$P(\text{car}) = \frac{4}{12} = \frac{1}{3} = 0.33$$

$$P(\text{truck}) = \frac{1}{2} = 0.50$$

$$= \frac{\frac{1}{6} \times \frac{01}{10}}{\frac{1}{6} \times \frac{01}{10} \times \frac{03}{10} \times \frac{1}{3} + \frac{015}{100} \times \frac{1}{2}}$$

0.0017

0.0099

0.075

= 0.087.

$$= \frac{\frac{1}{60}}{\frac{1}{60} + \frac{1}{10} + \frac{15}{200}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6} + 1 + \frac{3}{4}}$$

$$= \frac{\frac{1}{6}}{\frac{2+12+9}{12}}$$

$$= \frac{1}{6} \times 12$$

Q. 3 urn contains 6R, 4B ; 4R, 6B ; 5R, 5B. One of the urns is selected at random & a ball is drawn from it. If the ball drawn is red find the probability that it is drawn from first urn.

Sol ⇒

$$= \frac{\frac{1}{3} \times \frac{6}{10}}{\frac{1}{3} \times \frac{6}{10} + \frac{1}{3} \times \frac{4}{10} + \frac{1}{3} \times \frac{5}{10}}$$

$$= \frac{6}{6+4+5} = \frac{6}{15} = \frac{2}{5}$$

⇒ Correlation ⇐

Two variables x and y are so related that an increase in the one is accompanied by an increase or decrease in the other then the variable are said to be correlated. E.g. The yield of crop varies with amount of rainfall.

Types of correlation ⇒

① positive Correlation ⇒ If an increase in the value of X, results in the corresponding increase in the value of other variable Y on an average or if a decrease in value of one variable X, results in the corresponding ~~the~~ decrease in the value of y on an average. Then the correlation is said to positive correlation.

② Negative Correlation ⇒ If an increase in the value of one variable X results in a corresponding decrease in the other value Y. or If the ↓ in the value of one variable X results in the ↑ of one or two values of Y. The correlation said to be -ve.

③ Linear Correlation ⇒ All the plotted points lie approximately on a stgt line then the correlation is said to be linear.

④ perfect correlation ⇒

If ~~the~~ two variable vary in such a way that their ratio always remains constant then the correlation is said to be perfect.

⇒ ~~Good~~ Karl - pearson's coefficient of correlation

Let r, ~~the~~ b/w two variable 'x' & 'y' is defined by,

$$\boxed{r = \frac{\Sigma xy}{\sqrt{\Sigma(x^2)(\Sigma y^2)}}} \rightarrow \text{Coefficient of correlation.}$$

$$\boxed{r = \frac{P}{\sigma x \sigma y} = \frac{cov(x,y)}{\sqrt{var\,x}\sqrt{var\,y}}}$$

P = variance.
cov = covariance.

x, y are deviation measured by their ~~mean~~ mean.

$$X = x - \bar{x} \quad , \quad Y = y - \bar{y}$$

→ r always equals ~~the~~ ±1 including +1 & -1.

Q. 10 Students got the following percentage of marks in economics & statics.

Roll No.	Marks in economics (x)	Marks in statistics (y)
1	78	84
2	36	51
3	98	91
4	25	60
5	75	68
6	82	62
7	90	86
8	62	58
9	65	53
10	39	47.

Q ⇒ Let the marks of two subjects be denoted by x & y respectively.

The mean for x marks is $\bar{x}$

$$\bar{x} = \frac{\Sigma x}{n}$$

$$\bar{x} = 65$$

The mean for y marks is $\bar{y}$

$$\bar{y} = \frac{\Sigma y}{n}$$

$$\bar{y} = 66.$$

If X & Y are deviations of x & y from their respective mean then the data may be arranged in the following form.

P.No.	Marks in Gr (X)	Mat Stat (Y)	$X = x - \bar{x}$	$Y = y - \bar{y}$	XY	X^2	Y^2
1	78	84	13	18	234	169	324
2	36	51	-29	-15	435	841	225
3	98	91	33	25	825	1089	625
4	25	60	-40	-6	240	1600	36
5	75	68	10	2	20	100	4
6	82	62	17	-4	-68	289	16
7	90	86	25	20	500	625	400
8	62	58	-3	-8	24	9	64
9	65	53	0	-13	0	0	169
10	39	47	-26	-19	494	676	361
					$\Sigma xy = 2704$	$\Sigma x^2 = 5398$	$\Sigma y^2 = 2124$

