Unit - 3 Integral calculus

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- Beta & gamma funᶜ ~

1 <u>Beta funᶜ</u> : Let $m, n > 0$ then $\beta(m,n)$ is denoted by $\beta(m,n)$ and is given by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

# Properties of $\beta$ - funᶜ

i) $\beta(m,n) = \beta(n,m)$

ii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

iii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$

2 <u>Gamma funᶜ</u> : If $n > 0$, then

$$\Gamma n = \int_0^\infty e^{-t} \cdot t^{n-1}\,dt$$

# Properties of $\Gamma$-funᶜ

i) $\Gamma 1 = 1$

ii) $\Gamma n+1 = n\Gamma n$

iii) $\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} \cdot dx$

iv) $\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$

v) $\Gamma n+1 = \int_0^\infty e^{-y^{1/n}} \cdot dy$

- Establish relation b/w $\beta$ & $\Gamma$ funᶜ.

(or)

prove that $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$, where $m,n > 0$

Sol we know that

$$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} \cdot dx$$

$$\Gamma n = \int_0^\infty e^{-zx} x^{n-1} \cdot z^n \cdot dx$$

multiply $e^{-z} \cdot z^{m-1}$ both side, then

$$\Gamma n \cdot e^{-z} \cdot z^{m-1} = \int_0^\infty e^{-zx} \cdot x^{n-1} \cdot z^n \cdot e^{-z} \cdot z^{m-1} \, dx$$

$$= \int_0^\infty e^{-z(1+x)} x^{n-1} z^{m+n-1} \cdot dx$$

now, integrating both the side w.r.t $z$ in limit 0 to $\infty$.

$$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty \int_0^\infty e^{-z(1+x)} \cdot x^{n-1} z^{m+n-1} dx \cdot dz$$

$$\Gamma n \cdot \Gamma m = \int_0^\infty x^{n-1} \left[ \int_0^\infty e^{-z(1+x)} \cdot z^{m+n-1} dz \right] dx$$

$$\Gamma n \cdot \Gamma m = \int_0^\infty x^{n-1} \frac{\Gamma m+n}{(1+x)^{m+n}} \cdot dx \quad \left[ \because \int_0^\infty e^{-zx} x^{n-1} dx = \frac{\Gamma n}{z^n} \right]$$

$$\Gamma n \cdot \Gamma m = \Gamma m+n \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \cdot dx$$

$$\frac{\Gamma m \; \Gamma n}{\Gamma m+n} = B(m,n)$$

Hence, proved.

Q Prove that $\Gamma \frac{1}{2} = \sqrt{\pi}$.

Sol. we know that,

$$B(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n} \quad \text{--- i,}$$

put $m = n = \frac{1}{2}$ in eq i,

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma \frac{1}{2} \cdot \Gamma \frac{1}{2}}{\Gamma \frac{1}{2} + \frac{1}{2}}$$

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma \frac{1}{2} \cdot \Gamma \frac{1}{2}}{\Gamma 1} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2$$

again, $B(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx$

put $m = n = \frac{1}{2}$, then.

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1} \cdot (1-x)^{\frac{1}{2}-1} dx$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^1 x^{-\frac{1}{2}} \cdot (1-x)^{-\frac{1}{2}} \cdot dx.$$

put $x = \sin^2\theta$

$dx = 2\sin\theta \cdot \cos\theta \, d\theta.$

$x = 0$ & $x = 1$

$\theta = 0$ $\quad$ $\theta = \pi/2$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = \int_0^{\pi/2} (\sin^2\theta)^{-1/2} \cdot (1-\sin^2\theta)^{-1/2} \cdot 2\sin\theta\cdot\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin\theta\cdot\cos\theta}{\sin\theta\cdot\cos\theta}\cdot d\theta$$

$$= 2\int_0^{\pi/2} 1\, d\theta$$

$$= 2\left[\frac{\pi}{2} - 0\right]$$

$$= \pi$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$ Hence, proved

Q State & prove Legendre's duplication formula

or

prove : $\Gamma m\, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\, \Gamma 2m$, where $m > 0$.

Sol. we know that

$$B(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$$

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma m\, \Gamma n}{\Gamma m+n} \quad \text{—(i)}$$ [by β-3 property]

put $n = \frac{1}{2}$ in eq (i)

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{1-1}\theta \cdot d\theta = \frac{\Gamma m \cdot \Gamma \frac{1}{2}}{2\Gamma m+\frac{1}{2}}$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cdot d\theta = \frac{\sqrt{\pi}\cdot \Gamma m}{2\Gamma m+\frac{1}{2}} \quad \text{—(ii)}$$ [$\because \Gamma\frac{1}{2} = \sqrt{\pi}$]

put $n = m$, in eq (i)

