

UNIT - 4

Probability and Statistics.

\Rightarrow Probability \Leftarrow

If an event 'A' can happen in m ways and fail in n ways all these ways are equally likely to occur then the probability of the happening of 'A' is,

$$P(A) = \frac{\text{no. of fav. cases}}{\text{total no. of mutually exclusive & equally likely cases}}$$

$$P(A) = \frac{n_A}{n}$$

Probability is measure of success or failure of an event if occurs.

\Rightarrow From a pack of 52 cards one is drawn at random. Find the probability of getting a king.

Given, A = taking king from 52 cards.
 no. of fav. outcomes = 4
 total no. of mutually exclusive &
 equally likely outcomes = 52.

So,

$$P(A) = \frac{\text{no. of fav. cases}}{\text{total no. of mutually exclusive & equally likely cases}} \\ = \frac{4}{52} = \frac{1}{13}.$$

\Rightarrow Addition law of probability \Leftarrow

If A & B are two events associated with an experiment then,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Prepare its perform proof by your own.

\rightarrow Multiplication law of probability \Rightarrow If there are two independent events, the respective probability of which are known, then the probability that both will happen is the product of probabilities of their happening respectively.

$$\text{i.e., } P(A \cap B) = P(A)P(B)$$

\Rightarrow conditional probability

Let 'A' and 'B' be two events of a sample space. Probability of 'B' given that 'A' has occurred is called conditional probability of event 'B' given 'A' denoted by, $P(A|B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

\Rightarrow Baye's theorem
If B_1, B_2, \dots, B_n are mutually exclusive events with $P(B_i) \neq 0$, ($i = 1, 2, 3, \dots, n$) of random experiment, then for any arbitrary event of A of sample space of the above experiment, we have $P(A) > 0$

$$P(A|B_i) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^n P(B_i)P(A|B_i)}$$

prepare for proof by your own.

\Rightarrow A bag contains 4 white & 2 black balls. and a second bag contains 3 of each colour. A bag is selected at random and a ball is drawn from it from chosen bag. What is probability that the ball drawn is white.

(i) Given, event A = bag 1 is chosen

event B = bag 2 is chosen.

$$P(A) = \frac{1}{2}$$

$$\begin{aligned} P(\text{not defective}) &= \frac{91 \times 95}{100 \times 100} \\ &= 8.6 \times 10^3 \times 10^{-4} \\ &= 8.64 \times 10^{-1} = 0.8645 \end{aligned}$$

Now, let E_1 = white ball chosen from bag 1 and E_2 = white ball chosen from bag 2.

So, E_1 = white ball chosen.

So, $P(E_1) = \frac{4}{6} = \frac{2}{3} = \text{white ball in bag 1}$

$$\begin{aligned} P(E_2) &= \frac{3}{6} = \frac{1}{2} = \text{white ball in bag 2} \\ P(E_1 \cap E_2) &= \frac{2}{6} = \frac{1}{3} = \text{white ball in both bags} \end{aligned}$$

\Rightarrow An article manufactured by a company consists of two parts 'A' and 'B'. In the process of manufacture of part A, 3 out of 100 are likely to be defective similarly 5 out of 100 are likely to be defective in manufacture of part B. Calculate the probability that the assembled article not be defective, i.e., of finding no part A, non-defective and part B is independent.

(ii) Given, let E_1 = event non-defective part of A.

E_2 = non-defective part of B.

$$P(E_1) = 1 - P(\text{defective of A})$$

$$P(E_2) = 1 - P(\text{defective of B})$$

$$P(E_1) = 1 - \frac{9}{100} \left(\frac{\text{defective of A}}{\text{total part}} \right)$$

$$P(E_2) = \frac{81}{100}$$

$$P(E_1 \cap E_2) = 1 - \frac{81}{100} = \frac{19}{100}$$

Q.3. An urn containing 2 balls, 1 of which are red, 3 blue & 4 black. If 3 balls are drawn from the urn at random what is the probability that:

- (1) The three balls are of different colours!
- (2) The three balls are of same colour!

S)

$$\text{Urn} \begin{array}{|c|c|} \hline & 2 \text{ Red} \\ \hline & 3 \text{ Blue} \\ \hline & 4 \text{ Black} \\ \hline \end{array}$$

case I = $\frac{2}{9} * \frac{3}{8} * \frac{4}{7} =$

$$= \frac{12}{8 * 7} = \frac{1}{42} = \frac{1}{2}$$

$$= 0.48.$$

$$\text{Case II} = \frac{3}{9} * \frac{2}{8} * \frac{1}{7} + \frac{4}{9} * \frac{3}{8} * \frac{2}{7}$$

$$= \frac{6+24}{9*8*7} = \frac{30}{504} = \frac{5}{84}.$$

$$= \frac{5}{84} = 0.06.$$

Q. An urn first contains three white & 4 red balls & on urn 2nd contains 5 other & 6 red balls. One ball is drawn at random from one of the urns and is found to be white. find the probability that it was drawn from urn 1.

Given, Urn 1 Urn 2 $\rightarrow \frac{1}{2}$

$\frac{3}{7} \leftarrow 3 \text{ white}$ $\frac{5}{11} \leftarrow 6 \text{ red}$

$$= \frac{\frac{3}{7} * \frac{3}{11}}{\frac{3}{7} + \frac{5}{11}} = \frac{9}{28} = 0.49$$

Q. An insurance company insured 2000 motorists who can claim to have truck drivers.

The probability of accidents are 0.09, 0.03 & 0.15 respectively. One of the insure persons make with an accident. What is the probability that he is a scooter driver?

