e^2

* Unit - 4

Multiple Integral.
- → Double Integral.
- → Triple Integral.

(1) $\int_0^1 \int_{y=0}^{x^2} e^{y/x} \, dy \, dx$

$$\int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx \qquad = \int_0^1 \underset{\mathrm{I}}{x} \, \underbrace{(e^{x} - 1)}_{\mathrm{II}} \, dx$$

$$x(e^x - x) - \int e^x - x \, dx.$$

$$\left[x(e^x - x) - e^x + \frac{x^2}{2}\right]_0^1$$

$$1(\cancel{e}-\cancel{1}) - \cancel{e} + \frac{1}{2} + \cancel{1} \qquad = 1/2.$$

(2) $\int_0^1 \int_0^1 \frac{dy \, dx}{\sqrt{1-y^2}\,\sqrt{1-x^2}}$

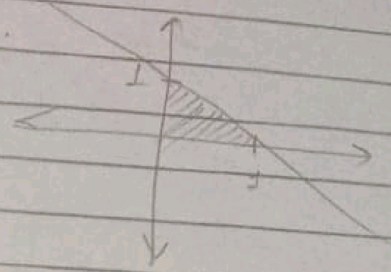
$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} \left[\sin^{-1} y\right]_0^1$$

$$\frac{\pi}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \qquad = \qquad \frac{\pi}{2} \times \left[\sin^{-1} x\right]_0^1$$

$$= \frac{\pi}{2} \times \frac{\pi}{2} \qquad = \frac{\pi^2}{4}.$$

(5) $\iint_R xy\,dy\,dx$ $\quad x \geq 0, \quad y \geq 0, \quad x + y \leq 1$

Taking a strip parallel to y axis
$\therefore$ Limit will be 0 to $1-x$



Taking a strip parallel to x axis
$\therefore$ Limit will be 0 to 1.

$$\int_0^1 \int_0^{1-x} xy\,dy\,dx$$

$$\frac{1}{2}\int_0^1 x\left[y^2\right]_0^{1-x} dx \Rightarrow \frac{1}{2}\int_0^1 x(1-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1 x(1+x^2-2x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3}\right]_0^1$$

$$= \frac{1}{2} \times \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right)$$

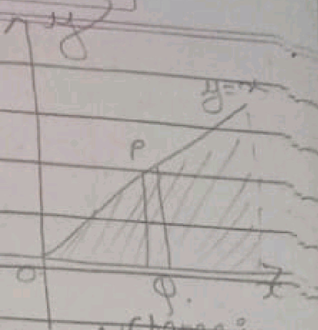
$$= \frac{1}{2}\left(\frac{6+3-8}{12}\right) = \frac{1}{24}.$$

(6) $\iint_R (x+y)^2\,dy\,dx$. Over the region boundaries by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

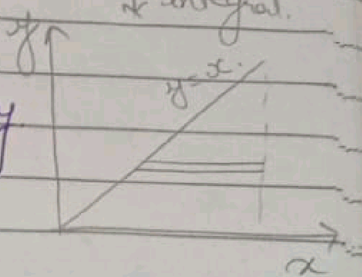
Taking a strip $\parallel$ to y axis
$\therefore$ Limit will be $-\frac{b}{a}\sqrt{a^2-x^2}$ to $\frac{b}{a}\sqrt{a^2-x^2}$

Taking a strip $\parallel$ to x axis
$\therefore$ Limit will be 0 to a.

