Matrix-

systematic arrangement of data-

Suppose we have simaltaneous set of equation.

$$x + 2y + 3z + 4t = 0$$
$$2x + 3y + 5z + 5t = 0$$
$$6x + 6y + 8z + 7t = 0$$

$$A = \begin{array}{cccccc} & C_1 & C_2 & C_3 & C_4 & \\ \left[\begin{array}{c} \\ \\ \\ \end{array}\right. & \begin{array}{c} 1 \\ 2 \\ 6 \end{array} & \begin{array}{c} 2 \\ 3 \\ 6 \end{array} & \begin{array}{c} 3 \\ 5 \\ 8 \end{array} & \begin{array}{c} 4 \\ 5 \\ 7 \end{array} & \left.\begin{array}{c} \\ \\ \\ \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}_{3 \times 4}$$

This system of no. arrangend in rect. array in rows & columns and bounded by the brackets is called a MATRIX-

$$\boxed{A_{m \times n} = [a_{ij}] \begin{array}{l} i = 1 \text{ to } m \\ j = 1 \text{ to } n \end{array}}$$

Types of matrices :-

1. Row matrix-

If a matrix has only one row and any no of column, it is called a row matrix.

For example,

$$A_{1 \times m} = [a$$

Matrix-

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Suppose we have simaltaneous set of equation.

$$x + 2y + 3z + 4t = 0$$
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$$A = \begin{array}{cccccc} & C_1 & C_2 & C_3 & C_4 & \\ \left[\begin{array}{c} \\ \\ \\ \end{array}\right. & 1 & 2 & 3 & 4 & \left.\begin{array}{c}\\ \\ \\ \end{array}\right] R_1 \\ & 2 & 3 & 5 & 5 & R_2 \\ & 6 & 6 & 8 & 7 & R_3 \end{array}_{3 \times 4}$$

This system of no. arrangend in rect. array in rows & columns and bounded by the brackets is called a MATRIX-

$$\boxed{A_{m \times n} = [a_{ij}] \quad \begin{matrix} i = 1 \text{ to } m \\ j = 1 \text{ to } n \end{matrix}}$$

Types of matrices :-

1. Row matrix-

If a matrix has only one row and any no. of column, it is called a row matrix.

For. example,

$A_{1 \times m} = [a$

2. Column Matrix

A matrix having only one column & any no. of row, is a column matrix.

Eg, $A_{m \times 1} = [a_{i1}]_{i = 1 \text{ to } m}$

3. Null matrix -

Any matrix in which all the elements are zero is called a null matrix or zero matrix.

Eg. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4. Square Matrix -

A matrix in which no. of rows is equal to no. of column, is called a sq. matrix.

$A_{m \times m} = [a_{ij}]_{\substack{i = 1 \text{ to } m \\ j = 1 \text{ to } m}}$

5- Diagonal matrix -

A sq. matrix is called a diagonal matrix if all its non diagonal elements are zero.

Eg. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

6. Scalar Matrix -

A diagonal matrix in which all the diagonal element is equal to a scalar, then it a scalar matrix -

Eg. $A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

7. Unit Matrix-
A sq. matrix is called unit matrix if all the diagonal elements are unity and non diagonal elements are zero.

Eg. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Symmetric Matrix-
~~If~~ A sq. matrix is called symmetric if for all values of i and j, $a_{ij} = a_{ji}$
i.e. $A = A'$

For example,
$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

9. Skew Symmetric Matrix -
A sq. matrix is called skew symmetric matrix if -

(i) $a_{ij} = -a_{ji}$, for all values of i & j.
(ii) All the diagonal elements are zero.

For example-
$$A = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

10. Triangular Matrix-
A sq. matrix all of whose elements below the leading diagonal are zero, is upper dia. matrix. A sq matrix all whose elements above the leading diagonal are zero, is lower dia.

