# Unit:- 2

## Ordinary Differential Equations

1) $\frac{dy}{dx} = \tan x$  $O = 1, D = 1$  1, 2, 3 → ODE

2) $\frac{d^2y}{dx^2} + y = x$  $O = 2, D = 1$  4 → PDE.

(3) $\frac{d^2y}{dx^2} + \sqrt{1 + \frac{dy}{dx}} = 0$  $O = 2, D = 2$

(4) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x$.  $O = 1, D = 1$

An equation containing dependent, independent ~~dept~~ variables with their derivatives.

Solⁿ:- An expression which satisfies the differential equation containing constants.

Solution
- General
- Particular

## Exact Differential Equations

$Mdx + Ndy = 0$ is said to be Exact if $\frac{\partial M}{\partial dy} = \frac{\partial N}{\partial x}$

Exact Differential Equations are those which are obtained from their solutions by direct differentiation and are not reduced or simplified further.

Condition for differential equation of first order and first degree to be Exact is that if the differential equation can be rearranged as $Mdx + Ndy = 0$, $M, N \to x, y$ is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution of Exact Differential Equation

$$\int_{y=\text{constant}} M\,dx + \int \left(\text{those terms of } N \text{ not containing } x\right) dy = c.$$

(i) Solve the following differential Equation

$$(1+4xy+2y^2)\,dx + (1+4xy+2x^2)\,dy = 0$$

$M = 1+4xy+2y^2$

$N = (1+4xy+2x^2)$

$\frac{\partial M}{\partial y} = 4x+4y$, $\frac{\partial N}{\partial x} = 4y+4x$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So, equation is exact.

Solution: $\int M\,dx +$

$$\int (1+4xy+2y^2)\,dx + \int 1\,dy = c$$

$$x + 2x^2y + 2y^2x + y = c$$

$(x+y)(2xy+1) = c$

(2) $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

$M = 1+e^{x/y}$

$N = e^{x/y}\left(1-\frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = -e^{x/y}\cdot\frac{x}{y^2}$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(0-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\cdot\frac{1}{y}$$

$$= \cancel{\frac{-e^{x/y}}{y}} + \cancel{\frac{e^{x/y}}{y}} - \frac{x}{y^2}e^{x/y}$$

$$= -e^{x/y}\frac{x}{y^2}$$

Here,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution = $\int\left(1+e^{x/y}\right) + 0 = c$

$$x + \frac{e^{x/y}}{y} = c$$

(3) $(e^y+1)\cos x\,dx + e^y\sin x\,dy = 0$

$M = (e^y+1)\cos x, \quad N = e^y\sin x$

$$\frac{\partial M}{\partial y} = e^y \cos x \quad , \quad \frac{\partial N}{\partial x} = e^y \cos x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\cancel{\sin x\, e^y} \int (e^y + 1) \cos x \, dx$$

$$(e^y + 1) \sin x = c_1$$

(4) $(x^2y - 2xy^2)dx - (x^3 + 3x^2y)dy = 0 \rightarrow (1)$

$M = x^2y - 2xy^2$

$N = 3x^2y - x^3$

$$\frac{\partial M}{\partial y} = x^2 - 4xy$$

$$\frac{\partial N}{\partial x} = 6xy - 3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Rules for converting non-Exact Differential Equation into Exact

(1) If the Equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$ then the integrating factor (I.F) is $\frac{1}{Mx + Ny}$

$$Mx + Ny \Rightarrow (x^2y - 2xy^2)x + (3x^2y - x^3)y$$

$$\Rightarrow x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$$

$$\Rightarrow x^2y^2$$

$\therefore$ If is $\dfrac{1}{x^2y^2}$

Multiplying 1 by I.f

$$\left(\frac{x^2y - 2xy^2}{x^2y^2}\right) dx - \left(\frac{x^3 - 3x^2y}{x^2y^2}\right) dy = 0$$

$$\left[\frac{1}{y} - \frac{2}{x}\right] dx - \left[\frac{x}{y^2} - \frac{3}{y}\right] dy = 0$$

$$M = \frac{1}{y} - \frac{2}{x}, \quad N = \frac{3}{y} - \frac{x}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2}, \quad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

So, Eqn is Exact now

General solution is

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y} dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$Mx + Ny \Rightarrow (x^2y - 2xy^2)x + (3x^2y - x^3)y$$

$$\Rightarrow x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$$

$$\Rightarrow x^2y^2$$

$\therefore$ If is $\dfrac{1}{x^2y^2}$

Multiplying 1 by I.F

$$\left(\frac{x^2y - 2xy^2}{x^2y^2}\right) dx - \left(\frac{x^3 - 3x^2y}{x^2y^2}\right) dy = 0$$

$$\left[\frac{1}{y} - \frac{2}{x}\right] dx - \left[\frac{x}{y^2} - \frac{3}{y}\right] dy = 0$$

$$M = \frac{1}{y} - \frac{2}{x}, \quad N = \frac{3}{y} - \frac{x}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2}, \quad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

So, Eq$^n$ is Exact now

General solution is

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y} dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = c$$

$$x^2y\,dx - (x^3+y^3)\,dy = 0.$$

$$M = x^2y$$

$$N = -x^3-y^3$$

$$\frac{\partial M}{\partial y} = x^2 \quad , \quad \frac{\partial N}{\partial x} = -3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$Mx + Ny \neq 0$$

$$\cancel{x^3y}\; x^3y - x^3y - y^4 \neq 0$$

$$-y^4$$

$$\text{If} = -\frac{1}{y^4}$$

Multiplying (1) by I.f

$$-\frac{x^2y}{y^4}\,dx + \left(\frac{x^3+y^3}{y^4}\right)dy = 0$$

$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

$$M = \frac{-x^2}{y^3}, \quad N = \frac{x^3}{y^4} + \frac{1}{y}$$

$$\frac{\partial M}{\partial y} = \cancel{\frac{-2x}{y^3}} \; \frac{3x^2}{y^4}, \quad \frac{\partial N}{\partial x} = \frac{-3x^2}{y^4}.$$

So, the equation is exact now

Solution $\Rightarrow$ $\int M dx + \int \frac{1}{y} dy$

$$\int \frac{-x^2}{y^3} dx + \int \frac{1}{y} dy$$

$$\cancel{\frac{-2x}{y^2}} + \frac{-x^3}{3y^3} + \log y = c$$

(2) If the equation $Mdx + Ndy = 0$ is of the form

$$f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$$

$$Mx - Ny \neq 0$$

$$\text{I.F.} = \frac{1}{Mx - Ny}$$

Q. Solve the equation

$$(xy \cdot \sin(xy) + \cos(xy))\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0.$$

$$M = [xy \cdot \sin(xy) + \cos(xy)]\, y$$
$$N = [xy \sin(xy) - \cos xy]\, x$$

$$\frac{\partial N}{\partial y} = [xy\sin(xy) + \cos(xy)] + \sin xy\,(1 \cdot x) + \cos(xy)\cdot x$$

$$= 2xy\sin xy$$

$$Mx - Ny$$

$$x^2y^2\sin xy + xy\cos y - x^2y^2\sin xy + xy\cos xy$$

$$2xy\cos xy$$

$$\frac{1}{Mx - Ny} = \frac{1}{2xy\cos xy}$$

Multiplying by $\frac{1}{2xy\cos xy}$

$$\left(\frac{y}{2}\tan xy + \frac{1}{2xy}\right)dx + \left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = 0$$

$$M = \frac{y}{2}\tan xy + \frac{1}{2x}$$

$$N = \frac{x}{2}\tan xy - \frac{1}{2y}$$

$$\frac{\partial M}{\partial y} = \frac{1}{2}\tan xy + \frac{y}{2}\cdot \sec^2 xy \cdot x$$

$$\Rightarrow \frac{1}{2}\tan xy + \frac{xy}{2}\sec^2 xy$$

$$\frac{\partial N}{\partial x} = \frac{1}{2}\tan xy + \frac{x}{2}\sec^2 xy \cdot y$$

$$\Rightarrow \frac{1}{2}\tan xy + \frac{xy}{2}\sec^2 xy.$$

Here,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

so,

$$\int\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \int\left(\frac{-1}{2y}\right)dy = c$$

$$\frac{y}{2}\cdot\frac{-\log(\cos(xy))}{y} + \frac{1}{2}\log x - \frac{1}{2}\log y = \log c$$

$$\frac{\cancel{y}}{2}\cdot\frac{-\log(\cos(xy))}{\cancel{y}} + \frac{1}{2}\log\left(\frac{x}{y}\right) = \log c$$

$$-\frac{1}{2}\log(\cos(xy)) + \frac{1}{2}\log\left(\frac{x}{y}\right) = \log c$$

$$\frac{1}{2}\log\left(\frac{x}{y\cos(xy)}\right) = \log c$$

$$\log\left(\frac{x}{y\cos(xy)}\right) = \log c^2$$

$$\frac{x}{y\cos(xy)} = c^2$$

$$x = y\cos(xy)\,c^2.$$

Rule:- (3) $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ only

$I.F = e^{\int f(x)dx}$

Q $(x^2+y^2+2x)dx + 2y\,dy = 0$

Here, $M = x^2+y^2+2x$

$N = 2y$

$\frac{\partial M}{\partial y} = 2y$

$\frac{\partial N}{\partial x} = 0.$

Here,

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

$\frac{1}{2y}(2y)$

$\Rightarrow 1 = f(x)$

$I.f = e^{\int 1\,dx} \Rightarrow e^{x}$

Multiply (1) by I.f

$$(x^2+y^2+2x)e^x\,dx + 2ye^x\,dy = 0.$$

Solving for exactness with new M and N,

$$M = (x^2+y^2+2x)e^x$$

$$N = 2ye^x$$

$$\frac{\partial M}{\partial y} = 2ye^x$$

$$\frac{\partial N}{\partial x} = 2ye^x$$

Here,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The solution is Exact

So,

General solution is

$$\Rightarrow \int (x^2+y^2+2x)e^x\,dx + \int 0\,dy$$

$$\Rightarrow e^x(x^2+y^2+2x) - \int e^x(2x+2)$$

$$\Rightarrow e^x(x^2+y^2+2x) - 2\left[e^x(x+1) - e^x\right]$$

$$\Rightarrow e^x(x^2+y^2+2x) - 2e^x(x+1) + 2e^x + c$$

$$\Rightarrow e^x x^2 + e^x y^2 + \cancel{2xe^x} - \cancel{2xe^x} - \cancel{2e^x} + \cancel{2e^x} + c$$

$$\Rightarrow e^x x^2 + e^x y^2 + c$$

Rule 4:- $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ only

$I.F = e^{\int f(y)dy}$

Q. Solve:-

$$(xy^3+y)dx + 2(x^2y^2+x+y^4)\,dy = 0$$

$M = xy^3+y$

$N = 2(x^2y^2+x+y^4)$

$\frac{\partial M}{\partial y} = 3xy^2+1$

$\frac{\partial N}{\partial x} = 2(2xy^2+1)$

$\frac{\partial N}{\partial x} = 4xy^2+2$

Here,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So,

Equation is not Exact

So,

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\frac{1}{(xy^3+y)\cancel{2(x^2y^2+x+y^4)}}\left[(4xy^2+2)-(3xy^2+1)\right]$$

$$\frac{1}{xy^3+y}\left[xy^2+1\right]$$

$$\Rightarrow \frac{1}{y(xy^2+1)}(xy^2+1)$$

$$\Rightarrow \frac{1}{y}$$

$$I.F = e^{\int \frac{1}{y}dy}$$

$$e^{\log y}$$

$$y$$

Multiplying by $y$

$$(xy^4+y^2)\,dx + 2(x^2y^3+xy+y^5)\,dy = 0$$

Now,

checking the exactness for new M and N

$$M = xy^4+y^2 \quad , \quad N = 2(x^2y^3+xy+y^5)$$

$$\frac{\partial M}{\partial y} = 4xy^3+2y$$

$$\frac{\partial N}{\partial x} = 2\left[2x^2y^2+y\right]$$

$$\Rightarrow 4x^2y^2+2y$$

Here,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So,

the equation is Exact now

So,

particular solution will be

$$\int M dx + \int$$

$$\Rightarrow \int (xy^4 + y^2) dx + 2\int y^5 dy$$

$$\Rightarrow \frac{x^2}{2}y^4 + xy^2 + \frac{y^6}{3} = c$$

$$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c.$$

Q. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}\left(\cancel{x}\, 1 + y^2\right) x\, dy = 0$

$$M = y + \frac{y^3}{3} + \frac{x^2}{2}$$

$$N = \frac{x}{4} + \frac{xy^2}{4}$$

$$\frac{\partial M}{\partial y} = 1 + y^2 + \cancel{x}$$

$$\frac{\partial N}{\partial x} = \frac{1}{4} + \frac{y^2}{4}$$

Here

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So,

Equation is not exact

So,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$\frac{4}{x(1+y^2)}\left[1+y^2 - \frac{1}{4} - \frac{y^2}{4}\right]$$

$$\frac{4}{x(1+y^2)}\left[\frac{3}{4} + \frac{3y^2}{4}\right]$$

$$\Rightarrow 3/x$$

Now,

$$I.F = e^{\int 3/x\, dx}$$

$$\Rightarrow e^{3\log x}$$

$$\Rightarrow e^{\log x^3}$$

$$\Rightarrow x^3$$

Multiply by $x^3$ in eqn (1)

$$\Rightarrow \int\left(yx^3 + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \cancel{\frac{1}{4}\left(x^4 + x^4y^2\right)dy}$$

$$\frac{yx^4}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12} = c$$

$$3yx^4 + x^4y^3 + x^6 = c.$$

Q. $(3x^2y^4+2xy)dx + (2x^3y^3 - x^2)dy = 0$

$M = 3x^2y^4 + 2xy$

$N = 2x^3y^3 - x^2$

$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$

$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$

Here

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So,

Equation is not exact.