So,

$$r = \frac{\Sigma xy}{\sqrt{(\Sigma x^2)(\Sigma y^2)}}$$

$\sqrt{\Sigma x^2} = 73.471$

$\sqrt{\Sigma y^2} = 47.159$

So,

$$r = \frac{2704}{73.471 \times 47.159} = \frac{2704}{3464.818} = 0.78$$

Q. Calculate the coefficient of correlation b/w marks obtained by 8 students in mathematics & statistics

Sl. No. Student	(X) Maths	(Y) Statistics	$X = x - \bar{x}$	$Y = y - \bar{y}$	XY	X^2	Y^2
A	25	8	-10.75	-9.75			
B	30	10	-5.75	-7.5			
C	32	15	-3.75	-2.75			
D	35	17	-0.75	-0.75			
E	37	20	1.25	2.25			
F	40	23	4.25	5.25			
G	42	24	6.25	6.25			
H	45	25	9.25	7.25			
Total	286	142	0				

A ⇒ Let the marks of x,

$$\bar{x} = \frac{\Sigma x}{n} = \frac{25+30+32+35+37+40+42+45}{8}$$

$$= \frac{286}{8} = 35.75$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{8+10+15+17+20+23+24+25}{8}$$

$$= 17.75.$$

⇒ ~~Bion~~ Binomial Distribution:

Let there be 'n' independent trials in an experiment, let the random variable X denote the number of success in this (n) trials, let ~~small~~ 'p' be the probability of a success & 'q' the prob. of a failure in a single trial. So that $p+q=1$, let the ~~trials~~ trials be independent and p the constant for every trial. Let us find the probability of r success in 'n' trials. 'r' success can be obtained in 'n' trials in ${}^nC_r$ ways.

$$P(x=r) = {}^nC_r\, P(\underbrace{SSSS\ldots}_{r \text{ times}}\ \underbrace{FFF\ldots F}_{(n-r)\text{ times}})$$

$$= {}^nC_r\, \underbrace{P(S)(PS)\ldots P(S)}_{r \text{ factors}}\ \underbrace{P(F)P(F)\ldots P(F)}_{(n-r)\text{ factors}}$$

$$= {}^nC_r\, \underbrace{ppp\ldots p}_{r\text{-factors}}\ \underbrace{qqq\ldots q}_{(n-r)\text{ factors}}$$

$$\boxed{P(x=r) = {}^nC_r\, p^r q^{n-r}} \qquad r=0,1,2,3\ldots n.$$

$$\boxed{p+q=1}$$

NOTE ⇒ The successive probabilities p^r in ~~one~~ 1 for $r=0,1,2,\ldots n$, ${}^nC_0 q^n, {}^nC_1 pq^{n-1}, {}^nC_2 q^{n-2}p^2 + \ldots, {}^nC_n p^n$ which are the successive terms of the binomial expansion of $(q+p)^n$ therefore this distribution is k/a binomial probability distribution.

Q. Prove that in Binomial P.D. mean is ~~np~~ np and variance is npq respectively.

Sol ⇒ r success in B.P.D is given by,

$$P(r) = {}^nC_r\, p^r q^{n-r}, \qquad r = 0,1,2,\ldots n.$$

Mean ⇒ $\mu = \sum_{r=0}^{n} r P(r)$ $\qquad \left[\because \mu = \sum_{x=0}^{n} x f(x)\right]$

$$\Rightarrow \mu = \sum_{r=0}^{n} r\left[{}^nC_r p^r q^{n-r}\right]$$

$$= 0 + \left[{}^nC_1 p^1 q^{n-1}\right] + {}^nC_2 q^{n-2}p^2 + \ldots {}^nC_n p^n.$$

$$= \frac{np q^{n-1}}{1} + \frac{n(n-1)p^2 q^{n-2}}{2} + \frac{(n)(n-1)(n-2)}{6} p^3 q^{n-3} + \ldots + np^n.$$

$$= np\left[q^{n-1} + \frac{(n-1)p q^{n-2}}{2} + \frac{(n-1)(n-2)p^2 q^{n-3}}{6} + \ldots p^{n-1}\right]$$

$$= np\left[{}^{n-1}C_0 p^0 q^{n-1} + {}^{n-1}C_1 p q^{n-2} + \ldots + {}^{n-1}C_{n-1} p^{n-1}\right]$$

$$= np\,[q+p]^{n-1}$$

$$= np \times 1$$

$$\boxed{\mu = np}$$

Hence proved.

⇒ Variance is denoted by σ^2,

$$\sigma^2 = \sum_{r=0}^{n} r^2 P(r) - \mu^2$$

$$= \sum_{r=0}^{n}\left[r + r(r-1)\right]P(r) - \mu^2$$

$$= \sum_{r=0}^{n} r P(r) + \sum_{r=0}^{n} r(r-1)P(r) - \mu^2$$

$$= np + \sum_{r=0}^{n} \underbrace{r(r-1)P(r)}_{G} - \mu^2$$

Here,

$$G = \sum_{x=0}^{n} x(x-1)\, {}^nC_x\, p^x q^{n-x}$$

$$= 0 + 0 + \frac{n(n-1)}{2!} p^2 q^{n-2} + \frac{n(n-1)(n-2)}{6} p^3 q^{n-3} + \dots + n(n-1)\, p^n$$