Total no. of insur = 2000 + 4000 = 6000

$$P(\text{Scooter ins}) = \frac{2}{6} = \frac{1}{3} = 0.33$$

$$P(\text{Car}) = \frac{4}{12} = \frac{1}{3} = 0.33$$

$$P(\text{Truck}) = \frac{1}{2} = 0.50$$

$$= \frac{1}{6} * \frac{4}{10} = \frac{1}{15} = 0.0666\overline{6}$$

$$= \frac{1}{60} = 0.0166\overline{6}$$

$$= \frac{1}{60} + \frac{1}{10} + \frac{15}{200} = 0.075$$

$$= 0.095$$

$$= 0.095$$

$$= 0.095$$

$$= 0.095$$

$$= 0.095$$

$$= 0.095$$

$$= 0.095$$

$$= \frac{1}{6} + \frac{1}{12} + \frac{3}{4}$$

$$= \frac{1}{6}$$

$$= \frac{1+2}{12} = \frac{3}{12} = \frac{1}{4}$$

$$= \frac{1}{6}$$

Q. A win contains 6P, 4B; 4R, 6B; 5P, 5B. One of the wins is selected at random. If a ball is drawn from it, find the probability that it is drawn from first win.

$$\text{Sol} \rightarrow = \frac{1}{3} \times \frac{6}{10} + \frac{1}{3} \times \frac{4}{10} + \frac{1}{3} \times \frac{5}{10}$$

$$= \frac{6}{30} + \frac{4}{30} + \frac{5}{30} = \frac{15}{30} = \frac{1}{2}$$

\Rightarrow Correlation \Leftarrow

Two variables x and y are said to be correlated if an increase in one is accompanied by an increase or decrease in the other than the variables are said to be correlated. E.g.: The yield of crop varies with amount of rainfall.

Types of correlation \Rightarrow

① Positive Correlation If an increase in the value of X , results in the corresponding increase in the value of other variable Y on an average or if a decrease in value of one variable X , results in the corresponding decrease in the value of Y on an average. Then the correlation is said to be positive correlation.

② Negative Correlation If an increase in the value of one variable X results in a decrease in the value of other variable Y or if the value of one variable X increases in the \uparrow of one or two values of Y . The correlation said to be negative.

⑤ Linear Correlation \Rightarrow All the plotted points lie in a straight line when the correlation is said to be linear.

④ Perfect Correlation If two variables vary in such a way that their ratio always remains constant then the correlation is said to be perfect.

③ Karl - Pearson's coefficient of correlation
Let x_1, x_2, \dots, x_n be two variables x & y is defined by,

$$r = \frac{\sum xy}{\sqrt{\sum (x^2)(y^2)}} \rightarrow \text{Coefficient of correlation}$$

$$r = \frac{p}{\sqrt{\text{var}(x)\text{var}(y)}} \quad p = \text{covariance}$$

x_i, y_i are deviation measured by their mean.

$$x_i = x - \bar{x}, \quad y_i = y - \bar{y}$$

\Rightarrow It always equals ± 1 including $+1 - 1$.

Q. 10 Students got the following percentage of marks in Economics & Statistics.

Roll No.	Maths in Economics	Maths in Statistics
1	78	84
2	36	51
3	38	60
4	25	68
5	75	62
6	82	86
7	30	58
8	62	53
9	65	47
10	39	59

Q) Let the marks of two subjects be denoted by x & y respectively.

The mean marks for x marks is \bar{x}

$$\bar{x} = \frac{\sum x}{n} = 65$$

The mean marks for y marks is \bar{y}

$$\bar{y} = \frac{\sum y}{n} = 66.$$

If x & y are deviations of X & Y from their respective mean then the data may be arranged in the following form:

P.NO.	Marks in x	Marks in y	$x = x - \bar{x}$	$y = y - \bar{y}$	xy	x^2	y^2
			x	y	xy	x^2	y^2
1	78	84	13	18	234	169	324
2	36	51	-29	-15	435	841	225
3	98	91	33	25	825	961	625
4	25	60	-40	-6	240	1600	36
5	75	68	10	2	140	5625	4
6	82	62	17	-4	-68	676	16
7	90	86	25	20	500	8100	400
8	62	58	-3	-8	24	36	64
9	65	53	0	-13	0	4225	169
10	39	47	-26	-19	741	1521	361
					2384	3388	104

Q) Let the marks of two subjects be denoted by x & y .

$$\bar{x} = \frac{\sum x}{n} =$$

$$= \frac{25+30+32+35+37+40+42+44}{8} =$$

$$\bar{y} = \frac{\sum y}{n} = \frac{286}{8} = 35.75$$

$$= 17.75$$

Q) Calculate the coefficient of correlation between the marks obtained by 8 students in mathematics & statistics.

