Differentiable extensions with uniformly continuous derivatives

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Problem ($C^{1,\omega}$ extension of 1-jets)

Let X be a Hilbert space. Let $E \subset X$ be arbitrary, let $(f, G) : E \to \mathbb{R} \times X$ be a 1-jet, and let $\omega : [0, \infty) \to [0, \infty)$ be a modulus of continuity.

- Find necessary and sufficient conditions on (f, G) for the existence of $F \in C^{1,\omega}(X)$ such that $(F, \nabla F) = (f, G)$ on E.
- Construct such extension F (if it exists), estimate the seminorm

$$M_\omega(\nabla F) := \sup_{x,y \in X;\, x \neq y} \frac{|\nabla F(x) - \nabla F(y)|}{\omega(|x - y|)},$$

and compare it to the $C^{1,\omega}$-trace seminorm of (f, G) on E :

$$\|(f, G)\|_{E,\omega} := \inf \left\{ M_\omega(\nabla H) \ : \ H \in C^{1,\omega}(X),\ (H, \nabla H) = (f, G) \text{ on } E \right\}.$$

Previous results

• H. Whitney-G. Glaeser (1934-1958) solved this problem when $X = \mathbb{R}^n$, obtaining extensions F with $M_\omega(\nabla F) \leq k(n)\|(f,G)\|_{E,\omega}$. Here $k(n)$ only depends on n, but $\lim_{n\to\infty} k(n) = \infty$. The proof relies on Whitney decomposition into cubes and partitions of unity.

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• As a consequence of our solution to the extension problem for $C^{1,1}$ convex functions, we proved the Wells-Le Gruyer's theorem, via simple and explicit formulas.

Any $F \in C^{1,1}(X)$ can be written as $F=\widetilde{F}-\frac{M}{2}|\cdot|^2$, with $\widetilde{F} \in C^{1,1}_{\text{conv}}(X)$.

Theorem (Wells 1973, Le Gruyer 2009, Azagra-Le Gruyer-M. 2017)

Let E be a subset of a Hilbert space X, and let $(f, G) : E \to \mathbb{R} \times X$ be a jet. There exists $F \in C^{1,1}(X)$ with $(F, \nabla F) = (f, G)$ on E if and only if there exists $M > 0$ such that

$$f(z) \leq f(y) + \frac{1}{2}\langle G(y) + G(z), z - y\rangle + \frac{M}{4}|y - z|^2 - \frac{1}{4M}|G(y) - G(z)|^2$$

for all $y, z \in E$. Moreover,

$$F = \mathrm{conv}(g) - \frac{M}{2}|\cdot|^2,$$

$$g(x) = \inf_{y \in E}\left\{f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2\right\} + \frac{M}{2}|x|^2, \quad x \in X,$$

defines a $C^{1,1}(X)$ function with $(F, \nabla F) = (f, G)$ on E, and $\mathrm{Lip}(\nabla F) \leq M$.

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$$f(z) \leq f(y) + \frac{1}{2}\langle G(y)+G(z), z-y\rangle + \frac{M}{4}|y-z|^2 - \frac{1}{4M}|G(y)-G(z)|^2$$

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The function F can be taken so as to satisfy

$$\mathrm{Lip}(\nabla F) = \inf\left\{ \mathrm{Lip}(\nabla H) \ : \ H \in C^{1,1}(X),\ (H,\nabla H) = (f,G) \text{ on } E \right\}.$$

Corollary (Kirszbraun's theorem via an explicit formula; Azagra-Le Gruyer-M.; 2017)

Let X, Y two Hilbert spaces, $E \subset X$ and $G : E \to Y$ a Lipschitz mapping. The following mapping $\widetilde{G} : X \to Y$ satisfies $\widetilde{G} = G$ on E and $\mathrm{Lip}(\widetilde{G}, X) = \mathrm{Lip}(G, E)$:

$$\widetilde{G}(x) := \nabla_Y(\mathrm{conv}(g))(x,0) \quad \textit{for every} \quad x \in X; \quad \textit{where}$$

$$g(x,y) = \inf_{z \in E} \left\{ \langle G(z), y \rangle_Y + \frac{\mathrm{Lip}(G)}{2} |x-z|_X^2 \right\} + \frac{\mathrm{Lip}(G)}{2} |x|_X^2 + \mathrm{Lip}(G) |y|_Y^2$$

for every $(x,y) \in X \times Y$.

$C^{1,\omega}$ extensions of 1-jets, arbitrary ω

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- Wells' and Le Gruyer's proof cannot be adapted to $C^{1,\omega}$.
- Unlike for $C^{1,1}$, there is no relation between $C^{1,\omega}$ and $C^{1,\omega}_{\mathrm{conv}}$.
- More sofisticated techniques are needed: paraconvex functions and paraconvex envelopes.