$$= n(n-1)\left[p^2 q^{n-2} + (n-2)p^3 q^{n-3} + \dots (n-1)p^n\right]$$

$$= n(n-1)\left[p\, q^{n-2} + (n-2)(n-1)p^2 q^{n-3} + \dots (n-1)p^{n-1}\right]$$

$$= np\left[\sum_{x=1}^{n-1} (p+q)\right]$$

$$= npq\left[(n-1)pq^{n-1} + (n-1)(n-2)p^2 q^{n-2} + \dots (n-1)p^{n-1}\right]$$

$$= npq \cdot \left[\sum_{x=0}^{n-1} (p+q)\right]$$

$$G = npq$$

So,

$$\sigma^2 = np + npq - n^2p^2$$

$$= np(1+q-np)$$

Q. Assume that on the average one telephone number out of 15 called b/w 2pm & 3pm on week days is busy. What is the probability that of 6 randomly selected telephone num. are called

(i) not more than 3.

(ii) at least three of them will be busy.

Sol:

p, be the probability of a telephone number being busy b/w 2p.m & 3p.m. on week days which is

$$= \frac{1}{15} \qquad \boxed{p = \frac{1}{15}}$$

q, be the probability of a telephone number being not busy b/w 2p.m & 3p.m on week days i.e.

$$1 - \frac{1}{15} = \frac{14}{15}.$$

So,

$$\boxed{q = \frac{14}{15}}$$

$$p = \frac{1}{15}, \quad q = \frac{14}{15}, \quad n = 6$$

(i) This implies that the probability that not more than 3 will be busy.

$$x \le 3$$

$$x = 0, 1, 2 \text{ \& } 3.$$

So,

$$P(x) = P(0) + P(1) + P(2) + P(3)$$

$$P(x) = {}^6C_0\left(\frac{1}{15}\right)^0\left(\frac{14}{15}\right)^6 + {}^6C_1\left(\frac{1}{15}\right)^1\left(\frac{14}{15}\right)^5 + {}^6C_2\left(\frac{1}{15}\right)^2\left(\frac{14}{15}\right)^4 + {}^6C_3\left(\frac{1}{15}\right)^3\left(\frac{14}{15}\right)^3$$

$$P(x) = \frac{(14)^3}{(15)^6}\left[{}^6C_0(14)^3 + {}^6C_1(14)^2 \times 1 + {}^6C_2 \times (14)^1 + {}^6C_3\, 1\right]$$

$$= \left[(14)^3 + 6(14)^2 + 15(14) + 20\right] \frac{(14)^3}{(15)^6}$$

$$= 4150 \times \frac{(14)^3}{(15)^6}$$

$$= \frac{4150 \times 2744}{(15)^6} = \frac{11387600}{11390625} = 0.9997.$$

(0.005)
(AP)

$r \geq 3.$

$$p(r) = p(3) + p(4) + p(5) + p(6)$$

$$= \cancel{p(3) +} \; 0.995 \; \cancel{(1 - 0.9997)}$$

$$= p(3) + p(4) + p(5) + p(6)$$

$$= {}^6C_3 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3 + {}^6C_4 \left(\frac{1}{15}\right)^4 \left(\frac{14}{15}\right)^2$$

$$+ {}^6C_5 \left(\frac{1}{15}\right)^5 \left(\frac{14}{15}\right)^1 + {}^6C_6 \left(\frac{1}{15}\right)^6$$

$$p(r) = \frac{20(14)^3 + 15(14)^2 + 6(14) + 1}{15^6}$$

$$p(r) = \frac{20 \times 2744 + 15 \times 196 + 85}{15^6}$$

$$\boxed{p(r) = 0.0050}$$

Q. If the probability of hitting a target is 10% and ten shots are fired independently what is the probability that the target will be hit atleast once.

Sol. $p = \frac{1}{10}, \; q = \frac{9}{10}, \; n = 10, \; r \geq 1$

$\cancel{p(r) =}$

$$p(r) = 1 - p(0)$$

$$= 1 - {}^{10}C_0 \left(\frac{9}{10}\right)^{10}$$

$$= 1 - \left(\frac{9}{10}\right)^{10} = 0.6513.$$

⇒ Poisson's probability distribution

Q. Prove that poisson's probability distribution is a limiting case of binomial probability distribution.

Sol. If the parameters 'a' and 'b' of a ~~bin~~ binomial distribution are known we can find the distribution but in a ~~such~~ situation that 'n' is very large & 'p' is very small application of binomial distribution become complicated however if we assume ~~that~~ that $n \to \infty$, $p \to 0$ such that np always remains finite say, $np = \lambda$ then we get the poissons approximation to the bino. prob. distr.