3rd Rule

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$\frac{1}{2x^3y^3 - x^2}\left(12x^2y^3 + 2x - 6x^2y^3 + 2x\right)$$

$$\Rightarrow \frac{1}{2x^3y^3 - x^2}\left(6x^2y^3 + 4x\right)$$

4th Rule

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\frac{1}{3x^2y^4+2xy}\left(6x^2y^3 - 2x - 12x^2y^3 - 2x\right)$$

$$\Rightarrow \frac{1}{3x^2y^4+2xy}\left(-6x^2y^3 - 4x\right)$$

$$\Rightarrow \frac{-2}{y}\left[\frac{3x^2y^3+2x}{3x^2y^3+2x}\right]$$

$$\Rightarrow \frac{-2}{y}$$

$$I.F. = e^{-2\int \frac{1}{y}dy}$$

$$\Rightarrow e^{-2\log y}$$

$$\Rightarrow e^{\log y^{-2}}$$

$$\Rightarrow \frac{1}{y^2}$$

$$\int\left(3x^2y^2 + \frac{2x}{y}\right)dx$$

$$x^3y^2 + \frac{2}{y}\frac{x^2}{2} = c$$

$$\Rightarrow \boxed{x^3y^2 + \frac{x^2}{y} = c}$$

Q $(1+xy)y\,dx + (1-xy)x\,dy = 0.$

$M = (1+xy)y$

$= y + xy^2$

$\frac{\partial M}{\partial y} = 1 + 2xy.$

$N = (1-xy)x$

$= x - x^2y$

$\frac{\partial N}{\partial x} = 1 - 2xy$

Here,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So,

Rule no. 2

$$\cancel{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$$

$$\cancel{\frac{1}{1-2xy}\left(1+2xy - (1-2\right.}$$

$Mx - Ny = (1+2xy)x - (1-2xy)y^2$

$= \cancel{x + 2x^2y - y + 2xy^2}$

$\Rightarrow (y+xy^2)x - (x-x^2y)y$

$\Rightarrow \cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2$

$\Rightarrow 2x^2y^2$

$$\frac{1}{\cancel{I.F}} = \frac{1}{2x^2y^2}$$

Multiplying Eqⁿ by ~~Q~~ I.F.

$$\frac{(1+xy)}{2x^2y^2}\, y\, dx + \frac{(1-xy)}{2x^2y^2}\, x\, dy = 0$$

$$\left(\frac{1+xy}{2x^2y}\right)dx + \left(\frac{1-xy}{2xy^2}\right)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$

$$M = \frac{1}{2x^2y} + \frac{1}{2x}$$

$$N = \frac{1}{2xy^2} - \frac{1}{2y}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{2x^2y^2}$$

$$\frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$$

Here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So,

the Equation is Exact now.

$$\int M\,dx + \int N \text{ not containing the term of } x = C$$

$$\int\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int\left(\frac{-1}{2y}\right)dy = c$$

$$\frac{1}{2y}\int \frac{1}{x^2}dx + \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int y\,dy = c$$

$$\frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\frac{-1}{2xy} + \frac{1}{2}\log \frac{x}{y} = c$$

Given this Equation if we can arrange in the form

$$x^a y^b (my\,dx + nx\,dy) + x^r y^s (py\,dx + qx\,dy) = 0$$

Then the Integrating factor is

$$I.F = x^h . y^k \text{ where}$$

h, k are to be found as by using condition of Exactness.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Q. Solve :-

$$(2y\,dx + 3x\,dy) + 2xy\,(3y\,dx + 4x\,dy) = 0.$$

$$M = 2y + 6xy^2$$

$$N = 3x + 8x^2y$$

$$\frac{\partial M}{\partial y} = 2 + 12xy$$

$$\frac{\partial N}{\partial x} = 3 + 16xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Let, $x^h y^k$ be the Integrating factor

Multiplying (1) by I.F

$$(2y^{k+1}x^h dx + 3x^{h+1}y^k dy) + (6x^{h+1}y^{k+2}dx + 8x^{h+2}y^{k+1}dy$$

As, 2 is exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M = 2y^{k+1}x^h + 6x^{h+1}y^{k+2}$$

$$\frac{\partial M}{\partial y} = 2(k+1)y^k x^h + 6x^{h+1}(k+2)y^{k+1}$$

$$N = 3x^{h+1}y^k + 8x^{h+2}y^{k+1}$$

$$\frac{\partial N}{\partial x} = 3(h+1)x^h y^k + 8(h+2)x^{h+1}y^{k+1}$$

$$2(k+1) = 3(h+1)$$

$$2k+2 = 3h+3$$

$$3h-2k+1=0 \rightarrow (1)$$

$$6(k+2) = 8(h+2)$$

$$6k+12 = 8h+16$$

$$3k+6 = 4h+8$$

$$4h-3k+2=0$$

$$12h-8k+4=0$$

$$12h-9k+6=0$$

$$- \quad + \quad -$$

$$k-2=0$$

$$\boxed{k=2}$$

$$\boxed{h=1}$$

$$2y^3x\,dx + 3x^2y^2dy + 6x^2y^4dx + 8x^3y^3dy$$

$$(2y^3x + 6x^2y^4)\,dx + (3x^2y^2 + 8x^3y^3)\,dy = 0$$

$$\int (2y^3x + 6x^2y^4)\,dx$$

$$\cancel{2}y^3 \cdot \frac{x^2}{\cancel{2}} + \frac{\cancel{6}x^3y^4}{\cancel{3}_1} = c$$

$$y^3x^2 + 2x^3y^4 = c$$

$$\boxed{y^3x^2 + 2x^3y^4 = c}$$

. Solve

$$(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$$

$$M = y^3 - 2x^2y \quad , \quad N = 2xy^2 - x^3$$

$$\frac{\partial N}{\partial y} = 3y^2 - 2x^2$$

$$\frac{\partial N}{\partial x} = 2y^2 - 3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Rearranging,

$$y^3dx + 2xy^2dy - (2x^2y\,dx + x^3dy) = 0$$

$$y^2(y\,dx + 2x\,dy) - x^2(2y\,dx + x\,dy) = 0$$

Let,

$x^h \cdot y^k$ be the Integrating factor:

$\cancel{x^h y^1}$

$$(x^h y^{k+3} - 2x^{h+2} y^{k+1})\,dx + (2x^{h+1} y^{k+2} - x^{h+3} y^k)\,dy = 0$$

$$\frac{\partial M}{\partial y} = (k+3) y^{k+2} x^h - 2(k+1) y^k x^{h+2}$$

$$\frac{\partial N}{\partial x} = 2(h+1) x^h y^{k+2} - (h+3) x^{h+2} y^k$$

$$(k+3) = 2(h+1)$$

$$k+3 = 2h+2$$

$$2h - k = 1 \rightarrow (1)$$

$$2k+2 = h+3$$

$$\cancel{2k-1}\ h - 2k = -1 \rightarrow (2)$$

$$\begin{aligned} 2h - k &= 1 \\ 2h - 4k &= -2 \\ - \quad + \quad &+ \\ \hline 3k &= 3 \end{aligned}$$

$$\boxed{k = 1}$$

$$\boxed{h = 1}$$

$$(xy^4 - 2x^3y^2)\,dx + (2x^2y^3 - x^4y^1)\,dy = 0$$

$$\int (xy^4 - 2x^3y^2)\,dx = 0$$

$$\int xy^4 dx - 2\int x^3y^2 dx = c$$

$$\frac{x^2y^4}{2} - \frac{x^4y^2}{2} = c.$$

Linear Differential Equation of Higher Order with constant coefficient

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_ny = f(x) \rightarrow (1)$$

$$D = \frac{d}{dx}$$

$$D^ny + a_1D^{n-1}y + a_2D^{n-2}y + \cdots + a_ny = f(x)$$

$$D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_ny = f(x) \rightarrow (2)$$

$$f(D)y = f(x)$$

Let $y = e^{mx}$ be trial sol$^n$.

(1) gives,

$$\Rightarrow m^ne^{mx} + a_1m^{n-1}e^{mx} + \cdots a_ne^{mx} = f(x)$$

$$\Rightarrow e^{mx}(m^n + a_1m^{n-1} + \cdots + a_n) = f(x) \rightarrow (3)$$

Auxiliary

Case 1:- $f(x) = 0$, (1) homogenous.

$\therefore$ (3) gives:-

$$e^{mx}(m^n + a_1m^{n-1} + \cdots + a_n) = 0$$

$\Rightarrow m^n + a_1 m^{n-1} + ___ + a_n = 0$

$n$ roots,

(1) roots are real solution :

$m_1, m_2 \ldots m_n$

$\therefore$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + ___$

(2) equal

2 roots are equal

$y = (c_1 + c_2 x) e^{m_1 x} + ___$

3 roots are equal

$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + ___$

(3) when roots are imaginary

$\alpha \pm i\beta$

$m_1 = \alpha + i\beta$

$m_2 = \alpha - i\beta$

$\therefore$ General solution is

$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$

$y = e^{\alpha x} [c_1 e^{i\beta x} + c_2 e^{-i\beta x}]$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[\cos\beta x(c_1 + c_2) + \sin\beta x(ic_1 - ic_2)\right]$$

$$\Rightarrow e^{\alpha x}\left[A_1\cos\beta x + A_2\sin\beta x\right].$$

(1) solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$

$$D^2y - 3Dy - 4y = 0$$

$$y(D^2 - 3D - 4) = 0$$

Auxiliary Eqn. $(D \to m)$

$\cancel{D^2 - 3D - 4 = 0}$ $\quad m^2 - 3m - 4 = 0$

$m^2 - 4m + m - 4 = 0$

$\cancel{D^2 - 4D + D - 4 = 0}$ $\quad (m-4)(m+1) = 0$

$m = -1, 4.$

General solution is.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y = c_1 e^{-x} + c_2 e^{4x}$$

(2) $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$

$$D^3y - 9D^2y + 23Dy - 15y = 0$$

$$y(D^3 - 9D^2 + 23D - 15) = 0$$

Auxiliary Eqⁿ is

$$m^3 - 9m^2 + 23m - 15 = 0$$

$$m^2(m-1) - 8m(m-1) + 15(m-1) = 0$$

$$(m-1)(m^2 - 8m + 15) = 0$$

$$(m-1)(m^2 - 5m - 3m + 15) = 0$$

$$(m-1)(m-5)(m-3) = 0$$

$$m = 1, 3, 5$$

So,

General solution will be

$$\boxed{y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x}.}$$

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$$

$$D^3y - 2D^2y - 4Dy + 8y = 0$$

$$y(D^3 - 2D^2 - 4D + 8) = 0$$

Auxiliary Eqⁿ is

$$m^3 - 2m^2 - 4m + 8 = 0$$

$$m^2(m-2) + 0(m-2) - 4(m-2) = 0$$

$$(m-2)(m^2-4) = 0$$

$$m = 2, \pm 2$$

$$m = 2, 2, -2$$

$$y = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x}.$$

(3) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

$D^2y + 2Dy + 5y = 0$

$y(D^2+2D+5) = 0$

$m^2+2m+5 = 0$

$m = \frac{-2 \pm \sqrt{4-20}}{2}$

$\Rightarrow \frac{-2 \pm 4i}{2}$

$\Rightarrow -1 \pm 2i$ $\quad \alpha = -1-2i,\ \beta = -1+2i.$

General solⁿ:-

$y = c_1 e^{(-1-2i)x} + c_1 e^{(-1+2i)x}$

or $y = e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$

(4) $(D^2+D+1)^2(D-2)y = 0$

$(m^2+m+1)^2(m-2) = 0$

2, $\quad m = \frac{-1 \pm \sqrt{1-4}}{2}$

$\Rightarrow \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$

$\alpha = \frac{-1}{2} \quad \beta = \frac{\sqrt{3}}{2}$

$y = c_1 e^{2x} + c_2 e^{-x/2}\left[(c_2 + c_3 x)\cos\left(\frac{\sqrt{3}x}{2}\right) + (c_4 + c_5 x)\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$

## Non Homogenous Linear Equations with constant coefficients :-

$$f(D)y = f(x) \quad (\neq 0)$$

General solution

C.f. + P.I →

C.f. → Complementary function

P.I → Particular Integral

In C.f. → R.H.S. should be 0.