Definition (Paraconvexity)

Let X be a Banach space and $\varphi : [0, \infty) \to [0, \infty)$. We say that $F : X \to \mathbb{R}$ is φ-paraconvex if

$$F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda)F(y) \leq \lambda(1-\lambda)\varphi(\|x-y\|)$$

for all $x, y \in X$ and all $\lambda \in [0, 1]$.

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• Let X be a Banach space, $F : X \to \mathbb{R}$ locally bounded, and $\varphi = \int \omega$. If F and $-F$ are φ-paraconvex, then $F \in C^{1,\omega}(X)$.

Let ω be a modulus of continuity, $\varphi_\omega = \int \omega$, and $(f, G) : E \to \mathbb{R} \times X$ a jet. We define the seminorm:

$$A_\omega(f, G) := \sup_{\substack{x \in X;\, y,z \in E \\ |x-y|+|x-z|>0}} \frac{|f(y) + \langle G(y), x-y\rangle - f(z) - \langle G(z), x-z\rangle|}{\varphi_\omega(|x-y|) + \varphi_\omega(|x-z|)}.$$

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• $A_\omega(f, G)$ is absolutely comparable to other seminorms, e.g. given by Whitney-Glaeser conditions.

• If X is Hilbert, $A_\omega(f, G) < \infty$ if and only if (f, G) admits a $C^{1,\omega}(X)$ extension.

Theorem (Azagra-M.; 2019)

A 1-jet (f,G) defined on a subset E of a Hilbert space X has an extension $(F,\nabla F)$ with $F \in C^{1,\omega}(X)$ if and only if $A_\omega(f,G) < \infty$. Moreover, we can take F such that

$$A_\omega(F,\nabla F) \leq 2\, A_\omega(f,G).$$

In addition, when $\omega(t) = t^\alpha$ with $0 < \alpha \leq 1$, we can arrange

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The extension F can be taken so that

$$M_\omega(\nabla F) \leq (16/\sqrt{15}) \|(f,G)\|_{E,\omega} \quad \text{and}$$

$$M_\omega(\nabla F) \leq \frac{2^{2-2\alpha}}{\sqrt{1+\alpha}} \left(1 + \frac{1}{\alpha}\right)^{\alpha/2} \|(f,G)\|_{E,\omega} \quad \text{when} \quad \omega(t) = t^\alpha.$$

Recall that

$$\|(f,G)\|_{E,\omega} := \inf\left\{ M_\omega(\nabla H) \ : \ H \in C^{1,\omega}(X),\ (H,\nabla H) = (f,G) \text{ on } E \right\}.$$

Extension formulas

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- The extension is defined as a *$2M\varphi$-paraconvex envelope of g* :

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- We have $F \in C^{1,\omega}(X)$, $(F, \nabla F) = (f, G)$ on E, etc

For $\omega(t) = \sqrt{t}$ and $E = \{-1, 0, 1\} \subset \mathbb{R}$, set $f(-1) = 1, f(0) = 0, f(1) = 1$ and $G \equiv 0$ on E. Then $A_\omega(f, G) = 3/\sqrt{2}$.

