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# Singular stochastic integral operators

## and their applications to SPDEs

Emiel Lorist

Joint work with M.C. Veraar

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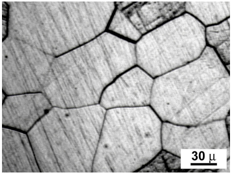
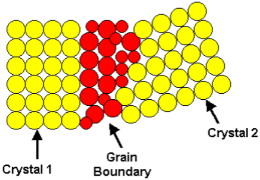
**University of Helsinki**

**Department of Mathematics and Statistics**

# Polycrystalline materials



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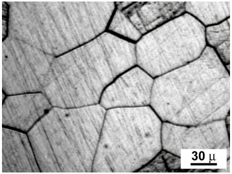
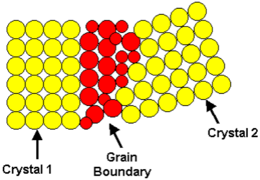
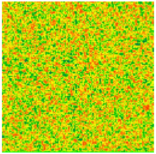
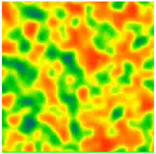


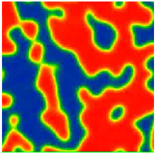

The Allen-Cahn equation is a prototype for the growth of grains:



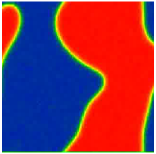
$t = 0$



$t = 1$

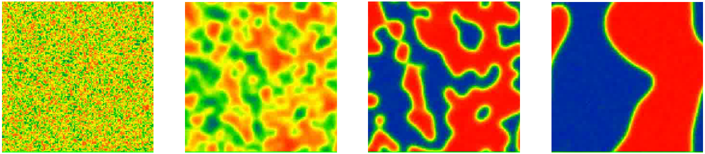


$t = 2$



$t = 50$

# The Allen-Cahn equation



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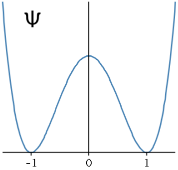
Given an initial state $u_0 \colon \mathbb{T}^2 \to \mathbb{R}$, look for a $u \colon \mathbb{R}_+ \times \mathbb{T}^2 \to \mathbb{R}$ satisfying

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} - \Delta u = -\Psi'(u) & \text{in } \mathbb{R}_+ \times \mathbb{T}^2, \\ u(0,\cdot) = u_0, \end{cases}$$

- $\Psi$ is a double well potential:



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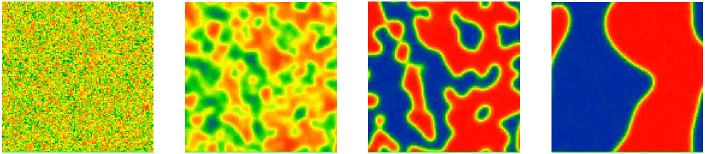

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Given an initial state $u_0 \colon \mathbb{T}^2 \to \mathbb{R}$, look for a $u \colon \Omega \times \mathbb{R}_+ \times \mathbb{T}^2 \to \mathbb{R}$ satisfying

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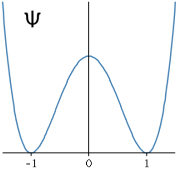
- $\Psi$ is a double well potential:
- $W$ is a Brownian motion.
- For example $B(u) = \varepsilon$.



## Stochastic evolution equations

"Hide" the space-variable in a Banach space $X$ to obtain an SDE:

Look for a function $u\colon \Omega \times \mathbb{R}_+ \to X$ satisfying

$$\begin{cases} \mathrm{d}u + Au\,\mathrm{d}t = F(u)\,\mathrm{d}t + G(u)\,\mathrm{d}W \quad \text{in } \mathbb{R}_+, \\ u(0) = u_0. \end{cases}$$

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For Allen-Cahn on $\mathbb{T}^2$:

- $X = L^q(\mathbb{T}^2)$
- $A = -\Delta$
- $F(u) = -\Psi'(u)$
  $G(u) = B(u)$
- $u_0 \in W^{2,q}(\mathbb{T}^2)$

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**Question**

Under what conditions does a unique solution exist?

# Abstract Cauchy problem

We first study the problem with $F$ and $G$ independent of $u$ and $u_0 = 0$.