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta = \frac{\Gamma m\cdot\Gamma m}{2\Gamma 2m}$$

$$\int_0^{\pi/2} \left(\frac{2\sin\theta\cdot\cos\theta}{2}\right)^{2m-1}\cdot d\theta = \frac{\Gamma m\, \Gamma m}{2\Gamma 2m}$$

$$\int_0^{\pi/2} \frac{(\sin 2\theta)^{2m-1}}{2^{2m-1}}\cdot d\theta = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

Kuch purna $\begin{cases} \int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx \\ \int_0^a f(x)\,dx = \int_0^a f(\theta)\,d\theta \end{cases}$

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put $2\theta = \phi$, then $2\,d\theta = d\phi$

L.L $\theta = 0$ & U.L $\theta = \pi/2$

$\phi = 0$ $\phi = \pi$

$$\int_0^{\pi} \frac{(\sin\phi)^{2m-1}}{2^{2m-1}} \cdot \frac{d\phi}{2} = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\cancel{\frac{1}{2}} \cdot \frac{1}{2^{2m-1}} \int_0^{\pi} (\sin\phi)^{2m-1} \cdot d\phi = \frac{(\Gamma m)^2}{\cancel{2}\Gamma 2m}$$

$$2\int_0^{\pi/2} (\sin\phi)^{2m-1}\, d\phi = \frac{2^{2m-1}(\Gamma m)^2}{\Gamma 2m}$$

replace $\phi$ by $\theta$

$$\int_0^{\pi/2} (\sin\theta)^{2m-1}\, d\theta = \frac{2^{2m-1}(\Gamma m)^2}{2\,\Gamma 2m} \quad \text{—— (iii)}$$

on compare eq ii & iii, then

$$\frac{\cancel{\Gamma m}\cdot\Gamma\pi}{\cancel{2}\,\Gamma m+\frac{1}{2}} = \frac{2^{2m-1}(\Gamma m)^{\cancel{2}}}{\cancel{2}\,\Gamma 2m}$$

$$2^{2m-1} \cdot \Gamma m \cdot \Gamma m+\tfrac{1}{2} = \Gamma 2m \cdot \sqrt{\pi}$$ Hence, proved.

Q prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$ where $m,n > 0$

Sol. $$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

taking RHS, then

$$= \beta(m+1,n) + \beta(m,n+1)$$

$$= \frac{\Gamma m+1\,\Gamma n}{\Gamma m+n+1} + \frac{\Gamma m\,\Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{\Gamma m+1\,\Gamma n + \Gamma m\cdot\Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{m\Gamma m\,\Gamma n + \Gamma m\,n\Gamma n}{m+n\,\Gamma m+n}$$

$$= \frac{\Gamma m\,\Gamma n\,\cancel{(m+n)}}{\cancel{m+n}\,\Gamma m+n}$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$$= \beta(m,n)$$

$$= \text{L.H.S}$$ Hence, proved.

Q find $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} \, dx$.

Sol $\int_0^\infty \frac{x^8 - x^{14}}{(1+x)^{24}} \, dx = \int_0^\infty \frac{x^8}{(1+x)^{24}} d - \frac{x^{14}}{(1+x)^{24}} \, dx$

$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{9+15}} dx$

$= \beta(9,15) - \beta(15,9)$

$= 0$ Ans

Q find $\int_0^\infty \sqrt{x} e^{-3\sqrt{x}} \, dx$

Sol put $3\sqrt{x} = t$

$\frac{3}{2} x^{-1/2} dx = dt$

$\frac{3}{2\sqrt{x}} dx = dt$.

using identity

$\int_0^\infty e^{-t} t^{m-1} dt = \Gamma n$

now, $\int_0^\infty \left(\frac{t}{3}\right) e^{-t} \frac{2t}{9} dt = \frac{2}{27} \int_0^\infty t e^{-t} t \, dt$

$= \frac{2}{27} \int_0^\infty t^2 e^{-t} dt$.

$= \frac{2}{27} \times \Gamma 3 = \frac{4}{27}$

Q find $\int_0^\infty e^{-4x} x^{5/2} dx$.

Sol put $t = 4x$

$dt = 4dx$

$\frac{dt}{4} = dx$

now, $\int_0^\infty e^{-t} \left(\frac{t}{4}\right)^{5/2} 4dx$

$= \frac{1}{4 \times 4^{5/2}} \int_0^\infty e^{-t} t^{5/2} dt$

$= \frac{1}{4 \times 4^{5/2}} \int_0^\infty e^{-t} t^{7/2-1} dt$

$= \frac{1}{4 \times 4^{5/2}} \Gamma \frac{7}{2}$

$= \frac{1}{4 \times 4^{5/2}} \frac{5}{2} \Gamma \frac{5}{2}$

$\frac{1}{4 \times 4^{5/2}} \times \frac{5}{2} \times \frac{3}{2} \Gamma \frac{3}{2}$

$\frac{1}{4 \times 4^{5/2}} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$

$\frac{15 \sqrt{\pi}}{4^{7/2} \times 2^3}$

$= \frac{15 \sqrt{\pi}}{2^{10}}$ Ans

Note = $\Gamma n+1 = n!$ (n is integer)
$\Gamma n+1 = n \Gamma n$ (n is fraction)

