$f(x) = a_0x^n + a_1x^{n-1} + \dots$
$(a_0 \neq 0), a_1, a_2, \dots a_n$
$f(x) = 0$ OR.
$a_0x^n + a_1x^{n-1} + \dots + a_n = 0$

PAGE NO.:
DATE: / /

Unit-V
Theory Of Equations

The expression $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ where a's are constants $(a_0 \neq 0)$ & n is a positive integer, is called a polynomial in x of degree n.

OR.

$f(x) = 0$

$\Rightarrow a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$

General Properties:-

i. If α is a root of the eq^n $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x-\alpha)$.

ii. Every equation of the n^{th} degree has n roots. it means that roots may be real or imaginary.

iii. Intermediate Value Property:- If $f(a)$ & $f(b)$ have different signs, then the eq^n $F(x)=0$ has atleast one root between $x = a$ & $x = b$.

iv. In an eq^n with real coefficients, imaginary roots occur in conjugate pairs, i.e. $\alpha + i\beta$ is a root then $\alpha - i\beta$ must be another root. Similarly, if $a + \sqrt{b}$ is an irrational root of an equation then $a - \sqrt{b}$ must also be its root.

v. Descarte's rule of signs:-

Relation's between Roots & Coefficients:-

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ ~~with~~ be the roots of the eq^n $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots a_nx^0 = 0$.

Then,

sum of the roots $= \Sigma\alpha_1 = \dfrac{-a_1}{a_0}$

Sum of product of two roots $= \sum \alpha_1\alpha_2 = \frac{a_2}{a_0}$

Sum of product of roots (3 at a time) $= \sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$

~~OR~~

Product of roots $= \alpha_1\alpha_2 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$

Examples

1. Solve the equation $2x^3+x^2-13x+6=0$

Solⁿ. Given that:- $2x^3+x^2-13x+6=0$ — ①

If we want to apply general properties,
By trial method. $x=2$ satisfies eqⁿ ①
So, $(x-2)$ must be factor
Then,

$$x-2 \,\big)\, 2x^3+x^2-13x+6 \,\big(\, 2x^2+5x-3$$
$$\underline{2x^3-4x^2}$$
$$5x^2-13x+6$$
$$\underline{5x^2-10x}$$
$$-3x+6$$
$$\underline{-3x+6}$$
$$0$$

Now, $2x^2+5x-3=0$

$\Rightarrow 2x^2+6x-x-3=0$

$\Rightarrow (2x-1)(x+3)=0$

$\Rightarrow x = 1/2, -3.$

Hence, $x = 2, 1/2, -3$

OR

By synthetic division method,

By trial ↑ 2	2	1	-13	6
		4	10	-6
	2	5	-3	0

} More reliable when higher degree polynomial ($x^5, x^6, \ldots$)

$(x-2)(2x^2+5x-3)=0$

2. Solve the equation $3x^3 - 4x^2 + x + 88 = 0$ one root being $2 + i\sqrt{7}$.

Solⁿ: Given that:- $3x^3 - 4x^2 + x + 88 = 0$ — ①

Since one root is $2 + i\sqrt{7} = x$

So, $x = 2 - i\sqrt{7}$ must be root of eqⁿ ①.

$\therefore$ $x - 2 - i\sqrt{7}$ and $x - 2 + i\sqrt{7}$ are factors.

$\therefore$ $(x - 2 - i\sqrt{7})(x - 2 + i\sqrt{7})$

$= (x-2)^2 + 7$

$= x^2 + 4 - 4x + 7$

$= x^2 - 4x + 11$ must be a factor.

$\therefore$

$$\begin{array}{r} 3x + 8 \\ x^2 - 4x + 11 \overline{) \; 3x^3 - 4x^2 + x + 88} \\ \underline{3x^3 - 12x^2 + 33x} \\ 8x^2 - 32x + 88 \\ \underline{8x^2 - 32x + 88} \\ R \to 0 \end{array}$$

$\therefore$ $3x + 8 = 0$

$\Rightarrow x = -8/3$

Hence, the roots are $x = -8/3$, $2 + i\sqrt{7}$ {given},
$2 - i\sqrt{7}$ {by property}

3. Solve the equation $x^3 - 7x^2 + 36x + 36 = 0$. given one root is double of another.

Solⁿ: Given that:- $x^3 - 7x^2 + 36x + 36 = 0$ — ①

Also, one root is double of another.

$\therefore$ $\alpha = 2\beta$

Let the roots be α, β & γ.

