Lecture 4 Notes

Kyle Chui

2021-1-11

1 Sets and Functions

1.1 Power Sets

Definition. *Power Set*
If X is a set, the *power set* of X, denoted $\mathscr{P}(X)$, is the set of subsets of X.

Example. *Power Sets*

- $\mathscr{P}(\varnothing) = \{\varnothing\}$
- $\mathscr{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
- $\mathscr{P}(\{a, b, c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

Definition. *Cardinality of Finite Sets*
If X has finitely many elements, then $|X|$ denotes the number of elements of X.

Theorem — *Cardinality of Power Sets*
If X is finite, then $|\mathscr{P}(X)| = 2^{|X|}$.

Proof. Let us induct on the cardinality of the set X. Suppose $|X| = 0$, so that $X = \varnothing$. Then $\mathscr{P}(X) = \{\varnothing\}$, so $|\mathscr{P}(X)| = 1 = 2^0$. Thus the statement is true when $|X| = 0$.

Suppose that the statement holds for some non-negative integer k. Let Y be a set such that $|Y| = k + 1$, and $y \in Y$. Observe that we may split $\mathscr{P}(Y)$ into two groups: the subsets containing y, and the subsets that do not contain y. A subset of Y that does not contain y is exactly $Y \setminus \{y\}$, which has k elements. By the inductive hypothesis, there exist 2^k such subsets. A subset of Y that does contain y is obtained by adding y to a subset of Y which does not contain y. Again, there are 2^k such subsets. Any subset of Y either does or does not contain y (but not both), so there are $2^k + 2^k = 2^{k+1}$ subsets of Y. Therefore $\mathscr{P}(X) = 2^{|X|}$ for all finite sets $|X|$. $\square$

1.2 Functions

Definition. *Function*
If X, Y are sets, a function f from X to Y, written $f\colon X \to Y$ is a subset of $X \times Y$ satisfying two properties:

- For all $a \in X$, there exists $b \in Y$ such that $(a, b) \in f$
  - Everything in the domain must get mapped to something in the codomain
- For all $a \in X$ and $b, b' \in Y$, if $(a, b), (a, b') \in f$, then $b = b'$
  - Every element in the domain can map to at most one element in the codomain

Note (Function Notation). If $(a, b) \in f$, we write $f(a) = b$.

Example. *Functions*

- $f\colon \mathbb{Z} \to \mathbb{N}$ such that $f(x) = x^2$
- $g\colon \mathbb{N} \to \mathbb{N}$ such that $g(x) = x^2$

Note that f and g are different functions.

Definition. *Domain and Codomain of a Function*
If $f\colon X \to Y$, X is the domain of f and Y is the codomain of f.

Definition. *Range of a Function*
For $f\colon X \to Y$, the range of f is:

$$\operatorname{range} f = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

Definition. *Surjectivity*
A function $f\colon X \to Y$ is *onto* or *surjective* if $\operatorname{range} f = Y$. In other words, a function is surjective if its range is equal to its codomain.

Example. *Surjective Functions*

- $f\colon \{a, b, c\} \to \{d, e, f\}$ defined by $f = \{(a, d), (b, e), (c, f)\}$
- $f\colon \mathbb{Z} \to \mathbb{N}$ defined by $f(x) = |x|$

Definition. *Injectivity*
A function $f\colon X \to Y$ is *one-to-one* or *injective* if, for all $x, y \in X$, $f(x) = f(y)$ implies that $x = y$. In other words, different elements in the domain map to different elements in the codomain.

Example. *Injective Functions*

- $g\colon \mathbb{N} \to \mathbb{N}$ defined by $g(x) = x^2$

Note (Properties of Functions). Observe that the both the domain and codomain of a function matter when it comes to determining whether the function satisfies certain properties. For instance, $f\colon \mathbb{Z} \to \mathbb{N}$ defined by $f(x) = x^2$ is not injective, but restricting the domain to $\mathbb{N}$ would make it injective. Similarly, a function $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$ is not surjective, but restricting the codomain to $\mathbb{N}$ would make it surjective.

Definition. *Composition of Functions*
If $f\colon X \to Y, g\colon Y \to Z$ are functions, then $g \circ f\colon X \to Z$ is a function defined by $(g \circ f)(x) = g(f(x))$.

Theorem — *Composition of Injective/Surjective Functions is Injective/Surjective*
Let $f\colon X \to Y$, $g\colon Y \to Z$.

- If f, g are injective, so is $g \circ f$
- If f, g are surjective, so is $g \circ f$

Proof. Suppose f, g are injective functions. Let $x, x' \in X$ such that $(g \circ f)(x) = (g \circ f)(x')$. Then

$$\begin{aligned} g(f(x)) &= g(f(x')) \\ f(x) &= f(x') && \text{(Because } g \text{ is injective)} \\ x &= x' && \text{(Because } f \text{ is injective)} \end{aligned}$$

Therefore $g \circ f$ is injective. $\square$

Proof. Suppose f, g are surjective functions. Let $z \in Z$. Because g is surjective, there exists some $y \in Y$ such that $g(y) = z$. Furthermore, because f is surjective, there exists some $x \in X$ such that $f(x) = y$. Thus, for every $z \in Z$, there exists some $x \in X$ such that $(g \circ f)(x) = g(f(x)) = g(y) = z$, so $g \circ f$ is surjective. $\square$

Definition. *Bijectivity*
If a function is both injective and surjective, then we say that it is *bijective*.

Note (Cardinality and Bijections). If there is a bijection between two sets, they have the same number of elements.