We draw the bounded region of given curve i.e limit of integral $x=0$ to ∞ and $y=0$ to x for the change of order of integration strip is $\parallel$ to x axis and limit of x is y to ∞ and limit of y is 0 to ∞.



change in order of integral.



$$\int_0^{\infty}\int_0^{x} x\, e^{-x^2/y}\, dy\, dx = \int_0^{\infty}\int_y^{\infty} x\, e^{-x^2/y}\, dx\, dy$$

$x^2 = t$

$2x\,dx = dt$

$$\int_0^{\infty}\int_{y^2}^{\infty} \frac{1}{2} e^{-t/y}\, dt\, dy$$

$$-\frac{1}{2}\int_0^{\infty} y\left[e^{-t/y}\right]_{y^2}^{\infty} dy$$

$$+\frac{1}{2}\int_0^{\infty} \underset{\mathrm{I}}{y}\; \underset{\mathrm{II}}{e^{-y}}\, dy$$

$$\frac{1}{2}\; -y\, e^{-y} + \int e^{-y}\, dy$$

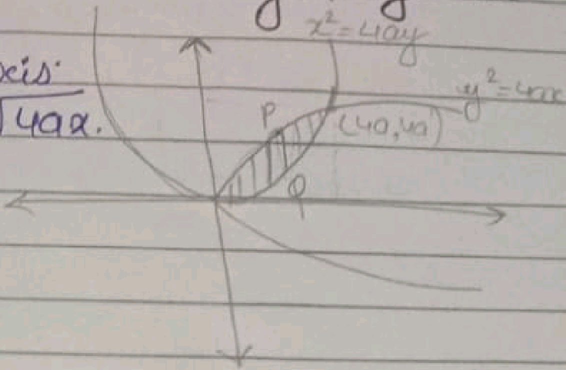
$$-\frac{1}{2}\left[+y e^{-y} + e^{-y}\right]_0^{\infty}$$

$$+\frac{1}{2}(1)\; (\cancel{ey+1})_0^{\infty} \qquad = \boxed{1/2}$$

~~(4) $x^2+y^2=a^2$~~ **Note:** $\underset{REA}{A} = \int\int_0 dx\,dy$; $\underset{REA}{A} = \int\int r\,dr\,d\theta$

~~(4) Prove that area of circle is πa^2.~~

(3) Find area of region by Double Integral $y^2 = 4ax$ and $x^2 = 4ay$



Taking strip parallel to y axis.

Limit of y. ~~$\sqrt{4ax}$ to~~ $\frac{x^2}{4a}$ to $\sqrt{4ax}$.

Limit of x :- 0 to $4a$.

$$\int_0^{4a}\int_{\sqrt{4ax}}^{x^2/4a} dx\,dy.$$

$$\int_0^{4a} [y]_{\sqrt{4ax}}^{x^2/4a}\,dx \Rightarrow \int_0^{4a} \frac{x^2}{4a} - \sqrt{4ax}\,dx.$$

$$= \frac{[x^3]_0^{4a}}{3\times 4a} - \sqrt{4a}\left[\frac{(x)^{3/2}}{3}\right]_0^{4a}\times 2$$

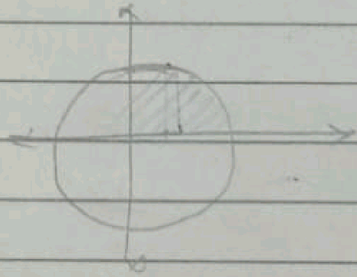
$$= \frac{(4a)^3}{3\times 4a} - \frac{\sqrt{4a}\times 2}{3}(4a)^{3/2}$$

$$= \frac{16a^2}{3} - \frac{4\sqrt{a}}{3}\times 8a\sqrt{a}$$

$$= \frac{16a^2}{3} - \frac{32a^2}{3}$$

$$= \frac{16a^2}{3}.$$

(4) Prove that area of circle is πa^2.



$$x^2+y^2=a^2.$$

Limit of y: 0 to $\sqrt{a^2-x^2}$

Limit of x :- 0 to a.

