Physics 1C Notes

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Contents

1 Lecture 1

1.1 Electromagnetism

In Physics 1B, we discussed how

$$\text{charge} \to \text{electric fields} \begin{cases} \text{Coulomb's Law} \\ \text{Gauss' Law} \end{cases}.$$

Furthermore, we know that moving charge $\to$ magnetic field (Ampere's Law).

$$\text{Gauss' Law: } \oint_{\text{surface}} \vec{E} \cdot \mathrm{d}\vec{A} = \frac{q}{\varepsilon_0} \qquad \text{Ampere's Law: } \oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I.$$

There are two other main laws for electromagnetism, and all of these laws give us magnets, motors, power generators, etc.

1.2 Electromagnetic Waves

These four laws also lead to electromagnetic waves, examples of which are radio waves, visible light, etc.

> **Definition.** *Speed of Light*
> We define the speed of light c to be
> $$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \cdot 10^8 \text{m/s}.$$

1.3 Light and Optics

Light and matter gives us the ideas of reflection and refraction, which lead to prisms, lens, and mirrors. In 1905, Einstein noticed that space and time depend on the observer, and that energy and momentum are different from what we thought.

2 Lecture 2

Definition. *Magnetic field*
The *magnetic field* is denoted by $\vec{B}$, and its units are given in Tesla (which is a large unit!).

2.1 Demonstrations

- A current through a magnetic field experiences a force (the jumping wire). We observed that the force $\vec{F}$ was perpendicular to the magnetic field $\vec{B}$, which was also perpendicular to the current $\vec{I}$.
- The lodestone is a magnet.
- Opposite poles attract, same poles repel (bar magnets).
- We visualised the magnetic field by looking at the alignment of iron filings in a viscous fluid.

2.2 Equations

1. The force due to a magnetic field is given by
$$\vec{F}_m = q\vec{v} \times \vec{B}.$$
This force is a part of the electromagnetic force,
$$\vec{F}_{EM} = q\vec{E} + q\vec{v} \times \vec{B}.$$

Note (Cross Products). Suppose we have two vectors $\vec{a}$ and $\vec{b}$. Then $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$. If θ is the angle between the two vectors, then
$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \left|\vec{b}\right| \sin\theta = ab\sin\theta.$$
The direction of the vector is given by the "right hand rule".

3 Lecture 3

3.1 Gauss' Law for Magnetic Fields

In Physics 1B, we learned Gauss' Law for electric fields:

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}.$$

We have a similar equation for magnetic fields:

$$\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0.$$

The reason why this is zero is because there are "no magnetic monopoles", i.e. every magnet has two poles.

> **Note.** This can also be interpreted to mean that for a given surface, the number of magnetic field lines coming out must equal the number of magnetic field lines going in.

3.2 The Motion of Charged Particles in a Magnetic Field

There are [some] cases:

i) When the particle is at rest, we have
$$\vec{F} = q\vec{v} \times \vec{B} = 0.$$

ii) For a particle in 2D motion with a constant magnetic field, we have
$$\vec{F} = q\vec{v} \times \vec{B},$$
which is perpendicular to both $\vec{v}$ and $\vec{B}$ (by the properties of the cross product). We use the right hand rule to get the actual direction of the force (when the charge is positive).

> **Note.** Since the magnetic force is perpendicular to the motion of the particle, we have uniform circular motion.

To get the radius of the motion, we solve:
$$\begin{aligned} F_C &= F_B \\ \frac{mv^2}{r} &= qvB \\ r &= \frac{mv^2}{qvB} \\ \Aboxed{r &= \frac{mv}{qB} = \frac{p}{qB}.} \end{aligned}$$

iii) For a particle in 3D motion with a constant magnetic field, $\vec{v}_0$ is not necessarily perpendicular to $\vec{B}$. Thus we first decompose $\vec{v}_0$ into $\vec{v}_0 = \vec{v}_{0_\parallel} + \vec{v}_{0_\perp}$. Then we have
$$\begin{aligned} \vec{F} &= q\left(\vec{v}_{0_\parallel} + \vec{v}_{0_\perp}\right) \times \vec{B} \\ &= qv_{0_\parallel} \times B + q\vec{v}_{0_\perp} \times \vec{B} \\ &= q\vec{v}_{0_\perp} \times \vec{B}. \end{aligned}$$

> **Note.** The movement of the particle is circular relative to the perpendicular plane to the magnetic field, but the velocity of the particle parallel to the magnetic field is constant. Thus the particle moves in a helix (or spiral) pattern.

3.3 Force Equation for a Current-Carrying Wire

We define $\vec{\ell}$ to be the length of the wire in the direction of $\vec{v}$. From earlier, we know that

$$\vec{F} = N \cdot q\vec{v} \times \vec{B}.$$

Furthermore, we have

$$I = \frac{Q}{t} = \frac{Nq}{t} = \frac{Nq}{\frac{\ell}{v}} = \frac{Nqv}{\ell}.$$

Thus we have

$$\begin{aligned} \vec{F} &= N \cdot q\vec{v} \times \vec{B} \\ &= N \cdot qv \cdot \frac{\vec{\ell}}{\ell} \times \vec{B} \\ &= I\vec{\ell} \times \vec{B}. \end{aligned}$$

Alternatively, if things are always changing, then

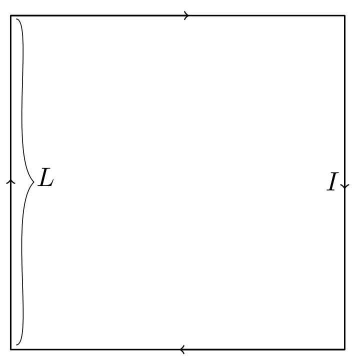
$$\vec{F} = \int_{\text{wire}} I \, \mathrm{d}\vec{\ell} \times \vec{B}.$$

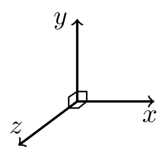
4 Lecture 4

4.1 Current Loops in Magnetic fields: Force and Torque

4.1.1 Uniform Magnetic Field Perpendicular to the Loop

Consider a fictional square loop of wire with side L, where the current flowing through the loop is I. We denote $\odot$ to be out of the page, and $\otimes$ to be into the page. We will use the equations $\vec{F} = I\vec{L} \times \vec{B}$ and $\vec{\tau} = \vec{r} \times \vec{F}$, and assume that we have a constant, uniform $\vec{B}$ field.





$\vec{B}$	$\vec{F}_{\text{left}}$	$\vec{F}_{\text{top}}$	$\vec{F}_{\text{right}}$	$\vec{F}_{\text{bottom}}$	$\vec{F}_{\text{net}}$	$\vec{\tau}_{\text{net}}$
$\otimes$	$\leftarrow$	$\uparrow$	$\rightarrow$	$\downarrow$	0	0
$\uparrow$	0	$\odot$	0	$\otimes$	0	$\hat{\imath}\left(\frac{L}{2}BIL \cdot 2\right) = (BIL^2)\hat{\imath}$

The magnetic moment μ is given by $\mu = IA$, where A is the area of the loop (in this case, $L = A^2$). We use this μ to find the torque, and we have $\tau = \mu B$ or $\vec{\tau} = \vec{\mu} \times \vec{B}$. The magnetic moment is defined as perpendicular to the surface of the loop.

For the example above, the magnetic moment must be *into the page*. We can find this by using the right hand rule, where you point your thumb in the direction of the current and curl your fingers around to find the direction of the moment.

4.1.2 Generalisation to a Loop of Any Shape

We can approximate an irregularly shaped loop of wire by breaking it into smaller, rectangular loops of wire.

Note. If the currents of the rectangular loops are all travelling in the same direction, the currents inside of the outer loop will cancel, leaving only the current around the perimeter of the outside loop.

5 Lecture 5

Last week, we studied the force due to a $\vec{B}$. This week, we will study how to produce a $\vec{B}$.

5.1 Biot-Savart Law

Point charge observations:

- When you have a point charge, you have an electric field $\vec{E}$ that radiates outwards from the point charge.
- This is slightly different from the magnetic field, which is generated by a *moving* point charge.
  - The closer you are to the point charge, the larger the magnetic field.

Theorem — *Coulomb's Law*
Given the charge and distance from a point charge, we may find the electric field:

$$\vec{E} = \frac{q\hat{r}}{4\pi\varepsilon_0 r^2}.$$

Biot-Savart Law precursor:

$$\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v} \times \hat{r}}{r^2}.$$

Note. In the equation for the magnetic field, we see that it is both orthogonal to the velocity vector and $\hat{r}$ (due to the cross product).