Sum of roots $= 7 = \alpha + \beta + \gamma = 3\beta + \gamma = 7$ — ②

Sum of product of roots $= \alpha\beta + \beta\gamma + \gamma\alpha = 36$

$2\beta^2 + \beta\gamma + 2\beta\gamma = 36 \quad -(2)$

Product of roots $= \alpha\beta\gamma = -36 = 2\beta^2\gamma = -36 \quad -(3)$

On solving (2), (3), (4) we get

$\gamma = 7 - 3\beta$

$\Rightarrow 2\beta^2(7-3\beta) = -36$

$\Rightarrow 7\beta^2 - 3\beta^3 + 18 = 0$

$\Rightarrow 3\beta^3 - 7\beta^2 - 18 = 0$

$\Rightarrow (\beta-3)\left(\beta - \left(\frac{-1}{3} + i\frac{\sqrt{17}}{3}\right)\right)\left(\beta - \left(\frac{-1}{3} - i\frac{\sqrt{17}}{3}\right)\right) = 0$

$\Rightarrow \beta = 3, \frac{-1}{3} \pm i\frac{\sqrt{17}}{3}$

$\therefore \alpha = 6, 2\left(\frac{-1}{3} \pm i\frac{\sqrt{17}}{3}\right), \gamma = -2, 8 \mp i\sqrt{17},$

4. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given that the sum of two of its roots is zero.

Soln. Given that :- $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0 \quad -(1)$

Let the roots be $\alpha, \beta, \gamma, \delta$

Then, $\alpha + \beta = 0$ {given}

$\alpha + \beta + \gamma + \delta = 2 \Rightarrow \gamma + \delta = 2 \Rightarrow \gamma = 2 - \delta$

$\beta\delta + \alpha\gamma + \alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha = \cancel{6}\ 4$

$\cancel{(\alpha+\beta)}(\gamma+\delta) + (\alpha\beta + \gamma\delta) = 4$

$\alpha\beta + \gamma\delta = 4$

$x^4 - 2x^3 + x^2 q + px^2 - 2px + pq$

OR

Quadratic factor corresponding to α, β is of the form $x^2 - 0x + p$; & that corresponding to γ, δ is of the form $x^2 - 2x + q$

$\therefore (x^4 - 2x^3 + 4x^2 + 6x - 21) = (x^2 + p)(x^2 - 2x + q) \quad -(2)$

$(x^4 - 2x^3 + 4x^2 + 6x - 21) = x^4 - 2x^3 + x^2(p+q) - 2px + pq$

$\therefore p + q = 4$ & $p = -3$ & $pq = -21$

$(-3)q = -21$

$\boxed{q = 7}$

Hence the given eqn is eq. to $(x^2 - 3)(x^2 - 2x + 7) = 0$

$x = \pm\sqrt{3}, 1 \pm i\sqrt{6}$

PAGE NO.:
DATE: / /

Solution of Cubic Equations by Cardan's Method:-

Suppose we have a cubic eqn. $ax^3+bx^2+cx+d=0$ —①
where a, b, c & d are constant coefficients.

Step:1. Identify coefficient of x^3 & divide whole eqn by a, we get.

$$x^3+lx^2+mx+n=0 \quad (l=b/a,\ m=c/a,\ n=d/a)$$

Step:2. Remove x^2 term by putting $x=y-l/3$, we have

$$y^3+py+q=0 \quad —②$$

Step:3 Now, to solve eqn ② by putting $y=u+v$ then cubing both sides,

$$y^3=(u+v)^3$$

$$y^3=u^3+v^3+3u.v.y$$

$$y^3-3uvy-(u^3+v^3)=0 \quad —③$$

Step:4 Compare eqn ② & ③

$$uv=-p/3, \quad u^3+v^3=-q$$

$$u^3v^3=-p^3/27, \quad u^3+v^3=-q$$

$$t^2-qt-p^3/27=0$$

$$u^3=\frac{1}{2}\left(-q+\sqrt{q^2+4p^3/27}\right)=\lambda^3$$

$$v^3=\frac{1}{2}\left(-q-\sqrt{q^2+4p^3/27}\right)$$

$$u \to \lambda, \lambda\omega, \lambda\omega^2 \qquad \{\omega^3=1\}$$

$$v=-p/3\lambda, -p\omega^2/3\lambda, -p\omega/3\lambda$$

$$y \to \lambda-p/3\lambda, \ \lambda\omega-p\omega^2/3\lambda, \ \lambda\omega^2-p\omega/3\lambda$$

Note:- If one value of u is found to be a rational number, find the corresponding value of v by $y=v+u$ then find the corresponding finally divide left side of one by $(x-\alpha)$ giving the remaining Q.E. from which the other two roots can be found.