-: Special methods to find Particular Integral :-

(i) $f(x) = e^{ax}$

$$P.I = \frac{1}{f(D)} \cdot f(x)$$

$$P.I = \frac{1}{f(a)\ (\neq 0)} \cdot e^{ax} \quad (D \to a)$$

$$G.S. = C.f. + P.I.$$

Q. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = e^{2x}$

$$4D^2y + 4Dx - 3y = e^{2x}$$

$$y(4D^2 + 4D - 3) = e^{2x}$$

$f(D) = 4D^2+4D-3$

Auxiliary eqn

$4m^2+4m-3=0$

$4m^2+6m-2m-3=0$

$2m(2m+3)-1(2m+3)=0$

$m = \frac{1}{2}, \frac{-3}{2}$

$C.F. = c_1 e^{x/2} + c_2 e^{-3x/2}$

$P.I = \frac{1}{f(a)} \cdot e^{ax}$

$\frac{1}{4D^2+4D-3} \cdot e^{2x}$

$\frac{1}{4\times 2^2+4\times 2-3} \cdot e^{2x}$

$\frac{e^{2x}}{16+8-3} = \frac{e^{2x}}{21}$

General solution is

$y = c_1 e^{x/2} + c_2 e^{-3x/2} + \frac{e^{2x}}{21}$

Q. $(D^3+1)y = (e^x+1)^2$

$D^3+1=0$

$m^3+1=0$

$m^3+1=0$

$(m+1)(m^2-m+1)=0$

$$m^2 - m + 1 = 0$$

$m = -1,$ $\quad \dfrac{1 \pm \sqrt{1-4}}{2}$

$$\Rightarrow \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

$$C.F = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

For $P.I =$ $\quad e^{2x} + 2e^{x} + 1$

$$\frac{e^{2x}}{D^3+1}, \quad \frac{2e^{x}}{D^3+1}, \quad \frac{e^{0x}}{1}$$

$$\frac{e^{2x}}{2^3+1} \Rightarrow \frac{e^{2x}}{9} \qquad \Rightarrow \frac{2e^{x}}{2} \Rightarrow e^{x} \qquad \Rightarrow 1$$

$$y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}x}{2}\right)\right] + \frac{e^{2x}}{9} + e^{x} + 1.$$

② $f(x) = \sin ax / \cos ax$

$$P.I = \frac{1}{f(D^2)} \sin ax$$

$$= \frac{1}{f(-a^2)} \cdot \sin ax \quad (\neq 0)$$

Q. 1 Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin 3x$.

$$D^2y - 5Dy + 6y = \sin 3x$$

Auxiliary eqn is

$$\cancel{CF} = m^2 - 5m + 6 = 0$$

$$m^2 - 3m - 2m + 6 = 0$$

$$(m-3)(m-2)$$

$$m = 2, 3.$$

$$C.F = c_1 e^{2x} + c_2 e^{3x}.$$

$$P.I = \frac{1}{D^2 - 5D + 6} \cdot \sin 3x$$

$$\frac{1}{-3^2 - 5D + 6} \cdot \sin 3x$$

$$\frac{\sin 3x}{-3 - 5D} = \frac{-\sin 3x}{(3+5D)} \times \frac{3-5D}{3-5D}$$

$$= \frac{-\sin 3x\,(3-5D)}{9 - 25D^2}$$

$$= \frac{-\sin 3x\,(3-5D)}{9+225}$$

$$-\frac{3\sin 3x}{234} + \frac{15\cos 3x}{234}$$

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{15}{234}\cos 3x - \frac{\sin 3x}{78}.$$

Q. $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$

$$D^2y - 4y = e^x + \sin 2x$$

Auxiliary Eqn is

$$m^2 - 4 = 0$$
$$m = 2, -2$$

$$C.F = c_1 e^{2x} + c_2 e^{-2x}$$

$$P.I = \frac{e^x}{D^2-4} + \frac{\sin 2x}{D^2-4}$$
$$= \frac{-e^x}{3} + \frac{\sin 2x}{(-8)}$$
$$= \frac{-e^x}{3} - \frac{\sin 2x}{8}$$

General solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{e^x}{3} - \frac{\sin 2x}{8}$$

Q $(D^3 - 2D^2 + 3)y = \cos x$

$$m^3 - 2m^2 + 3 = 0$$

$$m^2(m+1) - 3m(m+1) + 3(m+1) = 0$$
$$(m^2 - 3m + 3)(m+1) = 0$$
$$m = \frac{3 \pm \sqrt{9-12}}{2} \Rightarrow \frac{3 \pm \sqrt{3}i}{2} \Rightarrow \frac{3}{2} \pm \frac{\sqrt{3}}{2}i \quad \alpha = \frac{3}{2}, \beta = \frac{\sqrt{3}}{2}$$

$$C.F = c_1 e^{-x} + e^{3x/2}\left[c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right]$$

$$P.I = \frac{\cos x}{D^3 - 2D^2 + 3}$$

$$\frac{\cos x}{D^3 + 2D + 3}$$

$$\frac{\cos x}{D^3 + 5}$$

$$\frac{\cos x}{-D + 5} \times$$

$$\frac{\cos x}{5 - D} \times \frac{5 + D}{5 + D}$$

$$\frac{5\cos x - \sin x}{25 - D^2}$$

$$\frac{5\cos x - \sin x}{26}$$

$$\frac{5}{26}\cos x - \frac{\sin x}{26}$$

$$y = c_1 e^{-x} + e^{3x/2}\left[c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right] + \frac{5}{26}\cos x - \frac{1}{26}\sin x.$$

3rd Rule:- $f(x)$ = polynomial :-

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + ___$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + ___$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3$$

$P.I = \frac{1}{f(D)}$ polynomial

$[f(D)]^{-1}$ polynomial

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x$$

$$D^2y - 5Dy + 6y = x$$

$$(D^2 - 5D + 6)y = x$$

$$m^2 - 5m + 6 = 0$$

$$m = 2, 3$$

$$CF = c_1e^{2x} + c_2e^{3x}$$

$$P.I = \frac{1}{D^2 - 5D + 6}x$$

$$\Rightarrow (D^2 - 5D + 6)^{-1}x$$

$$\Rightarrow 6^{-1}\left(1 + \frac{D^2 - 5D}{6}\right)^{-1}x$$

$$\Rightarrow \frac{1}{6}\left[1 + \frac{D^2 - 5D}{6}\right]^{-1}x$$

$$\Rightarrow \frac{1}{6}\left[1 - \left(\frac{D^2 - 5D}{6}\right) + \left(\frac{D^2 - 5D}{6}\right)^2\right]x$$

$$\Rightarrow \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{D^4}{6} + \frac{25D^2}{36} - \frac{10D^3}{6}\right]x$$

$$\frac{1}{6}\left[1+\frac{5D}{6}\right]x$$

$$\frac{1}{6}x+\frac{5}{36}Dx$$

P.I. $\Rightarrow$ $\frac{1}{6}x+\frac{5}{36}$

General solution

$$y = c_1e^{2x}+c_2e^{3x}+\frac{x}{6}+\frac{5}{36}$$

Q. $\frac{d^2y}{dx^2}-4y = x^2$

$$D^2y-4y = x^2$$

$$(D^2-4)y = x^2$$

$$m = \pm 2$$

$$C.F = c_1e^{2x}+c_2e^{-2x}$$

P.I =

$$\frac{1}{(D^2-4)}x^2$$

$$\frac{1}{-4\left(\frac{D^2}{4}-1\right)}x^2$$

$$\frac{-1}{4\left(1-\frac{D^2}{4}\right)}x^2$$

$$\frac{-1}{4}\left(1-\frac{D^2}{4}\right)^{-1}x^2$$

$$\frac{-1}{4}\left[1+\frac{D^2}{4}\right]x^2$$

$$-\frac{x^2}{4}-\frac{2}{16}$$

$$-\frac{x^2}{4}-\frac{1}{8}$$

$$y = c_1e^{2x}+c_2e^{-2x}-\frac{x^2}{4}-\frac{1}{8}$$

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$$

$$D^3y + 3D^2y + 2Dy = x^2$$

$$y(D^3 + 3D^2 + 2D) = x^2$$

$$m^3 + 3m^2 + 2m = 0$$

$$m(m^2 + 3m + 2) = 0$$

$$m(m+1)(m+2) = 0$$

$$m = 0, 1, -2,$$

$$C.F = c_1 e^{0x} + c_2 e^{x} + c_3 e^{-2x}$$

$$c_1 + c_2 e^{x} + c_3 e^{-2x}$$

$$P.I = \frac{1}{D^3 + 3D^2 + 2D} x^2$$

$$\frac{1}{D(D^2 + 3D + 2)} x^2$$

$$\cancel{\frac{x^3}{3(D^2+3D+2)}}$$

$$\frac{1}{D}\left[\frac{1}{D^2 + 3D + 2} \cdot x^2\right]$$

$$\frac{1}{2D}\left[\frac{1}{1 + \frac{D^2 + 3D}{2}}\right] x^2$$

$$\frac{1}{2D}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}x^2$$

$$\frac{1}{2D}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{D^4}{4}+\frac{9D^2}{4}+\right]x^2$$

$$\frac{1}{2D}\left[1-\frac{D^2}{2}+\frac{9D^2}{4}\right]x^2$$

$$\frac{1}{2D}\left[1+\frac{7D^2}{4}\right]x^2$$

$$\frac{1}{2}\left[\frac{1}{D}+\frac{7D^2}{4D}\right]x^2$$

$$\frac{1}{2}\left[\frac{1}{D}+\frac{7}{4}D\right]x^2$$

$$\frac{1}{2}\left[\frac{1}{D}x^2+\frac{7}{4}Dx^2\right]$$

$$\frac{x^3}{6}+\frac{7x}{4}$$

$$\frac{1}{2D}\left[1-\frac{3D}{2}+\frac{7D^2}{4}\right]x^2$$

$$\left[\frac{1}{2D}-\frac{3}{4}+\frac{7}{8}D\right]x^2$$

$$\frac{1}{2D}x^2-\frac{3}{4}x^2+\frac{7}{8}Dx^2$$

$$\frac{x^3}{6}-\frac{3}{4}x^2+\frac{7}{4}x$$

$$y = c_1+c_2e^{x}+c_3e^{-2x}+\frac{x^3}{6}-\frac{3}{4}x^2+\frac{7}{4}x$$

(4) $f(x)=e^{ax}\cdot V$ where $V$ is some function of $x$

$$P.I = \frac{1}{f(D)}e^{ax}\cdot V$$

$$e^{ax}\frac{1}{f(D+a)}\cdot V$$

Q. $(D^2-2D+5)y = e^{2x}\cdot\sin x$

$m^2-2m+5=0$

$m = \frac{2\pm\sqrt{4-20}}{2}$

$\Rightarrow \frac{2\pm 4i}{2}$

$\Rightarrow 1\pm 2i$

$\alpha = 1, \quad \beta = 2$

$C.F = e^{x}\left[c_1\cos 2x + c_2\sin 2x\right]$

$P.I = \frac{1}{f(D)}\cdot e^{2x}\sin x$

$e^{2x}\cdot\frac{1}{f(D+2)}\sin x$

$e^{2x}\cdot\frac{1}{(D+2)^2-2(D+2)+5}\sin x$

$e^{2x}\cdot\frac{1}{D^2+4+4D-2D-4+5}\sin x$

$e^{2x}\cdot\frac{1}{D^2+5+2D}\sin x$

$e^{2x}\cdot\frac{\sin x}{D^2+2D+5}$

$e^{2x}\cdot\frac{\sin x}{-1+2D+5}$

$e^{2x}\cdot\frac{\sin x}{2D+4}$

$\frac{e^{2x}}{2}\frac{\sin x}{D+2}\cdot\frac{D-2}{D-2} \Rightarrow \frac{e^{2x}}{2}\cdot\frac{(\cos x-2\sin x)}{(-5)}$

$$-\frac{e^{2x}}{10}\left[\cos x - 2\sin x\right]$$

$$y = e^{x}\left[c_1\cos 2x + c_2\sin 2x\right] - \frac{e^{2x}}{10}\left[\cos x - 2\sin x\right].$$

$$(D^3+3D^2+3D+1)y = x^2e^{-x}$$

$$m^3+3m^2+3m+1$$

$$m^2(m+1) + 2m(m+1) + 1(m+1) = 0$$

$$(m^2+2m+1)(m+1) = 0$$

$$m = -1, -1, -1$$

$$CF = (c_1 + c_2x + c_3x^2)e^{-x}$$

$$P.I = \frac{1}{\cancel{f(D)}}\frac{1}{D^3+3D^2+3D+1}\cdot x^2e^{-x}$$

$$= e^{-x}\cdot\frac{1}{(D-1)^3+3(D-1)^2+3(D-1)+1}\,x^2$$

$$e^{-x}\cdot\frac{x^2}{(D+1-1)^3}$$

$$e^{-x}\cdot\frac{x^2}{D^3}$$

$$e^{-x}\cdot\iiint x^2$$

$$e^{-x}\cdot\iint\frac{x^3}{3}$$

$$e^{-x}\cdot\frac{1}{12}\iint x^4 = e^{-x}\cdot\frac{x^5}{60}$$

$$y = (c_1 + c_2 x + c_3 x^2) e^{-x} + e^{-x} \cdot \frac{x^5}{60}$$

Q. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

$$D^2 y - 2Dy + y = e^x$$

$$D^2 - 2D + 1 = 0$$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$C.F. = (c_1 + c_2 x) e^x$$

$$P.I = \frac{e^x \cdot 1}{(D-1)^2}$$

$$e^x \frac{1}{(D-1+1)^2}$$

$$e^x \cdot \frac{1}{D^2}$$

$$\frac{e^x \cdot x^2}{2}$$

Rule (5) $f(x) = x \cdot V$ where $V$ is the function of $x$ then particular Integral is

$$P.I = \frac{1}{f(D)} \cdot xV$$

$$= x \cdot \frac{1}{f(D)} \cdot V + \frac{d}{dD}\left(\frac{1}{f(D)}\right) \cdot V$$

$(D^2-2D+1)\,y = x\cdot\sin x.$

$CF = (c_1+c_2x)\,e^{x}.$

$P.I = \cancel{\frac{1}{(D-1)^2}}\cdot\frac{x\sin x}{D^2-2D+1}$

$x\cdot\frac{1}{D^2-2D+1}\cdot\sin x + \left(\frac{d}{dD}\left[\frac{1}{D^2-2D+1}\right]\right)\sin x \rightarrow (1)$

(i) $\frac{x}{2}\cdot\frac{-1}{D}\sin x$

$\frac{x\cos x}{2}$

(ii) $\left(\frac{d}{dD}(D^2-2D+1)^{-1}\right)\sin x$

$-(D^2-2D+1)^{-2}\cdot(2D-2)\sin x$

$\frac{-2(D-1)}{(D^2-2D+1)^2}\sin x \Rightarrow \frac{-2(D-1)\sin x}{4D^2}$

$= \frac{1}{2}\cdot(D-1)\sin x$

$\cancel{-2}(D-1)\frac{\cancel{\sin x}}{\cancel{-2}D^2}$

$\frac{1}{2}[\cos x-\sin x]$

$\cancel{-1}(D-1)\cos x$

$\cancel{(D-1)(-\cos x)}$

$\cancel{\sin x+\cos x}$

$y = C.F + P.I$

$= (c_1 + c_2 x)e^{x} + \frac{x}{2}\cos x + \frac{1}{2}\cos x - \frac{1}{2}\sin x$