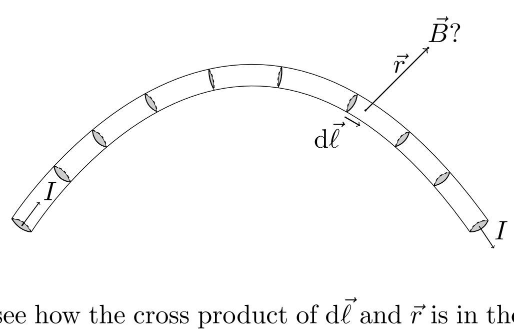
Points:

- Both are inversely proportional to the square of the distance.
- $\mu_0 \coloneqq 4\pi \cdot 10^{-7}$ Tm/A, and is used in the definition of the ampere (and so the Coulomb).

Note. This means that the $\vec{E}$ is very large, but $\vec{B}$ is quite small.

Theorem — *Biot-Savart Law*
Given the current through a wire, and the displacement from a point to the wire, we can find the magnetic field:

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{\ell} \times \hat{r}}{r^2}.$$

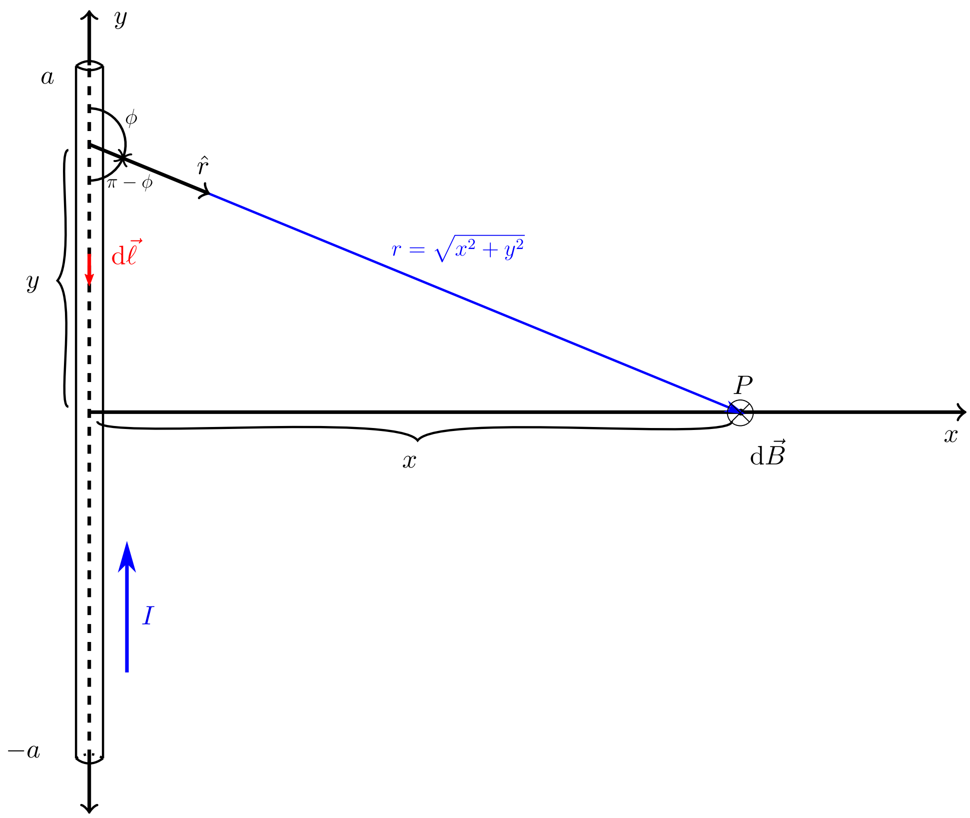


In the above diagram, you can see how the cross product of $\mathrm{d}\vec{\ell}$ and $\vec{r}$ is in the direction of the magnetic field. Furthermore, we can integrate the Biot-Savart Law to get

$$\vec{B} = \int_{\text{wire}} \mathrm{d}\vec{B}$$

5.2 Magnetic Field of a Straight Current-Carrying Wire

For a long, straight current-carrying wire, we try to find the field on some point P from locations $-a$ to a on the wire.



From the diagram, we can see that $r = (x^2 + y^2)^{\frac{1}{2}}$, and

$$\hat{r} - \frac{x\hat{\imath} - y\hat{\jmath}}{(x^2 + y^2)^{\frac{1}{2}}}. \qquad \text{(Scale down } \vec{r}\text{)}$$

Furthermore, $\mathrm{d}\vec{\ell} = \mathrm{d}y\hat{\jmath}$. The hard part is calculating the following:

$$\begin{aligned} \frac{\mathrm{d}\vec{\ell} \times \hat{r}}{r^2} &= \frac{\mathrm{d}y\hat{\jmath} \times (x\hat{\imath} - y\hat{\jmath})}{(x^2 + y^2)^{\frac{3}{2}}} \\ &= \frac{-\hat{k}x\,\mathrm{d}y}{(x^2 + y^2)^{\frac{3}{2}}}. \end{aligned}$$

We then integrate over the length of the wire:

$$\vec{B} = -\hat{k}\frac{\mu_0 I}{4\pi}\int_{-a}^{a} \frac{x\,\mathrm{d}y}{(x^2 + y^2)^{\frac{3}{2}}}$$

Letting $y = x\tan\theta$, and going through some algebra, we get

$$= -\hat{k}\frac{\mu_0 I}{4\pi}\left(\frac{2a}{x\sqrt{a^2 + x^2}}\right).$$

For an infinitely long wire, we see what happens when $a \to \infty$. In this case, we have

$$\vec{B} = -\hat{k}\frac{\mu_0 I}{4\pi}\left(\frac{2}{x}\right) = -\hat{k}\frac{\mu_0 I}{2\pi x}.$$

We started with a special point P on the wire, which is on the perpendicular bisector of the wire. However, when $a \to \infty$, we have an infinite amount of wire on either side, so the y becomes irrelevant and all we care about is the distance x to the wire. Thus to find the magnitude of the wire, we have

$$B = \frac{\mu_0 I}{2\pi r},$$

where r is the distance to the wire. If we were to look at the wire from the top (the xz-plane), we would have that $\vec{B}$ circles the wire.

Note. Another right-hand rule is to put your thumb in the direction of the current, and your thumbs will curl to form the magnetic field.

6 Lecture 6

The force between long parallel wires defines the ampere. As a reminder, last class we found the formula

$$B = \frac{\mu_0 I}{2\pi r}.$$

Consider two parallel, infinitely-long wires that are carrying current. If the currents are going in the same direction, then we use the right-hand rule to find that the wires have an attractive force between them. We find that the force acting on wire 2 is

$$\begin{aligned} \vec{F} &= I_2\vec{L} \times \vec{B}_1 \\ &= I_2 L\frac{\mu_0 I_1}{2\pi r} \text{ to the left,} \end{aligned}$$

so we have

$$F_2 = \frac{\mu_0}{2\pi} \cdot \frac{I_1 I_2 L}{r}.$$

Rearranging some more, we get

$$\frac{F_2}{L} = \left(2 \cdot 10^{-7}\right)\frac{I_1 I_2}{r} \text{ Newtons.}$$

What happens when we ave $I_1 = I_2$? Well, we get that

$$\frac{F}{L} = \left(2 \cdot 10^{-7}\right)\frac{I^2}{r},$$

which we can rearrange to get

$$I = \underbrace{\left(r \cdot \frac{F}{L} \cdot \frac{1}{2 \cdot 10^{-7}}\right)^{\frac{1}{2}}}_{\text{defines the Ampere}}.$$

6.1 Ampere's Law

Just like Coulomb's Law leads to Gauss' Law (which is an integral), the Biot-Savart Law leads to Ampere's Law (also an integral). Remember the Biot-Savart Law from before:

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{\ell} \times \hat{r}}{r^2}.$$

Theorem — *Ampere's Law*
The amount of magnetic field in a loop can give you the current, given by

$$\oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enc}}.$$

The analogue to the Gaussian surface is what we call an *Amperium loop*. There is no radial component to $\vec{B}$ because there are no monopoles. Thus we have that $\vec{B} \parallel \mathrm{d}\vec{\ell}$, and $\vec{B} \cdot \mathrm{d}\vec{\ell} = B\,\mathrm{d}\ell$. Then

$$\begin{aligned} \oint B\,\mathrm{d}\ell &= B\oint \mathrm{d}\ell \\ &= B \cdot 2\pi r \\ &= \mu_0 I, \end{aligned}$$

so

$$B = \frac{\mu_0 I}{2\pi r}.$$

The differential form of Ampere's Law is given by:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J},$$

where $\vec{J}$ is the current density.