Solution of Cubic Equations by Cardan's Method:-

Suppose we have a cubic eqn. $ax^3+bx^2+cx+d=0$ —①
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$$\Downarrow$$

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$$v^3=\frac{1}{2}\left(-q-\sqrt{q^2+4p^3/27}\right)-$$

$$u\to\lambda,\ \lambda\omega,\ \lambda\omega^2 \qquad \{\omega^3=1\}$$

$$v=-p/3\lambda,\ -p\omega^2/3\lambda,\ -p\omega/3\lambda$$

$$y\to\lambda-p/3\lambda,\ \lambda\omega-p\omega^2/3\lambda,\ \lambda\omega^2-p\omega/3\lambda$$

Note:- If one value of u is found to be a rational number, find the corresponding value of v by $y=v+u$ then find the corresponding finally divide left side of one by $(x-\alpha)$ giving the remaining Q.E. from which the other two roots can be found.

2. If u^3, v^3 turn out to be conjugate complex no.'s the roots of the given cubic can be obtained in need form by applying De-Morgan's Theoram.

Examples

1. Solve by Cardan's method $x^3-3x^2+12x+16=0$

Solⁿ. Given eqⁿ :- $x^3-3x^2+12x+16=0$ — (1)

To remove x^2 term, we substitute $x=y-l/3$

$$\left(y-\frac{l}{3}\right)^3-3\left(y-\frac{l}{3}\right)^2+12\left(y-\frac{l}{3}\right)+16=0 \quad (l=-3)$$

$\Rightarrow \quad (y+1)^3-3(y+1)^2+12(y+1)+16=0$

$\Rightarrow \quad y^3+9y+26=0$ — (2)

$\Rightarrow \quad y=u+v$

$\Rightarrow \quad y^3-3uv.y-(u^3+v^3)=0$ — (3)

compare (2) & (3)

we get, $uv=-3, \quad u^3+v^3=-26$

$u^3v^3=-27, \quad u^3+v^3=-26$

$\therefore u^3, v^3$ are the roots of the eqⁿ $t^2+26t-27=0$

$\therefore t^2+26t-27=0$

$(t-1)(t+27)=0$

$t=-27, 1$

$u^3=-27$ i.e., $u=-3$ & $v^3=1$, i.e. $v=1$

$y=u+v=-3+1=-2$ & $x=y+1=-1$

Dividing L.H.S of (i) by $x+1$, we obtain

$$\begin{array}{r} x^2-4x+16 \\ x+1 \overline{\big)\, x^3-3x^2+12x+16} \\ x^3+x^2 \\ \hline -4x^2+12x+16 \\ -4x^2-4x \\ + \quad + \\ \hline 16x+16 \\ 16x+16 \\ \hline R\to 0 \end{array}$$

$$x^2 - 4x + 16 = 0$$

or $x = \frac{4 \pm \sqrt{16-64}}{2} = 2 \pm i 2\sqrt{3}$

Hence the required roots of the given equatn are $-1, 2 \pm i2\sqrt{3}$.

Fuzzy Set Theory

o Definations:-

i. Characteristic Function- Let X be an Universal set & let A be a subset of X. The characteristic functn A of A is denoted by $\chi_A(x)$ & for each x,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \quad \forall x \in X$$

Eg: Let the universal set X be the set of all non-negative real no's & A is a set whose elements are $A = \{x : x \text{ is a real no. } \& \ 2 \leq x \leq 8\}$

The characteristic fnctn of A is

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases} \quad \forall x \in X$$

ii. Membership Function- A fnctn similar to the characteristic fnctn is called a membership fnctn. The membership fnctn assigns to each element $x \in X$ a number; $\mu_A(x)$ in the closed interval $[0, 1]$ that characterize the degree of membership of x in A. Thus membership functn are the functions of the form

$$\mu_A : X \rightarrow [0, 1]$$

PAGE NO.:
DATE:

If A, B are two sets {A & B are subsets of universal set} then

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$
$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$
$$\mu_{\bar{A}}(x) = 1 - \mu_A(x), \quad \forall x \in X.$$

iii. Fuzzy Set :-
Let X be a collection of objectsive (universal set) Then a fuzzy A in X is defined as a set of ordered pairs $A = \{(x, \mu_A(x)); x \in X\}$

- Fuzzy set can also be denoted by
$A = \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)) \ldots (x_n, \mu_A(x_n))\}$.

> Basic/Standard Operations on Fuzzy Set :-

i. Complement of A Fuzzy Set - Let A be a fuzzy set be defined on a Universal set X. The complement of A is denoted by $\bar{A}$, A' or A^c.

$$\bar{A} = \{(x, \mu_{\bar{A}}(x)); x \in X\}$$
$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

ii. Subset - Let A & B be two fuzzy set then $A \subseteq B$ iff and only if $\mu_A(x) \leq \mu_B(x), \forall x \in X$.

iii. Union of two fuzzy sets - Let A & B be two fuzzy sets defined on a universal set X, the union of A & B is denoted by $A \cup B$ and defined as

$$A \cup B = \{(x, \mu_{A\cup B}(x)) : x \in X\},$$

where $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}\ \forall\ x \in X$.

iv. Intersection of Two Fuzzy Sets - Let A & B be two fuzzy sets defined on a universal set X, the intersection of A & B is denoted by $A \cap B$ & defined as $A\cap B(x) = \{(x, \mu_{A\cap B}(x)) : x \in X\}$

where $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}\ \forall\ x \in X$.

v. Equality of Two Fuzzy Sets - If A & B are two fuzzy sets, then $A = B \Leftrightarrow \mu_A(x) = \mu_B(x),\ \forall\ x \in X$.