We know that in BPD,

$$p(r) = {}^nC_r q^{n-r} p^r$$

$$p(r) = \frac{n(n-1) \cdots [n-(r-1)]}{r!} q^{n-r} p^r.$$

$$p(r) = \frac{n(n-1) \cdots (n-r+1)}{r!} (1-p)^{n-r} p^r.$$

$[np = \lambda \Rightarrow p = \lambda/n]$

$$p(r) = \frac{n(n-1) \cdots (n-r+1)}{r!} \left(1 - \frac{\lambda}{n}\right)^{n-r} \left(\frac{\lambda}{n}\right)^r$$

$$p(r) = \frac{\lambda^r}{r!} \left\{\frac{n(n-1)(n-2) \cdots (n-r+1)}{n^r}\right\} \frac{(1 - \lambda/n)^n}{(1 - (\lambda/n))^r}$$

$$p(r) = \frac{\lambda^r}{r!}\left\{\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-r+1}{n}\right)\right\}\frac{\left[\left[1-(\lambda/n)\right]^{n/\lambda}\right]^{\lambda}}{(1-\lambda/n)^r}$$

$$p(r) = \frac{\lambda^r}{r!}\left\{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(r-1)}{n}\right)\right\}\frac{\left[(1-\lambda/n)^{n/\lambda}\right]^{\lambda}}{(1-\lambda/n)^r}$$

As $\lim_{n\to\infty}$ so, $1-\frac{1}{n}\to 1$, $1-\frac{2}{n}\to 1$ ..... $1-\frac{(r-1)}{n}\to 1$

& $\left(1-\frac{\lambda}{n}\right)^r \to 1$

So,

$$p(r) = \frac{\lambda^r}{r!}\lim_{n\to\infty}\left[\left[(1-\lambda/n)\right]^{-n/\lambda}\right]^{-\lambda}$$

$$p(r) = \frac{\lambda^r}{r!}e^{-\lambda} \quad \text{where } r = 0, 1, 2, \ldots \infty$$

So,

$$\boxed{p(r) = \frac{\lambda^r}{r!}e^{-\lambda}}$$

Q. Prove that in poissons Prob. distribution mean & variance both are equals to λ.

Ans) we know that r success in poissons prob. dist. given by,

$$p(r) = \frac{\lambda^r}{r!}e^{-\lambda}$$

$$\text{mean } \mu = \sum_{r=0}^{\infty} r\,p(r)$$

$$= \sum_{r=0}^{\infty}\frac{r e^{-\lambda}\lambda^r}{r!}$$

$$= e^{-\lambda}\sum_{r=0}^{\infty}\frac{\lambda^r}{(r-1)!}$$

$$= e^{-\lambda}\left[1+\lambda+\frac{\lambda^2}{1!}+\frac{\lambda^3}{2!}+\cdots\right]$$

$$= \lambda e^{-\lambda}\left[1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right]$$

$$= \lambda e^{-\lambda}e^{\lambda}$$

$$\boxed{\mu = \lambda}$$

Now,

$$\text{variance } \sigma^2 = \sum_{r=0}^{\infty} r^2 p(r) - \mu^2$$

$$= \sum_{r=0}^{\infty} r^2 p(r) - \sum_{r=0}^{\infty} r^2[p(r)]^2$$

$$= \sum_{r=0}^{\infty} r^2 p(r) - r^2 p(r)^2$$

$$= \sum_{r=0}^{\infty} r^2 p(r)\left[1-p(r)\right]$$

$$= \sum_{r=0}^{\infty} r^2 p(r) - \lambda^2$$

$$= \sum_{r=0}^{\infty} r^2 e^{-\lambda}\frac{\lambda^r}{r!} - \lambda^2$$

$$= e^{-\lambda}\sum_{r=0}^{\infty}\frac{r\lambda^r}{(r-1)!} - \lambda^2$$

$$= e^{-\lambda}\left[1+\lambda+\frac{2\lambda^2}{1!}+\frac{3\lambda^3}{2!}+\cdots\right] - \lambda^2$$

$$\boxed{\sigma^2 = \lambda}$$

Q. Assume that the probability of an individual coal miner will killed in a mine accident during a year is $\frac{1}{2400}$. Use poisson distribution to calculate the probability that in a mine employing 200 miners their will be atleast 1 fated accident in a year.

Sol. Given,

~~P(miner killed) = $\frac{1}{2400}$~~

p be the probability of an individual coal miner being killed $= \frac{1}{2400}$

no. of coal miners = 200

$$\lambda = np = \frac{1}{2400} \times 200 = \frac{1}{12} = 0.083$$

So,

$$p(r) = \frac{\lambda^r}{r!} e^{-\lambda}$$

p(at least one fated Accident) = 1 − p(No fatal accident)

$$= 1 - p(0)$$

$$= 1 - \frac{\lambda^0}{0!} e^{-0.083}$$

$$= 1 - e^{-0.083}$$

$$= 1 - 0.9203$$

$$= 0.079$$

$$= 0.08.$$

$f(x) = a_0x^n + a_1x^{n-1} + \ldots$
$(a_0 \neq 0), a_1 a_2, \ldots a_n$
$f(x) = 0$ OR.
$a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$

Unit-V

Theory of Equations

The expression $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ where a's are constants $(a_0 \neq 0)$ & n is a positive integer, is called a polynomial in x of degree n.