For example,

Upper dia. Matrix -

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Lower Dia. Matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

11. Orthogonal Matrix -

A sq. matrix 'A' is orthogonal if $\boxed{A \cdot A' = I}$

12. Idempotant Matrix -

A matrix such that $\boxed{A^2 = A}$ is an idempotant

13. Periodic Matrix -

A matrix 'A' is periodic if

$\boxed{A^{k+1} = A}$, where k is a +ve integer.

For the least value of 'k' for which $A^{k+1} = A$ then k is period of 'A'.

14. Nilpotent Matrix

A matrix if,

$A^k = 0$, where k is a +ve integer

k is the least +ve inly. for which

$A^k = 0$, then k is called the index of the nilpotent matrix.

15. Involuntary Matrix -

A matrix, if,

$$\boxed{A^2 = I}$$

Singular when det value is zero.

Multiply -

$$\underbrace{A_{m \times \underbrace{n \cdot B_{x}}_{\text{same}} \times y}}_{\text{resultant order}}$$

Elementary Transformations -

Any one of the following operations on an matrix is called elementary transformations.

(1) Interchanging any two rows or columns.

Indication - $R_i \to R_i \longleftrightarrow R_j$ R_{ij}

or $C_i \to C_i \longleftrightarrow C_j$ C_{ij}

(2) Multiplication of the elements of any row, Rim or a column by a non-zero scalar quantity is denoted by -

kR_i or kC_i.

3) Addition of the const. multiplication of the elements of any row to the corresponding elements of another row.

Indication -

$R_i + kR_j$ or $C_i + k C_j$

If a matrix B is obtained by A. by one or more opertation the B is said to be equivalent to B. And the symbol is - $A \sim B$

Rank of Matrix - 19/09

→ The rank of a matrix 'r' if -

(1) It has one non-zero minor of order 'r'

(2) Every minor of A order higher than 'r' is zero.

Note :- (1) Non 0 row is in which all the elements are non-zero

(2) The rank of the product matrix A to B of two matrices A & B is less than the rank of either of the matrices A and B

→ Rank of a matrix is equal to no. of non zero rows in uper Δ matrix

Note :- This defination, we use only elementary row transformation

→ Normal form of a matrix, by using elementary transformation any non zero matrix 'A' can be reduced to one of the following four forms which we called as normal form of matrix -

(1) $[I_r]$

(2) $[I_r \; 0]$ where 'I' is the unit matrix of order is range.

(3) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

Rank is denoted by -

$\rho(A) = r$

(4) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

row

Inverse: $A^{-1} = QP$

Note -
corresponding to every matrix 'A' of rank 'r' there exists two non-singular matrices 'p' & 'Q' such that, PAQ is in the normal form -

Que: Find the rank of the given matrix -

(1) $A = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

From def. #1

Operation at,

$R_1 \to R_1 \leftrightarrow R_2$ [To make $A_{11} = 1$]

$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

operation at, [A_{11} ke neeche sab zero ana chaiye]

$R_2 \to R_2 - 4R_1$; $R_3 \to R_3 - 3R_1$; $R_4 \to R_4 - R_1$

$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

operation at,

$R_3 \to R_3 \mp R_2$ [To make upper Δ matrix]

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

From def 2.
3 non zero rows.
So rank is 3

operation at, (must use only row transformation)

$R_4 \rightarrow R_4 + \frac{1}{2} R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Combination -

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & -7 & 6 \\ 0 & 0 & -2 \end{vmatrix}_{3\times3}, \begin{vmatrix} 2 & -1 & 3 \\ -7 & 6 & -11 \\ 0 & -2 & 4 \end{vmatrix}_{3\times3},$$

$$\begin{vmatrix} 0 & -7 & 6 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} -7 & 6 & -11 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{vmatrix}$$

So, $\rho(A) = 3$

all minor of order greater than 3 must be zero.

Que, Find the rank of the given matrix converting it into the normal form $\rightarrow$ diagonal pe hi kam hoga. Sare diagonal 1 hone h!