6.1.1 Ampere's Law Example—Long Wire, Thin but Finite Thickness

Consider a wire with radius R that has uniform current density J. We define current density to be the current per unit area, in other words

$$J = \frac{I}{\pi R^2}, \text{ a constant.}$$

We have a few cases here:

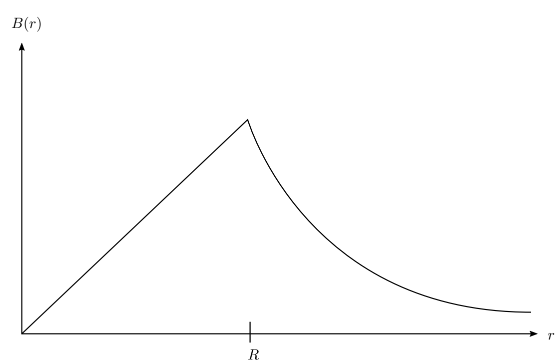
(a) Outside the wire, i.e. $r > R$.

$$\begin{aligned}\oint \vec{B} \cdot \mathrm{d}\vec{\ell} &= \mu_0 I_{\text{encl}} \\ 2\pi r B &= \mu_0 I \\ \boxed{B = \frac{\mu_0 I}{2\pi r}.}\end{aligned}$$

(b) Inside the wire, i.e. $r < R$.

$$\begin{aligned}\oint \vec{B} \cdot \mathrm{d}\vec{\ell} &= \mu_0 I_{\text{encl}} \\ 2\pi r B &= \mu_0 I \cdot \frac{\pi r^2}{\pi R^2} \\ 2\pi r B &= \mu_0 I \cdot \frac{r^2}{R^2} \\ \boxed{B = \frac{\mu_0 I r}{2\pi R^2}.}\end{aligned}$$

Pictorially, we can draw a graph for the magnetic field as a function of the distance to the centre of the wire as follows:



6.1.2 Ampere's Law Example—Plane of Current

If we have two parallel wires next to each other with the currents moving in the same direction, by superposition of field lines we generate a space with zero field between the wires. Extending this idea to more than just two wires (read: an infinite number, creating a plane of current), we create a planar magnetic field.

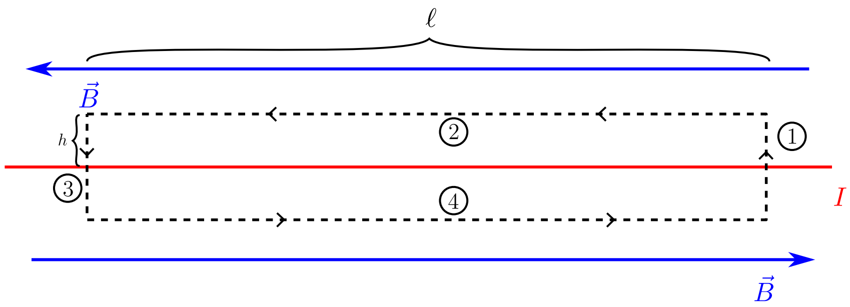


Figure 1: Plane of Current

Definition. *Winding Density*
We define the *winding density* n to be the number of wires per unit length, or

$$n = \frac{N}{\ell}.$$

Applying Ampere's Law, we have

$$\begin{aligned}
\oint \vec{B} \cdot \mathrm{d}\vec{\ell} &= \mu_0 I_{\text{encl}} \\
\int_1 + \int_2 + \int_3 + \int_4 &= \mu_0 N I \\
2 \int_4 \vec{B} \cdot \mathrm{d}\vec{\ell} &= \mu_0 N I \\
2B\ell &= \mu_0 N I \\
B &= \frac{\mu_0 n I}{2}.
\end{aligned}$$

7 Lecture 7

7.1 Magnetic Field from a Solenoid

> **Definition.** *Solenoid*
> A *solenoid* is the winding of wire in a helix shape.

If you take a 2D cross-section of the solenoid, you can see how the inside of the solenoid has parallel magnetic field lines due to the superposition of the field lines. This is analogous to the parallel electric field lines found between parallel, charged plates. In the infinite case, the magnetic field can be calculated with the equation

$$B = \mu_0 \cdot \frac{nI}{2}.$$

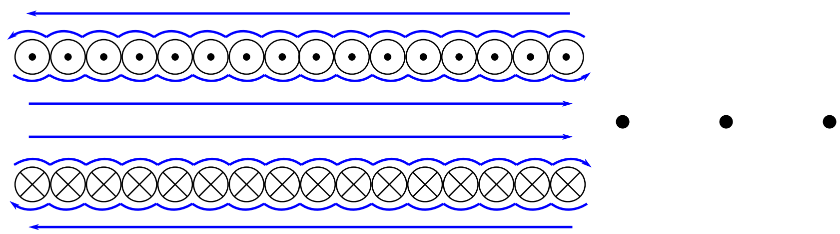


Figure 2: Cross Section of a Solenoid

Using Ampere's Law, with a box around all of the currents shown, we have

$$\begin{aligned}\oint_{\text{loop}} \vec{B}\cdot \mathrm{d}\vec{\ell} &= \mu_0 I_{\text{encl}}\\ &= \int_1 + \int_3\\ &= 0. && \text{(The currents cancel)}\end{aligned}$$

From our diagram we see that the magnetic field outside of the solenoid is in one direction, regardless where you are, so $\int_1 = \int_3 = 0$. Restricting our attention to only the top half of the currents, we have

$$\begin{aligned}\oint_{\text{loop}} \vec{B}\cdot \mathrm{d}\vec{\ell} &= \mu_0 NI\\ B\cdot L &= \mu_0 nLI\\ B &= \mu_0 nI.\end{aligned}$$

The magnetic field lines outside the solenoid cancel out.

7.2 Magnetic Field from a Toroid

To get a toroid, you basically bend a solenoid into a loop. Consider a toroid with inner radius a and outer radius b. Then applying Ampere's Law, we get

$$\begin{aligned}\oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} &= 2\pi r B \\ &= \mu_0 N I,\end{aligned}$$

so $B = \frac{\mu_0 N I}{2\pi r}$ (here we have $n = \frac{N}{2\pi a}$ when $a < r < b$). Outside of the toroid, the enclosed current is zero.

8 Lecture 8

8.1 Magnetism in Matter—Atoms

For a series of loops, we know that the magnetic moment is given by

$$\vec{\mu} = NIA \cdot \hat{n},$$

where N is the number of loops, I is the current, and A is the area enclosed by the loop. Furthermore, the torque on the loop(s) in an external $\vec{B}$ field is given by $\vec{\tau} = \vec{\mu} \times \vec{B}$. This is analogous to a dipole in an electric field, where we had

$$\vec{p} = q \cdot \vec{\ell} \quad \text{and} \quad \vec{\tau} = \vec{p} \times \vec{E}.$$

Question. What happens if dipoles go to minimum potential energy U?

If we call the *electric field generated by the dipole* $\vec{E}'$, then we see that it tends to point in the opposite direction to $\vec{E}_0$, the original electric field. Then we have that the net electric field is given by $\vec{E} = \vec{E}_0 + \vec{E}'$.

In contrast, the magnetic field created by the solenoid is usually in the *same* direction as $\vec{B}_0$, so the magnetic field usually gets *amplified*.

8.2 Describing Magnetic Materials

The textbook goes over the "Bohr magneton" for a single atom. We have the equations

$$\vec{M} = \frac{\vec{\mu}_{\text{total}}}{V} \quad \text{and} \quad \vec{B} = \vec{B}_0 + \mu_0 \vec{M}.$$

However, the above equations are usually not that useful, so we ignore them. More interestingly, we have

$$\begin{aligned} \mu_0 \to \mu &= K_m \cdot \mu_0 \\ \oint B \cdot \mathrm{d}\ell &= \mu I \\ B \cdot 2\pi r &= \mu I. \end{aligned}$$

The K_m constant referenced above is called "permeability". We replace all of the μ_0's in the previous equations with just μ.