$$4\int_0^a\int_0^{\sqrt{a^2-x^2}} dx\,dy$$

(3) $\int_2^4\int_0^x\int_0^{x+y} z\,dx\,dy\,dz.$

$\frac{1}{2}\int_2^4\int_0^x [z^2]_0^{x+y}\,dx\,dy$

$\frac{1}{2}\int_2^4\int_0^x (x+y)^2\,dx\,dy$

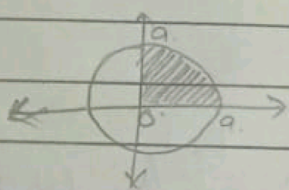
$\frac{1}{2}\int_2^4 \left[\frac{(x+y)^3}{3}\right]_0^x dx$

$\frac{1}{6}\int_2^4 8x^3 - x^3 \cdot dx \quad \Rightarrow = \frac{1}{6}\int_2^4 7x^3\,dx.$

$= \frac{7}{6}\left[\frac{x^4}{4}\right]_2^4$

$= \frac{7}{\cancel{24}\,8} \times \cancel{16}^{2} \times \cancel{15}^{5} \quad = 70.$

(4) $\iint_R xy\,dy\,dx \quad , x^2+y^2=a^2 \quad x \geq 0 \; , y \geq 0.$



Taking a strip parallel to y axis.
∴ Limit will be 0 to $\sqrt{a^2-x^2}$
Taking a strip parallel to x axis.
∴ Limit will be 0 to a.

$\int_0^a\int_0^{\sqrt{a^2-x^2}} x\,y\,dx\,dy$

$\frac{1}{2}\int_0^a x\left[y^2\right]_0^{\sqrt{a^2-x^2}} dx \quad \Rightarrow \frac{1}{2}\int_0^a x\,(a^2-x^2)\,dx$

$\Rightarrow \frac{1}{2}\left[\frac{x^2a^2}{2} - \frac{x^4}{4}\right]_0^a$

$= a^4/8.$

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4. $\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2\,dx\,dy.$

$$\frac{4}{3}\int_0^a \left[(x+y)^3\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx.$$

$$\frac{4}{3}\int_0^a \left(x+\frac{b}{a}\sqrt{a^2-x^2}\right)^3 - x^3\,dx$$

$$\frac{4}{3}\int_0^a \cancel{x^3}+\frac{b^3}{a^3}(a^2-x^2)^{3/2}+3x^2\frac{b}{a}(a^2-x^2)^{1/2}+3x\frac{b^2}{a^2}(a^2-x^2)$$

$$\frac{4}{3}\int_0^a \frac{b^3}{a^3}(a^2-x^2)^{3/2}dx + \int_0^a 3x^2\frac{b}{a}(a^2-x^2)^{1/2} - \cancel{x^3}\,dx + \int_0^a 3x\frac{b^2}{a^2}(a^2-x^2)$$

(1) (2)

(1) $\displaystyle\int_0^a \frac{b^3}{a^3}(a^2-x^2)^{3/2}dx$ $\quad x = a\sin\theta$, $dx = a\cos\theta\,d\theta$

$$\frac{b^3}{\cancel{a^3}}\int_0^{\pi/2} \cancel{a^3}\cos^3\theta\; a\cos\theta\,d\theta$$

$$b^3 a\,\frac{3\times1}{4\times2}\times\frac{\pi}{2} = \frac{3\pi ab^3}{16}$$

(2) $\displaystyle\int_0^a \frac{3b}{a}x^2(a^2-x^2)^{1/2}$ $\quad x = a\sin\theta$, $dx = a\cos\theta\,d\theta$

$$\frac{3b}{a}\int_0^a a^4\sin^2\theta\cos^2\theta\,d\theta$$

$$3a^3b\,\frac{\pi}{2}\times\frac{1}{4\times2} = \frac{3\pi a^3b}{16}$$

$16 \quad 4$ $\qquad\qquad\qquad\qquad$ 4

(2) Note :- $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(\theta, r)\, d\theta\, dr$

$-x^2) dx$ (7) $\int_0^{\pi}\int_0^{a(1-\cos\theta)} r^2 \sin\theta\, d\theta\, dr$