$f(x) = 0$ OR.

$\Rightarrow a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$

General Properties:-

i. If α is a root of the eq^n $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$.

ii. Every equation of the nth degree has n roots. it means that roots may be real or imaginary.

iii. Intermediate Value Property:- If $f(a)$ & $f(b)$ have different signs, then the eq^n $F(x) = 0$ has atleast one root between $x = a$ & $x = b$.

iv. In an eq^n with real coefficients, imaginary roots occur in conjugate pairs, i.e. $\alpha + i\beta$ is a root then $\alpha - i\beta$ must be another root. Similarly, if $a + \sqrt{b}$ is an irrational root of an equation then $a - \sqrt{b}$ must also be its root.

v. Descarte's rule of signs:-

Relation's between Roots & Coefficients:-

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ ~~with~~ be the roots of the eq^n $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots a_nx^0 = 0$.

Then,

sum of the roots $= \Sigma\alpha_1 = -\dfrac{a_1}{a_0}$

sum of product of two roots $= \sum \alpha_1\alpha_2 = \frac{a_2}{a_0}$

sum of product of roots (3 at a time) $= \sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$

~~OR~~

Product of roots $= \alpha_1\alpha_2 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$

Examples

1. Solve the equation $2x^3+x^2-13x+6=0$

Solu^n. Given that :- $2x^3+x^2-13x+6=0$ — ①

If we want to apply general properties,

By trial method. $x=2$ satisfies eq^n ①

So, $(x-2)$ must be factor

Then,

$$
\begin{array}{r|l}
x-2 & 2x^3+x^2-13x+6 \quad (2x^2+5x-3 \\
 & \underline{2x^3-4x^2} \\
 & \quad 5x^2-13x+6 \\
 & \quad \underline{5x^2-10x} \\
 & \qquad -3x+6 \\
 & \qquad \underline{-3x+6} \\
 & \qquad\quad 0 \rightarrow 0
\end{array}
$$

Now, $2x^2+5x-3=0$

$\Rightarrow 2x^2+6x-x-3=0$

$\Rightarrow (2x-1)(x+3)=0$

$\Rightarrow x = 1/2, -3.$

Hence, $x = 2, 1/2, -3$

OR.

By synthetic division method,

By trial ↑

2	2	1	-13	6
		4	10	-6
	2	5	-3	0

} More reliable when higher degree polynomial ($x^5, x^6, \ldots$)

$(x-2)(2x^2+5x-3)=0$

2. Solve the equation $3x^3 - 4x^2 + x + 88 = 0$ one root being $2 + i\sqrt{7}$.

Solⁿ. Given that :- $3x^3 - 4x^2 + x + 88 = 0$ — ①

Since one root is $2 + i\sqrt{7} = x$

So, $x = 2 - i\sqrt{7}$ must be root of eqⁿ ①,

$\therefore$ $x - 2 - i\sqrt{7}$ and $x - 2 + i\sqrt{7}$ are factors.

$\therefore = (x - 2 - i\sqrt{7})(x - 2 + i\sqrt{7})$

$= (x-2)^2 + 7$

$= x^2 + 4 - 4x + 7$

$= x^2 - 4x + 11$ must be a factor.

$\therefore$

$$
\begin{array}{r|l}
 & 3x + 8 \\
x^2 - 4x + 11 & 3x^3 - 4x^2 + x + 88 \\
 & \underline{\cancel{3x^3} \mp 12x^2 \mp 33x} \\
 & 8x^2 - 32x + 88 \\
 & \underline{\cancel{8x^2} \mp \cancel{32x} \mp \cancel{88}} \\
 & R \to 0
\end{array}
$$

$\therefore$ $3x + 8 = 0$

$\Rightarrow x = -8/3$

Hence, the roots are $x = -8/3$, $2 + i\sqrt{7}$ {given}, $2 - i\sqrt{7}$ {by property}

3. Solve the equation $x^3 - 7x^2 + 36x + 36 = 0$. given one root is double of another.

Solⁿ. Given that :- $x^3 - 7x^2 + 36x + 36 = 0$ — ①

Also, one root is double of another.

$\therefore$ $\alpha = 2\beta$

Let the roots be α, β & γ.

