Second variation techniques for stability in geometric inequalities

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Plan of the talk

General problem

$I : \mathcal{A} \to \mathbb{R}$ ($\mathcal{A}$ - smooth sets/$C^{1,\vartheta}$ sets/open sets).
Consider the following minimization problem:

$$\min \{I(\Omega) : \Omega \in \mathcal{A}\}$$

We want to investigate the behavior of I around a strongly stable critical point.

Definition.

$\Omega^* \in \mathcal{A}$ is a strongly stable critical point for I, if for any $X \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ we have

$$\frac{d}{dt} I(\Phi_t(\Omega^*))|_{t=0} = 0, \qquad \frac{d^2}{dt^2} I(\Phi_t(\Omega^*))|_{t=0} > 0,$$

where $\Phi_t := Id + tX$.

General problem

Possible questions:

- Can we prove

$$I(\Omega) - I(\Omega^*) \geq \omega\left(\mathrm{dist}(\Omega, \Omega^*)\right), \tag{1}$$

  where Ω is in a small neighborhood of Ω^*, 'dist' denotes some suitable notion of distance, and ω is some modulus of continuity?
- And if so, does (1) hold for every Ω, thus making Ω^* a global minimum?
- Which ω make the inequality sharp? That is, we wish to have a sequence Ω_ε such that

$$I(\Omega_\varepsilon) \to I(\Omega^*) \text{ as } \varepsilon \to 0, \qquad I(\Omega_\varepsilon) - I(\Omega^*) \sim \omega\left(\mathrm{dist}(\Omega_\varepsilon, \Omega^*)\right).$$

History

- Bonnesen '1924: I = perimeter and $\mathcal{A} = \{\text{convex sets in } \mathbb{R}^2\}$ (Ω^* in this case is a ball)
- Fusco, Maggi, Pratelli '2008: I = perimeter, sharp
- Cicalese, Leonardi '2012: I = perimeter, sharp (different approach)
- Fusco, Maggi, Pratelli '2009: quantitative Faber-Krahn, Cheeger, isocapacitary inequality
- Brasco, De Philippis, Velichkov '2015: sharp quantitative Faber-Krahn inequality

Possible approaches: symmetrization, mass transportation, and second variation techniques

Second variation approach

Selection Principle by Cicalese and Leonardi:

- Get a contradicting sequence.
  We argue by contradiction and for any $c > 0$ we get a sequence of sets Ω_h, such that

  $$I(\Omega_h) - I(\Omega^*) < c\,\mathrm{dist}(\Omega_h, \Omega^*)^2$$

  and Ω_h converges to Ω^* in some (typically weak) topology.
- Improve the convergence.
  Perturb the sequence $\{\Omega_h\}$ to be a sequence of minimizers of some functionals. Then use regularity results.
- Prove the inequality for smooth sets (Fuglede's computation).
  Write Taylor expansion, bound the remainder.

Improving the convergence

We know:

$$\Omega_h \to \Omega^* \text{ weakly.}$$

To improve convergence:

$$\tilde{\Omega}_h \in \operatorname{argmin}\{I(\Omega) : \Omega \text{ is such that } \operatorname{dist}(\Omega^*, \Omega_h) = \operatorname{dist}(\Omega^*, \Omega)\}.$$

Then

$$\begin{aligned} I(\tilde{\Omega}_h) - I(\Omega^*) &\leq I(\Omega_h) - I(\Omega^*) \\ &< c \operatorname{dist}(\Omega_h, \Omega^*)^2 = c \operatorname{dist}(\tilde{\Omega}_h, \Omega^*)^2. \end{aligned}$$

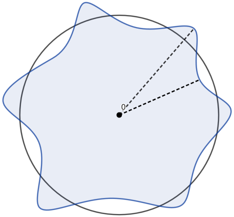
For technical reasons we can't use this exact formula, we need to perturb it.

Fuglede computation

Definition.

An open bounded set $\Omega \subset \mathbb{R}^N$ is called nearly-spherical of class $C^{2,\vartheta}$ parametrized by φ, if there exists $\varphi \in C^{2,\vartheta}$ with $\|\varphi\|_{L^\infty} < \frac{1}{2}$ such that

$$\partial\Omega = \{(1+\varphi(x))x : x \in \partial B_1\}.$$



We want:

$$\begin{aligned} I(\Omega) = I(\Omega^*) + \frac{1}{2} I''(\Omega^*)\|\varphi\|^2 \\ + \omega(\|\varphi\|)\|\varphi\|^2 \end{aligned}$$

Capacity: definition

Definition.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be an open set. We define the absolute capacity of Ω as

$$\operatorname{cap}(\Omega) = \inf_{u \in C_c^\infty(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : u \geq 1 \text{ on } \Omega \right\}.$$

Moreover, for $\Omega \subset\subset B_R$ (B_R denotes the ball of radius R centered at the origin) we denote by $\operatorname{cap}_R(\Omega)$ the relative capacity of Ω with respect to B_R defined as

$$\operatorname{cap}_R(\Omega) = \inf_{u \in C_c^\infty(B_R)} \left\{ \int_{B_R} |\nabla u|^2 dx : u \geq 1 \text{ on } \Omega \right\}.$$

Quantitative isocapacitary inequality

Classical isocapacitary inequality:

$$\mathrm{cap}(\Omega) - \mathrm{cap}(B_r) \geq 0,$$

with r such that $|B_r| = |\Omega|$. The proof is a simple application of Pólya-Szegö principle.
Do we have

$$\mathrm{cap}(\Omega) - \mathrm{cap}(B_r) \geq \omega\left(\mathrm{dist}(\Omega, B_r)\right)$$

for some ω and dist? Yes!

Definition.

Let Ω be an open set. The Fraenkel asymmetry of Ω, $\mathcal{A}(\Omega)$, is defined as:

$$\mathcal{A}(\Omega) = \inf\left\{\frac{|\Omega \Delta B|}{|B|} \,:\, B \text{ is a ball with the same volume as } \Omega\right\}.$$

Sharp quantitative isocapacitary inequality

Theorem (De Philippis, Marini, M '2019)

Let Ω be an open set such that $|\Omega| = |B_r|$. Then

1. *if Ω is compactly contained in B_R, there exists a constant $c_1 = c_1(N, R)$ such that the following inequality holds:*

$$\frac{\mathrm{cap}_R(\Omega) - \mathrm{cap}_R(B_r)}{r^{N-2}} \geq c_1(N,R)\left(\frac{|\Omega \Delta B_r|}{|B_r|}\right)^2 .$$

2. *there exists a constant $c_2 = c_2(N)$ such that the following inequality holds:*

$$\frac{\mathrm{cap}(\Omega) - \mathrm{cap}(B_r)}{r^{N-2}} \geq c_2(N)\mathcal{A}(\Omega)^2.$$

Testing on ellipsoids reveals that the inequalities are sharp.

Quantitative isocapacitary inequality: p-capacity case

Definition.

Let $\Omega \subset \mathbb{R}^N$, be an open set. We define the p-capacity of Ω as

$$\mathrm{cap}_p(\Omega) = \inf_{u \in C_c^\infty(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\nabla u|^p dx : u \geq 1 \text{ on } \Omega \right\}$$

for $1 < p < N$.

Theorem (M '2020)

Let Ω be an open set such that $|\Omega| = |B_r|$. Then there exists a constant $c = c(N, p)$ such that the following inequality holds:

$$\frac{\mathrm{cap}_p(\Omega) - \mathrm{cap}_p(B_1)}{r^{N-p}} \geq c\, \mathcal{A}(\Omega)^2.$$

Thank you for your attention!