Note. This is analogous to the dielectric constant for capacitors.

Example. The permeability of iron is approximately 1000, so when you fill a solenoid with iron, you increase its magnetic field output by a factor of 1000. This is similar to filling the space between the plates of a capacitor with a dielectric, which increased the capacitor's capacitance.

Note. Most materials have a permeability K_m that is close to 1.

We define a new constant called "susceptibility" given by $\chi_m \coloneqq K_m - 1$. This makes it such that have a "susceptibility" close to zero are the items that are not very magnetic.

If atomic loops perfectly align, then the currents inside the magnet perfectly cancel out, leaving only a net current along the boundary of the magnet. This generates the same effect as a solenoid, which just has a current looping around and around its boundary.

8.3 Three Types of Magnetic Materials

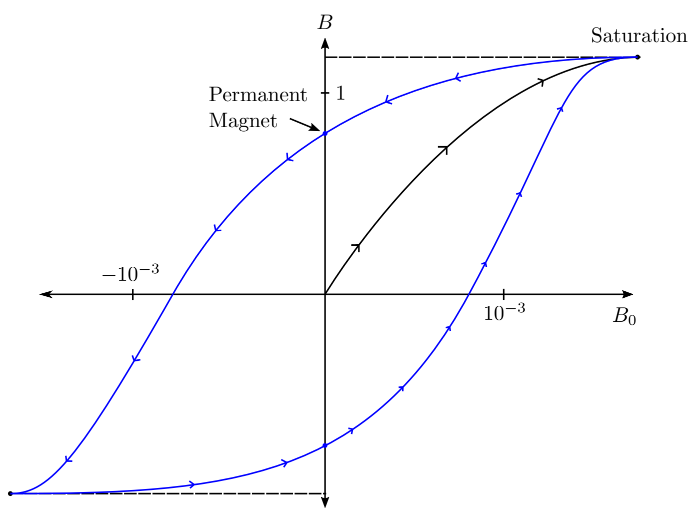
1. Iron-like: Ferromagnetic materials have $K_m \gg 1$ (much greater than one).
2. Paramagnetic materials have $K_m = 1 + \underbrace{\chi_m}_{\text{around } 10^{-4}}$.

3. Diamagnetic materials have $K_m = 1 - \underbrace{|\chi_m|}_{\text{around } 10^{-5}}$.

Ferromagnetic materials are useful:

- They have big K_m due to atoms aligning better than thermally expected (they are usually in disarray).

In the following diagram (hysteresis curve), the x-axis represents the applied magnetic field, and the y-axis the induced magnetic field. If you apply a strong magnetic field to a ferromagnetic material, it will also create its own magnetic field, amplifying the effects of the original magnetic field. What's interesting here is that when you remove the external magnetic field, the ferromagnetic field will settle into being a permanent magnet. If you apply a negative magnetic field to the object, the same happens, but in the other direction.

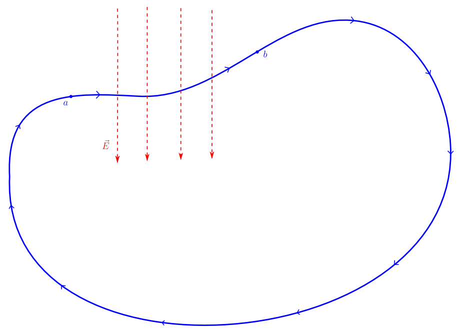


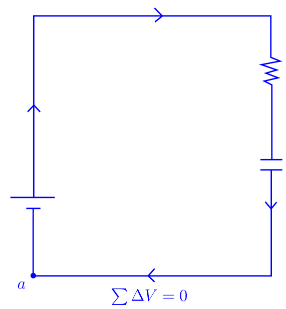
The black curve above traces the induced field as you apply an external $\vec{B}$, and the blue curves show how the induced magnetic field changes with the external field after the initial application.

9 Lecture 9

9.1 A New Fundamental Law of Nature—Faraday's Law

A changing magnetic field produces *loops* in the electric field. Recall for voltage:





$$\Delta V_{ab} = -\int_a^b \vec{E}\,\mathrm{d}\vec{\ell} \quad \text{and} \quad \oint_{\text{loop}} \vec{E} \cdot \mathrm{d}\vec{\ell} = 0.$$

Theorem — *Faraday's Law*
If Φ_m is the magnetic flux, given by

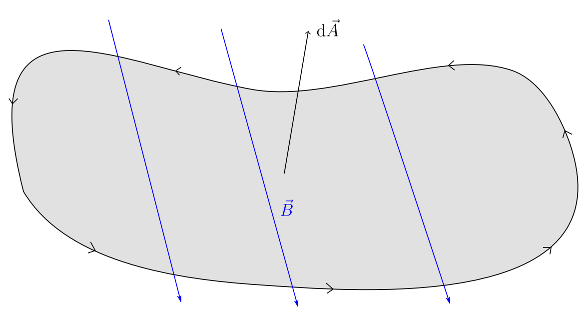
$$\Phi_m = \oint_{\text{surface}} \vec{B} \cdot \mathrm{d}\vec{A},$$

then we have

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_m = -\frac{\mathrm{d}\Phi_m}{\mathrm{d}t}.$$

Alternatively, we have the differential form of Faraday's Law, given by

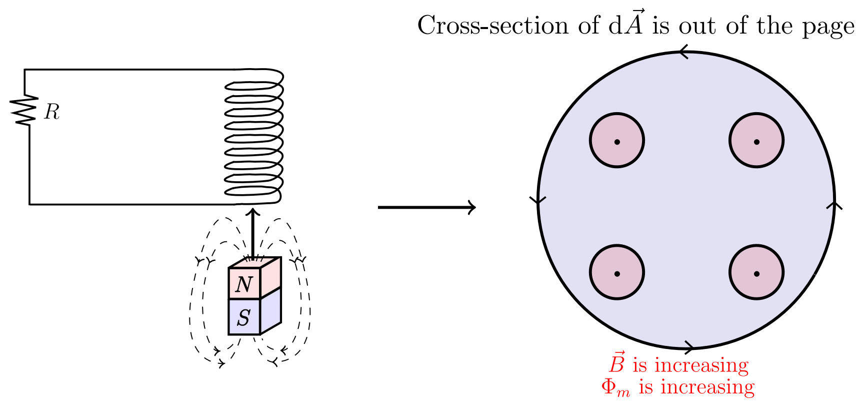
$$\vec{\nabla} \times \vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}.$$



Note. Note that for Gauss' Law, we needed a potential difference in order to be able to generate an electric field, but for Faraday's Law we just need a changing magnetic flux.

9.2 Moving a Magnet Into Coil: EMF

When we move the north pole of a magnet into a coil of wire, we are increasing the magnetic field through the same area, so the magnetic flux is increasing.



Applying Faraday's Law, we have

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} < -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_m < 0.$$

From this, we know that $\vec{E}$ is clockwise, and the current is also in the clockwise direction.

9.3 Rotating Coil in a Magnetic Field

Suppose we were to rotate a square coil in a magnetic field, as follows:

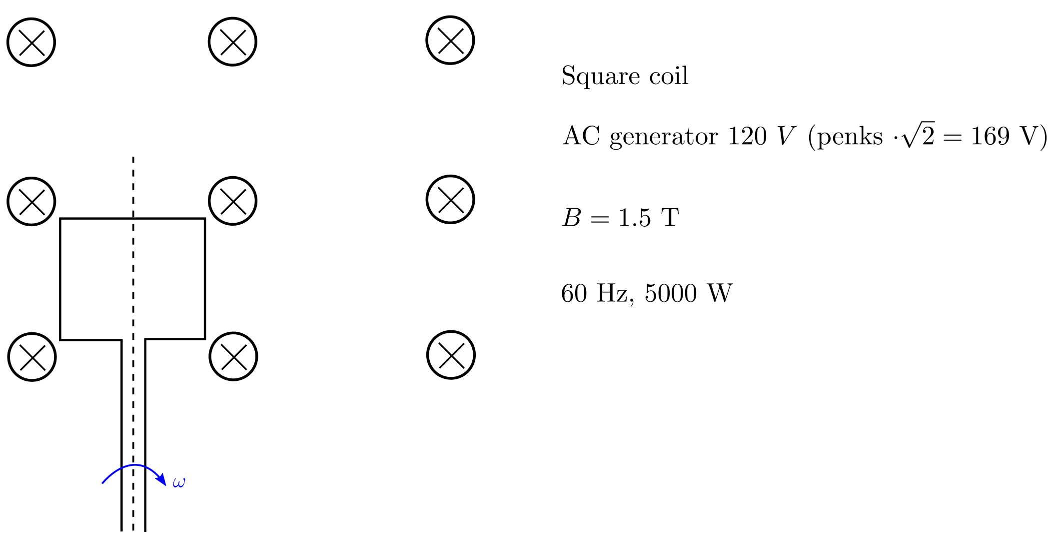


Figure 3: Top view of the loops of wire

Suppose that there are $N = 169$ loops of wire, so there is one loop of wire per volt needed. We can find the

