Characterizations of weak reverse Hölder inequalities on metric spaces

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AGENT Forum, 27 October 2021

The whole story

J. Kinnunen, E.-K. K. and C. Mudarra (2021): *Characterizations of weak reverse Hölder inequalities on metric measure spaces*, available at https://arxiv.org/abs/2107.05022

Warm-up: (Strong) RHI and Muckenhoupt A_∞ weights

(X, d, μ) complete metric space, where μ is doubling: $\mu(2B) \leq C_d \mu(B)$.
w is a *weight*, i. e. nonnegative locally integrable function on open sets $\Omega \subset X$.

w satisfies the reverse Hölder inequality (RHI), if there exist $p > 1$ and C such that for every ball B with $B \Subset \Omega$

$$\left(\fint_B w^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \leq C \fint_B w \, \mathrm{d}\mu. \tag{RHI}$$

Assuming (X, d, μ) satisfies the so-called annular decay property (Kinnunen–Shukla 2014), RHI is equivalent to $w \in A_\infty(\Omega)$: there exist $0 < \varepsilon, \eta < 1$ such that for any $B \subset \Omega$ and measurable set $A \subset B$,

$$\mu(A) \leq \varepsilon \mu(B) \quad \Longrightarrow \quad w(A) \leq \eta w(B)$$

...and a number of other characterizations.

Weak RHI and "weak A_∞" weights

w satisfies the weak reverse Hölder inequality (WRHI; Giaquinta–Modica 1979), if there exist $p > 1$ and C such that for every ball B with $2B \Subset \Omega$

$$\left(\fint_B w^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \leq C \fint_{2B} w \, \mathrm{d}\mu. \tag{WRHI}$$

Proposition

(WRHI) $\Leftrightarrow$ *There exist* $0 < \varepsilon, \eta$ *with* $\eta < C_d^{-5}$*, such that for any* B *with* $2B \Subset \Omega$ *and measurable set* $F \subset B$*,*

$$\mu(F) \leq \varepsilon \mu(B) \quad \Longrightarrow \quad w(F) \leq \eta w(2B)$$

…and a number of other characterizations (we show 10 in total: likely not exhaustive).

Weak A_∞ is not A_∞

- No extra assumptions on the space
- Allows for nondoubling weights
- Weights may vanish on a set of positive measure (i. e. truly nonnegative)
- The upper bound for η cannot be removed (example!)
- Not all conditions for A_∞ apply to the weak case (examples!)

Weak A_∞ is not A_∞: example on the Hölder side

Example

Let $w(x) = e^x$ in $\mathbb{R}$ with the Lebesgue measure. Then $w \in WRH_p(\mathbb{R})$ (in fact, for every $p > 1$). To see this, let $B = (a - r, a + r)$ be an interval. Then

$$\frac{\left(\fint_B w^p \right)^{\frac{1}{p}}}{\fint_{2B} w} \leq \frac{4re^{3r}}{e^{4r} - 1},$$

which is a bounded function of $r > 0$.

However, $w \notin RH_p(\mathbb{R})$ for any $p > 1$ (consider intervals $(-r, r)$ as $r \to \infty$).

Main theorem

Theorem

(X, d, μ) is a metric measure space with a doubling measure μ, $\Omega \subset X$ an open set, and w a weight on Ω. The following assertions are equivalent.

(a) *For every $\eta > 0$ there exists $\varepsilon > 0$ such that if B is a ball with $2B \Subset \Omega$ and $F \subset B$ is a measurable set, then $\mu(F) \leq \varepsilon\mu(B)$ implies that $w(F) \leq \eta w(2B)$.*

(b) *There exist $\eta, \varepsilon > 0$ with $\eta < C_d^{-5}$ such that if B is a ball with $2B \Subset \Omega$ and $F \subset B$ is a measurable set, then $\mu(F) \leq \varepsilon\mu(B)$ implies that $w(F) \leq \eta w(2B)$.*

(c) (Weak reverse Hölder inequality) *There exist $p > 1$ and a constant $C > 0$ such that for every ball B with $2B \Subset \Omega$, it holds that*

$$\fint_B w^p \,\mathrm{d}\mu \leq C \left(\fint_{2B} w \,\mathrm{d}\mu \right)^p .$$

※ (a) ⇔ (c) shown by Spadaro (2012) in $\mathbb{R}^n$.

Theorem (continued)

(d) (Quantitative nondoubling A_∞) *There exist constants* $C, \alpha > 0$ *such that for every ball* B *with* $2B \Subset \Omega$ *and every measurable set* $F \subset B$*, it holds that*

$$w(F) \leq C \left(\frac{\mu(F)}{\mu(B)} \right)^{\alpha} w(2B).$$

(e) (Fujii–Wilson condition; Anderson–Hytönen–Tapiola 2017) *There exists a constant* $C > 0$ *such that for every ball* B *with* $2B \Subset \Omega$

$$\int_B M(w\mathcal{X}_B) \, \mathrm{d}\mu \leq Cw(2B).$$

(f) *There exist* $\alpha, \beta > 0$ *with* $\beta < C_d^{-5}$ *such that for every ball* B *with* $2B \Subset \Omega$

$$w\left(B \cap \{w \geq \alpha w_{2B}\}\right) \leq \beta w(2B).$$

(g) *There exists* $C > 0$ *such that for every ball* B *with* $2B \Subset \Omega$

$$\int_B w \log^+ \left(\frac{w}{w_{2B}} \right) \mathrm{d}\mu \leq Cw(2B).$$

Theorem (end)