# $\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin n\pi}$

iii,

red.

ere m,n > 0

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Q prove that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

Sol · Given, $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

Let $x^2 = t \Rightarrow \sqrt{t} = x$

$2x\,dx = dt$

$dx = \frac{1}{2x} dt$

$\int_0^\infty e^{-t} \cdot \frac{1}{2\sqrt{t}} \cdot dt$

$\frac{1}{2} \int_0^\infty e^{-t} \cdot t^{-1/2} dt$

$\frac{1}{2} \Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$

(a) $\int_0^1 x^3 (1-x^2)^4 dx$

Sol. $m = \frac{3+1}{2}$ $n = 4+1$

Thus, $\frac{1}{2} \frac{\Gamma 2 \Gamma 5}{\Gamma 7} = \frac{1}{2} \times \frac{1 \times \cancel{4!}}{6 \times 5 \times \cancel{4!}}$

$= \frac{1}{60}$ Ans

Q Express the integral $\int_0^1 x^m (1-x^n)^p dx$ in terms of Grama fun

Sol. Given, $\int_0^1 x^m (1-x^n)^p dx$

put $x^n = y \Rightarrow y^{1/n} = x$

$n x^{n-1} dx = dy$

$dx = \frac{1}{n x^{n-1}} \cdot dy$

$dx = \frac{x}{n x^n} \cdot dy$

$dx = \frac{y^{1/n}}{n y} \cdot dy = \frac{1}{n} y^{1/n - 1} dy$

now, $\int_0^1 (y^{1/n})^m (1-y)^p \frac{1}{n} y^{1/n - 1} dy$

$\frac{1}{n} \int_0^1 y^{m/n} (1-y)^p \cdot y^{1/n - 1} dy$

$\frac{1}{n} \int_0^1 y^{\frac{m-n+1}{n}} (1-y)^p \cdot dy$

$\frac{1}{n} \int_0^1 y^{\frac{m+1}{n} - 1} (1-y)^{(p+1)-1} dy$

$\because M = \frac{m+1}{n}$ & $N = p+1$

then, $\frac{1}{n} \beta\left(\frac{m+1}{n}, p+1\right)$

$= \frac{1}{n} \frac{\Gamma\frac{m+1}{n} \Gamma p+1}{\Gamma \frac{m+1}{n} + p+1}$ Ans.

Q $\int_0^\infty \frac{x^c}{c^x} dx = \frac{\Gamma c+1}{(\log c)^{c+1}}$, $c > 1$

Sol Let $I = \int_0^\infty \frac{x^c}{c^x} \cdot dx$

Let $c^x = e^t$

$\log c^x = t$

$x \log c = t$

$x = \frac{t}{\log c}$

$dx = \frac{dt}{\log c}$

and L.L $x=0$ $t=0$

U.L $x=\infty$ $t=\infty$

$I = \int_0^\infty \left(\frac{t}{\log c}\right)^c \cdot \frac{1}{e^t} \cdot \frac{dt}{\log c}$

$= \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-t} t^c \cdot dt$

$= \frac{\Gamma c+1}{(\log c)^{c+1}}$ Ans

Q $\int_0^1 y^{q-1}\left(\log \frac{1}{y}\right)^{p-1} dy = \frac{\Gamma p}{q^p}$

~~Sol we know that~~

$\cancel{\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx}$

~~put~~ $\cancel{n = p \ \& \ x = qt}$

$\cancel{\Gamma p = \int_0^\infty e}$

Sol we know that

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

put $n = p$ & $x = qt$ then, $dx = q\,dt$

$$\Gamma p = \int_0^\infty e^{-qt}\, qt^{p-1}\, q\,dt$$

$$\Gamma p = \int_0^\infty e^{-qt}\, q^{p-1} \cdot q\, t^{p-1}\, dt$$

$$\Gamma p = \int_0^\infty e^{-qt}\, q^p \cdot t^{p-1} \cdot dt$$

$$\frac{\Gamma p}{q^p} = \int_0^\infty e^{-qt} \cdot t^{p-1} \cdot dt$$

put $t = \log \frac{1}{y} \Rightarrow e^t = \frac{1}{y} \Rightarrow y = e^{-t}$

$\therefore dy = -e^{-t} \cdot dt$

then, $t = 0 \rightarrow y = 1$

$t = \infty \rightarrow y = 0$

$$\frac{\Gamma p}{q^p} = \int_1^0 e^{-q\cancel{\log \frac{1}{y}}} \cdot \left(\log \frac{1}{y}\right)^{p-1} \cdot \cancel{e^{\log \frac{1}{y}}} \cdot \cancel{dt} \cdot \frac{dy}{-e^{-t}}$$

$$\frac{\Gamma p}{q^p} = -\int_1^0 \frac{(e^{-t})^q}{e^{-t}} \cdot \cancel{e^t} \cdot \left(\log \frac{1}{y}\right)^{p-1} dy$$

$$\frac{\Gamma p}{q^p} = \int_0^1 y^{q-1} \cdot \left(\log \frac{1}{y}\right)^{p-1} dy \qquad \text{H.P.}$$

# Tracing of curve

i) cartesion curve , ii) polar curve.