Q. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x\cos x$

$m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

$C.F. = c_1 + c_2 e^{-x}$

$P.I : \frac{1}{D^2+D} x\cos x$

$x \frac{\cos x}{D^2+D} + \frac{d}{dD}\left[\frac{1}{D^2+D}\right] \cdot \cos x$

$x \frac{\cos x}{D-1} \times \frac{D+1}{D+1}$

$x \frac{(D+1)\cos x}{D^2-1}$

$x \frac{(D+1)\cos x}{-2}$

$-\frac{x}{2}[-\sin x + \cos x)$

$\frac{x}{2}[\sin x - \cos x]$

(ii) $\frac{d}{dD}\left[(D^2+D)^{-1}\right]\cos x$

$-1(D^2+D)^{-2}\cdot \cancel{\cos x}\,(2D+1)\cos x$

$-\frac{(2D+1)}{(D^2+D)^2}\cos x$

$-\frac{(2D+1)}{(D-1)^2}\cos x$

$+\frac{1}{2}(2D+1)(D+1)\cos x$

$\frac{1}{2}(2D^2+2D+D+1)\cos x$

$\frac{1}{2}(2D^2+3D+1)\cos x$

$-\cos x - \frac{3}{2}\sin x$

$-\frac{(2D+1)\cos x}{D^4+D^2+2D^3}$

$-\frac{(2D+1)\cos x}{1-1+2D^3}$

$-\frac{1}{2}(2D+1)\frac{1}{D^3}\cos x$

$+\frac{1}{2}(2D+1)\frac{1}{D}\cos x$

$\frac{1}{2}(2D+1)\sin x$

$\cos x + \frac{\sin x}{2}$

$y = c_1 + c_2e^{-x} + \frac{x\sin x}{2} - \frac{x\cos x}{2} + \cos x + \frac{\sin x}{2}$

Q. $\cancel{(D^2+2D+1)y}$ $(D^2+3D+2)y = x\cos 2x$

$m^2+3m+2=0$

$m^2+2m+m+2=0$

$m(m+2)+1(m+2)=0$

$m_2 -1, -2.$

$C.F = c_1e^{-x}+c_2e^{-2x}.$

$$P.I = \frac{1}{D^2+3D+2} x\cos 2x$$

$$\frac{x\cos 2x}{D^2+3D+2} + \frac{d}{dD}\left[\frac{1}{D^2+3D+2}\right]\cos 2x$$

(i) $$\frac{x\cos 2x}{-4+2+3D}$$

$$\frac{x\cos 2x}{3D-2}$$

(~~ii~~) $$x\,\frac{\cos 2x\,(3D+2)}{9D^2-4}$$

$$x\,\frac{(3D+2)\cos 2x}{-36-4}$$

$$\Rightarrow \frac{-x}{40}(3D+2)\cos 2x$$

$$\frac{-x}{40}(-6\sin 2x+2\cos 2x)$$

$$\Rightarrow \frac{3x}{20}\sin 2x - \frac{\cos 2x}{20}.$$

(iii) $$\cancel{x}\,\frac{d}{dD}\left[\frac{1}{D^2+3D+2}\right]\cos 2x$$

$$\frac{d}{dD}\left[\frac{1}{D^2+3D+2}\right]\cdot\cos 2x$$

$$\frac{(D^2+3D+2)\,0 - (2D+3)\cos 2x}{(D^2+3D+2)^2}$$

$$-(2D+3)\cdot\frac{\cos 2x}{(D^2+3D+2)^2}$$

$$-(2D+3)\frac{\cos 2x}{(-4+3D+2)^2}$$

$$-(2D+3)\cdot\frac{\cos 2x}{(3D-2)^2}$$

$$-(2D+3)\cdot\frac{\cos 2x}{9D^2+4-12D}$$

$$-(2D+3)\cdot\frac{\cos 2x}{-36+4-12D}$$

$$-(2D+3)\cdot\frac{\cos 2x}{-32-12D}$$

$$\frac{1}{4}(2D+3)\frac{\cos 2x}{(8+3D)}$$

$$\frac{1}{4}(2D+3)\cdot\frac{\cos 2x}{8+3D}\cdot\frac{8-3D}{8-3D}$$

$$\frac{1}{4}(2D+3)\frac{\cos 2x\,(8-2D)}{64-9D^2}$$

$$\frac{1}{4}(2D+3)\frac{\cos 2x\,(8-2D)}{64+36}$$

$$\frac{1}{4}(2D+3)(8-2D)\frac{1}{100}\cos 2x$$

$$\frac{1}{400}(16D-4D^2+24-6D)\cos 2x$$

$$\frac{-1}{400}\cdot 32\sin 2x + \frac{4\times 4}{400}\cos 2x + \frac{24\cos 2x}{400} + \frac{12\sin 2x}{400}$$

$$\frac{-2}{25}\sin 2x + \frac{1}{25}\cos 2x + \frac{3}{50}\cos 2x + \frac{3}{100}\sin 2x$$

$$-\frac{8}{100}\sin 2x + \frac{3}{100}\sin 2x + \frac{2}{50}\cos 2x + \frac{3}{50}\cos 2x$$

$$\Rightarrow \quad -\frac{1}{20}\sin 2x + \frac{1}{10}\cos 2x$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{3x}{20}\sin 2x - \frac{1}{20}\cos 2x - \frac{1}{20}\sin 2x + \frac{1}{10}\cos 2x$$

$$= c_1 e^{-x} + c_2 e^{-2x} + \frac{3x}{20}\sin 2x + \frac{1}{20}\cos 2x - \frac{1}{20}\sin 2x.$$

## Cases of failure :-

(1) $f(x) = e^{ax}$

$$= \frac{1}{f(D)} \cdot e^{ax}$$

$$= \frac{1}{0} e^{ax}$$

It is a case of failure

(i) $\frac{1}{D-a} e^{ax} = x e^{ax}$

(ii) $\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax}$

(iii) $\frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$

Q $\frac{d^2y}{dx^2} - a^2y = e^{ax}$

$m = \pm a$

$CF = c_1 e^{ax} + c_2 e^{-ax}$

$\frac{1}{(D-a)(D+a)} \cdot e^{ax}$

$\frac{1}{2a} \frac{1}{(D-a)} e^{ax}$

$\frac{x e^{ax}}{2a}$

Q. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 2e^{-3x}$

$m^2 + 6m + 9 = 0$

$(m+3)^2 = 0$

$m = -3, -3.$

$C.F = (c_1 + c_2 x) e^{-3x}$

$P.I. = \frac{1}{(D+3)^2} \cdot 2e^{-3x}$

$2 \frac{1}{(D+3)^2} e^{-3x}$

$2 \cdot \frac{x^2}{2!} e^{-3x}$

$= x^2 e^{-3x}.$

Q (2) $f(x) = \sin ax$

$P.I = \frac{1}{f(D^2)} \sin ax$

$\frac{1}{f(-a^2)} \sin ax$

$f(-a^2) = 0$

case fails.

$P.I = \frac{-x \cos ax}{2a}$

$\frac{1}{f(D^2)} \cdot \cos ax$

$\frac{1}{f(-a^2)} \cos ax$

$f(-a^2) = 0$

$P.I. = \frac{x \sin ax}{2a}$

Q $\frac{d^2y}{dx^2} + a^2y = \cos ax$

$c_1 \cos ax + c_2 \sin ax$

$C.F = \cancel{c_1 e^{-ax} + c_2 e^{ax}}$

$\frac{1}{D^2 + a^2} \cdot \cos ax$

$P.I = \frac{x}{2a} \sin ax$

$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{2a} \sin ax$

$$\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$

$$P.I = \frac{e^x}{5} - \frac{x}{4} \cos 2x$$

## General Method:-

① $P.I = \frac{1}{D-a} f(x)$

$$e^{ax} \int f(x) \cdot e^{-ax} dx$$

② $P.I = \frac{1}{(D-a_1)(D-a_2)} f(x)$

$$= \frac{1}{(D-a_1)} e^{a_2 x} \int f(x) e^{-a_2 x} dx$$

$$= e^{a_1 x} \int g(x) \cdot e^{-a_1 x} dx$$

Apply General Method to solve.

① $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$m = 1, 2$

C.F $= c_1 e^x + c_2 e^{2x}$

$$\frac{1}{D^2 - 3D + 2} e^{5x}$$

$$\frac{1}{(D-1)(D-2)} e^{5x}$$

$$\frac{1}{(D-1)(D-2)} = \frac{A}{D-1} + \frac{B}{D-3}$$

$$\frac{1}{(D-1)(D-2)} = \frac{AD - 3A + BD - B}{(D-1)(D-3)}$$

$1 = D(A+B) - 3A - B$

$A = -B$

$-3A - B = 1$

$3B - B = 1$

$B = 1$

$A = -1$

$$\frac{-1}{(D-1)} e^{5x} + \frac{1}{D-2} e^{5x}$$

$$-\left[e^x \int e^{5x} \cdot e^{-x} dx\right] + \left[e^{2x} \int e^{5x} \cdot e^{-2x} dx\right]$$

$$-\left[e^x \cdot \int e^{4x} dx\right] + \left[e^{2x} \frac{e^{3x}}{3} dx\right]$$

$$-\left[\frac{e^x \cdot e^{4x}}{4}\right] \quad \frac{e^{5x}}{3}$$

$$= -\frac{e^{5x}}{4}$$

$$-\frac{e^{5x}}{4} + \frac{e^{5x}}{3}$$

(2) $\frac{d^2y}{dx^2} - y = 2 + 5x$

$(D^2-1)\,y = 2+5x$

$m = \pm 1$

$C.F = c_1 e^{-x} + c_2 e^{x}$

$P.I = \frac{1}{(D-1)(D+1)}(2+5x)$

$\frac{A}{D-1} + \frac{B}{D+1} = \frac{1}{(D-1)(D+1)}$

$AD + A + BD - B = 1$

$A = -B$

$A - B = 1$

$-2B = 1$

$B = -1/2$

$A = 1/2$

$\frac{1}{2}\left[\frac{1}{(D-1)}(2+5x) - \frac{1}{(D+1)}(2+5x)\right]$

$\frac{1}{2}\left[e^{x}\int(2+5x).e^{-x}dx - e^{-x}\int(2+5x)e^{x}dx\right]$

$\cancel{\frac{1}{2}\left[e\right.}$ $e^{x}\int(2+5x)\,e^{-x}dx$

$e^{x}\left[-e^{-x}(2+5x) + \int 5e^{-x}dx\right.$

$e^{x}\left[-e^{-x}(2+5x) - 5e^{-x}\right]$

$e^{x}\left[-e^{-x}(2+5x) - 5e^{-x}\right]$

$\Rightarrow -(2+5x)-5$

$\Rightarrow e^{-x}\int(2+5x)e^{x}dx$

$\Rightarrow e^{-x}\left[(2+5x)e^{x}-5\int e^{x}dx\right]$

$e^{-x}\left[(2+5x)e^{x}-5e^{x}\right]$

$(2+5x)-5$

$\frac{1}{2}\left[-2-5x-5-((2+5x)-5)\right]$

$\frac{1}{2}\left[-7-5x-2-5x+5\right]$

$\frac{1}{2}\left[-4-10x\right]$

$-2-5x$

☆ <u>Linear Differential Equations with Variable coefficients</u>

$$x^{n}\frac{d^{n}y}{dx^{n}}+a_{1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n}y=f(x)$$

variable coefficients should reduce in constant coeffi..

Let,

$x=e^{z}$

$z=\log x$

$\frac{dz}{dx}=\frac{1}{x}$

$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$ ; $x\frac{dy}{dx}=\frac{dy}{dz}=Dy$ where $D=\frac{d}{dz}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dz}\right) = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2} = (D^2 - D)\,y \quad , D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)\,y \quad , D = \frac{d}{dz}$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)$$

Q.Q. solve :- $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$

Let, $x = e^z$

$z = \log x$

~~dz~~ and $x\frac{dy}{dx} = Dy$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

where $D = \frac{d}{dz}$

$$D(D-1)y - 3Dy + 4y = 2x^2$$

$$y(D^2-D-3D+4) = 2x^2$$

$$y(D^2-4D+4) = 2e^{2z}$$

$$m^2-4m+4=0$$

$$(m-2)^2=0$$

$$m=2,2$$

$$C.F = (c_1+c_2z)e^{2z}$$

$$P.I = \frac{2e^{2z}}{(D-2)^2}$$

$$2\frac{e^{2z}}{(D-2)^2}$$

$$\cancel{2}\cdot\frac{z^2}{\cancel{2}}e^{2z}$$

$$\therefore \quad P.I. = z^2\cdot e^{2z}.$$

General solution is

$$y = (c_1+c_2z)e^{2z} + z^2\cdot e^{2z}$$

$$(c_1+c_2\log x)e^{2\log x} + (\log x)^2\cdot e^{2\log x}$$

$$(c_1+c_2\log x)\cdot x^2 + (\log x)^2\cdot x^2.$$

Q. $x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 13y = \log x$

Let :

$x = e^z$

$z = \log x$

$x\frac{dy}{dx} = Dy$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$(D(D-1))y + 7Dy + 13y = z$