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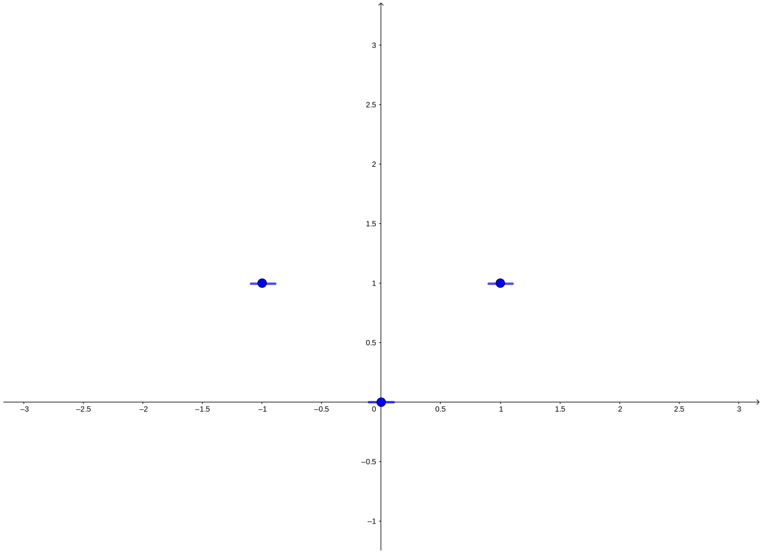
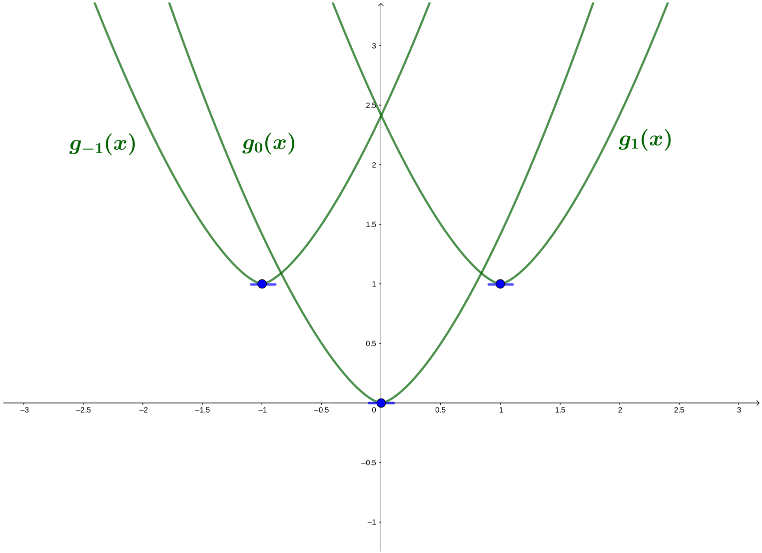
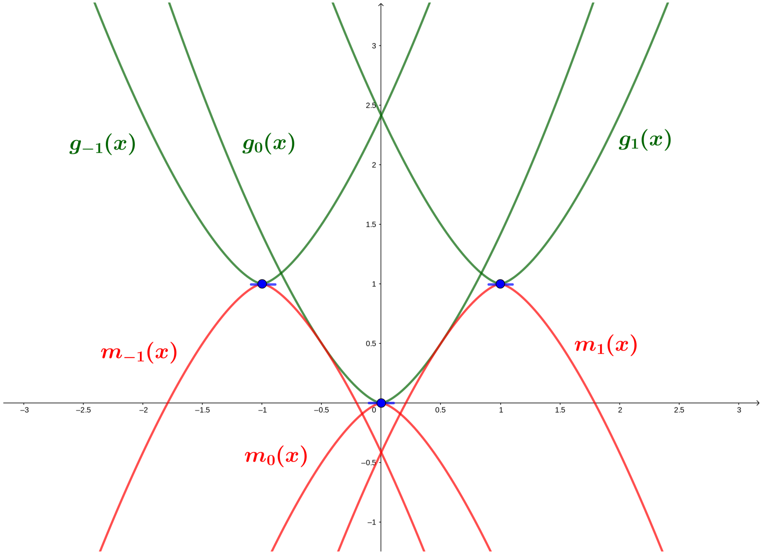


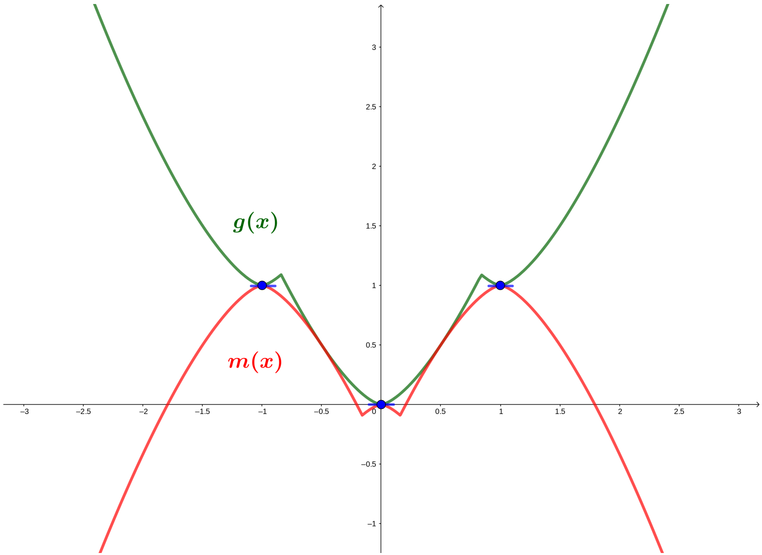
For every $y \in E$, set $g_y(x) := f(y) + G(y)(x-y) + \frac{2A(f,G)}{3}|x-y|^{3/2}$, $x \in \mathbb{R}$.