$\int_0^{\pi} \left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)} \sin\theta\, d\theta$

$\frac{1}{3}\int_0^{\pi} a^3 (1-\cos\theta)^3 \sin\theta\, d\theta$ $\qquad$ Put $1-\cos\theta = t$

$\qquad\qquad\qquad\qquad \sin\theta\, d\theta = dt$

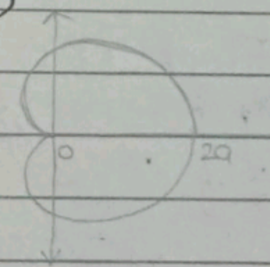
$\frac{a^3}{3}\int_0^{2} t^3 dt \qquad = \left[\frac{t^4}{4}\right]_0^2 \times \frac{a^3}{3}$

$= \frac{a^3}{\cancel{12}_3} \times \cancel{16}\,4 \qquad = \frac{4a^3}{3}$

(8) $\iint r \sin\theta\, dr\, d\theta \quad ; \qquad r = a(1+\cos\theta)$

R $a(1+\cos\theta)$

$\int_0^{\pi}\int_0^{a(1+\cos\theta)} r \sin\theta\, dr\, d\theta$



$\int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta \sin\theta$

$$\int_0^{\pi} a^2 (1+\cos\theta)^2 \sin\theta \, d\theta$$

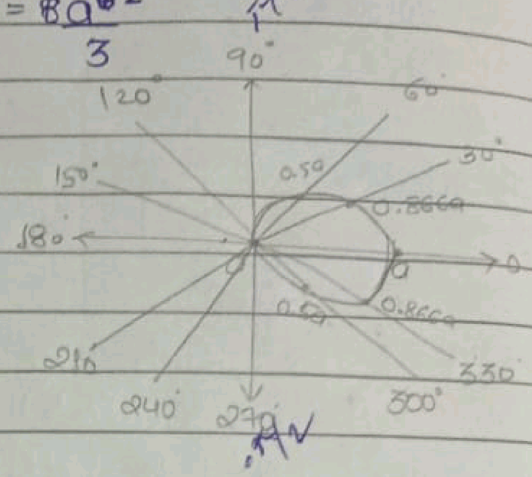
$$1+\cos\theta = t$$

$$-\sin\theta \, d\theta = dt$$

$$-\int_2^0 a^2 t^2 \, dt = a^2 \int_0^2 t^2 \, dt$$

$$= a^2 \left[\frac{t^3}{3}\right]_0^2 = \frac{8a^2}{3}$$

(9) $\iint r^2 \, d\theta \, dr \quad ; \quad r = a\cos\theta$



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} r^2 \, d\theta \, dr$$

$$\int_{-\frac{\pi}{2}}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{a\cos\theta} d\theta$$

$$\frac{1}{3} a^3 \int_{-\frac{\pi}{2}}^{\pi/2} \cos^3\theta \, d\theta$$

$$\frac{2a^3}{3} \int_0^{\frac{\pi}{2}} \cos^3\theta \, d\theta = \frac{2a^3}{3} \times \frac{2}{3} = \frac{4a^3}{9}$$

*** Change in order of Integration.**

$$\int_{x=a}^{b} \left[\int_{y=c}^{d} f(x,y) \, dy\right] dx = \int_{y=c}^{d} \left[\int_{x=a}^{b} f(x,y) \, dx\right] dy$$

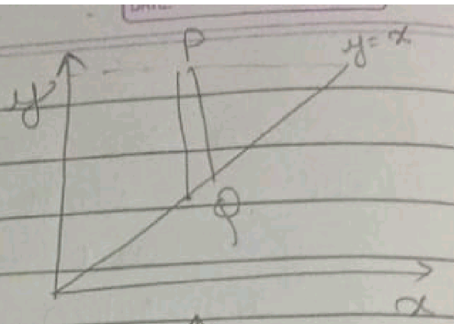
Q) Find the value of integral $\int_0^{\infty}\int_0^{x} x e^{-x^2/y} \, dy \, dx$ by changing the order of integral.