$y(D^2 - D + 7D + 13) = z$

$(D^2 + 6D + 13)y = z$

$m^2 + 6m + 13 = 0$

$\frac{-6 \pm \sqrt{36-52}}{2}$

$\frac{-6 \pm \sqrt{-16}}{2}$

$\frac{-6 \pm 4i}{2}$

$= -3 \pm 2i$

$\alpha = -3 \quad \beta = 2$

C.F. $= e^{-3z}(c_1 \cos 2z + c_2 \sin 2z)$

$$\frac{}{D^2+6D+13} \cdot z$$

$$\frac{}{13\left(1+\left(\frac{D^2+6D}{13}\right)\right)} z$$

$$\frac{1}{13}\left(1+\left(\frac{D^2+6D}{13}\right)\right)^{-1} \cdot z$$

$$\frac{1}{13}\left(1-\left(\frac{D^2+6D}{13}\right)+\left(\frac{D^2+6D}{13}\right)^2\right) \cdot z$$

$$\frac{1}{13}\left(z-\frac{6D \cdot z}{13}\right)$$

$$\frac{1}{13}\left(z-\frac{6}{13}\right)$$

$$P.I = \frac{z}{13} - \frac{6}{169}$$

$$\frac{\log x}{13} - \frac{6}{169}$$

$$\frac{1}{13}\left(\log x - \frac{6}{13}\right)$$

☆ General solution :-

$$y = e^{-3\log x}\left(c_1\cos(2(\log x)) + c_2\sin(2\log x)\right) + \frac{1}{13}\left(\log x - \frac{6}{13}\right)$$

$$y = x^{-3}\left(c_1\cos(2\log x) + c_2\sin(2\log x)\right) + \frac{1}{13}\left(\log x - \frac{6}{13}\right)$$

Q. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$

let, $x = e^z$

$z = \log x$

$$D(D-1)y - Dy + 4y = \cos z + e^z \sin z$$

$$y(D^2 - D - D + 4) = \cos z + e^z \sin z$$

$$y(D^2 - 2D + 4) = \cos z + e^z \sin z$$

Auxiliary Equation :-

$$m^2 - 2m + 4 = 0$$

$$m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} \Rightarrow \frac{2 \pm 2i\sqrt{3}}{2} \Rightarrow 1 \pm i\sqrt{3}$$

$$m = 1 \pm i\sqrt{3}$$

$$\alpha = 1, \ \beta = \sqrt{3}$$

So,

$$CF = e^z [c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z]$$

Now,

$$P.I = \frac{1}{D^2 - 2D + 4} (\cos z + e^z \sin z)$$

$$= \underbrace{\frac{1}{D^2 - 2D + 4} \cos z}_{I} + \underbrace{\frac{1}{D^2 - 2D + 4} e^z \sin z}_{II}$$

Taking Part I

$$= \frac{1}{D^2 - 2D + 4} \cos z$$

$$= \frac{1}{-1 - 2D + 4} \cos z$$

$$= \frac{1}{3 - 2D} \cos z$$

$$= \frac{1}{3 - 2D} \times \frac{3 + 2D}{3 + 2D} \cos z$$

$$= \frac{(3 + 2D)\cos z}{9 - 4D^2}$$

$$\frac{3+2D}{9+4}\cos z = \frac{(3+2D)\cos z}{13} = \frac{1}{13}(3\cos z - 2\sin z)$$

Taking Part II

$$= \frac{1}{D^2-2D+4} e^z \sin z$$

$$= e^z \frac{1}{(D+1)^2-2(D+1)+4} \sin z$$

$$= e^z \frac{1}{D^2+1+2D-2D-2+4} \sin z$$

$$\Rightarrow e^z \frac{1}{D^2+3} \sin z$$

$$\Rightarrow e^z \frac{1}{-1+3} \sin z$$

$$= \frac{1}{2} e^z \sin z$$

So, $P.I. = \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2} e^z \sin z$

So,

General solution is :-

$$C.F + P.I$$

$$\Rightarrow e^z[c_1\cos(\sqrt{3}z) + c_2\sin(\sqrt{3}z)] + \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2}e^z \sin z.$$

$$\Rightarrow e^{\log x}[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)] + \frac{1}{13}(3\cos(\log x) - 2\sin(\log x)) + \frac{1}{2}e^{\log x}\sin(\log x)$$

$$\Rightarrow x[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x) + \frac{1}{13}(3\cos x(\log x) - 2\sin(\log x)) + \frac{1}{2}x\sin(\log x).$$

# Method of Variation of Parameters

$$\cancel{P}\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = f(x)$$

**Working Rule :-**

P and Q are constants, Given this

① To find C.F

$C.F = c_1 u + c_2 v$

② Let the complete solution be $y = Au + Bv$

where, $\frac{dA}{dx} = \frac{-v f(x)}{w}$, $\frac{dB}{dx} = \frac{u f(x)}{w}$

where, $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

Wronskian's Determinant.

$A = \int ___ \, dx + c_1$

$B = \int ___ \, dx + c_2$

$\therefore$

Q. Solve:- $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

$(D^2+1)y = \operatorname{cosec} x$

$m^2+1 = 0$

$m = \pm i$

$C.F. = c_1 \cos x + c_2 \sin x$

$u = \cos x \quad v = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$W = 1$

$$\frac{dA}{dx} = \int -\sin x . \operatorname{cosec} x \, dx$$

$$\Rightarrow -\int 1 \, dx$$

$$\Rightarrow -x + c_1$$

$$\frac{dB}{dx} = \int \cos x . \operatorname{cosec} x \, dx$$

$$\int \cot x \, dx$$

$$dB \Rightarrow \log \sin x + c_2$$

General solution:-

$Au + Bv$

$(-x+c_1)u + (\log\sin x + c_2)v$

$(-x+c_1)\cos x + (\log\sin x + c_2)\sin x$

$y = c_1\cos x + c_2\sin x - x\cos x + \sin x\log\sin x$

2. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = \frac{e^x}{2x}$

$D^2y - 2Dy + y$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$xe^x$

$e^x + xe^x$

$e^x(1+x)$

$C.F = (c_1 + c_2x)e^x$

$c_1e^x + c_2e^x \cdot x$

$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(1+x) \end{vmatrix}$

$u = e^x \quad v = xe^x. \quad e^{2x}(1+x) - e^{2x}x$

$W = e^{2x}$

$A = \int \frac{-v \cdot f(x)}{W}$

$= -\int \frac{xe^x \cdot e^x}{W \; 2x} = \frac{-1}{2}\int \frac{\cancel{x}\, e^{2x}}{\cancel{e^{2x}} \cdot \cancel{x}} = \frac{-1}{2}x + c_1$

$B = \int \frac{e^x e^x}{e^{2x}\, 2x} = \frac{1}{2}\int \frac{1}{x}dx = \frac{1}{2}\log x + c_2$

$$\left(-\frac{x}{2}+c_1\right)e^x + \left(\frac{1}{2}\log x + c_2\right)xe^x$$

$$y = c_1e^x + c_2xe^x - \frac{xe^x}{2} + \frac{xe^x}{2}\log x$$

$$y = (c_1+c_2x)e^x + \frac{1}{2}xe^x(\log x - 1)$$

Q. $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$

$$m = \pm 2i$$

$$C.F. = c_1\cos 2x + c_2\sin 2x$$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$2\cos^2 2x + 2\sin^2 2x$$

$$W = 2$$

$$A = -\int \frac{\sin 2x \cdot 4\tan 2x}{2}$$

$$\Rightarrow -2\int \sin 2x \tan 2x \, dx$$

$$\Rightarrow -2\int \frac{\sin 2x \sin 2x}{\cos 2x} dx$$

$$\Rightarrow -2\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$\Rightarrow -2\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$A = -2\int(\sec 2x - \cos 2x)\,dx$$

$$\Rightarrow -2\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$\Rightarrow -2\left[\frac{1}{2}\log|\sec 2x + \tan 2x| - \frac{1}{2}\sin 2x\right]$$

$$\Rightarrow -\log|\sec 2x + \tan 2x| + \sin 2x + c_1$$

$$B = \int \frac{u \cdot f(x)}{w}$$

$$4\int \frac{\cos 2x \cdot \tan 2x}{2}\,dx$$

$$\Rightarrow 2\int \cos 2x \cdot \frac{\sin 2x}{\cos 2x}\,dx$$

$$\Rightarrow 2\int \sin 2x\,dx$$

$$\Rightarrow \frac{-2}{2}\cos 2x + c_2$$

$$\Rightarrow -\cos 2x + c_2$$

So,
General solution is

$$Au + Bv$$

$$(-\log|\sec 2x + \tan 2x| + \sin 2x + c_1)\cos 2x + (-\cos 2x + c_2)\sin 2x$$

$$\Rightarrow c_1\cos 2x + c_2\sin 2x + \cancel{\sin 2x\cos 2x} - \cancel{\sin 2x\cos 2x} - \log|\sec 2x + \tan 2x|$$

$$\Rightarrow c_1\cos 2x + c_2\sin 2x - \log|\sec 2x + \tan 2x|$$

# Simultaneous Diff. Equations

(1) $\frac{dx}{dt} - 7x + y = 0$ , $\frac{dy}{dt} - 2x - 5y = 0$

$\cancel{Dx}(D-7)x + y = 0$, $(D-5)y - 2x = 0$

$2(D-7)x + 2y = 0$, $y = -(D-7)x$

$-2(D-7)x + (D-7)(D-5)y = 0$ $-(D-5)(D-7)x - 2x = 0$

$x((D-5)(D-7)+2) = 0$

$(D-7)(D-5)y + 2y = 0$ $\cancel{D^2}(D^2 - 12D + 37) = 0$

$y(D^2 - 12D - 33) = 0$

$m^2 - 12m - 33 = 0$

$\frac{12 \pm \sqrt{144 + 132}}{2}$

$\Rightarrow \frac{12 \pm \sqrt{12}}{2}$

$\Rightarrow \frac{12 \pm \sqrt{276}}{2}$

let $\frac{d}{dx} = D$

$$\frac{dy}{dx} + y = z + e^x$$

$$\frac{dz}{dx} + z = y + e^x$$

$$Dy + y - z = e^x$$

$$y(D+1) - z = e^x$$

$$-y + (D+1)z = e^x$$

$$\cancel{y(D+1)} - z = e^x$$

$$\cancel{-y(D+1)} + (D+1)^2 z = e^x(D+1)$$

$$(D+1)^2 z - z = e^x(D+2)$$

$$z(D^2 + \cancel{1} + 2D - \cancel{1}) = e^x(D+2)$$

$$z(D^2 + 2D) = e^x(D+2)$$

$$z = \frac{e^x(D+2)}{D^2+2D}$$

$$z = \frac{(D+2)\ e^x}{1+2}$$

$$z = \frac{(D+2)\ e^x}{\cancel{1}\ 3}$$

$$z = \frac{1}{3}(e^x + 2e^x)$$

$$z = \cancel{e^x}\ \frac{3e^x}{3} = e^x$$

$$D(D+2) = 0$$

$$D = 0 \quad D = -2$$

$$C.F = c_1 + c_2 e^{-2x}$$

General solution for $z$ =

$$z = c_1 + c_2 e^{-2x} + e^x$$

$$(D+1)y - z = e^x$$
$$-y + (D+1)z = e^x$$

$$(D+1)^2 y - \cancel{(D+1)z} = e^x(D+1)$$
$$-y + \cancel{(D+1)z} = e^x$$

$$((D+1)^2 - 1)y = 3e^x$$

$$y(D^2 + \cancel{1} + 2D - \cancel{1})y = 3e^x$$

$$D(D+2) = 3e^x$$
$$D = 0, \quad D = -2$$

C.f =

$$y = c_1 + c_2 e^{-2x}$$

P.I =

$$\frac{3e^x}{\cancel{e^x} D^2 + 2D}$$

$$\Rightarrow \frac{3e^x}{3}$$

$$\Rightarrow e^x$$

$$y = c_1 + c_2 e^{-2x} + e^x.$$

$$\frac{dx}{dt} + 4x + 3y = t$$

$\frac{d}{dt} = D$

$$\frac{dy}{dt} + 2x + 5y = e^t$$

$$Dx + 4x + 3y = t$$
$$(D+4)x + 3y = t$$
$$2x + (D+5)y = e^t$$

$$2(D+4)x + 6y = 2t$$
$$2(D+4)x + (D+4)(D+5)y = (D+4)e^t$$
$$- \qquad - \qquad -$$

$$6y - (D+4)(D+5)y = 2t - (D+4)e^t$$

$$y(6 - (D^2+9D+20))y = 2t - e^t - 4e^t$$

$$y(-D^2-9D-14) = 2t - e^t - 4e^t$$

$$y(D^2+9D+14) = e^t + 4e^t - 2t$$

$$y(D^2+9D+14) = 5e^t - 2t$$

C.F = $m^2+9m+14=0$

$m^2+7m+2m+14=0$

$(m+7)(m+2)=0$

$m=-2, m=-7$

C.F = $c_1e^{-2t} + c_2e^{-7t}$

$$\frac{5e^t}{D^2+9D+14} - \frac{2t}{D^2+9D+14}$$

$$\frac{5e^t}{1+9+14} \qquad \frac{-2}{14}\left(1+\frac{D^2+9D}{14}\right)^{-1} t$$

$$\frac{5e^t}{24} \qquad \frac{-1}{7}\left(1-\left(\frac{D^2+9D}{14}\right)\right) t$$

$$\frac{-1}{7}\left(t - \frac{9}{14}Dt\right)$$

$$\frac{-1}{7}\left(t - \frac{9}{14}\right)$$

$$\frac{1}{12} - \frac{t}{7} \qquad \frac{9}{14\times 7} - \frac{t}{7}$$

$$\boxed{y = c_1 e^{-2t} + c_2 e^{-7t} + \frac{5}{24} e^{t} + \frac{9}{128} - \frac{t}{7}}$$

$$(D+4)x + 3y = t$$
$$2x + (D+5)y = e^{t}$$

$$(D+4)(D+5)x + \cancel{3(D+5)y} = t(D+5)$$
$$6x + \cancel{3(D+5)y} = 3e^{t}$$

$$(D^2+9D+14)x = 1 + 5t - 3e^{t}$$

$$C.F = c_1 e^{-2x} + c_2 e^{-7x}$$

$$\frac{t^{0x}}{D^2+9D+14} + \frac{5}{D^2+9D+14} t - \frac{3}{D^2+9D+14} e^{t}$$

$$\frac{1}{14} + \frac{5}{14}\left(1 + \left(\frac{D^2+9D}{14}\right)\right)^{-1} t - \frac{3}{24} e^{t}$$

$$\frac{1}{14} - \frac{3}{24} e^{t} + \frac{5}{14}\left(t - \frac{9D}{14} t\right)$$

$$\frac{14}{196} - \frac{3}{24} e^{t} + \frac{5t}{14} - \frac{45}{196}$$

$$\frac{5t}{14} - \frac{3}{24} e^{t} + \frac{1}{14}\left(1 - \frac{45}{14}\right)$$

$$\frac{-31}{196}$$

$$\boxed{x = c_1 e^{-2x} + c_2 e^{-7x} + \frac{5t}{14} - \frac{3}{24} e^{t} - \frac{31}{196}}$$