S. Student	Marks	Maths	Statistics	$x = x - \bar{x}$	$y = y - \bar{y}$	xy	x^2	y^2
A	25	8	10	-10.75	-5.75	-55	100	30.25
B	30	10	15	-5.75	-7.5	-82.5	225	56.25
C	32	15	17	-3.75	-7.5	-28.125	100	56.25
D	35	17	17	-0.75	-0.75	1.25	25	0.5625
E	37	20	23	1.25	2.25	2.75	49	5.0625
F	40	23	24	2.25	5.25	12	169	27.0625
G	42	24	25	6.25	6.25	31.25	144	39.0625
H	45	25	25	9.25	7.25	68.125	225	51.0625
Total.	286	142	142	0	0	0	0	0

$$S_9 = \frac{\sum xy}{\sqrt{\sum x^2} \sqrt{\sum y^2}} = \frac{2384}{\sqrt{3388} \sqrt{104}} = \frac{2384}{73.47 \cdot 10.2} = 0.78$$

$$S_9 = \frac{\sum xy}{\sqrt{\sum x^2} \sqrt{\sum y^2}} = \frac{2384}{\sqrt{3388} \sqrt{104}} = 0.78$$

\Rightarrow Binomial Distribution

Let there be 'n' independent trials in an experiment. Let the random variable X denote the number of success in these n trials. Let p be the probability of success in a single trial. So that of failure in a single trial will be $q = 1 - p$. Now let us find the constant for every trial. Let us find the probability of r success in n trials. i.e. the success can be obtained in n ways.

$$P(X=r) = {}^n C_r P(S)S^r(1-P(S))^{n-r}$$

$$\begin{aligned} &= {}^n C_r \underbrace{P(S) \cdots P(S)}_{r \text{ times}} \underbrace{P(1-P) \cdots P(1-P)}_{(n-r) \text{ times}} \\ &= {}^n C_r P(S)^r (1-P)^{n-r} \end{aligned}$$

\downarrow
in factors.

$$\begin{aligned} &= {}^n C_r \underbrace{p \cdot p \cdots p}_{r \text{ factors}} \underbrace{q \cdot q \cdots q}_{(n-r) \text{ factors}} \\ &= n! / (r!(n-r)!)$$

\downarrow
a-factors
(n-r) factors

$$P(X=r) = {}^n C_r p^r q^{n-r} \quad r=0, 1, 2, 3, \dots, n.$$

$$\boxed{P(X=r) = {}^n C_r p^r q^{n-r}}$$

$$\boxed{p+q=1}$$

Note \Rightarrow The successive probabilities p^n for n for

$$n=0, 1, 2, \dots, n. \Rightarrow p^0, p^1, p^2, \dots, p^n$$

which are the successive terms of the binomial expansion of $(p+q)^n$ therefore this distribution is called binomial probability distribution.

Prove that in Binomial P.D. mean is np and variance is npq respectively.

$$P(X)= {}^n C_r p^r q^{n-r}, \quad r=0, 1, \dots, n.$$

$$\text{Mean} \Rightarrow \mu = \sum_{r=0}^n r P(X) \quad \therefore \mu = \sum_{r=0}^n r {}^n C_r p^r q^{n-r}$$

$$\begin{aligned} &= 0 + [{}^n C_1 p^1 q^{n-1}] + {}^n C_2 p^2 q^{n-2} + \dots \\ &= np \left[q^{n-1} + \frac{n(n-1)p^2 q^{n-2}}{2} + \frac{(n(n-1)(n-2))}{6} p^3 q^{n-3} + \dots \right] \end{aligned}$$

$$\begin{aligned} &= np \left[q^{n-1} + \frac{npq^{n-2}}{2} + \frac{np^2 q^{n-3}}{6} + \dots \right. \\ &\quad \left. + \dots + np^n \right] \end{aligned}$$

$$\begin{aligned} &= np \left[q + npq + \frac{np^2 q^2}{2} + \dots \right. \\ &\quad \left. + \dots + np^n \right] \end{aligned}$$

$$\begin{aligned} &= np [q + np]^n \\ &= np x 1 = \\ &= np \quad \square \end{aligned}$$

$$\boxed{\mu = np}$$

Hence proved.

\Rightarrow Variance is denoted by σ^2

$$\sigma^2 = \sum_{r=0}^n r^2 P(X) - \mu^2$$

$$\begin{aligned} &= \sum_{r=0}^n [r^2 + 2r(r-1)] P(X) - \mu^2 \\ &= \sum_{r=0}^n r^2 P(X) + \sum_{r=0}^n 2r(r-1) P(X) - \mu^2 \\ &= np + \sum_{r=0}^n r(r-1) P(X) - \mu^2 \end{aligned}$$

Here,

$$\begin{aligned}A_n &= \sum_{k=0}^n \Pr_{\text{wk}}^{(k)} \Pr_{\text{bus}}^{(n-k)} \\&= \cancel{\Pr_0} + 0 + \frac{\Pr_1(n-1) p q}{2!} + \frac{\Pr_2(n-2) p^2 q^2}{3!} + \dots + \frac{\Pr_{n-1}(n-1) p^n}{n!} \\&= n(n-1) \left[p^2 q^{n-2} + (n-2) p^3 q^{n-3} + \dots + (n-1) p^{n-1} \right] \\&= n(n-1) \left[p^2 q^{n-2} + (n-2)(n-1)p^2 q^{n-3} + \dots + (n-1)p^{n-1} \right] \\&= n p^2 \left[\sum_{k=1}^{n-1} (k+1) \right] \\&= n p^2 \left[(n-1) p q^{n-1} + (n-1)(n-2) p^2 q^{n-2} + \dots + (n-1)p^{n-1} \right] \\&= n p q \left[(n-1) p q^{n-1} + (n-1)(n-2) p^2 q^{n-2} + \dots + (n-1)p^{n-1} \right] \\&= n p q \cdot \left[\sum_{k=0}^{n-1} (k+1) \right]\end{aligned}$$

(1) Assume that on the average one telephone number out of 15 called busy 2pm & 3pm on week days i.e. randomly selected telephone num. called if not more than 5. at least three of them will be busy.