## Tracing of cartesion curve :-

following points are helpful in tracing of cartesion curve $y = f(x)$

I Symmetry

(a) A curve is symmetry about the x-axis if all power of y in the eq. of curve are even for

Example :- $y^2 = 4ax$

(b) A curve is symmetry about the y-axis if all the power of x in the eq. of curve even

for Ex :- $x^2 = 4ay$.

(c) A curve is symmetry about the line $y = x$ if the eq of curve remains unchanged on the Interchanging of x & y    Ex - $x^2 + y^2 = a^2$

(d) if the $eq^n$ of curve remain unchanged when x is chaned to $-x$ & y is changed in $-y$ then curve is symmetric about opposite quadrant.

Ex - $x^3 + y^3 = 3x^2y$.

II origin & tangent.

if the point (0, 0) satisfied the eq. of given curve then the curve passes through the origin.

if the curve is passes through the origin then. we check the nature of the origin (node, cust) etc. for this we find tangent at origin by equating the lowest degree term to zero

ex - $x^2 + y^2 + x + y = 0$

$x + y = 0 \Rightarrow x = -y$ tangent

if we get two tangent at origin then we find its nature.

(a) if both tangent are real and distinct then origin is called node

(b) if both tangent are coincident then origin is called cust.

(c) if both tangent are imaginary then origin is called isolated point.

III point of intersection of curve with coordinate axis
put $x=0$ & $y=0$ in the eq$^n$ of the curve respectively
it gives point of intersection of the curve on x-axis and y-axis resp.

IV Asymptotes
straight line at finite distance from the origin to which the curve does not intersect it but touches it at infinity it means that asymptotes is the boundary line and limiting line for the curve.

(a) asymptotes parallel to x-axis - equating the coefficient of highest power of $x$ to zero. in the given eq$^n$ of curve
(b) asymptotes equating the coefficient of highest power of $y$ to zero. in the given eq$^n$ of curve

note
- it is not nessacary ~~is~~ that every curve has asymptotes.

for EX the curve $x^2+y^2=a^2$ has no asymptotes

V Region.
To find the region of curve we solve the eq$^n$ of curve for the $y$ or $x$ which every is conviniunt.
suppose we solve the eq$^n$ of curve for $y$ then we examine the following point.

(a) we find those value of $x$ for which $y \to \infty$
(b) we find the interval for $x$ in which the value of $y$ become imaginary.
c we find the interval of $x$ in which the value of incre

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VI Special point.

find $\frac{dy}{dx}$ by the eq$^n$ of curve.

(a) If $\frac{dy}{dx} = 0$, for some value of $x$ & $y$, then the tangent is parallel to $x$-axis

(b) If $\frac{dy}{dx} = \infty$, for some value of $x$ then, then the tangent is parallel to $y$-axis.

(c) if $\frac{dy}{dx}$ is +ve, for some interval of $x$ then, the curve is inc. in the interval.

(d) if $\frac{dy}{dx}$ is -ve, for some interval of $x$, then the curve is decreasing in the interval.

Q trace the curve $y^2(a-x) = x^2(a+x)$

Sol Given $ay^2 - xy^2 = ax^2 + x^3$ — i,

i, Symmetry - the curve is symmetric about $x$-axist

ii, origin & tangent - the curve is passes through the origin, the equating lowest degree term to zero.

$$ay^2 = ax^2 \Rightarrow y = \pm x$$

Two real & distinct tangent at origin, therefore origin is node.

iii, point of intersection of curve on coordinate axis.

put $x = 0$ in eq$^n$ i, then

$$ay^2 - 0 = 0 + 0$$

$$ay^2 = 0 \Rightarrow y = 0$$

again, put $y = 0$ in eq$^n$ i, then

$$0 - 0 = ax^2 + x^3$$

$$x^2(a+x) = 0 \Rightarrow x = -a, 0$$

Therefore, point are $(0,0)$ & $(-a,0)$

iv, Asymptotes - parallel to $y$-axis

$$(a-x)y^2$$

$$a - x = 0 \Rightarrow x = a$$

v, region - solve eq$^n$ for $y$.

$$y^2 = \frac{x^2(a+x)}{(a-x)} \Rightarrow y = \pm x\sqrt{\frac{a+x}{a-x}}$$

Observation ;

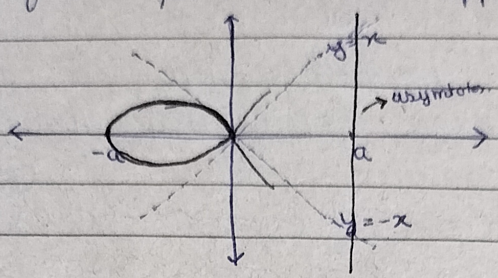
i, $y$ become imaginery when $x > a$

ii, $y$ become imaginery when $x < -a$

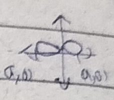
iii, $y$ exist b/w $-a \le x < a$

iv, for each $x$ $(-a \le x < a)$ $y$ has two equal or opposite (sign) value.

using above five point the approx shape of curve.