Sum of roots $= 7 = \alpha + \beta + \gamma = 3\beta + \gamma = 7$ — ②

Sum of product of roots $= \alpha\beta + \beta\gamma + \gamma\alpha = 36$

$2\beta^2 + \beta\gamma + 2\beta\gamma = 36$ — (2)

Product of roots $= \alpha\beta\gamma = -36 = 2\beta^2\gamma = -36$ — (3)

On solving (2), (3), (4) we get

$\gamma = 7 - 3\beta$

$\Rightarrow 2\beta^2(7 - 3\beta) = -36$

$\Rightarrow 7\beta^2 - 3\beta^3 + 18 = 0$

$\Rightarrow 3\beta^3 - 7\beta^2 - 18 = 0$

$\Rightarrow (\beta - 3)\left(\beta = \left(\frac{-1}{3} + \frac{i\sqrt{17}}{3}\right)\right)\left(\beta - \left(\frac{-1}{3} - \frac{i\sqrt{17}}{3}\right)\right) = 0$

$\Rightarrow \beta = 3, \frac{-1}{3} \pm \frac{i\sqrt{17}}{3}$ $\quad \therefore \alpha = 6, 2\left(-\frac{1}{3} \pm \frac{i\sqrt{17}}{3}\right), \gamma = -2, 8 \mp i\sqrt{17},$

4. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given that the sum of two of its roots is zero.

Solⁿ. Given that :- $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ — (1)

Let the roots be $\alpha, \beta, \gamma, \delta$

Then, $\alpha + \beta = 0$ {given}

$\alpha + \beta + \gamma + \delta = 2 \Rightarrow \gamma + \delta = 2 \Rightarrow \gamma = 2 - \delta$

$\beta\delta + \delta\gamma + \alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha = \cancel{-6}\ 4$

$(\cancel{\alpha + \beta})(\gamma + \delta) + (\alpha\beta + \gamma\delta) = 4$

$\alpha\beta + \gamma\delta = 4$ $\qquad x^4 - 2x^3 + x^2 q + px^2 - 2px + pq$

OR

Quadratic factor corresponding to α, β is of the form $x^2 - 0x + p$; & that corresponding to γ, δ is of the form $x^2 - 2x + q$

$\therefore (x^4 - 2x^3 + 4x^2 + 6x - 21) = (x^2 + p)(x^2 - 2x + q)$ — (i)

$(x^4 - 2x^3 + 4x^2 + 6x - 21) = x^4 - 2x^3 + x^2(p + q) - 2px + pq$

$\therefore p + q = 4$ & $p = -3$ & $pq = -21$

$(-3)q = +21$

$\boxed{q = 7}$

Hence the given eqⁿ is eq. to $(x^2 - 3)(x^2 - 2x + 7) = 0$.

$x = \pm\sqrt{3}, 1 \pm i\sqrt{6}$

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Solution of Cubic Equations by Cardan's Method:-

Suppose we have a cubic eq^n $ax^3+bx^2+cx+d=0$ — ①
where a, b, c & d are constant coefficients.

Step:1. Identify coefficient of x^3 & divide whole eq^n by a, we get.

$$x^3+lx^2+mx+n=0 \quad (l=b/a,\ m=c/a,\ n=d/a)$$

Step:2. Remove x^2 term by putting $x=y-l/3$, we have

$$y^3+py+q=0 \text{ — ②}$$

Step:3 Now, do solve eq^n ② by putting $y=u+v$ the cubing both sides,

$$y^3=(u+v)^3$$

$$y^3=u^3+v^3+3u.v.y$$

$$y^3-3uvy-(u^3+v^3)=0 \text{ — ③}$$

Step:4 Compare eq^n ② & ③

$$uv=-p/3,\quad u^3+v^3=-q$$

$$\Downarrow$$

$$u^3v^3=-p^3/27,\quad u^3+v^3=-q$$

$$t^2-qt-p^3/27=0$$

$$u^3=\frac{1}{2}\left(-q+\sqrt{q^2+4p^3/27}\right)=\lambda^3$$

$$v^3=\frac{1}{2}\left(-q-\sqrt{q^2+4p^3/27}\right)-$$

$$u\to\lambda,\ \lambda\omega,\ \lambda\omega^2 \qquad \{\omega^3=1\}$$

$$v=-p/3\lambda,\ -p\omega^2/3\lambda,\ -p\omega/3\lambda$$

$$y\to\lambda-p/3\lambda,\ \lambda\omega-p\omega^2/3\lambda,\ \lambda\omega^2-p\omega/3\lambda$$

Note:- If one value of u is found to be a rational number, find the corresponding value of v by $y=v+u$ then find the corresponding finally divide left side of one by $(x-d)$ giving the remaining Q.E. from which the other two roots can be found.

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Solution of Cubic Equations by Cardan's Method:-

Suppose we have a cubic eq^n. $ax^3+bx^2+cx+d=0$ — ①
where a, b, c & d are constant coefficients.