Take $f, g \colon \Omega \times \mathbb{R}_+ \to X$. We look for a function $u \colon \Omega \times \mathbb{R}_+ \to X$ satisfying

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$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = f, \\ u(0) = 0. \end{cases} \qquad \begin{cases} \mathrm{d}v + Av\,\mathrm{d}t = g\,\mathrm{d}W, \\ v(0) = 0. \end{cases}$$

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The solutions $u$ and $v$ are given by the variation of constants formulas:

$$u(t) = \int_0^t \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s, \qquad v(t) = \int_0^t \mathrm{e}^{-(t-s)A} g(s)\,\mathrm{d}W(s),$$

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**Deterministic question**

If

$$f \in L^p(\mathbb{R}_+; X),$$

do we have

$$Au \in L^p(\mathbb{R}_+; X)?$$

$$f \in L^p(\mathbb{R}_+; X)$$

$$\Leftrightarrow$$

$$\|f\|_{L^p(\mathbb{R}_+;X)} = \left(\int_0^\infty \|f(t)\|_X^p\,\mathrm{d}t\right)^{\frac{1}{p}} < \infty$$

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**Stochastic question**

If

$$g \in L^p(\Omega \times \mathbb{R}_+; X),$$

do we have

$$A^{\frac{1}{2}} v \in L^p(\mathbb{R}_+; X)?$$

# From SPDEs to harmonic analysis

Note that

$$Au(t) = \int_0^t A\mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s, \qquad A^{\frac{1}{2}} v(t) = \int_0^t A^{\frac{1}{2}} \mathrm{e}^{-(t-s)A} g(s)\,\mathrm{d}W(s).$$

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So we can reformulate our questions as:

**Deterministic question**

Does

$$T_K f(t) = \int_0^\infty K(t-s) f(s)\,\mathrm{d}s$$

for $\qquad K(t) = A\mathrm{e}^{-tA}\mathbf{1}_{t>0}$

define a bounded operator on

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These kernels $K$ are singular:

$$\|K(t)\| \leq \frac{C}{t} \qquad\qquad \|K(t)\| \leq \frac{C}{t^{\frac{1}{2}}}$$

# (Stochastic) singular integral operators

The operator

$$T_K f(t) = \int_{\mathbb{R}^d} K(t-s) f(s) \, \mathrm{d}s$$

on $L^p$ for a singular kernel $K$ has been studied thoroughly:

- Hilbert transform $K(t) = \frac{1}{\pi} \frac{1}{t}$ ('28)
- Regularity theory for elliptic PDE ('40s)
- Calderón–Zygmund and Fourier multiplier theory ('50s)
- Regularity theory for parabolic PDE ('00s)

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$$S_K g(t) = \int_0^\infty K(t-s) g(s)\,\mathrm{d}W(s)$$

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## Stochastic Calderón–Zygmund theory

**Theorem (L., Veraar, Analysis & PDE '21)**

- Let $X$ be a Banach space with certain geometric properties.
- Let $K\colon \mathbb{R}_+ \to \mathcal{L}(X)$ be a smooth singular kernel.

Suppose that $S_K$ is bounded on $L^{p_0}(\mathbb{R}_+ \times \Omega; X)$ for **some** $p_0 \in [2, \infty)$.
Then $S_K$ is bounded on $L^p(\mathbb{R}_+ \times \Omega; X)$ for **all** $p \in (2, \infty)$.

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- $L^q$ is allowed for $q \in [2, \infty)$.
- It suffices to have

$$\|K(t)\| \leq \frac{C}{t^{\frac{1}{2}}}, \qquad \|K'(t)\| \leq \frac{C}{t^{\frac{3}{2}}}.$$

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**Corollary**

If our stochastic question holds for **some** $p_0 \in [2, \infty)$, then it holds for **all** $p \in (2, \infty)$.

For our deterministic question this was shown in:

(Dore, Adv. Differential Equations, '00).

# Back to Allen-Cahn

Given an initial state $u_0 \in L^q(\mathbb{T}^2)$, look for a $u\colon \Omega \times \mathbb{R}_+ \to L^q(\mathbb{T})$ satisfying

$$\begin{cases} \mathrm{d}u - \Delta u \,\mathrm{d}t = -\Psi'(u)\,\mathrm{d}t + B(u)\,\mathrm{d}W \quad \text{in } \mathbb{R}_+, \\ \qquad\quad u(0,\cdot) = u_0. \end{cases}$$

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| Positive answers to our det. and stoch. question | $\Rightarrow$ | Banach fixed point theorem | $\Rightarrow$ | Unique solution $u$ in $L^p(\Omega \times \mathbb{R}_+; L^q(\mathbb{T}))$ |

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Our theorem allows one to study the stochastic question in

$$L^q(\Omega \times \mathbb{R}_+; L^q(\mathbb{T}^2)) = L^q(\Omega \times \mathbb{R}_+ \times \mathbb{T}^2),$$

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## Why not just analyze $p = q = 2$?

- More classical smoothness through Sobolev embeddings with large $p, q$
- Scaling of the nonlinearities can dictate $p \neq q$.

**Thank you for your attention!**