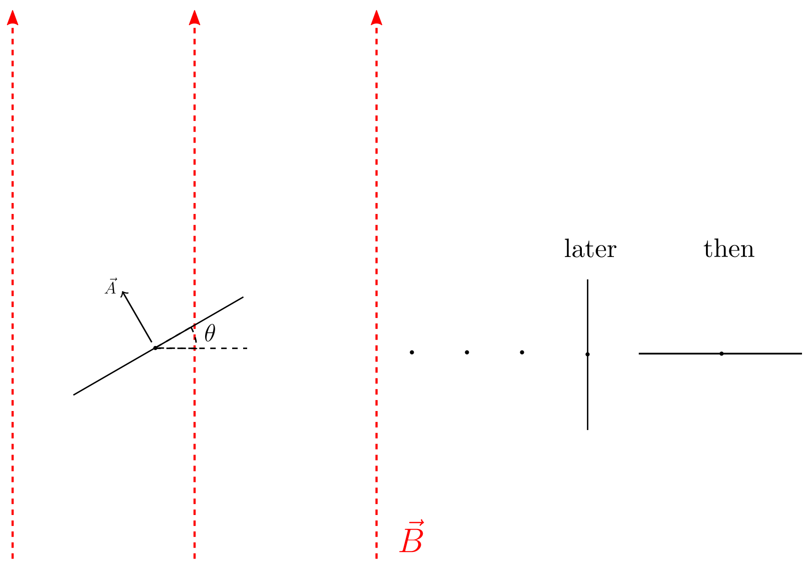


Figure 4: Side view of the loops of wire

angular velocity from the frequency, given by $\omega = 2\pi f = 377$ rad/s. We see from our diagram that

$$\Phi_m = BA \cdot \cos\theta = BA\cos(\omega t),$$

so

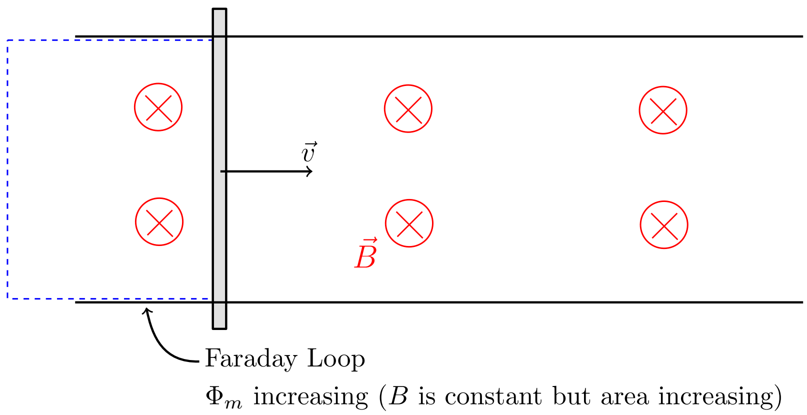
$$\frac{\mathrm{d}\Phi_m}{\mathrm{d}t} = -BA\omega\sin(\omega t).$$

The electromotive force $\varepsilon = BA\omega$ should be equal to 1 volt (because each of the 169 loops contributes 1 volt to the net potential). Thus we can solve for the area:

$$A = \frac{\varepsilon}{B\omega} \approx 26.5\ \text{cm}^2.$$

9.4 Motional EMF

Consider a wire laying on rails, with magnetic field between the rails, as shown:



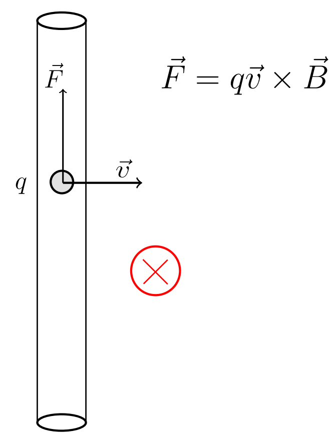
If the width of the Faraday loop is x, then the area is

$$A = xL, \text{ and } \Phi_m = B \cdot A = B \times L.$$

Furthermore, we have

$$\varepsilon = -\frac{\mathrm{d}\Phi_m}{\mathrm{d}t} = -BL \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = -BLv$$

across the end of the slider. From another point of view, we see



The magnetic force on the charges inside the rod move the positive charges upwards, and the negative charges downwards. This creates an electric field in the rod from top to bottom, which counteracts the magnetic force. From either perspective, we reach the same conclusion that there is a potential difference across the rod.

10 Lecture 10

10.1 Lenz's Law, Eddy Currents

This is useful for determining the sign and direction of things in a system involving magnetic fields/forces.

> **Note.** When changing a magnetic system, forces and currents act in the direction to *oppose* the change.

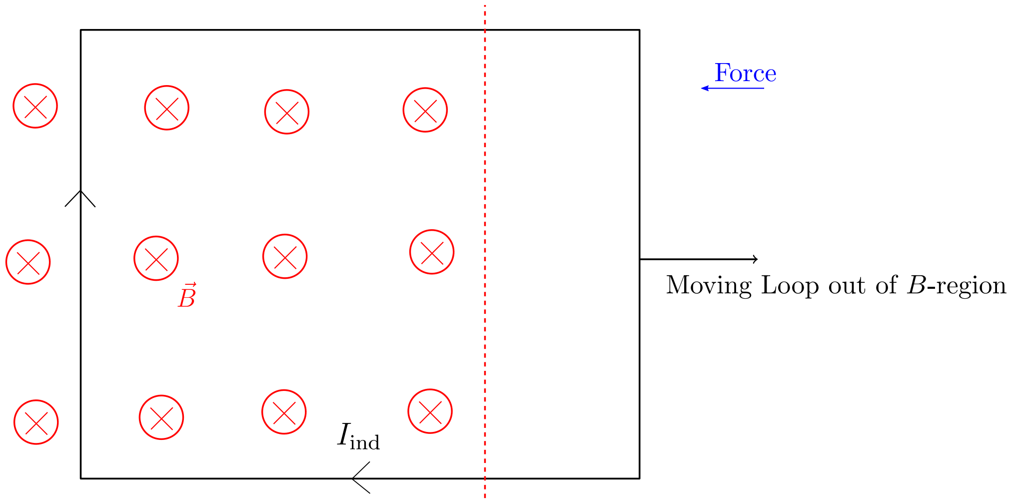


Figure 5: Pulling a Loop of Wire out of a Magnetic Field

As you pull the loop of wire out of the magnetic field that is into the page, the area of the loop that is in the constant magnetic field will decrease. Thus the loop of wire will have an induced current such that it will generate more field into the page. By the right-hand rule, we can see that the induced current should run clockwise. Furthermore, the magnetic forces on the top and bottom portions of the wire are equal and in opposite directions, and so cancel out. There is a force on the left part of the loop that is pointing to the left, which opposes the force that is pulling the loop out of the magnetic field.

11 Lecture 11

11.1 Motional EMF—Conductors Moving in Magnetic Fields

See the diagram in section 9.4. From the Lorentz force equation, we have

$$\vec{F} = q\vec{v} \times \vec{B}.$$

The electric field generated by the separation of charges (positives to the top, negatives to the bottom) in the conductor will create a force equal to and opposing the magnetic force. In other words,

$$\vec{F}_E = q\vec{E} = -q\vec{v} \times \vec{B} = -\vec{F}_M,$$

so $\vec{E} = -\vec{v} \times \vec{B}$. If the conductor is connected to metal rails, we see that this electric field will actually send the charges through the loop. The magnitude of this EMF is given by $\varepsilon = BLv$, which agrees with what we got in section 9.4.

