UNIT - 5

Q. if $r = (t^3 + t^2 + t)\hat{i} + (t^2 + t)\hat{j} + (t+1)\hat{k}$ find $\frac{dr}{dt}$ & $\frac{d^2r}{dt^2}$

Sol. $\frac{dr}{dt} = (3t^2 + 2t)\hat{i} + (2t)\hat{j} + \hat{k}$

$\frac{d^2r}{dt^2} = (6t + 2)\hat{i} + 2\hat{j}$ Ans

Q A partical moving along. $x = 4\cos t$, $y = 4\sin t$, $z = 6t$ find valocity & acc at time $= 0$ & $t = \pi/2$.

Sol we know that

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 6t\hat{k}$

$\therefore \frac{d\vec{r}}{dt} = -4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 6\hat{k}$

put $t = 0$

$\left.\frac{d\vec{r}}{dt}\right]_{t=0} = 4\hat{j} + 6\hat{k}$

$V = \sqrt{16+36} =$

$V = 2\sqrt{13}$

& Put $t = \pi/2$

$\left.\frac{d\vec{r}}{dt}\right]_{t=\pi/2} = -4\hat{i} + 6\hat{k}$

$V = \sqrt{16+36}$

$V = 2\sqrt{3}$

$\therefore \frac{d^2r}{dt^2} = -4\cos t\,\hat{i} + 4\sin t\,\hat{j}$

put $t = 0$

$\frac{d^2r}{dt^2} = -4\hat{i}$

$a = \sqrt{16}$

$a = 4$

Put $t = \pi/2$

$\frac{d^2r}{dt^2} = 4\hat{j}$

$a = \sqrt{16}$

$a = 4$

Q $r = (2x^2y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + x^2\cos y\,\hat{k}$, then prove $\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial^2 r}{\partial y \partial x}$.

Sol $\frac{\partial r}{\partial y} = 2x^2\hat{i} + (e^{xy}.x - \sin x)\hat{j} - x^2\sin y\,\hat{k}$

$\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial y}\right) = 4x\hat{i} + (ye^{xy}.x + e^{xy} - \cos x)\hat{j} - 2\sin y\, x\,\hat{k}$

$(e^{xy}.y - y(\cos x))\hat{j} + 2(\cos y x)\hat{k}$

$$\frac{\partial r}{\partial x} = (4xy - 4x^3)\hat{i} + (e^{xy}.y - y(\cos x))\hat{j} - 2x\sin y\,\hat{k}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial r}{\partial x}\right) = 4x\hat{i} + (ye^{xy}.x + e^{xy} - \cos x)\hat{j}$$

no

Vector differentiation operator

The operator ∇ (delta or nevla) is known as vector differentiation operator is defined as.

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Gradient of scalar point func.

Let f be a scalar point func then, gradient of f is denoted by ∇f & it is defined as

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)f$$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Q find gradient of scalar point func $\phi(x,y,z) = x^2+y^2-z$ at the point $(1,2,5)$.

Sol. Given $\phi = x^2+y^2-z$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-z)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2-z) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2-z) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2-z)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

put $(x,y,z) = (1,2,5)$, then

$$\nabla\phi = 2\hat{i} + 4\hat{j} - \hat{k} \quad \text{Ans.}$$

Q

Sol

Q If $r = x\hat{i} + y\hat{j} + z\hat{k}$, then show gradient $\vec{r} = \hat{r}$

Sol. Given, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r^2 = x^2 + y^2 + z^2 \quad \text{—(i)}$$

$$\because\ 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

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now, Gradient of $\gamma = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\gamma$

$$\nabla\gamma = \left(\hat{i}\frac{\partial\gamma}{\partial x} + \hat{j}\frac{\partial\gamma}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= \left(\frac{x}{\gamma}\hat{i} + \frac{y}{\gamma}\hat{j} + \frac{z}{\gamma}\hat{k}\right)$$

$$= \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\gamma}\right)$$

$$\boxed{\nabla\gamma = \hat{\gamma}} \quad \text{H.P.}$$

Q If $\vec{\gamma} = x\hat{i} + y\hat{j} + z\hat{k}$ show that gradient $\cdot\, \gamma^n = n\gamma^{n-2}\gamma$

Sol. Given, $\vec{\gamma} = x\hat{i} + y\hat{j} + z\hat{k}$.

~~partial diff of γ^n w.r.t x, y, z, then~~

$$\cancel{n\gamma^{n-1}}\left(\frac{\partial}{\partial x}(\gamma^n)\hat{i} + \frac{\partial}{\partial y}(\gamma^n)\hat{j} + \frac{\partial}{\partial z}(\gamma^n)\hat{k}\right.$$

$$= n\gamma^{n-1}\frac{\partial\gamma}{\partial x}\hat{i} + n\gamma^{n-1}\frac{\partial}{\partial y}(\gamma)\hat{j} + n\gamma^{n-1}\frac{\partial}{\partial z}(\gamma)\hat{k}$$

$$= n\gamma^{n-1}\frac{x}{\gamma}\hat{i} + n\gamma^{n-1}\frac{y}{\gamma}\hat{j} + n\gamma^{n-1}\frac{z}{\gamma}$$

$$= n\gamma^{n-2}x\hat{i} + n\gamma^{n-1}y\hat{j} + n\gamma^{n-2}z\hat{j}.$$

$$= n\gamma^{n-2}(x\hat{i} + y\hat{j} + z\hat{j})$$

$$\boxed{\nabla\gamma^n = n\gamma^{n-2}\vec{\gamma}} \quad \text{H.P}$$

$$\left\{\begin{aligned} \nabla\gamma^n &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\gamma^n \\ \nabla\gamma^n &= \left(\frac{\partial(\gamma^n)}{\partial x}\hat{i} + \frac{\partial}{\partial y}(\gamma^n)\hat{j} + \frac{\partial}{\partial z}(\gamma^n)\hat{k}\right) \end{aligned}\right.$$