$y \to 0$ to x

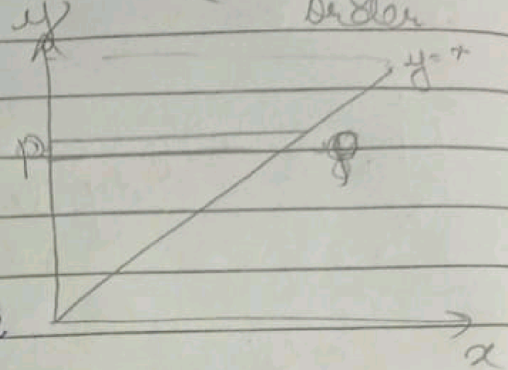
$x \to 0$ to ∞

(2) Find the value of integral

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$$



Change in order



Initial limits of y is x to ∞.

~~y~~ of x is 0 to ∞.

After changing order of integral.

limit of x is 0 to y

limit of y is 0 to ∞.

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dy\, dx$$

$$= \int_0^\infty \frac{e^{-y}}{y} [x]_0^y\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{\cancel{y}} \cancel{y}\, dy$$

$$= -\left[e^{-y}\right]_0^\infty$$

$$= 1$$

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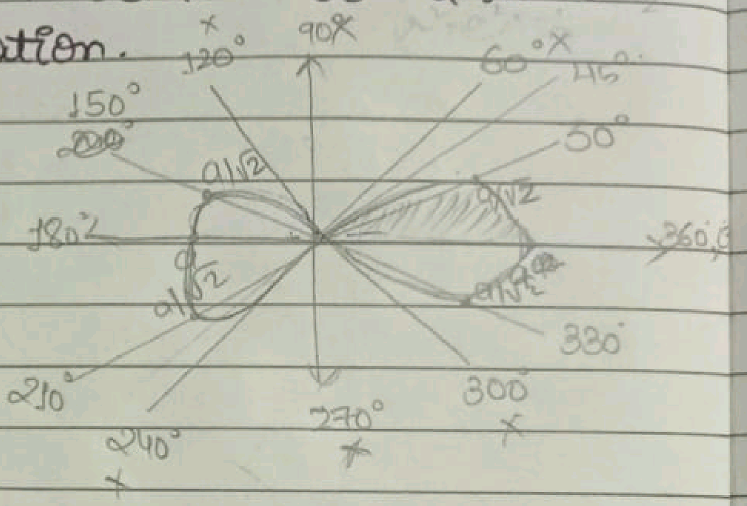
$$4\int_0^a \sqrt{a^2-x^2}\, dx =$$

$$4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$4\times\frac{a^2}{2}\times\frac{\pi}{2}$$

$$=\pi a^2$$

5) Find the area the LEMINISCATE $r^2 = a^2\cos 2\theta$ using double integration.



~~Area~~ Limit of r :-

0 to $r\sqrt{\cos 2\theta}$

Limit of θ = 0 to $\pi/4$

$$\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\, dr\, d\theta$$

$$\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$\frac{1}{2}\int_0^{\pi/4} a^2 \cos 2\theta\, d\theta \qquad 2\theta=\phi$$

$$d\theta = \frac{d\phi}{2}$$

$$\frac{a^2}{2\times 2}\int_0^{\pi/2}\cos\phi\, d\phi$$

$$\frac{a^2}{2\times 2}\left[\sin\phi\right]_0^{\pi/2}$$

$$=\frac{a^2}{2\times 2}\times 4$$

$$\boxed{\text{Area} = 2a^2}$$

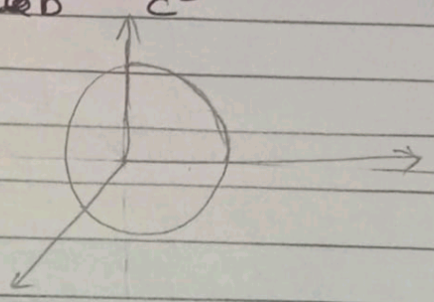
* Volume using double and triple integration

