Navier–Stokes equations on Riemannian manifolds

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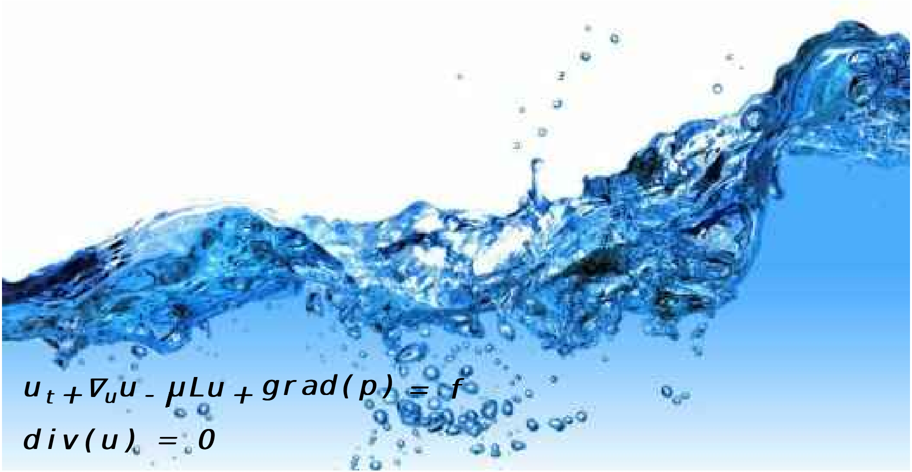

Diffusion operator in the Navier–Stokes equations

The standard way to write the Navier–Stokes equations in $\mathbb{R}^n$:

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\begin{aligned}
&u_t + u\nabla u - \mu\Delta u + \nabla p = f \\
&\nabla \cdot u = 0
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(iii) Linear equation: One dimensional $\rightarrow$ Stokes equation

$$\begin{aligned} &u_t - \Delta_B u + \operatorname{grad}(p) = f \\ &\nabla \cdot u = 0 \end{aligned}$$

Diffusion operator in the Navier–Stokes equations

The Navier–Stokes equations on Riemannian manifold (M, g) :

The nonlinear term: $(u\nabla u)^k = (\nabla_u u)^k = u^k_{;i} u^i$

The gradient: $(\mathrm{grad}(p))^i = g^{ij} p_{;j}$

The divergence: $\nabla \cdot u = \mathrm{div}(u) = \mathrm{tr}(\nabla u) = u^i_{;i}$

The diffusion term Δu:

(i) *Bochner Laplacian*:

$$(\Delta_B u)^k = \mathrm{div}(g^{ij} u^k_{;i}) = g^{ij} u^k_{;ij}$$

deformation rate tensor:

$$(Su)^{kj} = (\nabla u + (\nabla u)^T)^{kj} = g^{ki} u^j_{;i} + g^{ij} u^k_{;i}$$

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The diffusion operator $Lu = \mathrm{div}(Su)$.

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Lemma

$$Lu = \Delta_B u + \mathrm{grad}(\mathrm{div}(u)) + \mathrm{Ri}(u)$$

Diffusion operator in the Navier–Stokes equations

The Navier–Stokes equations on Riemannian manifold (M, g) :

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\begin{aligned}
&u_t + \nabla_u u - \mu L u + \mathrm{grad}(p) = f \\
&\mathrm{div}(u) = 0\,.
\end{aligned}
$$

Diffusion operator in the Navier–Stokes equations

The Navier–Stokes equations on Riemannian manifold (M, g) :

(ii) *Hodge Laplacian*:

$$L_A u = \mathrm{div}(Au) = \Delta_B u - \mathrm{grad}(\mathrm{div}(u)) - \mathrm{Ri}(u)$$

where

$$(Au)^{kj} = (\nabla u - (\nabla u)^T)^{kj} = g^{ki} u^j_{;i} - g^{ij} u^k_{;i}$$

Diffusion operator in the Navier–Stokes equations

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$$\Delta_H u = \operatorname{div}(Au) + \operatorname{grad}(\operatorname{div}(u)) = \Delta_B u - \operatorname{Ri}(u)$$

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$$\Delta_H u = \operatorname{div}(Au) + \operatorname{grad}(\operatorname{div}(u)) = \Delta_B u - \operatorname{Ri}(u)$$