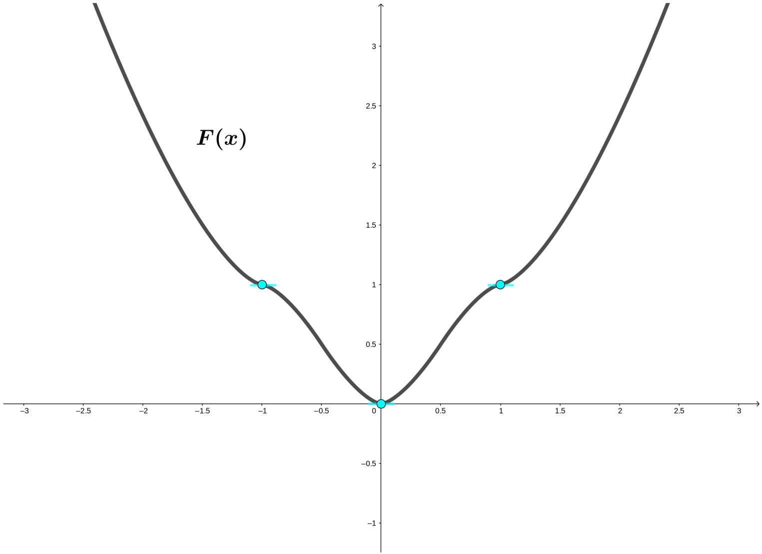
For every $y \in E$, set $m_y(x) := f(y) + G(y)(x-y) - \frac{2A(f,G)}{3}|x-y|^{3/2}$, $x \in \mathbb{R}$.



Define $g = \inf(g_y)_{y \in E}$ and $m = \sup(m_y)_{y \in E}$.



A suitable paraconvex envelope F of g defines a $C^{1,1/2}$ extension of (f, G).



Fix $M > 0$, and a modulus of continuity ω, $\varphi = \int \omega$. Let $\mathcal{F}(M, \omega)$ be the family of functions of the form

$$X \ni z \longmapsto h(z) = \text{affine} - \sum_{i=1}^{n} \lambda_i M\varphi(|z - p_i|),$$

where $p_i \in X$, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, and $n \in \mathbb{N}$.

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For a jet $(f, G) : E \to \mathbb{R} \times X$ such that $M := A_\omega(f, G) < \infty$, let

$$g(x) = \inf_{y \in E} \{f(y) + \langle G(y), x - y\rangle + M\varphi(|x - y|)\}, \quad x \in X.$$

Then the following formula defines an extension too:

$$F(x) := \sup\{h(x) \;:\; h \leq g, \;\; h \in \mathcal{F}(M, \omega)\}.$$

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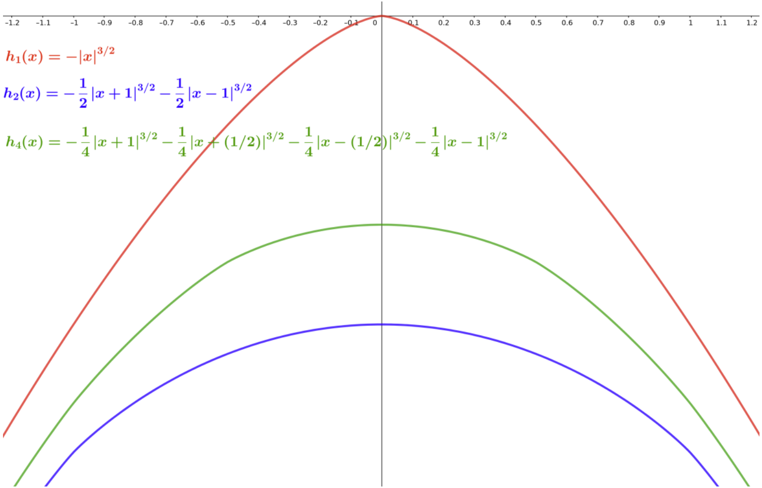

Bounded and/or Lipschitz functions. Banach spaces

• If (f, G) is also bounded, there are extensions $F \in C^{1,\omega}(X)$ with $(F, \nabla F)$ bounded, and there is $C > 0$ absolute with

$$A_\omega(F, \nabla F) + \|F\|_\infty + \|\nabla F\|_\infty \leq C \left(A_\omega(f, G) + \|f\|_\infty + \|G\|_\infty \right).$$

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• If the sequence of jets $\{(f_n, G_n)\}_n$ is A_ω-uniformly bounded, and (f_n, G_n) converges to (f, G) uniformly on E, then the corresponding sequence of $C^{1,\omega}(X)$ extensions $(F_n, \nabla F_n)$ converges to $(F, \nabla F)$ uniformly on X.

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• If f is Lipschitz and G bounded, we can find construct extensions $F \in C^{1,\omega}(X)$ with F Lipschitz, and there is $C > 0$ absolute with

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• If $(X, \|\cdot\|)$ is a Banach space, ω is a modulus of continuity, and $\varphi = \int \omega$. All the results are true for $C^{1,\omega}(X)$ extensions, provided $\varphi \circ \|\cdot\| \in C^{1,\omega}(X)$.

Thank you for your attention!