* $V = \iint_D z \, dx \, dy$ * $V = \iint_D f(x, y) \, dx \, dy$

* $V = \iiint_D dx \, dy \, dz$

(1) Find the volume of Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$V = \iint z \, dx \, dy.$



$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$y = 0$ to $b\sqrt{1 - \frac{x^2}{a^2}}$.

$x = 0$ to a.

$b\sqrt{a^2 - x^2}$

$4 \pi abc$

$$V = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy.$$

$$\text{Put } b\sqrt{1-\frac{x^2}{a^2}} = t$$

$$V = 8c\int_0^a \int_0^t \sqrt{\frac{t^2}{b^2}-\frac{y^2}{b^2}}\,dx\,dy$$

$$= \frac{8c}{b}\int_0^a \int_0^t \sqrt{t^2-y^2}\,dx\,dy$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{t^2-y^2}+\frac{t^2}{2}\sin^{-1}\frac{y}{t}\right]_0^t dx.$$

$$= \frac{8c}{b}\int_0^a \frac{t^2}{2}\frac{\pi}{2}\,dx$$

$$= \frac{8c\pi}{4b}\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$= \frac{2c\pi b}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= \frac{2c\pi b}{a^2}\times\frac{2a^3}{3} \qquad = \frac{4\pi abc}{3}.$$

* ~~Using~~ Centre of gravity using $\iint$ & $\iiint$.

Density,

$$m = \iint_D f(x, y)\, dx\, dy$$

$$m = \iint_D \rho\, dx\, dy$$ where $\rho = f(x, y)$

Centre of gravity = (x_c, y_c)

$$x_c = \frac{\iint_D x f(x, y)\, dx\, dy}{m}$$

$$y_c = \frac{\iint_D y f(x, y)\, dx\, dy}{m}$$

Let (x_c, y_c, z_c) be the coordinate of centroid.

$$m = \iiint_V f(x, y, z)\, dx\, dy\, dz$$

$$m = \iiint_V \rho\, dx\, dy\, dz$$

$$x_c = \frac{\iiint_V x \rho\, dx\, dy\, dz}{m}$$ $y_c / = \int$

$$y_c = \frac{\iiint_V y \rho\, dx\, dy\, dz}{m}$$

$$z_c = \frac{\iiint_V z \rho\, dx\, dy\, dz}{m}$$

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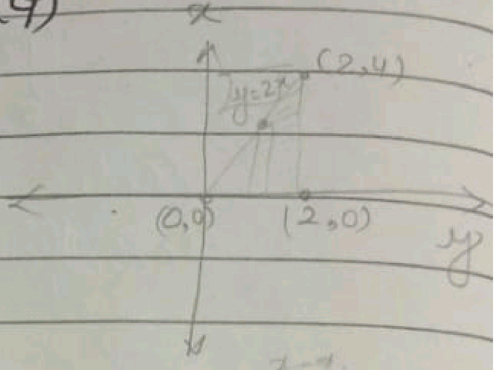
(1) A triangular plate its vertices (0,0), (2,0) and (2,4) has density $\rho = 1+x+y$ then find
(i) the mass of plate
(ii) the position of its center of gravity (G)



$\rho = 1+x+y$

$m = \iint_D f(x,y)\, dx\, dy$

$= \iint_D \rho \, dx\, dy$

• Limit of y is $2x$ to 0.
x is 0 to 2.