Q find directional derivative of $\phi = xy + yz + zx$ on the direction of the vector $\vec{n} = \hat{i} + 2\hat{j} + 2\hat{k}$ at the point $(1,1,2)$

Sol. Directional derivative = (gradient of ϕ at point) $\hat{n}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = (y+z)\hat{i} + (z+x)\hat{j} + (y+x)\hat{k}$$

$$\nabla\phi = 2\hat{i} + \hat{j} + 3\hat{k}$$

$$DD = 2\hat{i} + \hat{j} + 3\hat{k} \cdot \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

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$DD = \frac{1}{3}(2+2+6) = \frac{10}{3}$ Ans

• Directional Derivative

for any scalar point function at any point $P(x, y)$ that is $x\hat{i}+y\hat{j}+z\hat{k}$ is the direction of $\vec{P}$ is defined as

$$\text{Directional Derivative} = (\text{gradiet at point } P) \cdot \frac{\vec{N}}{|\vec{N}|}$$

Q find the directional derivatève of function $\phi = x^2 - y^2 + 2z^2$ at the point $(1,2,3)$ in the direction of line $\vec{pq}$ when q has coordinate $(5,0,4)$.

Sol $\phi = x^2 - y^2 + 2z^2$

$\vec{P} = \hat{i} + 2\hat{j} + 3\hat{k}$ $\quad \vec{q} = 5\hat{i} + 4\hat{k}$

$\vec{Pq} = \cancel{\hat{i} + 2\hat{j} + 3\hat{k} + x\hat{t}} \quad 4\hat{i} - 2\hat{j} + \hat{k}$

and $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$= 2\hat{i} - 4\hat{j} + 12\hat{k}$

then, $D.D = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$

$D.D = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$

• Divergence -

Divergence of a vector point func.

Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and vector point function defined in a certain field then

$$\nabla \cdot \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$\text{Div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

which is a scalar point function called Divergence of f

when divergence of $f = 0$ the f is said to be solenoidal vector.

- Curl of a vector point function.

Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any vector point function of defined as a certain field that,

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

If $\nabla \times F = 0$, then vector is irrotational.

Q Show that the vector $\nabla(x^3 + y^3 + z^3 - 3xyz)$ is irrotational also find divergence of function.

Sol.
$$\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$$
$$\vec{F} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= (-3x + 3x)\hat{i} - (3y - 3y)\hat{j} + (-3z + 3z)\hat{k}$$

$\text{curl } \vec{F} = 0$, then

$\vec{F}$ is irrotational.

and
$$\text{Div } \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 6x + 6y + 6z$$ Ans

Q Prove that the vector $\vec{V} = 8y^2z^2\hat{i} + 4x^2z^2\hat{j} - 3x^2y^2\hat{k}$ is a solenoid vector. & find curl of $\vec{V}$

Sol.
$$\text{Div } \vec{V} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 0 + 0 + 0$$

then, $\vec{V}$ is solenoid vector.

$$\text{curl } \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(3y^2z^2\hat{i} + 4x^2z^2\hat{j} - 3x^2y^2\hat{k}\right)$$

and

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y^2z^2 & 4x^2z^2 & 3x^2y^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(3x^2y^2) - \frac{\partial}{\partial z}(4x^2z^2)\right)\hat{i} - \left(\frac{\partial}{\partial x}(3x^2y^2) - \frac{\partial}{\partial z}(3y^2z^2)\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}(4x^2z^2) - \frac{\partial}{\partial y}(3y^2z^2)\right)\hat{k}$$

$$= (-6x^2y - 8zx^2)\hat{i} - (6xy^2 - 6y^2z)\hat{j} + (8xz^2 - 6yz^2)\hat{k}$$

Q prove that divergence of $\hat{r} = \frac{2}{r}$ & curl of $\hat{r} = 0$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol. Given, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

now,

$$\text{Div } \hat{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \hat{r}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \left(\frac{\sqrt{x^2+y^2+z^2} - x \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x}{x^2+y^2+z^2}\right) + \left(\frac{\sqrt{x^2+y^2+z^2} - y \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x}{x^2+y^2+z^2}\right)$$

$$+ \left(\frac{\sqrt{x^2+y^2+z^2} - z \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2z}{x^2+y^2+z^2}\right)$$

$$= \frac{x^2+y^2+z^2 - x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2 - y^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2 - z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{y^2+z^2 + x^2+z^2 + x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2}{|\vec{r}|}$$ Hence, proved.

and $\quad \text{curl } \hat{r} = \nabla \times \hat{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) \hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \hat{k}$$

$$\cancel{= \left(\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}} - \left(\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} \right) \right) \hat{i} - \left(\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}} - \left(\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} \right) \right) \hat{j}}$$

$$\cancel{+ \left(\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} - \left(\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} \right) \right) \hat{k}}$$

$$= \left(z \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2y - y \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2z \right) \hat{i} + \ldots$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= 0$$

$\therefore \quad \text{curl } \hat{r} = 0 \qquad$ Hence, proved.