(2) P_i be the probability of a telephone number being busy b/w 2pm & 3pm. on week days which is

$$= \frac{1}{15} \quad \boxed{P = \frac{1}{15}}$$

Let be the probability of a telephone number being not busy b/w 2pm & 3pm on week days i.e.

$$\therefore \frac{14}{15} = \frac{14}{15}.$$

So,

$$\boxed{q = \frac{14}{15}}$$

$$p = \frac{1}{15}, \quad q = \frac{14}{15}, \quad n = 6$$

(1) This implies that the probability that not more than 3 will be busy.

$$\text{on } \leq 3$$

$$n = 0, 1, 2, 3.$$

Now,

$$P(\text{bus}) = P(0) + P(1) + P(2) + P(3)$$

$$\begin{aligned}P(0) &= {}^6C_0 \left(\frac{1}{15}\right)^0 \left(\frac{14}{15}\right)^6 + {}^6C_1 \left(\frac{1}{15}\right)^1 \left(\frac{14}{15}\right)^5 \\&\quad + {}^6C_2 \left(\frac{1}{15}\right)^2 \left(\frac{14}{15}\right)^4 + {}^6C_3 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3\end{aligned}$$

$P(0) =$

$$\frac{(14)^3}{15^6} \left[{}^6C_0 (14)^3 + {}^6C_1 (14)^2 \cdot 1 + {}^6C_2 \cdot 1^2 + {}^6C_3 \cdot 1^3 \right]$$

$$= \left[(1.14)^3 + 3(1.14)^2 + 3(1.14) + 1.14 \right] \frac{(1.14)^3}{(1.15)^3}$$

$$= 4150 \times \frac{(1.14)^3}{(1.15)^3}$$

$$= \frac{4150 \times 2.744}{(1.15)^6} = \frac{11383600}{13310625} = 0.8557.$$

\therefore

$$P(x) = \frac{1}{10}, \quad q = \frac{9}{10}, \quad n = 10, \quad x = 1$$

(0.005)

(10)

$$P(x) = P(3) + P(4) + P(5) + P(6)$$

$$= C_3 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3 + C_4 \left(\frac{1}{15}\right)^4 \left(\frac{14}{15}\right)^4$$

$$+ C_5 \left(\frac{1}{15}\right)^5 \left(\frac{14}{15}\right)^5 + C_6 \left(\frac{1}{15}\right)^6$$

$$P(x) = \frac{20(1.14)^3 15(1.14)^2 + 6(1.14)^4 + 1}{15^6}$$

We know that in BPD, $P(x) = n! q^x p^{n-x}$

$$P(x) = 20 \times 2.744 + 15 \times 1.96 + 1$$

15^6

$$\boxed{P(x) = 0.0050}$$

$$P(x) = \frac{n(n-1)\dots(n-(x-1))}{x!} q^{n-x} p^x$$

$$[np = \lambda \Rightarrow p = \frac{\lambda}{n}]$$

$$P(x) = \frac{n!}{x!(n-x)!} \left(1 - \frac{1}{n}\right)^{n-x} \left(\frac{1}{n}\right)^x$$

$$P(x) = \frac{\lambda^x}{x!} \left\{ \frac{n(n-1)\dots(n-(x-1))}{x!} \left(1 - \frac{1}{n}\right)^{n-x} \left(\frac{1}{n}\right)^x \right\}$$

$$P(n) = \frac{\lambda^n}{n!} \left\{ \binom{n}{n} \left(\frac{\lambda^{n-1}}{n} \right)^1 + \cdots + \binom{n-1}{n-1} \left(\frac{\lambda^{n-1}}{n-1} \right)^1 + \cdots + \binom{n-1}{1} \left(\frac{\lambda^{n-1}}{1} \right)^1 \right\}$$

$$= e^{-\lambda} \left[1 + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots \right]$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}$$

$$P(n) = \frac{\lambda^n}{n!} \left\{ 1 + \left(1 - \frac{1}{n} \right) \left(1 - \frac{2}{n} \right) \cdots \left(1 - \frac{(n-1)}{n} \right) \right\}$$

$$= \lambda e^{-\lambda} \left[1 + 1 + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$= \lambda e^{-\lambda} e^\lambda$$

$$\text{As } \lim_{n \rightarrow \infty} n! \cdot \left(1 - \frac{1}{n} \right)^{n-1} \rightarrow 1$$

$$\text{variance } \sigma^2 = \sum_{n=0}^{\infty} n^2 P(n) - \mu^2$$

$$P(n) = \frac{\lambda^n}{n!} \left[(\lambda - \lambda n) \cdot n! \right]^{-1}$$

$$= \sum_{n=0}^{\infty} n^2 P(n) - \sum_{n=0}^{\infty} n^2 P(n) [1 - P(n)]^2$$

$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{where } n = 0, 1, 2, \dots$$

$$= \sum_{n=0}^{\infty} n^2 P(n) - \lambda^2$$

$$\int P(n) = \frac{\lambda^n e^{-\lambda}}{n!}$$

$$= \sum_{n=0}^{\infty} n^2 \frac{e^{-\lambda} \lambda^n}{n!} - \lambda^2$$

prove that in poisson prob. distribution

mean & variance both are equals to λ .