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 3 & 4 & 2 \\ 8 & -5 & 9 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$$

Operation at,

$R_2 \to R_2 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$$ A_{11} ke neeche sab zero-

Operation at,

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix}$$

Operation at,

$C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 4C_1$, $C_4 \to C_4 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix}$$ # To make $A_{22} \to 1$

Operation at,

$C_2 \to -\frac{1}{7} C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & -2 \\ 0 & 2 & -10 & -4 \\ 0 & 3 & -16 & -6 \end{bmatrix}$$

Operation at,

$R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operation at,

$C_3 \to C_3 + 5C_2 \qquad C_4 \to C_4 + 2C_2$

Meow

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -01 & 0 \end{bmatrix}$$

Operation at,

$R_3 \to R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operation at,

$R_3 \to -1R_3$

unit matrix of order 3

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to 0$$

0

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ order of unit matrix is 3

$\therefore$ Rank is 3

Que. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, Find two non singular matrices P & Q such that PAQ is in the normal form, hence find A^{-1} if it exists, oterwise give reason?

$A_{3 \times 3} = I_{3 \times 3} \, A \, I_{3 \times 3}$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operation at, # row opt. pre factor me lagega

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate while we get one of those forms

Operation at,

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operation at,

$C_2 \to C_2 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

operation at, $C_3 \to C_3 - 4C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

$I_3 = PAQ$, where P is the matrix

$P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, is non singular,

and $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & -3 \end{bmatrix}$, is also non singular.

$\rho(A) = 3 \to$ order of unit

Now inverse is given by,

$$A^{-1} = QP \quad (\text{As } I_{3\times3} \text{ and } A_{3\times3} \text{ is same})$$

[→ P & Q can vary depending on operation but should be non singular]

[→ P or Q any one is singular the inverse doesn't exists]

20/09

Solution of System of linear eqs- Using Rank theory-

RHS non zero

Homogeneous

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_n x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

→ This above system can be written in matrix eqn as-

$$AX = B \longrightarrow \text{eq } ①$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{m} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{rr} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Since, we want to use rank theory,

So we construct a new matrix with the help of the elements of A & B which is known as augmented matrix, i.e. C-

$$C = [A, B]$$

(1) If Rank of A is equal to Rank of C then system is consistent.

(a) If Rank of A is equal to Rank of C is equal to R, $(R=n)$ then system is consistent to uniqe sol^n.

(b) If $(R<n)$ then system is consistent with infinitely many sol^n.

(2) If Rank of A is not equal to Rank of C then system is inconsistent and there is no solution.

(Q.) Solve the given sys. of eq^n.

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + x_2 - 2x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 3 \\ 2x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}\right] \text{eq } 1$$

given sys. of equations is written in the matrix form.

i.e. $AX = B \rightarrow$ eq 2

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$\because$ We want to use rank theory so we construct a new matrix with the help of the elements of A & B.

$C = [A, B]$

$$C = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{array}\right]$$

Row transformation → Upar △ matrix.

Operation at,

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Operation at,

$R_2 \to -\frac{1}{5} R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Operation at,

$R_3 \to R_3 + 11R_2$.

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Operation at,

$R_3 \to \frac{1}{6} R_3$

$$\sim \left[\begin{array}{c|c} \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{A} & \begin{matrix} 2 \\ 1 \\ 1 \\ 0 \end{matrix} \end{array}\right]$$

where,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Rank of A, $\rho(A) = 3 = \rho(C)$

$\therefore$ System is consistem.

System has unique sol^n, i.e. 3.

For the sol^n replace A by $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

and B by $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ in eq (2)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

eq^n formed-

$$x_1 + 2x_2 + x_3 = 2$$
$$0x_1 + x_2 + x_3 = 1$$
$$0x_2 + 0x_2 + x_3 = 1$$

on solving,

$$x_1 = 1$$
$$x_2 = 0$$
$$x_3 = 1$$

Linear dependance & independance of vectors

Vectors : $X_1, X_2, X_3 \ldots X_n$
are said to be dependent if -
(1) All the vectors are of the same order.
(2) n scalars $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ (not all zero) exists such that $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \ldots \lambda_n x_n = 0$
otherwise they are linearly independent -

Question :-
Examine the following vectors for linear dependau
and find the relation if exists -