Q trace the curve $a^2y^2 = x^2(a^2-x^2)$



Sol Given $a^2y^2 = x^2a^2 - x^4$ —— i,

i, Symmetry - both the power of $x$ & $y$ is even, then symmetry about both axis.

ii, origin.- put $x=0$ & $y=0$ $\Rightarrow$ The curve is passes through origin.

iii, tangent of origin - equating lowest degreen term to zero, then $a^2y^2 = x^2a^2$

$$y = \pm x$$

Two real & distict tagnest at origin, therefore origin is node

iv, point of intersection with coordinate axis.-

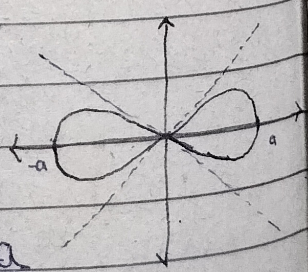
put $y=0$ in eq i, | put $x=0$ in eq i,

$x^2(a^2-x^2) = 0$ | $a^2y^2 = 0$

$x=0$ & $x = \pm a$ | $y = 0$

Thus, The point of —"— is $(0,0)$, $(-a,0)$ & $(a,0)$.

v, asymptotes - There is no asymptotes.




vi, region - $y = \frac{x}{a}\sqrt{a^2-x^2}$

observation:

i, $y$ become imaginary when $x > a$ ~~& $x < -a$~~

ii, $y$ become imaginary when $x < -a$

iii, $y$ exist b/w $-a \le x \le a$

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# Tracing of ~~pattern~~ curve

To find shape of curve $r = f(\theta)$, the following procedure is adopted

or opposite (i) Symmetry -

(A) when $\theta$ is replaced by $-\theta$ then, curve equation remain unchanged then curve is symmetrical about 'intial line' i.e $x$-axis.

(B) if $\theta$ is replaced by $(\pi - \theta)$ then, curve eq remain same then it is symmetrical about '$\theta = \frac{\pi}{2}$ line' i.e $y$-axis

(C) if ~~or~~ replaced by $-\theta$ & $-r$, then curve eq remain same, then curve is symmetrical about '$\theta = \pi/2$' i.e $y$ axis

(D) if $r$ is replaced by $-r$, eq remain same, then the curve is symmetrical about pole i.e origin $(0,0)$.

(ii) Pole & tangent - if put $r = 0$ in given eq$^n$ of curve & $\theta = \alpha$ (finite value) is the tangent of curve at pole.

(iii) Table :- solve eq$^n$ of curve for $r$, prepare table of corr. values of $\theta$ & $r$, this table gives a no of point on the curve. the shape of curve is obtained by ploting these points.

(iv) if the value of $r$ comes out to be imaginary or $-ve$ in the region given by, then the curve will not exist in regions. - Region.

Q Trace the curve $r = a(1 + \cos\theta)$

Sol. Given $r = a(1 + \cos\theta)$ — (i)

(i) Symmetry - put $\theta = -\theta$ in eq (i), then

$r = a(1 + \cos(-\theta)) \Rightarrow a(1 + \cos\theta)$

The eq$^n$ of remain same, then curve is symmetrical about '$\theta = 0$' 'initial line'

(ii) Pole & tangent - Put $r = 0$, $a(1+\cos\theta) = 0 \Rightarrow \cos\theta = -1$

$\theta = \pi$

curve passing through the pole & $\theta = \pi$ is tangent to curve at pole.

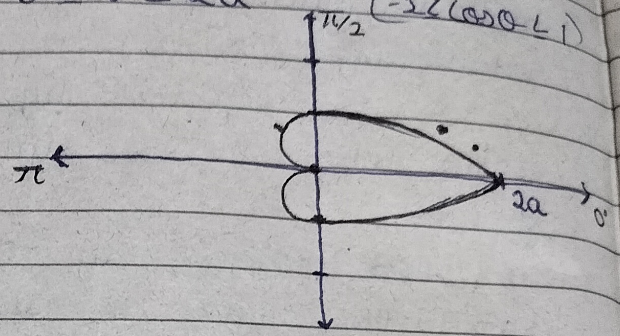
iii, Table -

| $\theta$ | $0°$ | $30°$ | $45°$ | $60°$ | $90°$ | $120°$ | $135°$ | $150°$ | $180°$ |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | $2a$ | $1.86a$ | $1.70a$ | $1.5a$ | $a$ | $0.5a$ | $0.3a$ | $0.14a$ | $0$ |

iv, Region -

curve exists for Region $0 \le r \le 2a$ $(-1 < \cos\theta < 1)$

so the approx shape of the curve is




Q Trace the curve $r = 2a\cos\theta$

Sol Given $r = 2a\cos\theta$

i, Symmetry - put $\theta = -\theta$, The eq$^n$ of curve remain same then, it is symmtrical about initial line.