(4) $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$

$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$

$Dx + (D-2)y = 2\cos t - 7\sin t \quad \times D$

$(D+2)x - Dy = 4\cos t - 3\sin t \quad \times (D-2)$

$D^2x + D(D-2)y = -2\sin t - 7\cos t$

$(D^2-4)x - D(D-2)y = -4\sin t - 3\cos t - 8\cos t + 6\sin t$

$(D^2+D^2-4)x = \cancel{-14\cos} -18\cos t$

$2m^2 - 4 = 0$

$m^2 = 2$

$m = \pm\sqrt{2}$

$c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x}$

$\frac{-18\cos t}{2D^2-4}$

$= \frac{-18\cos t}{-2-4}$

$= \frac{-18\cos t}{-6}$

$= 3\cos t$

$x = c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x} + 3\cos t.$

$(D+2)\cancel{D}x + (D^2-4)y = -2\sin t - 7\cos t + 4\cos t - 14\sin t$

$D(D+2)x - D^2y = -4\sin t - 3\cos t$

$- \quad + \quad + \quad +$

$\cancel{D^2} - D(D^2-4+D^2)y = -12\sin t$

$2D^2 - 4 = 0$

$m^2 = 2$

$m = \pm\sqrt{2}$

$C.F = c_1e^{-\sqrt{2}x} + c_2e^{\sqrt{2}x}$

$$\frac{-12\sin t}{2D^2-4}$$

$$\Rightarrow \frac{-12\sin t}{-2-4}$$

$$\Rightarrow \frac{-12}{-6}\sin t$$

$$\Rightarrow 2\sin t$$

$$y = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} + 2\sin t$$

## Differential Equations of 1st order and $n^{th}$ degree

Let, $p = \frac{dy}{dx}$

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \text{———} + A_{n-1} p + A_n = 0$$

$$A_i \rightarrow f(x, y)$$

(1) Solvable for p.

Solve:- $p^2 - 5p + 6 = 0$

$p^2 - 3p - 2p + 6 = 0$

$p(p-3) - 2(p-3) = 0$

$(p-3)(p-2) = 0$

$p = 2, \; p = 3$

$p - 2 = 0$

$\frac{dy}{dx} = 2$

$\int dy = \int 2dx \Rightarrow y = 2x + c \Rightarrow (y - 2x - c) = 0$

$p - 3 = 0$

$\frac{dy}{dx} = 3$

$y = 3x + c$

$(y - 3x - c) = 0$

G.S. $(y - 2x - c)(y - 3x - c) = 0$

(2) $x^2p^2 + xyp - 6y^2 = 0$

$x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$

$xp(xp + 3y) - 2y(xp + 3y) = 0$

$(xp - 2y)(xp + 3y) = 0$

$p = \frac{2y}{x}, \frac{-3y}{x}$

$\frac{dy}{dx} = \frac{2y}{x}$

$\frac{1}{2}\int \frac{dy}{y} = \int \frac{dx}{x}$

$\frac{1}{2}\log y = \log x + c$

$\log\sqrt{y} = \log xc$

$\sqrt{y} = xc$

$y = x^2c^2$

$y - x^2c^2 = 0$

$\frac{dy}{dx} = \frac{-3y}{x}$

$-\frac{1}{3}dy = \frac{dx}{x}$

$y^{-1/3} = xc$

$y = x^{-3}c$

$yx^3 = c$

$(y - x^2c^2)(yx^3 - c) = 0$

(2) Solvable for y

$y = \sin p - p\cos p$

$\frac{dy}{dx} = \cos p \cdot \frac{dp}{dx} - \left(1\cos p\frac{dp}{dx} - p\sin p\frac{dp}{dx}\right)$

$p = \cancel{\cos p\frac{dp}{dx}} - \cancel{\cos p\frac{dp}{dx}} + p\sin p\frac{dp}{dx}$

$$\cancel{p} = \cancel{p}\sin p \frac{dp}{dx}$$

$$\int dx = \int dp \sin p$$

$$x = -\cos p + c$$

$$x + \cos p - c = 0$$

$$\cos p = c - x$$

$$p = \cos^{-1}(c-x)$$

Putting in 1:-

$$y = \sin(\cos^{-1}(c-x)) - \cos^{-1}(c-x) \cdot \cos(\cos^{-1}(c-x))$$

Q. $x^2 + p^2 x - yp = 0 \rightarrow (1)$

$$yp = x^2 + p^2 x$$

$$y = \frac{x^2}{p} + px \rightarrow \frac{dy}{dx} = \frac{p \cdot 2x - x^2 \frac{dp}{dx}}{p^2} + \frac{dp}{dx} \cdot x + p$$

$$y - px = \frac{x^2}{p} \qquad p = \frac{2xp}{p^2} - \frac{x^2}{p^2}\frac{dp}{dx} + x\frac{dp}{dx} + p$$

$$\frac{dp}{dx}\left(\frac{-x^2}{p^2} + x\right) + \frac{2x}{p} = 0 \qquad \cancel{p} = \frac{2x}{p} - \frac{x^2}{p^2}\frac{dp}{dx} + x\frac{dp}{dx} + \cancel{p}$$

$$-x + \frac{x^2}{p^2} = \frac{dx}{dp}\left(\frac{2x}{p}\right) \qquad -\frac{x^2}{p^2}\frac{dp}{dx} + x\frac{dp}{dx} + \frac{2x}{p} = 0$$

$$\left(\frac{x^2}{p^2} - x\right)\frac{p}{2x} = \frac{dx}{dp}$$

$$\left(\frac{x^2}{p^2} - x\right)\frac{p}{2x} = \frac{dx}{dp}$$

$$\frac{x}{2p} - \frac{p}{2} = \frac{dx}{dp}$$

$$\frac{dx}{dp} - \frac{x}{2p} = \frac{-p}{2}$$

$$I.F = e^{-\int \frac{1}{2p}}$$

$$e^{-\frac{1}{2}\int \frac{1}{p}}$$

$$e^{-\frac{1}{2}\log p}$$

$$p^{-1/2}$$

$$I.F = \frac{1}{\sqrt{p}}$$

$$x \cdot \frac{1}{\sqrt{p}} = -\int \frac{1}{\sqrt{p}} \cdot \frac{p}{2}\, dp$$

$$\frac{x}{\sqrt{p}} = -\frac{1}{2}\int \sqrt{p} + c$$

$$\frac{x}{\sqrt{p}} = -\frac{1}{2}\,\frac{p^{1/2+1}}{1/2+1} + c$$

$$-\frac{2}{2}\,\frac{p^{3/2}}{3}$$

$$\frac{x}{\sqrt{p}} = -\frac{1}{3}p^{3/2} + c$$

$$\frac{x}{\sqrt{p}} + \frac{p\sqrt{p}}{3} = c$$

$$x + \frac{p\sqrt{p}\sqrt{p}}{3} = c\sqrt{p}$$

$$x + \frac{p^2}{3} = c\sqrt{p} \rightarrow (2)$$

$$x = -\frac{p^2}{3} + c\sqrt{p} \rightarrow (2)$$

Eq$^n$ (1) and (2) collectively gives the required answer

## (4) Clairout's Equation

The Equation $y = px + f(p)$ where $p$ is $\frac{dy}{dx}$ is called Clairout's Equation.

Diff. w.r.t $x$

$$\cancel{\frac{dy}{dx}} = \cancel{p} + x\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}$$

$$\frac{dp}{dx}\left(x + f'(p)\right) = 0$$

Either $\frac{dp}{dx}=0$ or $x+f'(p)=0$, On solving ① one

$$\frac{dp}{dx}=0$$

$$dp=0$$

$$p=c$$

General solution is ~~$y=p$~~ $y=cx+f'(c)$

Q. find the general sol$^n$ of $y=px+\frac{a}{p}$

$$y=cx+\frac{a}{c}$$

The Given Eq$^n$ is in clairout's form. So, General solution is obtained by replacing p by c.

So,

G.S. is $y=cx+\frac{a}{c}$

$$y=px+p-p^2$$

$$y=cx+c-c^2$$

$$(y-px)(p-1)=p$$

$$y=px\ \cancel{p-1}+\frac{p}{p-1}$$

$$y=cx+\left(\frac{c}{c-1}\right)$$

(4) $\sin y \cdot \cos px - \cos y \cdot \sin px - p = 0$

$$\sin(y - px) = p$$

$$y - px = \sin^{-1} p$$

$$y = px + \sin^{-1} p$$

$$y = cx + \sin^{-1} c.$$

$$(x-a)p^2 + (x-y)p - y = 0$$

$$xp^2 - ap^2 + xp - yp - y = 0$$

$$x - y(p+1) + x(p+p^2) = ap^2$$

$$-y(p+1) + xp(1+p) = ap^2$$

$$-y + xp = \frac{ap^2}{1+p}$$

$$y - xp = \frac{-ap^2}{p+1}$$

$$y = xp - \frac{ap^2}{p+1}$$

$$y = xc - \frac{ac^2}{c+1}$$

(6) $axyp^2 + (x^2 - ay^2 - b)p - xy = 0$

Hint $x^2 = u$

$y^2 = v$

$2x\frac{dx}{dx} = \frac{du}{dx}$, $2y\frac{dy}{dx} = \frac{dv}{du}$

$\frac{y\,dy}{x\,dx} = \frac{dv}{du} \Rightarrow \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du} \Rightarrow p = \frac{x}{y}\frac{dv}{du} \Rightarrow p = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$

$$p = \frac{\sqrt{u}}{\sqrt{v}} P$$

$$a\sqrt{u}\sqrt{v}\,p^2 + (u - av - b)\,p - \sqrt{u}\sqrt{v} = 0$$

$$a\sqrt{u}\sqrt{v}\,p^2 + up - avp - bp - \sqrt{u}\sqrt{v} = 0$$

$$ap^2 + \frac{u}{\sqrt{u}\sqrt{v}}\,p - \frac{av}{\sqrt{u}\sqrt{v}}\,p - \frac{b}{\sqrt{u}\sqrt{v}}\,p - 1 = 0$$

# Unit :- 3

## Partial Differential Equation

$z = f(x, y)$ ☆ formation of P.D.E.

$\frac{\partial z}{\partial x} = p$ ☆ Elimination of Arbitrary Constants

$\frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial x^2} = r$

$\frac{\partial^2 z}{\partial x \partial y} = s$

$\frac{\partial^2 z}{\partial y^2} = t$

Q.1 find a P.D.E. by eliminating arbitrary constants h and k from the relation

$$(x-h)^2 + (y-k)^2 + z^2 = c^2$$

Differentiating partially w.r.t x

$$2(x-h) + 2z\frac{\partial z}{\partial x} = 0$$

$$\cancel{\frac{\partial z}{\partial x}} = 2(x-h) = -2zp$$

$$x - h = -zp$$

Differentiating w.r.t y partially

$$2(y-k) + 2z\frac{\partial z}{\partial y} = 0$$

$$y - k = -zq.$$

$\therefore$ (1) gives

$$(-zp)^2 + (zq)^2 + z^2 = c^2$$

$$z^2p^2 + z^2q^2 + z^2 = c^2$$

$$z^2(1 + p^2 + q^2) = c^2$$

Q.2 $z = ax + by + ab$

$\frac{\partial z}{\partial x} = a$

$p = a$

$\frac{\partial z}{\partial y} = b$

$b = q$

$$\boxed{z = px + qy + pq}$$

Q. 3. $z = A e^{-p^2 t} \cos px$

$$\frac{\partial z}{\partial x} = -pA e^{-p^2 t} \sin px$$

$$\frac{\partial^2 z}{\partial x^2} = -p^2 A e^{-p^2 t} \cos px$$

$$\frac{\partial^2 z}{\partial x^2} = -p^2 z$$

$$p^2 = -\frac{1}{z} \cdot \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial z}{\partial t} = -p^2 A \cos px \, e^{-p^2 t}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}}$$

Q. $z = f(x+iy) + F(x-iy)$

$$\frac{\partial z}{\partial x} = f'(x+iy) + F'(x-iy)$$

$$\frac{\partial z}{\partial y} = f'(x+iy)\, i - F'(x-iy)\, i$$

$$\frac{\partial z}{\partial y} = i\left(f'(x+iy) - F'(x-iy)\right)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+iy) + f''(x-iy)$$

$$\frac{\partial^2 z}{\partial y^2} = -f''(x+iy) - F''(x-iy)$$

$$r + t = 2F''(x-iy)$$

$$\frac{r+t}{2} = F''(x-iy)$$

$$\boxed{r + t = 0}$$

Q. Elimination of Arbitrary function

from the P.D.E. by eliminating arbitrary function 'f' from the relation

$$z = f\left(\frac{y}{x}\right)$$

$$p = \frac{\partial z}{\partial x} = -f'\left(\frac{y}{x}\right) \cdot \frac{y}{x^2}$$