We can generalise this to an arbitrary loop of wire with segments all moving in random directions, by cutting the loop up into small segments of length $\mathrm{d}\vec{\ell}$ and integrating:

$$\varepsilon = \oint (\vec{v} \times \vec{B})\,\mathrm{d}\vec{\ell}.$$

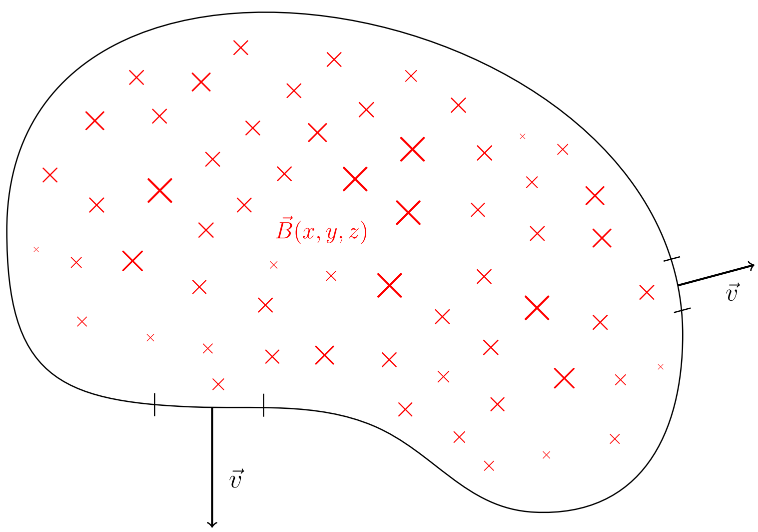


Figure 6: Arbitrary Loop of Wire in Magnetic Field

11.2 Eddy Current Demonstrations

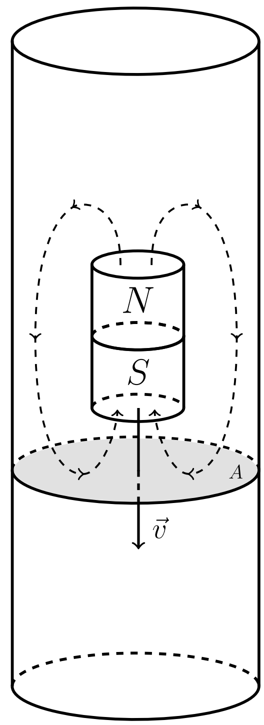


Figure 7: Magnet Falling Through a Conducting Tube

Notice that as the magnet falls, the magnetic flux through the area A is increasing upwards. By Lenz's Law, there will be an induced magnetic field pointing downwards, opposing the change in magnetic flux. This induced magnetic field exerts a force on the magnet, causing it to slow down and fall through the tube more slowly. As the tube gets thicker and more conductive, the eddy currents and induced magnetic field also get larger, causing the magnet to fall even more slowly.

12 Lecture 12

12.1 Magnetic Induction in Circuits

Take $\vec{B}$ uniform, increasing with $B = \alpha \cdot t$ into the page. We choose the area vector $\vec{A}$ to be out of the page, resulting in

$$\Phi = \int \vec{B} \cdot \mathrm{d}\vec{A} = \vec{B} \cdot \vec{A} = -BA < 0,$$

and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -A \cdot \frac{\mathrm{d}B}{\mathrm{d}t} = -A\alpha < 0.$$

Then $\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A \cdot \alpha > 0$. We know that ε is defined by

$$\varepsilon = \oint \vec{E} \cdot \mathrm{d}\vec{\ell} > 0,$$

and the direction of the loop is set by the direction of $\vec{A}$. By the right hand rule, this direction must be counter-clockwise. Thus having a positive ε means $\vec{E}$ is in the same direction as $\mathrm{d}\vec{\ell}$, so $\vec{E}$ is also counter-clockwise.

Note. Just like motional EMF: charges will accumulate almost instantaneously since $\vec{E} = 0$ inside the conductor.

Let's make this a circuit by attaching a resistor: If we look at the direction of this current, it will induce a magnetic field B' inside of the loop, that is pointing out of the page. This opposes the B that is going into the page.

12.2 Concept of Self-inductance L, Solenoid Example

We know that the magnetic field inside a solenoid is

$$B_{\text{in}} = \mu\frac{N}{\ell}I = \mu nI, \qquad (n \coloneqq \tfrac{N}{\ell})$$

where N is the number of turns, ℓ is the length of the solenoid, and I is the current. The flux inside the solenoid is given by

$$\Phi_B = B_{\text{in}} \cdot A \cdot N = (\mu nI)AN.$$

We differentiate this to get the electromotive force (and assume the current is the only one that changes),

$$\varepsilon = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\mu nAN \cdot \frac{\mathrm{d}I}{\mathrm{d}t}.$$

We define the self-inductance L by $L \coloneqq \mu nAN$, which simplifies the above into

$$\boxed{\varepsilon = -L \cdot \frac{\mathrm{d}I}{\mathrm{d}t}.}$$

For other geometries:

- $B = (\text{constant of geometry}) \cdot I$
- $\Phi_B = (\text{another constant of geometry}) \cdot I$
- $\varepsilon = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -(\text{some constant}) \cdot \frac{\mathrm{d}I}{\mathrm{d}t} = -L \cdot \frac{\mathrm{d}I}{\mathrm{d}t}$

12.3 Magnetic Field Energy

Consider a voltage V sending a current I through an inductor L. To find the power dispersed across the inductor, we use

$$P = |I\varepsilon| = IL \cdot \frac{\mathrm{d}I}{\mathrm{d}t} \qquad \text{(into/out of the inductor)}$$

Suppose V starts at zero, then increases. The back ε opposes the increase of current—power needs to be supplied. We use the equation

$$\text{Energy} = \int_0^t \text{Power} \cdot \mathrm{d}t' \qquad (t' \text{ is dummy variable})$$

to find the energy stored in the inductor (U). Thus we have

$$\begin{aligned} U &= \int_0^t P \,\mathrm{d}t' \\ &= \int_0^t IL \cdot \left(\frac{\mathrm{d}I}{\mathrm{d}t'}\right) \mathrm{d}t' \\ &= \int_0^I I'L \,\mathrm{d}I' && \text{(Change dummy variable)} \\ &= \frac{1}{2}I^2L. \end{aligned}$$

Note. This is similar to the equation for the energy of a capacitor, which was $U = \frac{1}{2}CV^2$.

Note. Just as how we think of energy stored in a capacitor as being stored in the electric field between the plates, we may think of energy stored in an inductor as being stored in the magnetic field generated by the inductor.

12.4 Magnetic Field Energy Density—Use Solenoid Calculation

We use the three equations we derived earlier:

$$\begin{aligned} \varepsilon &= -L \cdot \frac{\mathrm{d}I}{\mathrm{d}t} \\ U &= \frac{1}{2}LI^2 \\ B_{\text{in}} &= \mu nI \end{aligned}$$

Plugging in the definition of L, we have

$$\begin{aligned} U &= \frac{1}{2}\mu nNAI^2 \\ &= \frac{1}{2}\mu nNA \cdot \frac{B^2}{\mu^2 n^2} \\ &= \frac{1}{2}NA \cdot \frac{B^2}{\mu n} \\ &= \frac{\ell AB^2}{2\mu}. && (n = \tfrac{N}{\ell}) \end{aligned}$$

Since the volume of the solenoid is $A \cdot \ell$, we have

$$\frac{U}{\text{volume}} = \frac{B^2}{2\mu}.$$

Note. This is analogous to the capacitor's energy density, given by

$$\frac{U}{\text{volume}} = \frac{\varepsilon}{2}E^2.$$

13 Lecture 13

13.1 R-L Circuits

These are circuits with both a resistor (R) and an inductor (L). This is going to be a time dependent circuit with a switch. The switch, resistor, inductor, and battery are all in series. At time $t = 0$, the switch is closed. Let the voltage across the battery be V. We want to find the current i through the circuit as a function of time t. If there was no inductor, the current would be $i(t) = \frac{V}{R}$.