ii, pole & tangent - put $r = 0$, $2a\cos\theta = 0 \Rightarrow \cos\theta = 0$

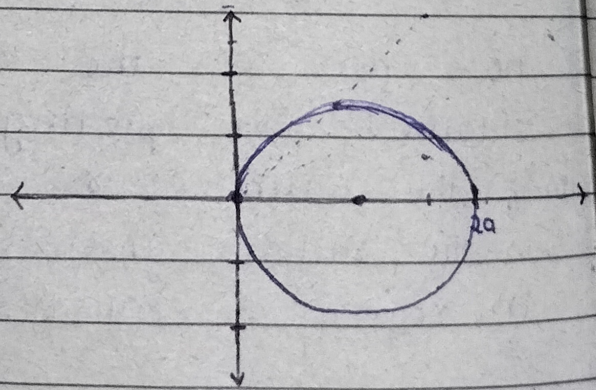
$\theta = \pi/2$

curve passing through the pole & $\theta = \pi/2$ is tangent the curve at pole.

iii, Table -

| $\theta$ | $0°$ | $30°$ | $45°$ | $60°$ | $90°$ |
|---|---|---|---|---|---|
| $r$ | $2a$ | $1.72a$ | $1.4a$ | $a$ | $0$ |

iv, Region.



- Area of bounded curve in cartesian coordinates

$$\text{Area of curve} = \int_a^b f(x)\cdot dx$$

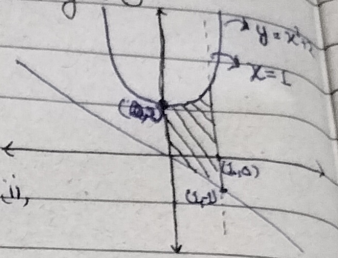
- Area b/w two curves / area enclosed b/w two curves $y_1 = f_1(x)$ & $y_2 = f_2(x)$ and the co-ordinates $x=a$ & $x=b$ is given by

$$A = \int_a^b (y_2 - y_1)\,dx = \int_a^b (f_2(x) - f_1(x))\,dx$$

Q Find the area bounded by the parabola $y = x^2+2$ and the straight line $x=0$, $x=1$ & $x+y=0$.

Sol Given, $y = x^2+2$ —i,

& $y = -x$ —ii, $x=0$ & $x=1$




point of intersemction

put $x=0$ in eq i,

$y = 2$

& put $x=1$ in eq i,

$y = 3$

put $x=1$ in eq ii,

$y = -1$

The parbola $y=x^2+2$ have vertex $(0,2)$ & axis along the y axis the line $x=1$ meet the parbola at $(1,3)$ and the line $x+y=0$ at $(1,-1)$. the required area is.

$$A = \int_a^b (f_2(x) - f_1(x))\,dx$$

$$= \int_0^1 (x^2+2-(-x))\,dx$$

$$= \int_0^1 (x^2+x+2)\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 + \left[2x\right]_0^1$$

$$A = \frac{1^3}{3} + \frac{1^2}{2} + 2 = \frac{14}{6} = \frac{7}{3} \text{ Ans}$$

Q Find the area of the loop of the curve $ay^2 = x^2(a-x)$

Sol. Given curve, $ay^2 = x^2(a-x) \Rightarrow ay^2 = x^2a - x^3$

By curve tracing,

i, Symmetry - The power of $y$ is even, therefore The curve is symmetrical about $x$-axis ii, origin & tangent -

put $x=0$ & $y=0$ in eq i,

$0=0$, then, the curve is passes through origin & equating lowest degree term to zero, then

$$ay^2 = ax^2 \Rightarrow y = \pm x$$

Two real & distinct tangents of curve at origin.

iii, Point of intersection -

put $y=0$ in eq i.

$x^2(a-x)=0$

$x=0$, $x=a$

put $x=0$ in eq i,

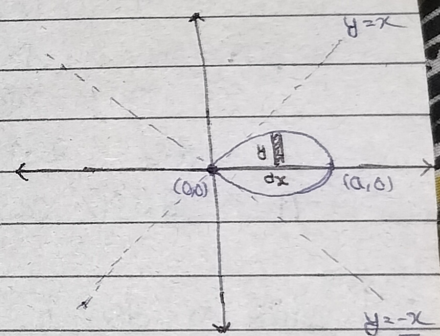
$ay^2=0$

$y=0$

Therefore, the point are $(0,0)$, $(a,0)$

iv, Asymptotes - There is no Asymptotes of a given curve.