(h) (Sawyer 1981, in $\mathbb{R}^n$) *There exists a nondecreasing* $\phi : (0,\infty) \to (0,\infty)$ *with* $\phi(0^+) = 0$ *such that for every ball* $B \subset X$ *with* $2B \Subset \Omega$ *and every* $F \subset B$ *measurable,*

$$w(F) \le \phi\left(\frac{\mu(F)}{\mu(B)}\right) w(2B).$$

(i) (Sawyer 1981, in $\mathbb{R}^n$) *There exists a* $C > 0$ *such that for every ball* B *with* $11B \Subset \Omega$ *and every* $f \in \mathrm{BMO}(\Omega)$ *with* $\|f\|_{\mathrm{BMO}(\Omega)} \le 1$*,*

$$\int_B |f - f_B|\, w \,\mathrm{d}\mu \le Cw(2B).$$

(j) *There exist* $C > 0$ *and a nondecreasing* $\phi : (1,\infty) \to (0,\infty)$ *with* $\phi(\infty) = \infty$ *such that for every ball* B *with* $2B \Subset \Omega$

$$\int_{B\cap\{w>w_{2B}\}} w\, \phi\left(\frac{w}{w_{2B}}\right) \mathrm{d}\mu \le Cw(2B).$$

※ $f \in L^1_{\mathrm{loc}} \in \mathrm{BMO}$ iff $\sup_{B\subset\Omega} ⨍_B |f - f_B|\, \mathrm{d}\mu < \infty$; also $\mathrm{BMO} = \alpha \log(A_p)$, $p > 1$, $\alpha \ge 0$.

Weak A_∞ is not A_∞: case of the constant η

Example (Sawyer 1981, $n = 2$)

Let $X = \mathbb{R}^n$ with the Lebesgue measure, and $w = \mathcal{X}_S$, where $S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_n \leq 1\}$. If $Q \subset \mathbb{R}^n$ is a cube with $|Q \cap S| > 0$, then

$$\frac{w(Q)}{w(2Q)} = \frac{|Q \cap S|}{|2Q \cap S|} \leq \frac{1}{2^{n-1}}.$$

Thus for a measurable subset $F \subset Q$ we have $w(F) \leq 2^{1-n} w(2Q)$. However, $w \notin WRH_p(\mathbb{R}^n)$ for any $p > 1$. To see this, let Q be centered at the origin with side length $l(Q) \geq 2$, and $F = Q \cap S$:

$$\mu(F) = w(F) = |Q \cap S| = l(Q)^{n-1} \quad \text{and} \quad w(2Q) = (2l(Q))^{n-1}.$$

The quantitative nondoubling A_∞ condition would give constants $c, \alpha > 0$ such that for every $l(Q) \geq 2$

$$\frac{1}{2^{n-1}} = \frac{w(F)}{w(2Q)} \leq c \left(\frac{|F|}{|Q|}\right)^{\alpha} = \frac{c}{l(Q)^{\alpha}}.$$

Weak A_∞ is not A_∞: Characterizations that fail (1/2)

Claim

There exists a $C > 0$ such that for every B with $2B \Subset \Omega$

$$\fint_B w \,\mathrm{d}\mu \le C \exp\left(\fint_{2B} \log w \,\mathrm{d}\mu \right).$$

Let $w(x) = e^x$ on $\mathbb{R}$, shown to satisfy (WRHI). Assume that there exist $C > 0$, $\alpha \in \mathbb{R}$ such that for every interval $B \subset \mathbb{R}$,

$$\fint_B w \,\mathrm{d}x \le C \exp\left(\alpha \fint_{2B} \log w \,\mathrm{d}x \right).$$

Consider intervals $B = (-r, r)$ with $r > 0$ Then

$$\fint_B w \,\mathrm{d}x = \frac{e^r - e^{-r}}{2r}, \quad \fint_{2B} \log w \,\mathrm{d}x = \frac{1}{4r} \int_{-2r}^{2r} x \,\mathrm{d}x = 0.$$

This would mean $(e^r - e^{-r})\,(2r)^{-1} \le C \exp(0) = C$ for every $r > 0$, while the left-hand side is an unbounded function of r.

Weak A_∞ is not A_∞: Characterizations that fail (2/2)

Claim

There exist $0 < \alpha, \beta < 1$ *such that for every* B *with* $2B \Subset \Omega$

$$\mu\left(B \cap \{w \le \beta w_{2B}\}\right) \le \alpha \mu(B).$$

Again let $w(x) = e^x$ on $\mathbb{R}$. Assume that there exist $0 < \alpha, \beta < 1$ such that the above is true. We have

$$\beta \fint_{2B} w \,\mathrm{d}x = \frac{\beta}{4r}\left(e^{2r} - e^{-2r}\right)$$

which is an unbounded function of r, so as $r \to \infty$ the claim becomes

$$\left|B \cap \{w \le \beta w_{2B}\}\right| = |B| \le \alpha\,|B| < |B|\,.$$