$m = \int_0^2 \int_0^{2x} 1+x+y \, dx\, dy$

$= \int_0^2 1+x + \left[\frac{y^2}{2}\right]_0^{2x} dx$

$= \int_0^2 1+x + \frac{4x^2}{2} dx \Rightarrow \left[x + \frac{x^2}{2} + \frac{2}{3}x^3\right]_0^2$

$= 2 + \frac{4}{2} + \frac{2\times 8}{3}$

$= 4 + \frac{16}{3} = \boxed{\frac{28}{3}}$

$m = \int_0^2 \left[y + xy + \frac{y^2}{2}\right]_0^{2x}$

$= \int_0^2 2x + 4x^2 + \ldots \, dx$

$= \left[x^2 + \frac{4}{3}x^3\right]_0^2 \Rightarrow 4 + \frac{4\times 8}{3} = 4 + \frac{32}{3}$

$= \frac{44}{3}$

Now.

x_c, y_c be the coordinate of centroid.

Then,

$$x_c = \frac{\iint_D x\rho\,dx\,dy}{\iint_D \rho\,dx\,dy}$$

$$= \frac{3}{44}\int_0^2\int_0^{2x} x(1+x+y)\,dx\,dy.$$

$$= \frac{3}{44}\int_0^2 \left[xy + x^2 y + x\frac{y^2}{2}\right]_0^{2x}$$

$$= \frac{3}{44}\int_0^2 2x^2 + 2x^3 + x\times\frac{\cancel{4}^{2}x^2}{\cancel{2}}\,dx.$$

$$= \frac{3}{44}\int_0^2 2x^2 + 4x^3\,dx.$$

$$= \frac{3}{44}\left[\frac{2}{3}x^3 + x^4\right]_0^2.$$

$$= \frac{3}{44}\times 8\left[\frac{2}{3}+2\right) \Rightarrow \frac{\cancel{3}}{\cancel{44}_{11}}\times \cancel{8}^{2}\times\frac{8}{\cancel{3}}$$

$$\boxed{x_c = 16/11}$$

$$y_c = \frac{\iint_D y\rho\,dx\,dy}{44/3}$$

$$= \frac{3}{44}\int_0^2\int_0^{2x} y + xy + y^2\,dy\,dx.$$

$$= \frac{3}{44}\int_0^2\left[\frac{y^2}{2} + \frac{xy^2}{2} + \frac{y^3}{3}\right]_0^{2x}.$$

$$= \frac{3}{44}\int_0^2 \frac{\cancel{4}^{2}x^2}{\cancel{2}} + \frac{x\times\cancel{4}^{2}x^2}{\cancel{2}} + \frac{8x^3}{3}\,dx.$$

$$= \frac{3}{44} \int_0^2 2x^2 + \cancel{2x^3} + \frac{14 x^3}{3} \, dx$$

$$= \frac{3}{44} \left[\frac{2x^3}{3} + \frac{14 x^4}{4 \times 3} \right]_0^2$$

$$= \frac{3}{44} \times 8 \left(\frac{2}{3} + \frac{14 \times 2}{3 \times 4} \right)$$

$$= \frac{3}{44} \times 8 \times 3 \qquad = \frac{18}{11}$$

Coordinate of centroid $\left(\frac{16}{11}, \frac{18}{11} \right)$

(2) Find the mass of elliptical plate $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

If the density of any point (x, y) is $\rho = 4xy$.

Calculate mass and centre of gravity

(3) Find the coordinates of centre of gravity of the +ve ~~quadrant~~ octant of the sphere $x^2 + y^2 + z^2 = a^2$, The density being given $\rho = k\, xyz$.

$$\text{mass} = \iiint_0 f(x, y, z) \, dx \, dy \, dz.$$

$$m = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \int^{\sqrt{a^2 - x^2 - y^2}} K\, xyz \, dx \, dy \, dz.$$

$$= K \int_0^a \int_0^{\sqrt{a^2 - x^2}} \left[\frac{z^2}{2} \right]_0^{\sqrt{a^2 - x^2 - y^2}} dx \, dy$$