In the inductor, we have an induced voltage $\varepsilon = -L \cdot \frac{\mathrm{d}i}{\mathrm{d}t}$. Furthermore, we know that the current $i = 0$ when $t = 0$. By Kirchoff's Loop Rule, we know that the sum of the voltages in the loop must be zero, in other words

$$\begin{aligned}
V - iR - \varepsilon &= 0\\
V - iR - L \cdot \frac{\mathrm{d}i}{\mathrm{d}t} &= 0\\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{V - iR}{L}.\\
\frac{\mathrm{d}i}{V - iR} &= \frac{1}{L}\,\mathrm{d}t\\
-\frac{1}{R}\ln(V - iR) &= \frac{1}{L}t + C_1\\
\ln(V - iR) &= -\frac{R}{L}t + C_2\\
V - iR &= C_3 e^{-\frac{R}{L}t}\\
iR &= V - C_3 e^{-\frac{R}{L}t}\\
i &= \frac{V}{R} - C_4 e^{-\frac{R}{L}t}.
\end{aligned}$$

We also know that $i(0) = 0$, so

$$\boxed{i(t) = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t} \quad \text{or} \quad \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right).}$$

If we define a new time constant $\tau = \frac{L}{R}$, then we have

$$i(t) = \frac{V}{R}\left(1 - e^{-\frac{t}{\tau}}\right).$$

Note. This is analogous to the RC time constant for capacitors.

If we were to instead connect a charged inductor to a resistor, closing the switch would *discharge* the inductor. In this case, we would have

$$i(t) = \frac{V}{R}e^{-\frac{t}{\tau}}.$$

13.2 L-C Circuits

These are circuits with both an inductor (L) and a capacitor (C). We know that the voltage drop across the circuit is zero, just like before, so

$$-L \cdot \frac{\mathrm{d}i}{\mathrm{d}t} - \frac{q}{C} = 0.$$

I'm too lazy to solve this differential equation, so just know that

$$q(t) = A\cos(\omega t + \phi),$$

where $\omega = \frac{1}{\sqrt{LC}}$. We may also write $q(t) = q_{\text{max}}\cos(\omega t)$. The current is then

$$i(t) = \frac{\mathrm{d}q}{\mathrm{d}t} = -q_{\text{max}} \cdot \omega \sin(\omega t).$$

14 Lecture 14

14.1 Transformers: Mutual Inductance

Consider a toroid with two different sections with loops of wire. Let the first section have N_1 turns and current i_1, and the second section have N_2 turns and current i_2. Since the toroid is made of iron, there is a magnetic field $\vec{B}$ flowing through it (it is "captured" in the iron because $\frac{\mu}{\mu_0} \approx 1000$).

For our ideal transformer, we assume that all of the magnetic field $\vec{B}$ goes around the toroid. The magnetic field is constant in the toroid, so Φ_m is the same for every loop. We have

$$\begin{aligned}\varepsilon_2 &= N_2 \cdot \frac{\mathrm{d}\Phi_m}{\mathrm{d}t} \\ \varepsilon_1 &= N_1 \cdot \frac{\mathrm{d}\Phi_m}{\mathrm{d}t} \\ \frac{\varepsilon_2}{\varepsilon_1} &= \frac{N_2}{N_1}.\end{aligned}$$

Note. This only works if the magnetic flux is time-varying (AC).

If we look at the power transfer, we get

$$P_{\text{in}} = VI = \varepsilon_1 i_1 = P_{\text{out}} = \varepsilon_2 i_2.$$

To find the current, we can manipulate the above equation to get

$$\frac{i_2}{i_1} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{N_1}{N_2}.$$

For mutual inductance, we have

$$N_2\Phi_{B2} \equiv M_{21}i_1 \quad \text{and} \quad N_1\Phi_{B1} \equiv M_{12}i_2.$$

We won't prove this in this class, but $M_{12} = M_{21} = M$, which we call the mutual inductance.

15 Lecture 15

15.1 Why Would We Want an RLC Circuit?

1. RLC Circuits are useful when you want to generate a specific frequency *or* you want to select a specific frequency.

> **Note.** Used in cell phones, internet, TV, radio, etc. for receiving and transmitting.

2. There is always some resistance in a circuit.
3. Electrical Engineering: All circuits also have capacitance and inductance.
4. Damped oscillation

15.2 RLC Circuits—Damped Oscillation

Suppose that at time $t = 0$, we close the circuit such that the charged capacitor, resistor, and inductor are in series. When the RLC circuit is underdamped, we get

$$-\frac{q}{c} - iR - L \cdot \frac{\mathrm{d}i}{\mathrm{d}t} = 0.$$

Substituting $i = \frac{\mathrm{d}q}{\mathrm{d}t}$, we have

$$L \cdot \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \underbrace{R \cdot \frac{\mathrm{d}q}{\mathrm{d}t}}_{\text{damping term}} + \frac{1}{c} \cdot q = 0.$$

The general solution to this differential equation is given by

$$q(t) = q_0 e^{-\frac{t}{2L/R}} \cos(\omega t + \phi),$$

where $\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$.

> **Note.** You have two parameters in the above equation, namely q_0 and ϕ.

15.3 RLC Circuits—Forced Oscillation

16 Lecture 16

16.1 AC Circuits—Motivation and History

There is power loss in normal circuits when the voltage is low, so we use transformers with "high-voltage lines" in order to reduce the power loss (not possible with DC power) when getting electricity from the source to destination. We have the equation

$$\frac{P_{\text{loss}}}{P} = \frac{PR}{\varepsilon^2}.$$

Note. The power loss decreases with the square of the voltage, so high-voltage lines are used.

16.2 RMS Voltages and Currents

Note. RMS stands for "root mean square".

Alternating currents follow a sinusoidal path, in other words

$$I = I_0 \sin(\omega t + \phi),$$

where $f = \frac{\omega}{2\pi}$, which is 60 Hz for Americans or 50 Hz for Europeans. If we square the current, we have

$$\begin{aligned} I^2 &= I_0^2 \sin^2(\omega t + \phi) \\ &= \frac{I_0^2}{2}\left(1 - \cos(2\omega t + 2\phi)\right), \end{aligned}$$

which is also sinusoidal (but stays positive). Furthermore, we know that the power in a resistor is $P = I^2 R$. The average power is thus

$$P_{\text{avg}} = (I^2)_{\text{avg}} R = \frac{I_0^2}{2} \cdot R = I_{\text{RMS}}^2 R. \qquad \left(I_{\text{RMS}} \coloneqq \frac{I_0}{\sqrt{2}}\right)$$

Similarly, we define V_{RMS} by $V_{\text{RMS}} \coloneqq \frac{V_0}{\sqrt{2}}$.

For $V_{\text{RMS}} = 120$ V, we have $V_0 = V_{\text{RMS}}\sqrt{2} \approx 170$ V (the voltage fluctuates between -170 and 170 volts).

16.3 The *Driven* RLC Circuit

We take a typical RLC circuit and add a time-varying EMF to it. There are two solutions to this:

1. Transitory (sinusoidal but amplitude approaching zero)
2. Steady-state (oscillates sinusoidally forever)

To make things simple for ourselves, we choose a phase such that $q = A\sin(\omega t)$. Then we know the voltage across the capacitor,

$$V_C = \frac{q}{C} = \frac{A}{C}\sin(\omega t).$$

The current can then be found by differentiating:

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} = A\omega\cos(\omega t) = A\omega\sin\left(\omega t + \frac{\pi}{2}\right).$$

And so the voltage across the resistor is given by

$$V_R = RI = RA\omega\sin\left(\omega t + \frac{\pi}{2}\right).$$

Finally, we find the voltage across the inductor:

$$V_L = L \cdot \frac{\mathrm{d}I}{\mathrm{d}t} = -LA\omega^2 \sin(\omega t) = LA\omega^2 \sin(\omega t + \pi).$$

We use the mnemonic "ELI the ICE man". For "ELI", we see that in an inductor (L) the EMF precedes the current. And for "ICE", we see that in a capacitor (C) the current precedes the EMF.