Examples

1. Let $X = \{x_1, x_2, x_3, x_4\}$ and two fuzzy sets be
$A = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.7), (x_4, 1)\}$
$B = \{(x_1, 0.6), (x_2, 1), (x_3, 0.4), (x_4, 0.3)\}$. Then find $A \cup B$, $A \cap B$ & is $A \subseteq B$.

Solⁿ - i. $A \cup B$; Let/Given A & B are two fuzzy sets defined on an universal set X, the union of A & B is defined as

$$A\cup B(x) = \{(x, \mu_{A\cup B}(x)) : x \in X\},$$

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\},\ \forall\ x \in X.$$

Now,

$$\mu_{A\cup B}(x_1) = \max\{\mu_A(x_1), \mu_B(x_1)\} = \max\{0.2, 0.6\} = 0.6$$

$$\mu_{A\cup B}(x_2) = \max\{\mu_A(x_2), \mu_B(x_2)\} = \max\{0.5, 1\} = 1$$

$$\mu_{A\cup B}(x_3) = \max\{\mu_A(x_3), \mu_B(x_3)\} = \max\{0.7, 0.4\} = 0.7$$

$$\mu_{A\cup B}(x_4) = \max\{\mu_A(x_4), \mu_B(x_4)\} = \max\{1, 0.3\} = 1$$

$$\therefore A\cup B(x) = \{(x_1, 0.6), (x_2, 1)(x_3, 0.7), (x_4, 1)\}$$

PAGE NO
DATE

ii. $A \cap B$; Given A & B are two fuzzy sets defined on an universal set X, the ~~union~~ intersection of A & B is defined as;

$$A \cap B(x) = \{(x, \mu_{A\cap B}(x)) : x \in X\}$$

where $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in X$

Now,

$\mu_{A\cap B}(x_1) = \min\{\mu_A(x_1), \mu_B(x_1)\} = \min\{0.2, 0.6\} = 0.2$

$\mu_{A\cap B}(x_2) = \min\{\mu_A(x_2), \mu_B(x_2)\} = \min\{0.5, 1\} = 0.5$

$\mu_{A\cap B}(x_3) = \min\{\mu_A(x_3), \mu_B(x_3)\} = \min\{0.7, 0.4\} = 0.4$

$\mu_{A\cap B}(x_4) = \min\{\mu_A(x_4), \mu_B(x_4)\} = \min\{0.3, 1\} = 0.3$

$\therefore A \cap B(x) = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.4), (x_4, 0.3)\}$

iii. Given A & B are two fuzzy set, then $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x), \forall x \in X$.

$\because \mu_A(x) = \mu_B(A)$

$\therefore A$ is a subset of B ($A \subseteq B$).

o De-Morgan's Law:-

If A & B are two fuzzy sets, then de-Morgan's law states that,

i. $(A \cup B)' = A' \cap B'$ ii. $(A \cap B)' = A' \cup B'$

& Some Fundamental Properties Of Fuzzy Sets:-

i. Identity Law:- (a) $A \cap X = A$ and $A \cup \emptyset = A$

(b) $A \cup X = X$ and $A \cap \emptyset = \emptyset$

ii. Idempotent Law:- $A \cap A = A$, $A \cup A = A$

iii. Commutative Law:- $A \cup B = B \cup A$, $A \cap B = B \cap A$

iv. Distributive Law:- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

v. Transitive Law:- $A \subseteq B$, $B \subseteq C$, $A \subseteq C$

* Algebraic Properties of Fuzzy Sets :-

i. Algebraic Sum :- The algebraic sum of two fuzzy set A & B is denoted by A + B & defined as

$A + B = \{ (x, \mu_{A+B}(x)) : x \in X \},$

$\mu(A+B(x)) = \mu_A(x) + \mu_B(x) - \mu_A(x) . \mu_B(x), \forall x \in X.$

ii. Algebraic Product : The algebraic ~~sum~~ product of two fuzzy set A & B is denoted by A . B & defined as

$A . B = \{ (x, \mu_{AB}(x)) . x \in X \}$

$\mu_{AB}(x) = \mu_A(x) . \mu_B(x), \forall x \in X.$