v, Region - $y = x\sqrt{\frac{a-x}{a}}$




Observation :

i, $y$ become imaginery when $x < a$

ii, $y$ is exist b/w $0 \le x \le a$

now, Area of curve $= \int_a^b y\,dx \times 2$

$$= 2\times\int_0^a \frac{x}{\sqrt{a}}\sqrt{a-x}\,dx$$

$$= \frac{2}{\sqrt{a}}\int_0^a x\sqrt{a-x}\,dx$$

Let $\sqrt{a-x}=t \Rightarrow a-x=t^2 \Rightarrow x=a-t^2$

$\therefore -1 = 2t\frac{dt}{dx}$

$dx = -2t\,dt$

The new limits are $x=0 \Rightarrow t=\sqrt{a}$

& $x=a \Rightarrow t=0$

$$A = \frac{2}{\sqrt{a}}\int_{\sqrt{a}}^{0}(a-t^2)t(-2t\,dt)$$

$$= \frac{4}{\sqrt{a}}\int_0^{\sqrt{a}} at^2 - t^4\,dt$$

$$= \frac{4}{\sqrt{a}}\left[\frac{at^3}{3} - \frac{t^5}{5}\right]_0^{\sqrt{a}}$$

$$= \frac{4}{\sqrt{a}}\left[\frac{a(\sqrt{a})^3}{3} - \frac{(\sqrt{a})^5}{5}\right]$$

$$= 4\left[\frac{a(\sqrt{a})^2}{3} - \frac{(\sqrt{a})^4}{5}\right]$$

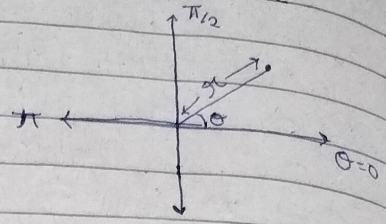
$$= 4\left[\frac{5a^2 - 3a^2}{15}\right]$$

$$= \frac{4}{15}(2a^2)$$

$$= \frac{8a^2}{15}$$ Ans

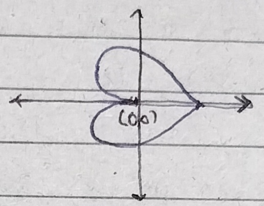
## Area of bounded curve in polar co-ordinate

Area of polar curve $= \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$




Q find the area of coordiae $r = a(1+\cos\theta)$

Sol i, Symmetry - curve is symatrical about initial line pole &




$$\text{Area} = \frac{1\times 2}{2}\int_0^{\pi} a^2(1+\cos\theta)^2 d\theta$$

$$= a^2\int_0^{\pi}(2\cos^2\theta/2)^2$$

$$= 4a^2\int_0^{\pi}\cos^4\theta/2 \, d\theta.$$

put $\theta/2 = \phi$

$\theta = 2\phi$

$d\theta = 2d\phi.$

$$A = 8a^2\int_0^{2}\cos^4\phi \, d\phi$$

$$\because \int_0^{\pi/2}\sin^m\theta\cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+1}{2}}$$

$$\Rightarrow \frac{8a^2\,\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{2\,\Gamma\frac{5}{2}}$$

$$\Rightarrow \frac{8a^2\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{2\times 2\times 1}$$

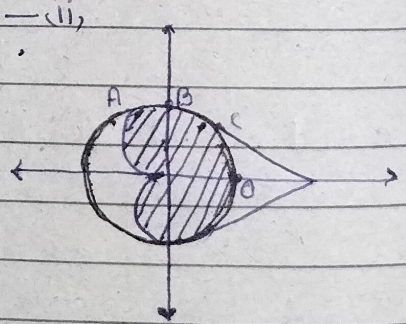
$$= \frac{3}{2}a^2\pi \quad \text{Ans}$$

Q find the area to circle

Sol the given eq$^n$ $r = a$ — i,

& $r = a(1+\cos\theta)$ — ii,

Solving eq-i, & eq-ii,




$a = a(1+\cos\theta)$

$1 = 1 + \cos\theta$

$\cos\theta = 0$

$\theta = \pi/2, 3\pi/2$

area of $A_1$ = Area (ODCBO)

$$= \frac{1}{2}\int_0^{\pi/2} r^2 \, d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2 \, d\theta.$$

$$A_1 = \frac{a^2\pi}{4}$$

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$$A_2 = \frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2 d\theta$$

$$A_2 = \frac{a^2}{8}(3\pi - 8)$$

$\theta = 0$

now, Required area $= 2(A_1 + A_2)$

$$= 2\left(\frac{a^2\pi}{4} + \frac{a^2}{8}(3\pi-8)\right)$$

$$= \frac{\cancel{2}a^2}{\cancel{4}2}\left(\pi + \frac{3\pi-8}{2}\right) = \left(\frac{5\pi}{4} - 2\right)a^2 \quad \text{Ans.}$$

l line

# Length of arc of curve

i, for cartesion curve

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$S = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy$$

ii, for polar curve

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$S = \int_{r_1}^{r_2} \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2} \cdot d\theta$$

Q find the length of arc of curve (semicubical parabola) $ay^2 = x^3$ from the vertex $(0,0)$ to the point $(a,a)$