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$-\frac{px^2}{y} = qx$$

$$-px = qy$$

$$px + qy = 0$$

Q. Construct a P.D.E.

$$f(x^2+y^2+z^2,\ z^2 - 2xy) = 0$$

let $x^2+y^2+z^2 = u$

$z^2 - 2xy = v$

$\phi(u, v) = 0$

Differentiating w.r.t u and v

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}\right)$$

$$+ \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x}\right)$$

$$\frac{\partial f}{\partial u}\left(2x\cdot 1 + 2y\cdot 0 + 2z\cdot p\right) + \frac{\partial f}{\partial v}\left(-2y + (-2x)0 + 2z\cdot p\right) = 0$$

$$\frac{\partial f}{\partial u}\left(2x + 2zp\right) + \frac{\partial f}{\partial v}\left(-2y + 2zp\right) = 0$$

$$\frac{\partial f}{\partial u}\left(2x + 2zp\right) = \frac{\partial f}{\partial v}\left(2y - 2zp\right)$$

$$\therefore \frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = \frac{y - zp}{x + zp}$$

Differentiating w.r.t. $y$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial f}{\partial u}\left(0 + 2y + 2z\cdot q\right) + \frac{\partial f}{\partial v}\left(-2y(0) + (-2x) + 2z\cdot q\right) = 0$$

$$\frac{\partial f}{\partial u}\left(2y + 2zq\right) = \frac{\partial f}{\partial v}\left(2x - 2zq\right)$$

$$\frac{\partial f}{\partial u}\left(y + zq\right) = \frac{\partial f}{\partial v}\left(x - zq\right)$$

$$\frac{\partial f/\partial u}{\partial f/\partial v} = \frac{x - zq}{y + zq}$$

$$\frac{y - zp}{x + zp} = \frac{x - zq}{y + zq}$$

$$(y - zp)(y + zq) = (x - zq)(x + zp)$$

$$\cancel{y^2 + z^2p^2 = x^2 - z^2p^2}$$

$$(y-zp)(y+zq) = (x-zq)(x+zp)$$

$$y^2 + yzq - zpy - \cancel{z^2pq} = x^2 + zxp - xzq - \cancel{z^2pq}$$

$$x^2 - y^2 + zp(x+y) - zq(x+y) = 0.$$

$$(x-y)(x+y) + zp(x+y) - zq(x+y) = 0$$

$$x - y + zp - zq = 0$$

Q find partial differentiation Equation $\phi(x^2-z^2, x^3-y^3) = 0$

$u = x^2 - z^2$

$v = x^3 - y^3$

$\phi(u,v) = 0 \rightarrow (1)$

Diff (1) w.r.t. $x$

$$\frac{\partial\phi}{\partial u}\left[2x - 2z\frac{\partial z}{\partial x}\right] + \frac{\partial\phi}{\partial v}\left[3x^2\right] = 0$$

$$\therefore \frac{\partial\phi/\partial u}{\partial\phi/\partial v} = \frac{-3x^2}{2x - 2zp} \rightarrow (2)$$

Partially Diff (2) w.r.t $y$

$$\frac{\partial\phi}{\partial u}\left[-2z\cdot\frac{\partial z}{\partial y}\right] + \frac{\partial\phi}{\partial v}\left[-3y^2\right] = 0$$

$$\therefore \frac{\partial\phi/\partial u}{\partial\phi/\partial v} = \frac{3y^2}{-2zq} \rightarrow (3)$$

from Eq$^n$ (2) & (3)

$$\frac{-3x^2}{2(x-zp)} = \frac{3y^2}{-2zq}$$

$$\frac{x^2}{x-zp} = \frac{y^2}{zq}$$

$$x^2zq = y^2x - y^2zp$$

or

$$z(x^2q - y^2p) = y^2x$$

# Lagrange's Equation:-

$$Pp + Qq = R$$

where, P, Q, R is $f(x,y)$

The subsidiary Equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$u(x,y,z) = a$$

$$v(x,y,z) = b$$

General solution is

$$\phi\, F(u,v) = 0$$

Q Solve, $xp + yq = z$

The subsidiary eq$^n$ are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking:-

$$\frac{dx}{x} = \frac{dy}{y}$$

On integration

$$\log x = \log y + \log a$$
$$\log \frac{x}{y} = \log a$$
$$\frac{x}{y} = a.$$

Taking :-

$$\frac{dy}{y} = \frac{dz}{z}$$

On integration

$$\log y = \log z + \log b$$
$$\log \frac{y}{z} = \log b$$
$$\frac{y}{z} = b$$

$\therefore$ General sol$^n$ is

$$f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

Q. $\left(\frac{y^2 z}{x}\right)p + xzq = y^2$

The subsidiary Eq$^n$ is

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Dividing by $x$ throughout

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

(i) $$\frac{dx}{y^2 z} = \frac{dy}{x^2 z}$$

$$\frac{dx}{y^2} = \frac{dy}{x^2}$$
$$x^2 dx = y^2 dy$$
$$\frac{x^3}{3} - \frac{y^3}{3} = a$$

or

$$x^3 - y^3 = a$$

(ii) $$\frac{dx}{y^2 z} = \frac{dz}{y^2 x}$$

$$x\,dx = z\,dz$$

On integrating

$$\frac{x^2}{2} - \frac{z^2}{2} = b$$

or

$$x^2 - z^2 = b$$

General sol$^n$ is

$$f(x^3 - y^3, x^2 - z^2) = 0$$

Q. $zp = -x$

The subsidiary Eq$^n$ is

$$\therefore \frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$$

(i) $$\frac{dx}{z} = \frac{dz}{-x}$$

$$x\,dx = -z\,dz$$

On integrating

$$\frac{x^2}{2} + \frac{z^2}{2} = a$$

or $(x^2 + z^2) = a$

(iii) $\frac{dx}{z} = \frac{dy}{0}$

$dy = 0$

$y = b$

General solⁿ is

$f(x^2+z^2, y) = 0$

Q. $(mz - ny)p + (nx - lz)q = ly - mx$

The subsidiary Eqⁿ is

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

(i) $l, m, n$ as multipliers

$$= \frac{ldx + mdy + ndz}{lmz - lny + mnx - mlz + lny - nmx}$$

$$= \frac{ldx + mdy + ndz}{0}$$

Taking,

$$\frac{dx}{mz-ny} = \frac{ldx + mdy + ndz}{0}$$

$$ldx + mdy + ndz = 0$$

On integrating

$$lx + my + nz = a$$

(ii) $x, y, z$ as multipliers

$$= \frac{x\,dx + y\,dy + z\,dz}{mxz - nxy + nxy - lzy + lyz - mxz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

Taking this with anyone of the ratios

$$\therefore \quad x\,dx + y\,dy + z\,dz = 0$$

$$x^2 + y^2 + z^2 = b$$

General solⁿ is

$$f(lx + my + nz,\ x^2 + y^2 + z^2) = 0$$

Q. $$\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$$

Multiplying by $xyz$ throughout,

$$x(y-z)p + y(z-x)q = (x-y)z$$

The subsidiary Eqⁿs is

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{(x-y)z}$$

(i) 1, 1, 1 as multipliers

$$\frac{dx + dy + dz}{xy - xz + yz - yx + xz - yz} = \frac{dx + dy + dz}{0}$$

Taking this with anyone of the ratios

$$dx + dz + dy = 0$$

On integrating

$$x + y + z = a$$

(iii) $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integrating

$$\log x + \log y + \log z = \log b$$

$$xyz = b$$

general solution is

$$f(x+y+z, xyz) = 0$$

Q $xzp + yzq = xy$

The subsidiary Eq^n's is

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

(i) $\frac{dx}{xz} = \frac{dy}{yz}$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

$$\log x - \log y = \log a$$

$$\log \frac{x}{y} = \log a$$

$$\frac{x}{y} = a$$

(ii) $\frac{dy}{yz} = \frac{dz}{xy}$

$$\frac{dy}{z} = \frac{dz}{x}$$

$$xdy = zdz$$

$$ay\,dy = z\,dz$$

$$\frac{ay^2}{2} = \frac{z^2}{2} + b$$

$$\frac{ay^2}{2} - \frac{z^2}{2} = b$$

$$\frac{x}{y}\frac{y^2}{2} - \frac{z^2}{2} = b$$

$$xy - z^2 = b$$

So, General solution is

$$f\left(\frac{x}{y}, xy - z^2\right) = 0$$

Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

Subsidiary Equation is

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

$1/x, 1/y, 1/z$ as multiplier

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$$

$$\frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{0}$$

Taking this with any one of the ratio.

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

On integrating

$$\log x + \log y + \log z = \log a$$

$$xyz = a$$

(ii) $x, y, z$ as multiplier

$$\frac{x\,dx + y\,dy + z\,dz}{0}$$

Taking this with any one of the ratio

$$x\,dx + y\,dy + z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b$$

or

$$x^2 + y^2 + z^2 = b$$

So, General solution is

$$f(\qquad\qquad, x^2+y^2+z^2) = 0$$

Q Solve

$$x(y^2+z)p - y(x^2+z)q = z(x^2-y^2) = 0$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

(i) Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

$$= \frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{0}$$

Taking this with one of the ratios

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

On integrating

$$\log x + \log y + \log z = \log a$$

$$\log xyz = \log a$$

$$a = xyz$$

(ii) Taking $x, y, -1$ as multiplier

$$\frac{x\,dx + y\,dy + dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$\frac{x\,dx + y\,dy - dz}{\cancel{x^2y^2} + \cancel{x^2z} - \cancel{y^2x^2} - \cancel{y^2z} - \cancel{zx^2} + \cancel{zy^2}}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

Taking this with one of the ratios

$$x\,dx + y\,dy - dz = 0$$

$$\frac{x^2}{2}+\frac{y^2}{2}-z = b$$

$$x^2+y^2-2z = b'$$

General solution is

$$f\left(xyz,\ x^2+y^2-2z\right) = 0$$

Q $z(xp-yq) = y^2-x^2$

$$zxp - zyq = y^2-x^2$$

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2}$$

(i) $\frac{dx}{x} = \frac{dy}{-y}$

$$\log x = -\log y + \log a$$

$$\log xy = \log a$$

$$\boxed{a = xy}$$

(ii) Taking $xyz$ as multipliers

$$\cancel{xdx + ydy + zdz}$$

$$0$$

Taking this with any one of the multiplier

$$xdx + ydy + zdz = 0$$

On integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=b$$

$$x^2+y^2+z^2=b$$

So,

General solution is :-

$$f(xy,\ x^2+y^2+z^2)=0.$$

## Homogenous linear Equation with constant coefficients

$z \to x, y$

$$\frac{\partial^n z}{\partial x^n}+\frac{A_1\partial^n z}{\partial z^{n-1}\partial y}+\frac{A_2\partial^n z}{\partial z^{n-2}\partial y^2}+\cdots+A_n\frac{\partial^n z}{\partial y^n}=f(x,y) \quad (1)$$

$$D=\frac{\partial}{\partial x},\quad D'=\frac{\partial}{\partial y}$$

$$(D^n+A_1D^{n-1}D'+A_2D^{n-2}D'^2+A_3D^{n-3}D'^3+\cdots+A_nD'^n)z=f(x,y)$$

Working Rule.

for CF :-

Replace D by m, D' by 1, we get auxiliary Equation as

$$m^n+A_1m^{n-1}+A_2m^{n-2}+A_3m^{n-3}+\cdots \quad A_n=f(x,y)=0$$

(i) Distinct

$$CF=f_1(y+m_1x)+f_2(y+m_2x)+\cdots$$

② Equal

$$CF. = f_1(y+m_1x) + x f_2(y+m_1x)$$

Q Find the General solution of following Equations

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$$

$$(D^2 - DD' - 6(D')^2)z = 0$$

Auxiliary Equation is

$$\cancel{D^2 - D} - m^2 - m - 6 = 0$$

$$m^2 - 3m + 2m - 6 = 0$$

$$m(m-3) + 2(m-3) = 0$$

$$(m-3)(m+2) = 0$$

$$m = 3, -2$$

General solution,

$$y \; \cancel{CF.}^{z} = f_1(y-2x) + f_2(y+3x)$$

Q $(D+2D')(D-3D')^2 z = 0$

$$(m+2)(m-3)^2 = 0$$

$$m = -2, 3, 3$$

$$z = f_1(y-2x) + f_2(y+3x) + x f_3(y+3x)$$

Q. $4r - 12s + 9t = 0$

$4m^2 - 12m + 9 = 0$

$4m^2 - 6m - 6m + 9 = 0$

$2m(2m-3) - 3(2m-3)$

$(2m-3)(2m-3) = 0$

$m = \frac{3}{2}, \frac{3}{2}$

$z = f_1\left(y + \frac{3}{2}x\right) + xf_2\left(y + \frac{3}{2}x\right)$

Q $\frac{\partial^2 z}{\partial x^2} + a^2 \frac{\partial^2 z}{\partial y^2} = 0$