we know that no success in poisson prob. dist. given

$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

$$= e^{-\lambda} \left[1 + 1 + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$\text{mean } \mu = \sum_{n=0}^{\infty} n P(n)$$

$$= \sum_{n=0}^{\infty} n \frac{\lambda^n e^{-\lambda}}{n!}$$

$$= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{(n-1)!}$$

$$\boxed{\lambda^2 = \lambda}$$

Q. Assume that the probability of an individual coal miner will killed in a mine accident during a year

$\frac{1}{2400}$. Use poisson distribution to calculate

The probability that in a mine employing 200 miners their will be at least 1 fatal accident in a year

(1) Given, $P(\text{miner killed}) = \frac{1}{2400}$

P be the probability of an individual Coal

miner being killed = $\frac{1}{2400}$

No. of coal miners = 200

$$\lambda = np = \frac{1}{2400} \times 200 = \frac{1}{12} = 0.083$$

So,
 $P(n) = \frac{\lambda^n}{n!} e^{-\lambda}$

$P(\text{at least one fatal accident}) = 1 - P(\text{NO fatal accident})$

$$\begin{aligned} &= 1 - P(0) \\ &= 1 - \frac{1^0}{0!} e^{-0.083} \\ &= 1 - e^{-0.083} \\ &= 1 - 0.9203 \\ &= 0.079 \\ &= 0.08. \end{aligned}$$

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

($a_0 \neq 0$), a_1, a_2, \dots, a_n .

$f(x) = 0 \text{ OR}$.

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$$

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Unit-V

Theory of Equations

The expression $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$

where a_i 's are constants ($a_0 \neq 0$) & n is a positive integer,
is called a polynomial in x of degree n .

$f(x) = 0$ OR.

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

General Properties:-

- i. If α is a root of the eqⁿ $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x-\alpha)$.
- ii. Every equation of the n^{th} degree has n roots,
it means that roots may be real or imaginary.
- iii. Intermediate Value Property:- If $f(a) \neq f(b)$ have
different signs, then the eqⁿ $F(x) = 0$ has at least
one root between $x = a$ & $x = b$.
- iv. In an eqⁿ with real coefficients, imaginary
roots occur in conjugate pairs, i.e. $\alpha + i\beta$
is a root then $\alpha - i\beta$ must be another root.
Similarly, if $\alpha + \sqrt{b}$ is an irrational root of an
equation then $\alpha - \sqrt{b}$ must also be its root.
- v. Descartes's rule of signs:-

Relation's between Roots & Coefficients:-

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ will be the
roots of the eqⁿ $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n x^0 = 0$,

Then,

$$\text{sum of the roots} = \sum \alpha_i = -\frac{a_1}{a_0}$$

$$\text{sum of product of two roots} = \sum d_1 d_2 = \frac{a_2}{a_0}$$

$$\text{sum of product of roots} = \sum d_1 d_2 d_3 = -\frac{a_3}{a_0}$$

OR

$$\text{Product of roots} = d_1 d_2 \dots d_n = (-1)^n \frac{a_n}{a_0}$$

Examples

1. Solve the equation $2x^3 + x^2 - 13x + 6 = 0$

Soln. Given that :- $2x^3 + x^2 - 13x + 6 = 0$ - (1)

If we want to apply general properties,
By trial method. $x=2$ satisfies eqⁿ (1)

So, $(x-2)$ must be factor

$$\begin{array}{r} x-2 \sqrt{2x^3 + x^2 - 13x + 6} \\ \underline{-2x^3 - 4x^2} \\ 5x^2 - 13x + 6 \\ \underline{-5x^2 - 10x} \\ -3x + 6 \\ \underline{-(-3x + 6)} \\ 0 \end{array}$$

$$\text{Now, } 2x^2 + 5x - 3 = 0$$

$$\Rightarrow 2x^2 + 6x - x - 3 = 0$$

$$\Rightarrow (2x-1)(x+3) = 0$$

$$\Rightarrow x = 1/2, -3.$$

Hence, $x = 2, 1/2, -3$

OR

By synthetic division method,

By trial ↑		2	1	-13	6	{ More reliable when higher degree polynomial (x^5, x^6, \dots) }
			4	10	-6	
		2	5	-3	0	

$$(x-2)(2x^2 + 5x - 3) = 0$$

a. Solve the equation $3x^3 - 4x^2 + x + 88 = 0$ one root being $2 + i\sqrt{7}$.

Soluⁿ Given that :- $3x^3 - 4x^2 + x + 88 = 0 \dots (1)$

Since one root is $2 + i\sqrt{7} = x$

So, $x = 2 - i\sqrt{7}$ must be root of eqⁿ (1),

$\therefore x - 2 - i\sqrt{7}$ and $x - 2 + i\sqrt{7}$ are factors.

$$\therefore (x - 2 - i\sqrt{7})(x - 2 + i\sqrt{7})$$

$$= (x - 2)^2 + 7$$

$$= x^2 + 4 - 4x + 7$$

$= x^2 - 4x + 11$ must be a factor.

$$\begin{array}{r} \therefore 3x+8 \\ x^2 - 4x + 11 \) \underline{3x^3 - 4x^2 + x + 88} \\ \underline{3x^3 - 12x^2 + 33x} \\ 8x^2 - 32x + 88 \\ \underline{8x^2 - 32x + 88} \\ R \rightarrow 0 \end{array}$$

$$\therefore 3x + 8 = 0$$

$$\Rightarrow x = -8/3$$

Hence, the roots are $x = -8/3, 2 + i\sqrt{7}$ given, $2 - i\sqrt{7}$ very properly?