Step:1. Identify coefficient of x^3 & divide whole eq^n by a, we get.

$$x^3+lx^2+mx+n=0 \quad (l=b/a,\ m=c/a,\ n=d/a)$$

Step:2. Remove x^2 term by putting $x=y-l/3$, we have

$$y^3+py+q=0 \text{ — ②}$$

Step:3 Now, do solve eq^n ② by putting $y=u+v$ the cubing both sides,

$$y^3=(u+v)^3$$

$$y^3=u^3+v^3+3uvy$$

$$y^3-3uvy-(u^3+v^3)=0 \text{ — ③}$$

Step:4 Compare eq^n ② & ③

$$uv=-p/3,\quad u^3+v^3=-q$$

$$\Downarrow$$

$$u^3v^3=-p^3/27,\quad u^3+v^3=-q$$

$$t^2-qt-p^3/27=0$$

$$u^3=\frac{1}{2}\left(-q+\sqrt{q^2+4p^3/27}\right)=\lambda^3$$

$$v^3=\frac{1}{2}\left(-q-\sqrt{q^2+4p^3/27}\right)$$

$$u\to\lambda,\ \lambda\omega,\ \lambda\omega^2 \qquad \{\omega^3=1\}$$

$$v=-p/3\lambda,\ -p\omega^2/3\lambda,\ -p\omega/3\lambda$$

$$y\to\lambda-p/3\lambda,\ \lambda\omega-p\omega^2/3\lambda,\ \lambda\omega^2-p\omega/3\lambda$$

Note:- If one value of u is found to be a rational number, find the corresponding value of v then find the corresponding by $y=v+u$ finally divide left side of one by $(x-d)$ giving the remaining Q.E. from which the other two roots can be found.

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2. If u^3, v^3 turn out to be conjugate complex no.'s the roots of the given cubic can be obtained in need form by applying De-Morgan's Theoram.

Examples

1. Solve by Cardan's method $x^3-3x^2+12x+16=0$

Solⁿ. Given eqⁿ :- $x^3-3x^2+12x+16=0$ — (1)

To remove x^2 term, we substitute $x=y-l/3$

$$\left(y-\frac{l}{3}\right)^3-3\left(y-\frac{l}{3}\right)^2+12\left(y-\frac{l}{3}\right)+16=0 \quad (l=-3)$$

$\Rightarrow (y+1)^3-3(y+1)^2+12(y+1)+16=0$

$\Rightarrow y^3+9y^2+26=0$ — (2)

$\Rightarrow y=u+v$

$\Rightarrow y^3-3uv.y-(u^3+v^3)=0$ — (3)

compare (2) & (3)

we get, $uv=-3$, $u^3+v^3=-26$

$u^3v^3=-27$, $u^3+v^3=-26$

$\therefore u^3, v^3$ are the roots of the eqⁿ $t^2+26t-27=0$

$\therefore t^2+26t-27=0$

$(t-1)(t+27)=0$

$t=-27, 1$

$u^3=-27$ i.e., $u=-3$ & $v^3=1$, i.e. $v=1$

$y=u+v=-3+1=-2$ & $x=y+1=-1$

Dividing L.H.S of (i) by $x+1$, we obtain

$$\begin{array}{r|l} & x^2-4x+16 \\ \hline x+1 & x^3-3x^2+12x+16 \\ & \underline{x^3+x^2} \\ & -4x^2+12x+16 \\ & \underline{-4x^2-4x} \\ & 16x+16 \\ & \underline{16x+16} \\ & R\to 0 \end{array}$$

$x^2 - 4x + 16 = 0$

or $x = \frac{4 \pm \sqrt{16-64}}{2} = 2 \pm i2\sqrt{3}$

Hence the required roots of the given equat^n are $-1, 2 \pm i2\sqrt{3}$.

Fuzzy Set Theory

0 Definations:-

i. Characteristic Function- Let X be an Universal set & let A be a subset of X. The characteristic funct^n χ_A of A is denoted by $\chi_A(x)$.

For each x,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \quad \forall x \in X$$

Eg: Let the universal set X be the set of all non-negative real no's & A is a set whose elements are $A = \{x : x \text{ is a real no. } \& \ 2 \leq x \leq 7\}$

The characteristic funct^n of A is

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases} \quad \forall x \in X$$

ii. Membership Function- A funct^n similar to the characteristic funct^n is called a membership funct^n. The membership funct^n assigns to each element $x \in X$ a number; $\mu_A(x)$ in the closed interval $[0, 1]$ that characterize the degree of number of x in A. Thus membership funct^n are the functions of the form

$$\mu_A : X \to [0, 1]$$

If A, B are two sets {A & B are subsets of universal set} then

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x), \quad \forall x \in X.$$

iii. Fuzzy Set:-

Let X be a collection of objective (universal set) Then a fuzzy A in X is defined as a set of ordered pairs $A = \{(x, \mu_A(x)); x \in X\}$

- Fuzzy set can also be denoted by $A = \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)) \ldots (x_n, \mu_A(x_n))\}$.