$X_1 = (1, 2, 4)$
$X_2 = (2, -1, 3)$
$X_3 = (0, 1, 2)$
$X_4 = (-3, 7, 2)$

Consider the matrix eq^n -

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0 \rightarrow \text{Eq } ①$$

$$\lambda_1 (1,2,4) + \lambda_2 (2,-1,3) + \lambda_3 (0,1,2) + \lambda_4 (-3,7,2)$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$$
$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$
$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Homo
Sy ste
→ Ran
fo

Operation at,
$R_2 \rightarrow R_2 - 2R_1$; $R_3 \rightarrow R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

operation at,

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 3 < 4$ (unknowns)

So, system is consistent with ∞ many soln.

Equation -

$0\lambda_1 + 0\lambda_2 + \lambda_3 + \lambda_4 = 0 \Rightarrow \lambda_3 + \lambda_4 = 0$

$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$

$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$

Let $\lambda_4 = t \quad (t \neq 0)$

$\lambda_3 = -t$

$\lambda_2 = 12t/5$

$\lambda_1 = -9t/5$

From eq ①.

$-\frac{9t}{5} x_1 + \frac{12t}{5} x_2 - t x_3 + t x_4 = 0$

$-\frac{t}{5}(9x_1 - 12x_2 + 5x_3 - 5x_4) = 0$

$\because t \neq 0 \quad \boxed{\therefore\ 9x_1 - 12x_2 + 5x_3 - 5x_4 = 0}$ relation

Hence, $\lambda_1 = 9$; $\lambda_2 = -12$; $\lambda_3 = 5$; $\lambda_4 = -5$

→ Not all zero.

So, given vectors are linearly dependent.

Eigen Values & Eigen Vectors-

sq matrix

Let
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Two vectors

$$AX = Y$$

where, A is the sq matrix.
X & Y is the column vector

→ 'X' is transformed into 'Y' by means of the sq matrix 'A'

Let X be a such vector which transforms λ into X, by means of the transformation.

Suppose the linear transformation-
$Y = AX$ transforms X into a scalar multiple of itself -

$$AX = Y = \lambda X$$

$$AX - \lambda I X = 0$$

$$[A - \lambda I][X] = [0]$$

Eigen values

Thus, the unknown scalar 'λ' is known as Eigen value of the matrix 'A' and the corresponding non-zero vector X is Eigen vector.

Characterstic Matrix -

This matrix $(A - \lambda I)$, is known as charasterstic Matrix.

For example,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Trace

Then,

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

Characterstic Polynomial -

The determinant $|A - \lambda I|$ when expanded will give a polynomial which we call as characterstic polynomial for the matrix 'A'.

For example -

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= 2-\lambda\left[(3-\lambda)(2-\lambda)-2\right] - \left[(2-\lambda)-1\right] + \left[2-(3-\lambda)\right]$$

$$= 2-\lambda\left[6-5\lambda+\lambda^2-2\right] - \left[1-\lambda\right] + \left[-1+\lambda\right]$$

as same as the order of A & I

$$|A - \lambda I| = -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

Characterstic Equation -

The eq^n $|A - \lambda I| = 0$ is called the characterstic eq^n of matrix A.

For example,

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

Characteristic roots for eigen value

The roots of the characteristic eqn $|A-\lambda I| = 0$ are called the roots for A.

For example,

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

By Trial & Error -

$\lambda = 1$ is one root

$\lambda = 1$ and $\lambda = 5$ are other roots (by solving)

Property

→ [Sum of eigen value is always equal to trace of A.]

→ [Product of Eigen value is always equal to $|A|$]

→ [Any sq matrix A and its transpose A' have the same Eigen values.]

→ [If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values an $n \times n$ matrix 'A' then the Eigen values of -

[(1) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \cdots \frac{1}{\lambda_n}$]

[(2) A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m \cdots \lambda_n^m$]

[(3) KA are $K\lambda_1, K\lambda_2, K\lambda_3 \cdots K\lambda_n$]

Property of Eigen Vector -

(1) The Eigen vector 'X' of a matrix 'A' is not unique.