3. Solve the equation $x^3 - 7x^2 + 36x + 36 = 0$. Given one root. Given that :- $x^3 - 7x^2 + 36x + 36 = 0 \dots (1)$ root is double of another. Also, one root is double of another.

$$\therefore d = 2B$$

Let the roots be a, b, c .

$$\text{Sum of roots} = a + b + c = d + B + C = 3B + C = 7 \dots (2)$$

$$\text{Sum of product of roots} = dB + BC + CA = 36$$

$$2\beta^2 + \beta\gamma + 2\beta\gamma = 36 \quad \text{--- (2)}$$

$$\text{Product of roots} = \alpha\beta\gamma = -36 = 2\beta^2\gamma = -36 \quad \text{--- (3)}$$

On solving (2), (3), (4) we get

$$\gamma = \gamma - 3\beta$$

$$\Rightarrow 2\beta^2(\gamma - 3\beta) = -36$$

$$\Rightarrow 7\beta^2 - 3\beta^3 + 18 = 0$$

$$\Rightarrow 3\beta^3 - 7\beta^2 - 18 = 0$$

$$\Rightarrow (\beta - 3)\left(\beta = \left(-\frac{1}{3} + i\sqrt{17}\right)\right)\left(\beta = \left(-\frac{1}{3} - i\sqrt{17}\right)\right) = 0$$

$$\Rightarrow \beta = 3, -\frac{1}{3} + i\sqrt{17}, -\frac{1}{3} - i\sqrt{17} \quad \therefore \alpha = 6, 2\left(-\frac{1}{3} + i\sqrt{17}\right), \gamma = -2, 8 + i\sqrt{17},$$

4. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given

that the sum of two of its roots is zero.

Soln: Given that: $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0 \quad \text{--- (1)}$

Let the roots be $\alpha, \beta, \gamma, \delta$

then, $\alpha + \beta = 0$ {given}

$$\alpha + \beta + \gamma + \delta = 2 \Rightarrow \gamma + \delta = 2 \Rightarrow \gamma = 2 - \delta$$

$$\beta\gamma + \beta\delta + \gamma\delta + \gamma\delta + \delta\gamma = -4$$

$$(\alpha + \beta)(\gamma + \delta) + (\beta\gamma + \gamma\delta + \delta\gamma) = 4$$

$$\alpha\beta + \gamma\delta = 4 \quad x^4 - 2x^3 + x^2 + px^2$$

$$\text{OR} \quad -2px + pq$$

Quadratic factor corresponding to α, β is of the form

$x^2 - px + q$; & that corresponding to γ, δ

is of the form $x^2 - 2x + a$

$$\therefore (x^4 - 2x^3 + 4x^2 + 6x - 21) = (x^2 + p)(x^2 - 2x + a) \quad \text{--- (2)}$$

$$(x^4 - 2x^3 + 4x^2 + 6x - 21) = x^4 - 2x^3 + x^2(p+q) - 2px + pq$$

$$\therefore p+q = 4 \quad \& \quad p = -3 \quad \& \quad pq = -21$$

$$(p+3)q = +21$$

Hence the given eqn is eqn to $(x^2 - 3)(x^2 - 2x + 7) [q = 7]$

$$x = \pm\sqrt{3}, 1 \pm i\sqrt{6}$$

Solution of Cubic Equations by Cardan's Method:-

Suppose we have a cubic eqⁿ. $ax^3 + bx^2 + cx + d = 0 \dots (1)$

where a, b, c & d are constant coefficients.

Step:1 Identify coefficient of x^3 & divide whole eqⁿ by a , we get.

$$x^3 + lx^2 + mx + n = 0 \quad (l = -b/a, m = c/a, n = d/a)$$

Step:2 Remove x^2 term by putting $x = y - l/3$, we have

$$y^3 + py + q = 0 \dots (2)$$

Step:3 Now, do solve eqⁿ (2) by putting $y = u+v$ the cubing both sides,

$$y^3 = (u+v)^3$$

$$y^3 = u^3 + v^3 + 3uv(u+v)$$

$$y^3 - 3uv(u+v) - (u^3 + v^3) = 0 \dots (3)$$

Step:4 Compare eqⁿ (2) & (3).

$$uv = -p/3, u^3 + v^3 = -q$$

$$u^3v^3 = -p^3/27, u^3 + v^3 = -q$$

$$t^2 - qt - p^3/27 = 0$$

$$u^3 = \frac{1}{2}(-q + \sqrt{q^2 + 4p^3/27}) = \lambda^3$$

$$v^3 = \frac{1}{2}(-q - \sqrt{q^2 + 4p^3/27}) -$$

$$u \rightarrow \lambda, \lambda w, \lambda w^2 \quad \{w^3 = 1\}$$

$$v = -p/3\lambda, -pw^2/3\lambda, -pw/3\lambda$$

$$y \rightarrow \lambda - p/3\lambda, \lambda w - pw^2/3\lambda, \lambda w^2 - pw/3\lambda$$

Note: If one value of u is found to be rational number, find the corresponding value of v by $y = v+u$ then find the corresponding value of w finally divide left side of one by $(x-a)$ giving the remainder Q.E. from which the other two roots can be found.

8 Solution Of Cubic Equations By Cardan's Method:-
Suppose we have a cubic eqⁿ. $ax^3 + bx^2 + cx + d = 0$ - ①
where a, b, c & d are constant coefficients.