By considering operator A in Navier–Stokes equations on a Riemannian manifold, Hodge Laplacian could construct the system of Navier–Stokes as follows:

$$\begin{aligned} & u_t + \nabla_u u - \mu \Delta_H u - 2\mu \operatorname{Ri}(u) + \operatorname{grad}(p) = f \\ & \operatorname{div}(u) = 0 \ . \end{aligned}$$

Vector fields and solutions for the Navier–Stokes equations

Definition

Vector field u is

Parallel: $\nabla u = 0$

Killing: $Su = \nabla u + (\nabla u)^T = 0 \quad \longrightarrow \quad \operatorname{div}(u) = \frac{1}{2}\operatorname{tr}(Su) = 0$

Harmonic: $\Delta_H u = \Delta_B u - \operatorname{Ri}(u) = 0 \quad \Leftrightarrow \quad \begin{cases} Au = \nabla u - (\nabla u)^T = 0 \\ \operatorname{div}(u) = 0 \end{cases}$

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(i) u parallel and p constant: (u, p) solution of homogeneous Navier–Stokes equations with Bochner Laplacian.

(ii) u Killing and $p = \frac{1}{2} g(u, u)$: (u, p) solution of homogeneous Navier–Stokes equations with diffusion term.

(iii) u harmonic and $p = -\frac{1}{2} g(u, u)$: (u, p) solution of homogeneous Navier–Stokes equations with Hodge Laplacian.

Killing vector fields as a solution to the Navier–Stokes system

$$
\begin{aligned}
&u_t + \nabla_u u - \mu L u + \mathrm{grad}(p) = 0 \\
&\mathrm{div}(u) = 0\,.
\end{aligned}
\tag{1}
$$

Theorem

(i) *u be a solution of (1) and v be Killing. Then*

$$\frac{d}{dt}\langle u, v\rangle = 0$$

(ii) *u be a solution of (1) with Hodge Laplacian and let v be harmonic. Then*

$$\frac{d}{dt}\langle u, v\rangle = 0$$

Killing vector fields as a solution to the Navier–Stokes system

$$u = u^K + u^\perp$$
$$p = p_K + p_\perp \,, \qquad p_K = \tfrac{1}{2}\, g(u^K, u^K)$$

Killing vector fields as a solution to the Navier–Stokes system

$$
\begin{aligned}
u &= u^K + u^\perp \\
p &= p_K + p_\perp\,, \qquad\qquad p_K = \tfrac{1}{2}\, g(u^K, u^K)
\end{aligned}
$$

$$
\begin{aligned}
& u^\perp_t + \nabla_{u^\perp} u^K + \nabla_{u^K} u^\perp + \nabla_{u^\perp} u^\perp - \mu\, L u^\perp + \mathrm{grad}(p_\perp) = 0 \\
& \mathrm{div}(u^\perp) = 0\,.
\end{aligned}
$$

Killing vector fields as a solution to the Navier–Stokes system

Theorem

Let $u^{\perp}$ be a solution of

$$
\begin{aligned}
& u^{\perp}_t + \nabla_{u^{\perp}} u^{K} + \nabla_{u^{K}} u^{\perp} + \nabla_{u^{\perp}} u^{\perp} - \mu\, L u^{\perp} + \operatorname{grad}(p_{\perp}) = 0 \\
& \operatorname{div}(u^{\perp}) = 0\,,
\end{aligned}
$$

then

$$\| u^{\perp} \|^2 \leq C\, e^{-\mu \alpha_K t}$$

Killing vector fields as a solution to the Navier–Stokes system

Theorem

Let $u^\perp$ be a solution of

$$u_t^\perp + \nabla_{u^\perp} u^K + \nabla_{u^K} u^\perp + \nabla_{u^\perp} u^\perp - \mu\, L u^\perp + \mathrm{grad}(p_\perp) = 0$$
$$\mathrm{div}(u^\perp) = 0\,,$$

then

$$\|u^\perp\|^2 \leq C\, e^{-\mu \alpha_K t}$$

$$V_K = \{u \in H^1(M) \mid \|u\| = 1\ ,\ \langle u, v\rangle = 0 \quad \text{for all Killing } v\}$$
$$\alpha_K = \inf_{u \in V_K} \int_M g(S_u, S_u)\, \omega_M \ : \quad S_u v = (g^{ki} u^j_{;i} g_{j\ell} + u^k_{;\ell})\, v^\ell$$

