Second order Sobolev-regularity of p-harmonic functions

Saara Sarsa

University of Helsinki

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Let $\Omega \subset \mathbb{R}^n$ be a domain and $1 < p < \infty$. Consider the Sobolev-space

$$W^{1,p}(\Omega) := \left\{ v\colon \Omega \to \mathbb{R} : \int_\Omega |v|^p dx < \infty \text{ and } \int_\Omega |Dv|^p dx < \infty \right\}.$$

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A function $u \in W^{1,p}(\Omega)$ is called p-harmonic, if it solves the p-Laplacian equation

$$\Delta_p u := \operatorname{div}\left(|Du|^{p-2} Du\right) = 0$$

in the weak sense, that is, if

$$\int_\Omega |Du|^{p-2} \langle Du, D\varphi \rangle dx = 0$$

for all test functions $\varphi \in C_c^\infty(\Omega)$.

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Theorem (Ural'tseva (1968), Uhlenbeck (1977), DiBenedetto (1983), Lewis (1983))

Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is p-harmonic. Then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, where $\alpha = \alpha(n,p) > 0$.

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Theorem (Bojarski, Iwaniec (1987))

Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is p-harmonic for $p > 2$. Then $|Du|^{\frac{p-2}{2}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

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$$\operatorname{div}\left(|Du|^{p-2}ADu_{x_k}\right) = 0. \tag{1}$$

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We test the weak form of (1) with a test function $\varphi = u_{x_k}\phi^2$, where $\phi \in C_c^\infty(\Omega)$ is suitable cutoff function:

$$\int |Du|^{p-2}\langle ADu_{x_k}, D(u_{x_k}\phi)\rangle dx = 0$$

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By Young's inequality, for any $\epsilon > 0$

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Since A is uniformly elliptic, we obtain

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Let $s \in \mathbb{R}$ be a parameter and define the mapping $V_s \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$V_s(z) := \begin{cases} |z|^{\frac{p-2+s}{2}} z & \text{for } z \in \mathbb{R}^n \setminus \{0\}; \\ 0 & \text{for } z = 0. \end{cases}$$

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Question: Does $V_s(Du)$ still belong to $W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ if $s \neq 0$?

Main results

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Theorem (S. 2020)

If $s > -1 - \frac{p-1}{n-1}$, then $V_s(Du) \in W^{1,2}_{\text{loc}}(\Omega; \mathbb{R}^n)$. Moreover, there exists a constant $C = C(n,p,s) > 0$ such that

$$\int_{B_r} |D(V_s(Du))|^2 dx \leq \frac{C}{r^2} \int_{B_{2r}} |V_s(Du) - z|^2 dx \tag{2}$$

for all vectors $z \in \mathbb{R}^n$ and all concentric balls $B_r \subset B_{2r} \subset\subset \Omega$.

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Corollary (Higher integrability)

Under the same hypothesis as in the above Theorem, there exists a constant $\delta = \delta(n,p,s) > 0$ such that $D(V_s(Du)) \in L^{2+\delta}_{\mathrm{loc}}(\Omega)$.

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These results improve earlier results by Dong, Peng, Zhang and Zhou (2020).

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$\underline{s = 2 - p} \implies V_{2-p}(Du) = Du \in W^{2,2}_{\mathrm{loc}}$ on the range $1 < p < 3 + \frac{2}{n-2}$ (Manfredi, Weitsman (1988))

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Thank you for attention!