Step: 1 Identify coefficient of x^3 & divide whole eqⁿ
by a , we get.

$$x^3 + lx^2 + mx + n = 0 \quad (l = -b/a, m = c/a, n = d/a)$$

Step: 2 Remove x^2 term by putting $x = y - l/3$, we have

$$y^3 + py + q = 0 \quad \text{--- ②}$$

Step: 3 Now, do solve eqⁿ ② by putting $y = u+v$ the
cubing
both sides,

$$y^3 = (u+v)^3$$

$$y^3 = u^3 + v^3 + 3uv(u+v)$$

$$y^3 - 3uv(u+v) - (u^3 + v^3) = 0 \quad \text{--- ③}$$

Step: 4 Compare eqⁿ ② & ③

$$uv = -p/3, \quad u^3 + v^3 = -q$$

$$u^3v^3 = -p^3/27, \quad u^3 + v^3 = -q$$

$$t^2 - qt - p^3/27 = 0$$

$$u^3 = \frac{1}{2}(-q + \sqrt{q^2 + 4p^3/27}) = \lambda^3$$

$$v^3 = \frac{1}{2}(-q - \sqrt{q^2 + 4p^3/27}) =$$

$$u \rightarrow \lambda, \lambda\omega, \lambda\omega^2 \quad \{ \omega^3 = 1 \}$$

$$v = -p/3\lambda, -p\omega^2/3\lambda, -p\omega/3\lambda$$

$$y \rightarrow \lambda - p/3\lambda, \lambda\omega - p\omega^2/3\lambda, \lambda\omega^2 - p\omega/3\lambda$$

Now: If one value of u is found to be rational number, find the corresponding value of v by $y = v+u$ then find the corresponding finally divide left side of one by $(x-a)$ giving the remainder Q.E. from which the other two roots can be found.

2. If u^3, v^3 turn out to be conjugate complex no.'s the roots of the given cubic can be obtained in need form by applying De-Morgan's theorem.

Examples

1. Solve by Cardan's method $x^3 - 3x^2 + 12x + 16 = 0$

Soln. Given eqn : $x^3 - 3x^2 + 12x + 16 = 0 \quad \textcircled{1}$
 To remove x^2 term, we substitute $x = y - l$,
 $(y - \frac{l}{3})^3 - 3(y - \frac{l}{3})^2 + 12(y - \frac{l}{3}) + 16 = 0 \quad (l = -3)$

$$\Rightarrow (y+1)^3 - 3(y+1)^2 + 12(y+1) + 16 = 0$$

$$\Rightarrow y^3 + 9y^2 + 26 = 0 \quad \textcircled{2}$$

$$\Rightarrow y = u+v$$

$$\Rightarrow y^3 - 3uv.y - (u^3 + v^3) = 0 \quad \textcircled{3}$$

Compare \textcircled{2} & \textcircled{3}

We get, $uv = -3, u^3 + v^3 = -26$

$$u^3v^3 = -27, u^3 + v^3 = -26$$

$\therefore u^3, v^3$ are the roots of the eqn $t^2 + 26t - 27 = 0$

$$\therefore t^2 + 26t - 27 = 0$$

$$(t-1)(t+27) = 0$$

$$t = -27, 1$$

$$u^3 = -27 \quad \text{i.e., } u = -3 \quad \& \quad v^3 = 1, \text{i.e., } v = 1$$

$$y = u+v = -3+1 = -2 \quad \& \quad x = y+1 = -1$$

Dividing L.H.S of \textcircled{1} by $x+1$, we obtain

$$x^2 - 4x + 16$$

$$x+1 \overline{) x^3 - 3x^2 + 12x + 16}$$

$$\underline{x^3 + x^2}$$

$$-4x^2 + 12x + 16$$

$$\underline{-4x^2 - 4x}$$

$$16x + 16$$

$$\underline{16x + 16}$$

R → 0

$$x^2 - 4x + 16 = 0$$

$$\text{or } x = \frac{4 \pm \sqrt{16 - 64}}{2} = 2 \pm i\sqrt{12}$$

Hence the required roots of the given equation are $-2, 2 \pm i\sqrt{12}$.

Fuzzy Set Theory

0 Definitions:

i. Characteristic Function- Let X be an universal set. Let A be a subset of X . The characteristic function of A is denoted by $\chi_A(x)$:

For each x ,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Eg: Let the universal set X be the set of all non-negative real no's. If A is a set whose elements are $a = \{x : x \text{ is a real no. } 2 \leq x \leq 8\}$

The characteristic function of A is

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$$

ii. Membership Function- A funcⁿ similar to the characteristic funcⁿ is called a membership funcⁿ. The membership funcⁿ assigns to each element $x \in X$ a number, $M_A(x)$ in the closed interval $[0, 1]$ that characterize the degree of member of x in A . Thus membership funcⁿ are the functions of the form

$$M_A : A \rightarrow [0, 1]$$

If A, B are two sets $\{A \subseteq B\}$ are subsets of universal set \mathcal{X} then

$$\mu_{A \cup B}^U(x) = \max \{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A \cap B}(x) = \min \{\mu_A(x), \mu_B(x)\}$$

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x), \quad \forall x \in \mathcal{X}.$$

iii. Fuzzy Set :-

Let \mathcal{X} be a collection of objective (universal set)

Then a fuzzy A in \mathcal{X} is defined as a set of ordered pairs $A = \{(x, \mu_A(x)); x \in \mathcal{X}\}$

- Fuzzy set can also be denoted by
 $A = \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), \dots, (x_n, \mu_A(x_n))\}$.