Linearized Navier–Stokes equations

$$
\begin{aligned}
& u_t + \nabla_u v + \nabla_v u - \mu L u + \mathrm{grad}(p) = 0 \\
& \mathrm{div}(u) = 0\,.
\end{aligned} \tag{2}
$$

Linearized Navier–Stokes equations

$$\begin{aligned} &u_t + \nabla_u v + \nabla_v u - \mu L u + \operatorname{grad}(p) = 0 \\ &\operatorname{div}(u) = 0\,. \end{aligned} \tag{2}$$

Theorem

Let u be a solution of (2) and w be Killing. Then

$$\frac{d}{dt}\langle u, w\rangle = 0$$

Linearized Navier–Stokes equations

$u = u^K + u^\perp$
$v = v^K + v^\perp$
$p = p_K + p_\perp = g(u, v) + p_\perp$
$f = -\nabla_{u^K} v^\perp - \nabla_{v^\perp} u^K$

Linearized Navier–Stokes equations

$u = u^K + u^\perp$
$v = v^K + v^\perp$
$p = p_K + p_\perp = g(u, v) + p_\perp$
$f = -\nabla_{u^K} v^\perp - \nabla_{v^\perp} u^K$

$$\begin{aligned} & u_t^\perp + \nabla_{v^K} u^\perp + \nabla_{u^\perp} v^K + \nabla_{u^\perp} v^\perp + \nabla_{v^\perp} u^\perp - \mu L u^\perp + \operatorname{grad}(p_\perp) = f \\ & \operatorname{div}(u^\perp) = 0\,. \end{aligned} \tag{3}$$

Linearized Navier–Stokes equations

$$
\begin{aligned}
u &= u^K + u^\perp \\
v &= v^K + v^\perp \\
p &= p_K + p_\perp = g(u,v) + p_\perp \\
f &= -\nabla_{u^K} v^\perp - \nabla_{v^\perp} u^K
\end{aligned}
$$

$$
\begin{aligned}
&u_t^\perp + \nabla_{v^K} u^\perp + \nabla_{u^\perp} v^K + \nabla_{u^\perp} v^\perp + \nabla_{v^\perp} u^\perp - \mu L u^\perp + \mathrm{grad}(p_\perp) = f \\
&\mathrm{div}(u^\perp) = 0\,.
\end{aligned}
\tag{3}
$$

Theorem

Let $u^\perp$ be a solution of (3). Then

$$\frac{d}{dt}\|u^\perp\|^2 \le -\mu\alpha_K\|u^\perp\|^2 + 2\langle f, u^\perp\rangle - 2\langle \nabla_{u^\perp} v^\perp, u^\perp\rangle$$

New solutions from old

$$u_t + \nabla_u v + \nabla_v u - \mu L u + \operatorname{grad}(p) = 0$$
$$\operatorname{div}(u) = 0\,.$$

Theorem

Let (u, p) be a solution of above system where we suppose that v is Killing. Then

$$(\hat{u}, \hat{p}) = \Big([u, v], -g\big(\operatorname{grad}(p), v\big)\Big)$$

is also a solution of the linearized Navier–Stokes equations.

Claim 1. $[\operatorname{grad}(p), v] = \operatorname{grad}(\hat{p})$.
Claim 2. $\nabla_v [u, v] = [\nabla_v u, v]$.
Claim 3. $\nabla_{[u,v]} v = [\nabla_u v, v]$.
Claim 4. $\Delta_B [u, v] = [\Delta_B u, v]$.
Claim 5. $\mathrm{Ri}[u, v] = [\mathrm{Ri}(u), v]$.