$m^2 + a^2 = 0$

$m = \pm ai$

$z = f_1(y + aix) + f_2(y - aix)$

Q. $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x^2 \partial y} - 6\frac{\partial^3 z}{\partial y^3} = 0$

$m^3 - 7m^2 - 6 = 0$

~~$m^2(m-3) - 4m(m-3) - 12(m-3)$~~

$m^2(m-1) - 6m(m-1) - 6(m-1) = 0$

$(m^2 - 6m - 1)(m-1) = 0$

$m = 1, 3 \pm \sqrt{15}$

$m = 1,\ 3+\sqrt{15},\ 3-\sqrt{15}$

$$z = f_1(y+x) + f_2(y+(3+\sqrt{15})x) + f_3(y+(3-\sqrt{15})x)$$

$$\phi(D, D')\ z = f(x, y)$$

$$G.S = \underset{D}{C.F} + \underset{E}{P.I}$$

Rules for P.I

(1) $f(x, y) = e^{ax+by}$

$$P.I = \frac{1}{\phi(D, D')} e^{ax+by}$$

$$P.I = \frac{1}{\phi(a, b)} e^{ax+by}$$

(1) $$\frac{\partial^3 z}{\partial x^3} + \frac{3\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$$

$$D^3 z - 3D^2 D' z + 4D'^3 = e^{x+2y}$$

$$m^3 - 3m^2 + 4 = 0$$

$$m^2(m+1) - 4m(m+1) + 4(m+1) = 0$$

$$(m^2 - 4m + 4)(m+1) = 0$$

$$m = \frac{4 \pm \sqrt{16-16}}{2} \Rightarrow 2, 2$$

$m = -1, 2, 2$

$$C.F = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$$

$$P.I = \frac{1}{D^3 - 3D^2D' + 4D'^3} \cdot e^{x+2y}$$

$$\frac{1}{1 - 3\times 2 + 4\times 8} e^{x+2y}$$

$$\frac{1}{33-6} e^{x+2y}$$

$$\frac{1}{27} e^{x+2y}$$

General solution is $= z = C.F. + P.I$

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

(2) $f(x,y) = \sin(ax+by)$

$$P.I = \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax+by)$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax+by)$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$$

$$(D^2 - DD')z = \sin x \cos 2y$$

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0, 1$

$C.F = f_1(y) + f_2(y+x)$

$$P.I = \frac{1}{D^2 - DD'}\left[\sin(x+2y) + \sin(x-2y)\right]$$

$$\frac{1}{D^2 - DD'}\sin(x+2y) + \frac{1}{D^2 - DD'}\sin(x-2y)$$

$$\frac{1}{-1+2}\sin(x+2y) + \frac{1}{-1-2}\sin(x-2y)$$

$$\frac{1}{2}\left(\sin(x+2y) - \frac{1}{3}\sin(x-2y)\right)$$

General solution =

$$f_1(y) + f_2(y+x) + \frac{1}{2}\sin(x-2y) - \frac{1}{6}\sin(x-2y)$$

Q. $(D^3 - 7D'D^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

$m^3 - 7m - 6 = 0$

$m^2(m+1) - m(m+1) - 6(m+1) = 0$

$(m^2 - m - 6)(m+1) = 0$

$(m-3)(m+2)(m+1) = 0$

$m = -1, -2, 3$

$m = -2, -1, 3$

$$C.F = f_1(y-2x) + f_2(y-x) + f_2(y+3x)$$

$$P.I_2 = \frac{1}{D^3 - 7D^2D' - 6D'^3} \sin(x+2y)$$

$$\frac{1}{-D + 7D^2 \times 4 + 24D'} \sin(x+2y)$$

$$\frac{1}{-D + 28D^2 + 24D'} \sin(x+2y)$$

$$\Rightarrow \frac{1}{27D + 24D'} \sin(x+2y)$$

$\Rightarrow$ Multiplying dividing by D

$$\frac{1}{D^3 - 7D^2D' - 6D'^3} e^{2x+y} \qquad \frac{D}{27D^2 + 24DD'} \sin(x+2y)$$

$$\frac{1}{8 - 14 - 6} \qquad \frac{D \sin(x+2y)}{-27 - 48}$$

$$P.I_1 = -\frac{1}{12} e^{2x+y} \qquad \frac{D \sin(x+2y)}{-75}$$

$$P.I_2 = -\frac{1}{75} \cos(x+2y)$$

General solution is

$$f_1(y-2x) + f_2(y-x) + f_3(y+3x) - \frac{1}{12} e^{2x+y} - \frac{1}{75} \cos(x+2y)$$

General Method :-

$$P.I = \frac{1}{D - mD'} f(x, y)$$

$$= \int f(x, c - mx)\, dx$$

$$c = y + mx.$$

Q Solve :- $r - 4s + 4t = e^{2x+y}$

$$(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$C.F = f_1(y+2x) + x f_2(y+2x)$$

$$P.I. = \frac{1}{(D^2 - 2D')^2} e^{2x+y}$$

Applying Rule no. 1

$$\frac{1}{(2 - 2\times 1)^2} e^{2x+y}$$

$$\frac{1}{0} e^{2x+y}$$

Case fails

Applying General Method:-

$$P.I = \frac{1}{(D-2D')(D-2D')} e^{2x+y}$$

$m = 2$

$$\frac{1}{(D-2D')} \int e^{2x+c-2x} dx$$

$$\frac{1}{(D-2D')} \int e^{c} dx$$

$$\frac{1}{(D-2D')} e^{c} \cdot x \, dx$$

$$\frac{1}{(D-2D')} e^{y+2x} \cdot x \, dx$$

$$\int e^{c-2x+2x} \cdot x \, dx$$

$$\int e^{c} \cdot x \, dx$$

$$e^{c} \int x \, dx$$

$$\Rightarrow e^{c} \cdot \frac{x^2}{2}$$

$$P.I \Rightarrow e^{y+2x} \cdot \frac{x^2}{2}$$

Q2. $(D^2 - 2DD' - 15D'^2) z = 12xy$

$m^2 - 2m - 15 = 0$

$m^2 - 5m + 3m - 15 = 0$

$m(m-5) + 3(m-5) = 0$

$(m-5)(m+3) = 0$

$m = -3, 5$

$$CF = f_1(y-3x) + f_2(y+5x)$$

$$PI = \frac{1}{(D^2-2DD'-15D'^2)} \cdot 12xy$$

$$= \frac{12}{D^2}\left[\frac{1}{1-2\frac{D'}{D}-15\left(\frac{D'}{D}\right)^2}\right] \cdot xy$$

$$= \frac{12}{D^2}\left[1-\left(\frac{2D'}{D}+15\left(\frac{D'}{D}\right)^2\right)\right]^{-1} xy$$

$$= \frac{12}{D^2}\left[1+\frac{2D'}{D}\right] xy$$

$$= \frac{12}{D^2}\left[xy + \frac{2D'}{D}xy\right]$$

$$\frac{12}{D^2}\left[xy + 2\frac{1}{D}\cdot x\right]$$

$$\frac{12}{D^2}\left[xy + 2\int x\right]$$

$$\frac{12}{D^2}\left[xy + 2\frac{x^2}{2}\right]$$

$$\frac{12}{D^2}\left[xy + x^2\right]$$

$$\frac{12}{D}\left[\frac{x^2}{2}y + \frac{x^3}{3}\right]$$

$$12\left[\frac{x^3}{6}y + \frac{x^4}{12}\right]$$

$$= 2x^3y + x^4$$

Q.8. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cdot \cos x$

$m^2 + m - 6 = 0$

$m^2 + 3m - 2m - 6 = 0$

$m(m+3) - 2(m+3) = 0$

$(m+3)(m-2) = 0$

$m = 2, -3.$

C.F : $f(y+2x) + f(y-3x)$

P.I : $\cancel{\frac{1}{(m+3)(m-2)} \cdot y\cos x}$

$$\frac{1}{D^2 + DD' - 6(D')^2} \cdot y\cos x$$

$$\frac{1}{D^2 + 3DD' - 2DD' - 6(D')^2} \cdot y\cos x$$

$$\frac{1}{D(D+3D') - 2D'(D+3D')} \cdot y\cos x$$

$$\frac{1}{(D+3D')(D-2D')} y\cos x$$

$$\frac{1}{(D+3D')} \int (c-2x)\cos x \, dx$$

$$\frac{1}{(D+3D')} \left[ c\int \cos x \, dx - 2\int x\cos x \, dx \right]$$

$$\frac{1}{(D+3D')} \left[ c\sin x - 2\left[ x\sin x + \cos x \right] \right]$$

$$\frac{1}{(D+3D')} \left[ c\sin x - 2x\sin x - 2\cos x \right]$$

$$\frac{1}{(D+3D')} \left[ (y+2x)\sin x - 2x\sin x - 2\cos x \right]$$

$$\frac{1}{(D+3D')} \left[ y\sin x + \cancel{2x\sin x} - \cancel{2x\sin x} - 2\cos x \right]$$

$$\frac{1}{(D+3D')} \left[ y\sin x - 2\cos x \right]$$

$$\int ((c+3x)\sin x - 2\cos x)\,dx$$

$$\int c\sin x\,dx + 3\int x\sin x\,dx - 2\int \cos x\,dx$$

$$c\int \sin x\,dx + 3\int x\sin x\,dx - 2\int \cos x\,dx$$

$$\Rightarrow -c\cos x + 3\left[-x\cos x + \sin x\right] - 2\sin x$$

$$\Rightarrow -c\cdot\cos x - 3x\cos x + 3\sin x - 2\sin x$$

$$\Rightarrow -c\cos x - 3x\cos x + \sin x$$

$$\Rightarrow -(y-3x)\cos x - 3x\cos x + \sin x$$

$$\Rightarrow -y\cos x + \cancel{3x\cos x} - \cancel{3x\cos x} + \sin x$$

$$\Rightarrow -y\cos x + \sin x$$

(4) $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

$m^2 - m - 2 = 0$

$m^2 - 2m + m - 2 = 0$

$m(m-2) + (m-2) = 0$

$(m-2)(m+1) = 0$

$m = -1, 2$

$C.F. = f(y-x) + f(y+2x)$

$$P.I = \frac{1}{D^2 - DD' - 2D'^2}\cdot(y-1)e^x$$

$$= \frac{1}{D^2 - 2DD' + DD' - 2D'^2}\cdot(y-1)e^x$$

$$= \frac{1}{D(D-2D') + D'(D-2D')}\cdot(y-1)e^x$$

Here $m=1$ let $y = c+x$

$$\Rightarrow \frac{1}{(D-2D')(D+D')} \cdot (y-1)e^x$$

$$\frac{1}{(D-2D')} \int (c+x-1)\, e^x$$

$$\frac{1}{(D-2D')} \cdot \left[ c\int e^x dx + \int x e^x dx - \int e^x dx \right]$$

$$\frac{1}{(D-2D')} \left[ c \cdot e^x + x e^x - e^x - e^x \right]$$

$$\frac{1}{(D-2D')} \left[ c e^x + x e^x - 2e^x \right]$$

$$\frac{1}{(D-2D')} \left[ (y-x)\, e^x + x e^x - 2e^x \right]$$

$$\frac{1}{(D-2D')} \left[ y e^x - \cancel{x e^x} + \cancel{x e^x} - 2e^x \right]$$

$$\frac{1}{(D-2D')} \left[ y e^x - 2e^x \right]$$

Here $m=2$

let $y = c-2x$

$$\int \left[ (c-2x)\, e^x - 2e^x \right] dx$$

$$\int c \cdot e^x dx - 2x e^x dx - 2 e^x dx$$

$$c\int e^x dx - 2\int x e^x dx - 2\int e^x dx$$

$$\Rightarrow c \cdot e^x - 2\left[ x e^x - e^x \right] - 2e^x$$

$$c e^x - 2x e^x + \cancel{2e^x} - \cancel{2e^x}$$

$$\Rightarrow (y+2x)\, e^x - 2x e^x$$

$$y e^x + \cancel{2x e^x} - \cancel{2x e^x}$$

$$\Rightarrow y e^x$$

(1) $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, $u(0,y) = 8e^{-3y} \to (1)$

Let, $u = XY \to (2)$ be solution of (1)
where $X$ is a function of $x$ only
and $Y$ is a function of $y$.

Only (2) in (1) gives :-

$X'Y = 4XY'$

$= \frac{X'}{X} = \frac{4Y'}{Y}$

$\frac{X'}{X} = \frac{4Y'}{Y} = K$ (say)

If $\frac{X'}{X} = K \Rightarrow X' - KX = 0$

auxiliary Eqn $m - K = 0$

$m = K$

Solution is $X = c_1 e^{Kx}$

If $\frac{4Y'}{Y} = K \Rightarrow 4Y' - KY = 0$

auxiliary Eqn

$4m - K = 0$

$m = K/4$

Solution is $Y = c_2 e^{\frac{Ky}{4}}$

General solution is

$u = c_1 c_2 e^{Kx} \cdot e^{\frac{Ky}{4}} \to (3)$

$u(x,y) = u = c_1 c_2 e^{Kx}$

Eqn (3) put $x = 0$

$8e^{-3y} = c_1 c_2 e^{Ky/4}$

$\boxed{c_1 c_2 = 8,}$

$\frac{Ky}{4} = -3$, $\boxed{K = -12}$

(2) $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

Given, $u = 3e^{-y} - e^{-5y}$
when $= 0$

Let $u = XY$
be solution of (1), (1) gives-

$\Rightarrow 4X'Y + XY' = 3XY$

$\Rightarrow 4X'Y = 3XY - XY'$

$\Rightarrow 4X'Y = X(-Y' + 3Y)$

$= \frac{4X'}{X} = \frac{-Y' + 3Y}{Y} = K$

If $\frac{4X'}{X} = K$

$\Rightarrow 4X' = KX$

$\Rightarrow 4X' - KX = 0$

$4m - K = 0$

$m = \frac{K}{4}$

Sol$^n$ is

$X = c_1 e^{Kx/4}$

If $\frac{-Y' + 3Y}{Y} = K$

$-Y' + 3Y = KY$

$-m + 3 = K$

$\therefore \quad m = 3 - K$

Sol$^n$ is $Y = c_2 e^{(3-K)y}$

G.S is

$u = c_1 c_2 e^{Kx/4} \cdot e^{(3-K)y}$

when $X = 0$, then on comparing with 1st term

$c_1c_2 e^{(3-k)y} = 3e^{-y}$

$\boxed{c_1c_2 = 3}$

$3-k = -1$

$\boxed{4 = k}$

$\boxed{u = 3e^{x-y}}$

→ when $x = 0$, then comparing with $2^{nd}$ term

$c_1c_2 e^{(3-k)y} = -e^{-5y}$

$c_1c_2 = -1$

$3-k = -5$

$k = 8$

$\boxed{u = e^{2x-5y}}$

(2) $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$

Q. $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u$

$u(x, 0) = 10e^{-x} - 6e^{-4x}$

$u = XT$

~~Hint~~ -