→ Basic / Standard Operations on Fuzzy Set :-

- Complement of A Fuzzy Set - Let A be a fuzzy set be defined on a universal set \mathcal{X} . The complement of A is denoted by \bar{A} , A' or A^c .
 $\bar{A} = \{(x, \mu_{\bar{A}}(x)); x \in \mathcal{X}\}$
 $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$.

- Subset - Let A & B be two fuzzy set then
 $A \subseteq B$ iff and only if $\mu_A(x) \leq \mu_B(x), \forall x \in \mathcal{X}$.

iii) Union of two fuzzy sets - Let A & B be two fuzzy sets defined on a universal set X , the union of A & B is denoted by $A \cup B$ and defined as

$$A \cup B = \{(x, \mu_{A \cup B}(x)) : x \in X\},$$

where $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} \forall x \in X$.

iv) Intersection of Two Fuzzy Sets - Let A & B be two fuzzy sets defined on a universal set X , the intersection of A & B is denoted by $A \cap B$ defined as

$$A \cap B(x) = \{\mu_A(x), \mu_B(x)\} \forall x \in X.$$

v) Equality of Two Fuzzy Sets - If A & B are two fuzzy sets, then $A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \forall x \in X$.

Examples

1. Let $X = \{x_1, x_2, x_3, x_4\}$ and two fuzzy sets be
 $A = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.7), (x_4, 1)\}$
 $B = \{(x_1, 0.6), (x_2, 1), (x_3, 0.4), (x_4, 0.3)\}$. Then find $A \cup B$, $A \cap B$ & is $A \subseteq B$.

Soln: i. $A \cup B$; Let / Given A & B are two fuzzy sets defined on an universal set X , the union of A & B is defined as

$$\begin{aligned} A \cup B(x) &= \{(x, \mu_{A \cup B}(x)) : x \in X\}, \\ \mu_{A \cup B}(x) &= \max\{\mu_A(x), \mu_B(x)\}, \forall x \in X. \end{aligned}$$

Now,

$$\mu_{A \cup B}(x_1) = \max\{\mu_A(x_1), \mu_B(x_1)\} = \max\{0.2, 0.6\} = 0.6$$

$$\mu_{A \cup B}(x_2) = \max\{\mu_A(x_2), \mu_B(x_2)\} = \max\{0.5, 1\} = 1$$

$$\mu_{A \cup B}(x_3) = \max\{\mu_A(x_3), \mu_B(x_3)\} = \max\{0.7, 0.4\} = 0.7$$

$$\mu_{A \cup B}(x_4) = \max\{\mu_A(x_4), \mu_B(x_4)\} = \max\{1, 0.3\} = 1$$

$$\therefore A \cup B(x) = \{(x_1, 0.6), (x_2, 1), (x_3, 0.7), (x_4, 1)\}$$

iii. $A \cap B$; Given A & B are two fuzzy sets defined on an universal set X , the intersection of A & B is defined as,

$$A \cap B(x) = \{ (x, \mu_{A \cap B}(x)) : x \in X \}$$

where $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$, $\forall x \in X$

Now,

$$\mu_{A \cap B}(x_1) = \min\{\mu_A(x_1), \mu_B(x_1)\} = \min\{0.2, 0.6\} = 0.2$$

$$\mu_{A \cap B}(x_2) = \min\{\mu_A(x_2), \mu_B(x_2)\} = \min\{0.5, 1\} = 0.5$$

$$\mu_{A \cap B}(x_3) = \min\{\mu_A(x_3), \mu_B(x_3)\} = \min\{0.7, 0.4\} = 0.4$$

$$\mu_{A \cap B}(x_4) = \min\{\mu_A(x_4), \mu_B(x_4)\} = \min\{0.3, 1\} = 0.3$$

$$\therefore A \cap B(x) = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.4), (x_4, 0.3)\}$$

iii. Given A & B are two fuzzy sets, then $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ $\forall x \in X$.

$$\therefore \mu_A(x) = \mu_B(x)$$

$\therefore A$ is a subset of B ($A \subseteq B$).

o De-Morgan's Law:-

If A & B are two fuzzy sets, then de-Morgan law states that,

$$i. (A \cup B)' = A' \cap B'$$

$$ii. (A \cap B)' = A' \cup B'$$

& Some Fundamental Properties of Fuzzy Sets:-

i. Identity Law:- a) $A \cap X = A$ and $A \cup \emptyset = A$
 b) $A \cup X = X$ and $A \cap \emptyset = \emptyset$

ii. Idempotent Law:- $A \cap A = A$, $A \cup A = A$

iii. Commutative Law:- $A \cup B = B \cup A$, $A \cap B = B \cap A$

iv. Distributive Law:- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

v. Transitive Law:- $A \subseteq B$, $B \subseteq C$, $A \subseteq C$

- Algebraic Properties Of Fuzzy Sets :-
- i) Algebraic sum :- The algebraic sum of two fuzzy set $A \otimes B$ is denoted by $A + B$ & defined as

$$A + B = \{(x, \mu_{A+B}(x)) : x \in X\},$$

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \forall x \in X.$$
 - ii) Algebraic Product: The algebraic sum of two fuzzy set $A \otimes B$ is denoted by $A \cdot B$ & defined as

$$A \cdot B = \{(x, \mu_{A \cdot B}(x)) : x \in X\}$$

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x), \forall x \in X.$$