Two–dimensional manifold as an atmospheric models

$$
\begin{aligned}
&u_t + \nabla_u u - \mu\Delta_B u - \mu\kappa\, u + \mathrm{grad}(p) = 0 \\
&-\Delta p - \mathrm{tr}((\nabla u)^2) - \kappa\, g(u,u) + 2\,\mu\, g(\mathrm{grad}(\kappa), u) = 0 \\
&\mathrm{div}(u) = 0\,.
\end{aligned} \tag{4}
$$

Two–dimensional manifold as an atmospheric models

$$\begin{aligned} & u_t + \nabla_u u - \mu \Delta_B u - \mu\kappa\, u + \operatorname{grad}(p) = 0 \\ & -\Delta p - \operatorname{tr}((\nabla u)^2) - \kappa\, g(u,u) + 2\,\mu\, g(\operatorname{grad}(\kappa), u) = 0 \\ & \operatorname{div}(u) = 0\,. \end{aligned} \tag{4}$$

Theorem

If u is the solution of (4) then

$$\zeta_t - \mu\Delta\zeta + g(\operatorname{grad}(\zeta), u) - 2\mu\, g(\operatorname{grad}(\kappa), Ku) - 2\mu\,\kappa\zeta = 0 \tag{5}$$

where the vorticity is:

$$\zeta = \operatorname{rot}(u) = \operatorname{div}(Ku) = \varepsilon^i_\ell u^\ell_{;i}$$

Let ζ be the solution of (5) on the sphere; then

$$\tfrac{d}{dt}\|\zeta\|^2 \leq 0$$

Two–dimensional manifold as an atmospheric models

$u = u^K + u^\perp$
$\zeta = \zeta^K + \zeta^\perp$

Lemma

On the sphere we have $\langle \zeta^K, \zeta^\perp \rangle = 0$.

Two–dimensional manifold as an atmospheric models

$u = u^K + u^\perp$
$\zeta = \zeta^K + \zeta^\perp$

Lemma

On the sphere we have $\langle \zeta^K, \zeta^\perp \rangle = 0$.

Theorem

Let $\zeta^\perp$ *be the solution of*

$$\zeta_t - \mu\Delta\zeta + g(\mathrm{grad}(\zeta), u) - 2\mu\, g(\mathrm{grad}(\kappa), Ku) - 2\mu\,\kappa\zeta = 0$$

on the sphere; then

$$\|\zeta^\perp\|^2 \leq Ce^{-8\mu\kappa t}$$

The unit sphere S^2

Function a in spherical coordinates (θ, φ) where θ is the longitude and φ is the colatitude. The unit sphere S^2 as a submanifold of $\mathbb{R}^3$ and choose x_3 the axis of rotation with the rotation vector $(0, 0, \omega)$: $a = 2\omega \cos(\varphi)$

$$
\begin{aligned}
& u_t + \nabla_u u - \mu L u + a\, K u + \operatorname{grad}(p) = 0 \\
& -\Delta p - \operatorname{tr}((\nabla u)^2) - g(u, u) - \operatorname{div}\big(a\, K u\big) = 0 \\
& \operatorname{div}(u) = 0 \,.
\end{aligned}
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\begin{aligned}
&u_t + \nabla_u u - \mu L u + a\,Ku + \mathrm{grad}(p) = 0 \\
&-\Delta p - \mathrm{tr}((\nabla u)^2) - g(u, u) - \mathrm{div}\big(a\,Ku\big) = 0 \\
&\mathrm{div}(u) = 0\,.
\end{aligned}
$$

Coriolis term has no effect on the norm:

$$
\frac{d}{dt}\,\|u\|^2 = -\mu \int_M g(S_u, S_u)\,\omega_M
$$

The unit sphere S^2

$$
\begin{aligned}
& u_t + \nabla_u u - \mu L u + a\, K u + \operatorname{grad}(p) = 0 \\
& -\Delta p - \operatorname{tr}((\nabla u)^2) - g(u,u) - \operatorname{div}\big(a\, K u\big) = 0 \\
& \operatorname{div}(u) = 0\,.
\end{aligned}
\tag{6}
$$

Theorem

Let $u = u^K + \hat{u}$, $p = p_K + \hat{p}$ be a solution to (6). Then

$$\tfrac{d}{dt} \|\hat{u}\|^2 \le 0$$

where

$$
\begin{aligned}
u^k &= c\, \partial_\theta \\
p_K &= \tfrac{1}{2} \left(c^2 \sin(\varphi)^2 + c\, \omega \cos(2\varphi)\right)
\end{aligned}
$$

Thanks for your attention!