> Basic/Standard Operations on Fuzzy Set:-

i. Complement of A Fuzzy Set - Let A be a fuzzy set be defined on a universal set X. The complement of A is denoted by $\bar{A}$, A' or A^c.

$$\bar{A} = \{(x, \mu_{\bar{A}}(x)); x \in X\}$$

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

ii. Subset - Let A & B be two fuzzy set then $A \subseteq B$ iff and only if $\mu_A(x) \leq \mu_B(x), \forall x \in X$.

iii. Union of two fuzzy Sets- Let A & B be two fuzzy sets defined on a universal set X, the union of A & B is denoted by $A \cup B$ and defined as

$$A \cup B = \{(x, \mu_{A \cup B}(x)) : x \in X\},$$

where $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} \ \forall \ x \in X$.

iv. Intersection of Two Fuzzy Sets- Let A & B be two fuzzy sets defined on a universal set X, the intersection of A & B is denoted by $A \cap B$ & defined as $A \cap B(x) = \{(x, \mu_{A \cap B}(x) : x \in X\}$

where $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \ \forall \ x \in X$.

v. Equality of Two Fuzzy sets- If A & B are two fuzzy sets, then $A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \forall x \in X$.

Examples

1. Let $X = \{x_1, x_2, x_3, x_4\}$ and two fuzzy sets be
$A = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.7), (x_4, 1)\}$
$B = \{(x_1, 0.6), (x_2, 1), (x_3, 0.4), (x_4, 0.3)\}$. Then find $A \cup B$, $A \cap B$ & is $A \subseteq B$.

Soln: i. $A \cup B$; Let/Given A & B are two fuzzy sets defined on an universal set X, the union of A & B is defined as

$$A \cup B(x) = \{(x, \mu_{A \cup B}(x)) : x \in X\},$$

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \ \forall x \in X.$$

Now,

$\mu_{A \cup B}(x_1) = \max\{\mu_A(x_1), \mu_B(x_1)\} = \max\{0.2, 0.6\} = 0.6$

$\mu_{A \cup B}(x_2) = \max\{\mu_A(x_2), \mu_B(x_2)\} = \max\{0.5, 1\} = 1$

$\mu_{A \cup B}(x_3) = \max\{\mu_A(x_3), \mu_B(x_3)\} = \max\{0.7, 0.4\} = 0.7$

$\mu_{A \cup B}(x_4) = \max\{\mu_A(x_4), \mu_B(x_4)\} = \max\{1, 0.3\} = 1$

$\therefore A \cup B(x) = \{(x_1, 0.6), (x_2, 1)(x_3, 0.7), (x_4, 1)\}$

ii $A \cap B$; Given A & B are two fuzzy sets defined on an universal set x, the ~~union~~ intersection of A & B is defined as,

$$A \cap B(x) = \{(x, \mu_{A \cap B}(x)) : x \in x\}$$

where $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in x$

Now,

$\mu_{A \cap B}(x_1) = \min\{\mu_A(x_1), \mu_B(x_1)\} = \min\{0.2, 0.6\} = 0.2$

$\mu_{A \cap B}(x_2) = \min\{\mu_A(x_2), \mu_B(x_2)\} = \min\{0.5, 1\} = 0.5$

$\mu_{A \cap B}(x_3) = \min\{\mu_A(x_3), \mu_B(x_3)\} = \min\{0.7, 0.4\} = 0.4$

$\mu_{A \cap B}(x_4) = \min\{\mu_A(x_4), \mu_B(x_4)\} = \min\{0.3, 1\} = 0.3$

$\therefore A \cap B(x) = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.4), (x_4, 0.3)\}$

iii Given A & B are two fuzzy set, then $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ $\forall x \in x$.

$\because \mu_A(x) = \mu_B(A)$

$\therefore$ A is a subset of B ($A \subseteq B$).

o De-Morgan's Law:-

If A & B are two fuzzy sets, then de-Morgan's Law states that,

i. $(A \cup B)' = A' \cap B'$ ii. $(A \cap B)' = A' \cup B'$

> Some Fundamental Properties Of Fuzzy Sets:-

i. Identity Law: (a) $A \cap x = A$ and $A \cup \emptyset = A$

(b) $A \cup x = x$ and $A \cap \emptyset = \emptyset$

ii. Idempotent Law:- $A \cap A = A$, $A \cup A = A$

iii. Commutative Law: $A \cup B = B \cup A$, $A \cap B = B \cap A$

iv. Distributive Law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

v. Transitive Law:- $A \subseteq B$, $B \subseteq C$, $A \subseteq C$

7 Algebraic Properties of Fuzzy Sets :-

i Algebraic Sum :- The algebraic sum of two fuzzy set A & B is denoted by A + B & defined as

$A + B = \{(x, \mu_{A+B}(x)) : x \in X\}$,

$\mu(A+B(x)) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \forall x \in X$.

ii Algebraic Product: The algebraic ~~sum~~ product of two fuzzy set A & B is denoted by A.B & defined as

$A \cdot B = \{(x, \mu_{AB}(x)) \cdot x \in X\}$

$\mu_{AB}(x) = \mu_A(x) \cdot \mu_B(x), \forall x \in X$.