(2) If $\lambda_1, \lambda_2, --- \lambda_n$ are distinct Eigen values of an $n \times n$ matrix then corresponding Eigen vectors $X_1, X_2, --- X_n$ form a linearly independent set -

(3) If two or more Eigen values are equal it may or may not be possible to get linearly independet Eigen vectors corresponding to Equal roots.

(4) Two Eigen vectors X_1 & X_2 are called orthogonal vectors if $X_1' X_2 = 0$

(5) Eigen vectors of a symmetrix corresponding to other Eigen values are pair wise Eigen orthogonal -

→ Non - symmetric matrices with non repeating Eigen values -

Find the Eigen value & Eigen vectors of the matrix 'A' = $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ ~~tet~~

Let the characterstic eqn for the given matrix be $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$3-\lambda\left[(2-\lambda)(5-\lambda)\right] + 4(0) = 0$

$(3-\lambda)(10+\lambda^2-7\lambda) = 0$

$\lambda = 3, \lambda = 2, \lambda = 5.$

The Eigen vectors of 'A' corresponding to λ is given by the non-zero sol^n of the eq^n

$$[A-\lambda I]X = 0$$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow eq \text{ (a)}$$

The 1^{st} Eigen vector corresponding to Eigen value $\lambda = 2$ is obtained - λ by 2 in eq (a)

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x + y + 4z = 0 \\ 0x + 0y + 6z = 0 \\ 0x + 0y + 3z = 0 \end{cases}$$

Identical
lete to
zero and not

[Take two eq^n → cross multiply]

$$\frac{x}{6} = \frac{y}{-6} = \frac{z}{0} = \frac{k}{6} \quad (k \neq 0)$$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad (k \neq 0)$$

The 2nd Eigen vector X_2 is obtained by corresponding to Eigen value by replacing $\lambda = 3$ in (a)

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x + y + 4z = 0$
$0x - y + 6z = 0$
$0x + 0y + 2z = 0$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (k \neq 0)$$

3rd Eigen vector, x^3 corresponding to Eigen value $\lambda = 5$ on ⓐ

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + 4z = 0$
$0x - 3y + 6z = 0$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad (k \neq 0)$$

→ Non Symmetric matrix with repeated Eigen values

Find the Eigen values and Eigen vectors of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let the characteristic eqn be $|A - \lambda I| = 0$

i.e $\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$

on solving,

$$2-\lambda\left[(2-\lambda)(1-\lambda)\right]-\left[(1-\lambda)\right]=0$$

$$(2-\lambda)(2-\lambda)(1-\lambda)-(1-\lambda)=0$$

$$1-\lambda\left((2-\lambda)^2-1\right)=0$$

$$\lambda=1,\ \lambda=1,\ \lambda=3$$

For 1st Eigen vector -

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x+y+z=0$$
$$x+y+z=0$$

[Unknowns #3, eqn 1]

∞ many soln

$$\cancel{x_1}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=k\left[\right.$$ (after 2nd vector)

For 2nd Eigen Vector - Put $\lambda = 3$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x+y+z=0$$
$$x-y+z=0$$

Let $y = k_1$ and $z = k_2$ $(k_1 \neq k_2 \neq 0)$

$\cancel{x} = k_1 + k_2$

$$x = \begin{bmatrix} -(k_1+k_2) \\ k_1 \\ k_2 \end{bmatrix}$$

$k_1 = 1$, $k_2 = 1$

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$k_1 = -1$, $k_2 = 2$

$$X_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

→ Symmetric matrix with non repeating Eigen values -

Que. Find the Eigen values & Eigen vector for the matrix -

$$A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$$

Characteristic Matrix $|A - \lambda I| = 0$

$$\begin{bmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{bmatrix} = 0$$

$$-2-\lambda\left[(7-\lambda)(-2-\lambda) - 25\right] + 4\left[25 - (7-\lambda)(4)\right]$$
$$- 5\left[25(-2-\lambda) - 20\right] = 0$$

$$(-2-\lambda)(-14+\lambda^2-25-9\lambda) + 4(25-28+4\lambda)$$
$$+ (5)(10+5\lambda+20) = 0$$

