Existence of parabolic minimizers to the total variation flow on metric measure spaces

Cintia Pacchiano Camacho
joint work with
Vito Buffa,
Michael Collins (University of Erlangen)

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Aalto University
School of Science

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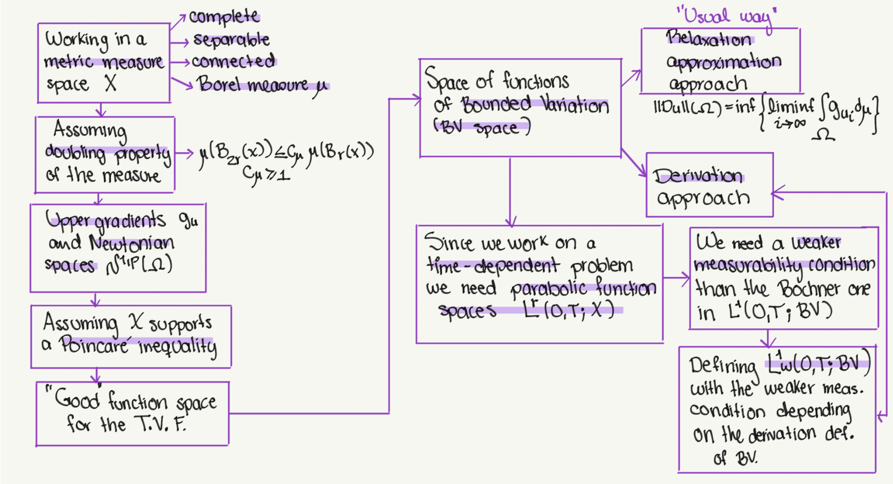
Setting



Working in a metric measure space X
complete
separable
connected
Borel measure μ
Assuming doubling property of the measure
$\mu(B_{2r}(x)) \leq C_\mu \mu(B_r(x))$
$C_\mu \geq 1$
Upper gradients g_u and Newtonian spaces $N^{1,p}(\Omega)$
Assuming X supports a Poincaré inequality
"Good" function space for the T.V.F.
Space of functions of Bounded Variation (BV space)
"Usual way"
Relaxation approximation approach
$\|Du\|(\Omega) = \inf\left\{ \liminf_{i\to\infty} \int_\Omega g_{u_i} d\mu \right\}$
Derivation approach
Since we work on a time-dependent problem we need parabolic function spaces $L^r(0,T;X)$
We need a weaker measurability condition than the Bochner one in $L^1(0,T;BV)$
Defining $L^1_w(0,T;BV)$ with the weaker meas. condition depending on the derivation def. of BV.

We assume Ω to be open and bounded, Ω^* open and bounded with $\Omega \Subset \Omega^*$ and

$$u_0 \in L^2(\Omega^*) \cap \mathrm{BV}(\Omega^*). \tag{0.1}$$

Definition 1

Assume that the Cauchy-Dirichlet datum u_0 fulfills (0.1). A map $u : \Omega_T^* \to \mathbb{R}$, $T \in (0, \infty)$ in the class

$$L^1_w\left(0, T; \mathrm{BV}_{u_0}(\Omega)\right) \cap C^0\left([0, T]; L^2(\Omega^*)\right)$$

will be referred to as a *variational solution* on Ω_T to the Cauchy-Dirichlet problem for the total variation flow if and only if the variational inequality

$$\begin{aligned} \int_0^T \|Du(t)\|(\Omega^*)\,\mathrm{d}t \le \int_0^T \left[\int_{\Omega^*} \partial_t v(v-u)\,\mathrm{d}\mu + \|Dv(t)\|(\Omega^*)\right] \mathrm{d}t \\ - \frac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega^*)} + \frac{1}{2}\|v(0)-u_0\|^2_{L^2(\Omega^*)} \end{aligned} \tag{0.2}$$

holds true for any $v \in L^1_w\left(0, T; \mathrm{BV}_{u_0}(\Omega)\right)$ with $\partial_t v \in L^2(\Omega_T^*)$ and $v(0) \in L^2(\Omega^*)$.

Main results

Theorem 2

Suppose that the Cauchy-Dirichlet datum u_0 fulfills the requirements of (0.1). Then, there exists a unique global variation solution in the sense of Definition 1.

Theorem 3

Suppose that the Cauchy-Dirichlet datum u_0 fulfills the requirements of (0.1). Then, any variational solution in the sense of Definition 1 on Ω_T with $T \in (0, \infty]$ satisfies

$$\partial_t u \in L^2(\Omega^*) \text{ and } u \in C^{0,\frac{1}{2}}\big([0,\tau]; L^2(\Omega^*)\big) \text{ for all } \tau \in \mathbb{R} \cap (0, T].$$

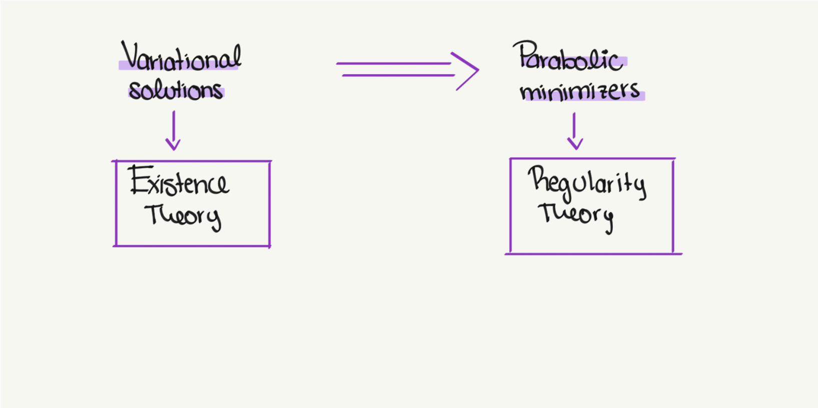
Furthermore, for the time derivative $\partial_t u$ there holds the quantitative bound

$$\int_0^T \int_{\Omega^*} |\partial_t u|^2 \, \mathrm{d}\mu \, \mathrm{d}t \leq \|Du_0\|(\Omega^*).$$

Finally, for any $t_1, t_2 \in \mathbb{R}$ with $0 \leq t_1 < t_2 \leq T$ one has the energy estimate

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \|Du(t)\|(\Omega^*) \, \mathrm{d}t \leq \|Du_0\|(\Omega^*). \tag{0.3}$$

What's next?


Variational solutions
Existence Theory
Parabolic minimizers
Regularity Theory

Thanks!

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