Sol Given, $ay^2 = x^3$

now, $a2y\frac{dy}{dx} = 3x^2$

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9x^4}{4a^2y^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9x^4}{4ax^3} = \frac{9x}{4a}$$

now, $$S = \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$= \int_0^a \sqrt{1+\frac{9x}{4a}} \cdot dx$$

$$= \int_0^a \sqrt{\frac{4a+9x}{4a}} \cdot dx$$

Let $1 + \frac{9x}{4a} = t^2$

LIMITS

$x=0 \Rightarrow t = \cancel{0}1$

$x=a \Rightarrow t = \sqrt{13}/2$

$$\frac{9}{4a}dx = 2t\,dt$$

$$dx = \frac{8at\,dt}{9}$$

$$S = \int_1^{\sqrt{13}/2} t \cdot \frac{8a}{9} t \cdot dt$$

$$S = \frac{8a}{9}\int_1^{\sqrt{13}/2} t^2 \cdot dt$$

$$S = \frac{8a}{9}\left[\frac{t^3}{3}\right]_1^{\sqrt{13}/2}$$

$$S = \frac{8a}{9}\left[\frac{(\sqrt{13}/2)^3}{3\times 2^3} - \frac{1}{3}\right]$$

$$S = \frac{\cancel{8}a}{27}\left[\frac{(\sqrt{13})^3}{\cancel{8}} - \frac{\cancel{8}}{\cancel{8}}\right]$$

Q find the entire length of cardioid $r = a(1+\cos\theta)$ Show that arc of upper half is bisected by the line $\theta = \pi/3$

now,

Sol. $r = a(1+\cos\theta)$

$\therefore \frac{dr}{d\theta} = a\sin\theta$

now, $S = 2\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$

$S = 2\int_0^{\pi} \sqrt{a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta} \cdot d\theta$

$S = 2\int_0^{\pi} \sqrt{2a^2 + 2a^2\cos\theta} \cdot d\theta$

$$\because \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos\theta = 2\cos^2\theta/2 - 1$$
$$2\cos\theta = 4\cos^2\frac{\theta}{2} - 2$$

$S = 2a\int_0^{\pi} \sqrt{2 + 2\cos\theta} \cdot d\theta$

$S = 2a\int_0^{\pi} \sqrt{\cancel{2} + 4\cos^2\frac{\theta}{2} - \cancel{2}} \cdot d\theta$

$S = 4a\int_0^{\pi} \cos\frac{\theta}{2} \cdot d\theta = 8a\int_0^{\pi/2} \cos t \cdot dt$

$S = 8a\left[-\sin t\right]_0^{\pi/2} = -8a = 8a$ Ans.

and The arc of the length of upper half of the curve is 4a

now limit in $0 \to \pi/3$

$S = \int_0^{\pi/3} a\sqrt{2+2\cos\theta} \cdot d\theta = 2a = 2\times 2a = 4a$

Q find the length of curve. $y = \log\left[\frac{e^x - 1}{e^x + 1}\right]$ from $x = 1$ to $x = 2$

Sol. Given $y = \log\left[\frac{e^x-1}{e^x+1}\right]$

$\frac{dy}{dx} = \frac{e^x+1}{e^x-1} \times \left(\frac{e^x(e^x+1) - e^x(e^x-1)}{(e^x+1)^2}\right)$

$\frac{dy}{dx} = \frac{2e^x}{e^{2x} - 1}$

$\left(\frac{dy}{dx}\right)^2 = \frac{4e^{x}}{e^{4x} + 1 - 2e^{2x}}$

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now, $S = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$

$$S = \int_1^2 \sqrt{1 + \frac{4e^{2x}}{e^{4x} - 2e^{2x} + 1}} \cdot dx$$

$$S = \int_1^2 \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{e^{4x} - 2e^{2x} + 1}} \cdot dx$$

$$S = \int_1^2 \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{e^{4x} - 2e^{2x} + 1}} \cdot dx$$

$$S = \int_1^2 \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}} \cdot dx$$

$$S = \int_1^2 \frac{e^{2x}+1}{e^{2x}-1} \cdot dx$$

Divide ~~both~~ ~~side~~ by $e^x$, then

$$S = \int_1^2 \frac{e^{2x}/e^x + 1/e^x}{e^{2x}/e^x - 1/e^x} \cdot dx$$

$$S = \int_1^2 \frac{e^x + e^{-x}}{e^x - e^{-x}} \cdot dx$$

put $e^x - e^{-x} = t$ now, limit

$\therefore (e^x + e^{-x})dx = dt \Rightarrow dx = \frac{dt}{e^x + e^{-x}}$

$$S = \int_1^2 \frac{\cancel{e^x + e^{-x}}}{t} \cdot \frac{dt}{\cancel{e^x + e^{-x}}} = \int_1^2 \frac{dt}{t}$$

$$S = [\log|t|]_1^2$$

$$S = [\log|e^x - e^{-x}|]_1^2 \quad \text{Ans}$$