$\lambda = -3, -6, 12.$

For 1^{st} Eigen vector Put $\lambda = -3$ in

$$\begin{bmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 5y + 4z = 0$

$5x + 10y + 5z = 0$

$4x + 5y + z = 0$

By Elimination method-

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad (k \neq 0)$$

For 2^{nd} eigen vector Put $\lambda = -6$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad (k \neq 0)$$

For 3^{rd} Eigen vector $P\lambda = 12$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad (k \neq 0)$$

when orthogonal,

$X_1' X_2 = 0 \qquad [1 \ -1 \ 1] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \cancel{1} + 0 - 1 = 0$

$X_2' X_3 = 0$

$X_3' X_1 = 0$

But,

$X_1 X_2' \neq 0 \rightarrow$ It'll give 3×3 matrix.

→ Symmetric with repeating

Que. Find all the Eigen value & vectors of 'A' = $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Characteristic matrix $|A - \lambda I| = 0$

$$\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} = 0$$

$$2-\lambda\left[(2-\lambda)(2-\lambda)-1\right] + 1\left[(2-\lambda)+1\right] + 1\left[1-(2-\lambda)\right] = 0$$

$$2-\lambda\left(4+\lambda^2-4\lambda-1\right) + (3-\lambda) + (-1-\lambda) = 0$$

$$(2-\lambda)(\lambda^2-4\lambda+3) + 2-2\lambda = 0$$

$$2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda + 2 - 2\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 13\lambda + 8 = 0$$

$$\lambda^3 - 6\lambda^2 + 13\lambda - 8 = 0$$

$$\lambda = 1, 1, 4$$

For 1st eigen vector, Put $\lambda = 4$

In $\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x - 2y + z = 0$$
$$-x - 2y - z = 0$$
$$x - y - 2z = 0$$

By cross multiplication

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad (k \neq 0)$$

For 2nd Eigen vector : Put $\lambda = 1$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. Operation as:-

$R_1 \to R_1 + R_2$; $R_2 \to R_2 + R_3$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

operation as

$C_1 \to C_1 + C_2$;

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eqn formed.

$$x - y + z = 0$$

If $x = k_1$; $y = k_2$

then $z = k_1 - k_2$

$$X = \begin{bmatrix} k_1 \\ k_2 \\ k_1 - k_2 \end{bmatrix}$$

Put $k_1 = 1$; $k_2 = 1$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as X_3 is orthogonal to X_1 and X_2.

$$\therefore X_1' X_3 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l - m + n = 0$$

$$X_2' X_3 = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + 0n = 0$$

Cayley- Hamilton Theorem-

→ Every sq. matrix satisfies its own characteristic eq^n if $|A - \lambda I| = (-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n)$ is a characteristic poly^n of a matrix $A_{n \times n}$ then characteristic eq^n given by-

$$|A - \lambda I| = 0$$

i.e.

$$(-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots a_n) = 0$$

Acc. to Cayley - Hamilton theorem, eq ⓐ satisfies by A in the place of λ.

$$(-1)^n (A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots a_n) = 0$$

(Que - verify C-H T)

↳ By equating LHS & RHS

Find the characteristic eq^n of $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$.

and hence find A^{-1}.

Character eq^n -

$$|A - \lambda I| = 0$$

$$(-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots a_n) = 0$$

$$(-1) (\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3) = 0$$

$$(\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3) = 0$$

$$\lambda^3 - 6\lambda^2 + 6\lambda \cancel{+} - 11 = 0 \quad \rightarrow \text{eq } ⓐ$$

Acc. to C-H theorem,

has to be satisfied by A.

i.e. $A^3 - 6A^2 + 6A - 11 = 0 \quad \longrightarrow$ eq ⓑ

For the A^{-1} we multiply eq ⓑ both side by A^{-1}.

$$A^{-1}(A^3 - 6A^2 + 6A - 11) = 0$$

$$A^2 - 6A + 6I - 11A^{-1} = 0$$

$$A^{-1} = \frac{A^2 - 6A + 6I_{3\times3}}{11} \qquad \text{(Find } A^2 \text{ \& } A \text{ --)}$$

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & 2 \end{bmatrix}$$

Find the characteristic eqn of the matrix - $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. Verify C-H theorem - Hence prove that

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Ch. eqn -

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3I = 0 \rightarrow (a)$$

LHS. $\lambda = 1, 1, 3$

Now for RHS.

$$A^3 - 5A^2 + 7A - 3I = 0 \rightarrow (b)$$

$$0 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}^3 - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}^2 + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

[Solving RHS - (If equal to zero then CH ✓)]

$\boxed{\because |A| = 3}$

$$RHS = (3)^3 - 5(3)^2 + 7(3) - 3$$

$$= 27 - 5(9) + 7(3) - 3$$

$$= 48 - 48$$

$$= 0$$

$$= LHS$$

Hence, CH theorem ✓.

Now multiply A^5 in eq (b)

$$A^5(A^3 - 5A^2 + 7A - 3I) = 0$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 = 0$$

$$A^8 - 5A^7 + 7A^6 + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

Complex Matrix -

Alingment of complex numbers in rows and columns -

$[z = x + iy]$

Conjucate of a matrix - $[\bar{z} = x - iy]$ conjugate.

$$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 4+5i & 6+7i & -3i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 4-5i & 6-7i & 3i \end{bmatrix}$$

A^θ Matrix -

Transpose of a conjugate of a matrix 'A' is denoted by A^θ.

$$A^\theta = (\bar{A})' \quad [\,] = \begin{bmatrix} 1-i & 4+5i \\ 2+3i & 6- \\ 4 & 3i \end{bmatrix}$$

Unitary Matrix -

A sq matrix A is said to be unitry if

$A^\theta . A = I$

Que Prove that - $A = \begin{bmatrix} \frac{1+i}{2} & -\frac{1+1}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is an unitary matrix.

We need to prove -

$A^\theta \cdot A = I$.

Solving LHS.

$$A^{\theta} = (\bar{A})' = \begin{bmatrix} \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}'$$

$$= \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+j}{2} \end{bmatrix}$$

$$A^{\theta} A = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+j}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right) + \left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right) & \left(\frac{1-i}{2}\right)\left(\frac{-1+i}{2}\right) + \left(\frac{1-i}{2}\right)\left(\frac{1-i}{2}\right) \\ \left(\frac{-1-i}{2}\right)\left(\frac{1+i}{2}\right) + \left(\frac{1+i}{2}\right)\left(\frac{1+i}{2}\right) & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

= RHS hence proved.

Harmition Matrix.

A square matrix 'A' = $[a_{ij}]$ if every i, j element of 'A' is equal to the conjugate complex j, i element of '$\bar{A}$'.

i.e. $a_{ij} = \bar{a}_{ji}$.

Que. Check the given matrix is harmition matrix.

$$A = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

Finding conjugate -

$$\bar{A} = \begin{bmatrix} 1 & 2-3i & 3-i \\ 2+3i & 2 & 1+2i \\ 3+i & 1-2i & 5 \end{bmatrix}$$

$$\bar{a}_{ji} = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix} = a_{ij}$$

Skew - harmition - Matrix -

A sq. matrix 'A' whose elements are a_{ij} will be called a skew harmition matrix if every $a_{i,j}$ element of A is equal to the -ve conjugate complex j,i element of A

i.e. ① $a_{ij} = -\bar{a}_{ji}$

② All the elements in the principal diagonal is either zero or purely imaginary.

Que - $A = \begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$ check - if skew harraition or not -

$$\bar{A} = \begin{bmatrix} -i & 2+3i & 4-5i \\ -2+3i & 0 & -2i \\ -4-5i & -2i & +3i \end{bmatrix}$$

$\because -\bar{a}_{ji} = a_{ij}$

$\therefore$ skew harraition