17 Lecture 17

17.1 Reactance in the Driven RLC Circuit

Using the equations from last class, notice that

$$\frac{V_C(\max)}{I(\max)} = \frac{A/C}{A\omega} = \frac{1}{\omega C} = \frac{V_C(\mathrm{RMS})}{I(\mathrm{RMS})}.$$

Definition. *Capacitive Reactance*
We define the *capacitive reactance* by the equation:

$$X_C \coloneqq \frac{1}{\omega C}.$$

Similarly for an inductor, we have

$$\frac{V_L(\max)}{I(\max)} = \frac{LA\omega^2}{A\omega} = \omega L.$$

Definition. *Inductive Reactance*
We define the *inductive reactance* by the equation:

$$X_L \coloneqq \omega L.$$

By Kirchoff's Loop Rule, we have

$$\begin{aligned}\varepsilon &= V_C + V_R + V_L \\ &= X_C I_{\max}\sin(\omega t) + I_{\max} R\sin\left(\omega t + \frac{\pi}{2}\right) + X_L I_{\max}\sin(\omega t + \pi) \\ &= I_{\max}\left(X_C\sin(\omega t) + R\sin\left(\omega t + \frac{\pi}{2}\right) + X_L\sin(\omega t + \pi)\right).\end{aligned}$$

Dividing both sides by $I_{\max}$, we get

$$\frac{\varepsilon}{I_{\max}} = X_C\sin(\omega t) + R\sin\left(\omega t + \frac{\pi}{2}\right) + X_L\sin(\omega t + \pi).$$

17.2 Phasors in the Driven RLC Circuit

If we graphed out all of these values as vectors and added them up, we would see that

$$\frac{\varepsilon_{\max}}{I_{\max}} = \sqrt{R^2 + (X_C - X_L)^2}.$$

Note. This is because if you pay attention to the angles in the original equation for $\frac{\varepsilon}{I_{\max}}$, you see that the middle term is perpendicular to the X_C and X_L terms, which are acting in opposition. Thus we may use the Pythagorean Theorem to get the above.

17.3 Impedance Z in the Driven RLC Circuit

Definition. *Impedance*
We define the *impedance* (units given in Ω) to be

$$Z \coloneqq \sqrt{R^2 + (X_C - X_L)^2}.$$

Furthermore, for the initial phase, we have

$$\tan\phi = \frac{X_L - X_C}{R}.$$

The power consumed by this circuit is given by

$$P_{\text{avg}} = I_{\text{RMS}} V_{\text{RMS}} \cdot \underbrace{\cos\phi}_{\text{“Power factor”}}.$$

Note. To maximise the power of the motor, we want $\cos\phi$ to be as close to 0 as possible. Thus we should add just the right amount of capacitance.

18 Lecture 18

Note. Impedance is a generalisation of resistance for AC circuits.

From last lecture, we learned that

$$I_{\text{max}} = \frac{\varepsilon_{\text{max}}}{Z} = \frac{\varepsilon_{\text{max}}}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}}.$$

If ω can vary, the maximum value of I_{max} occurs when $\frac{1}{\omega C} = \omega L$ (and so $I_{\text{max}} = \frac{\varepsilon_{\text{max}}}{R}$). Note that solving for ω yields

$$\omega = \frac{1}{\sqrt{LC}}, \quad \text{or} \quad f = \frac{1}{2\pi\sqrt{LC}}.$$

When R is small and $\omega = \frac{1}{\sqrt{LC}}$, I_{max} approaches infinity. Furthermore, when the frequency is high the inductor dominates, and when it is low the capacitor dominates.

18.1 Low-pass Filter

Consider an AC circuit that consists of a voltage source ε, and a resistor and capacitor in series (order matters!). We consider the output of this circuit to be the voltage drop across the capacitor, i.e. V_C. Since there is no inductor, we know that

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}.$$

Then we have

$$\begin{aligned} V_{C,\text{max}} &= X_C \cdot I_{\text{max}} \\ &= \frac{1}{\omega C} \cdot \frac{\varepsilon_{\text{max}}}{Z} \\ &= \frac{1}{\omega C} \cdot \frac{\varepsilon_{\text{max}}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \\ &= \frac{\varepsilon_{\text{max}}}{\sqrt{(RC\omega)^2 + 1}}. \end{aligned}$$

Notice that when ω is small ($RC\omega \ll 1$), we have that $V_{C,\text{max}} \approx \varepsilon_{\text{max}}$, and that when ω is large ($RC\omega \gg 1$), $V_{C,\text{max}} \approx 0$. Thus we see that this circuit is a "low-pass filter" because it allows lower frequencies to get through the circuit virtually unchanged, while mitigating or removing higher frequencies.

18.2 High-pass Filter

There are two examples of this: we could switch the order of the resistor and capacitor from the last section (so the output is V_R instead of V_C), or we could swap the capacitor with an inductor. Since there is no capacitor, we have

$$Z = \sqrt{R^2 + (\omega L)^2}.$$

Thus we have

$$\begin{aligned} V_{L,\text{max}} &= X_L \cdot I_{\text{max}} \\ &= \omega L \cdot \frac{\varepsilon_{\text{max}}}{\sqrt{R^2 + (\omega L)^2}} \\ &= \frac{\varepsilon_{\text{max}}}{\left(\frac{R}{\omega L}\right)^2 + 1}. \end{aligned}$$

Notice that when ω is small ($\frac{R}{\omega L} \gg 1$), we have $V_{L,\text{max}} \approx 0$, and when ω is large ($\frac{R}{\omega L} \ll 1$), $V_{L,\text{max}} \approx \varepsilon_{\text{max}}$. This circuit is thus a "high-pass filter" because it allows high frequencies through while blocking lower frequencies.

19 Lecture 19

19.1 Displacement Current

Recall Ampere's Law:

$$\oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{encl}}.$$

When we say I_{encl} to mean enclosed current, we mean that the current goes through the surface. The main thing to take note of is that the surface doesn't need to be planar—it can be any surface.

Consider a current flowing into a capacitor, with an amperium loop that goes around half of the capacitor. There is seemingly no current flowing across the capacitor, a contradiction. What Maxwell notices is that charges build up on the plates of the capacitor—maybe a changing $\vec{E}$ field could produce a current. He modified Ampere's Law to get

$$\oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 (I_{\text{encl}} + I_{\text{disp}}).$$

His idea was that I_{disp} should be proportional to $\frac{\mathrm{d}E}{\mathrm{d}t}$ and proportional to the area A of the parallel plates.

Remember that for capacitors, $E = \frac{V}{d}$ and $C = \varepsilon_0 \cdot \frac{A}{d}$. Thus

$$\begin{aligned} A \cdot \frac{\mathrm{d}E}{\mathrm{d}t} &= \left(\frac{Cd}{\varepsilon_0}\right) \cdot \frac{1}{d} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \\ &= \frac{1}{\varepsilon_0} C \cdot \frac{\mathrm{d}Q}{\mathrm{d}t} \cdot \frac{1}{C} \\ &= \frac{1}{\varepsilon_0} \cdot \frac{\mathrm{d}Q}{\mathrm{d}t}. \end{aligned}$$

Rearranging the above, we get

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = I_{\text{disp}} = \varepsilon_0 \cdot A \cdot \frac{\mathrm{d}E}{\mathrm{d}t}.$$

Thus Maxwell's modification of Ampere's Law takes the form

$$\oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 \left(I_{\text{encl}} + \varepsilon_0 A \cdot \frac{\mathrm{d}E}{\mathrm{d}t}\right).$$

Note. We can substitute $A \cdot \frac{\mathrm{d}E}{\mathrm{d}t}$ for $\frac{\mathrm{d}\Phi_E}{\mathrm{d}t}$.

This forms a parallel with Faraday's Law, which states that a changing Φ_B generates $\vec{E}$, whereas this says that a changing Φ_E generates $\vec{B}$.

Note. A magnetic field (although possibly not large) is generated when charging up a capacitor, parallel to the plates of the capacitor.

Note. This exhibits *exactly* the same behaviour as the wire of radius R from Lecture 6.

19.2 Maxwell's Equations

1. Gauss' Law for $\vec{E}$:

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0}$$

2. Gauss' Law for $\vec{B}$:

$$\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0$$

3. Faraday's Law for a stationary integration path:

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

4. Ampere's law for a stationary integration path:

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 \left(i_C + \varepsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t} \right)_{\text{encl}}$$

Note. In empty space (with no charges nor currents), Maxwell's equations are highly symmetric.

19.3 Plane Electromagnetic Waves (in Vacuum)

19.3.1 Recap on Waves

Given a "wave number" k, we usually describe sinusoidal waves by

$$f(x) = A\sin(kx).$$

We know that the wavelength must satisfy $k\lambda = 2\pi$, so $\lambda = \frac{2\pi}{k}$. If we take $f(x - vt)$, the wave is moving to the *right*. If the function is $f(x + vt)$, then the function is moving to the *left*. Waves can take other shapes, for example

$$f(x,t) = \frac{1}{1 + (x - vt)^2}. \qquad \text{(Moves to the right)}$$

20 Lecture 20

20.1 Describing a Wave in an Arbitrary Direction

We have the equation for a sine wave moving in the +x direction:

$$\begin{aligned} f(x,t) &= A\sin(k(x-vt)) \\ &= A\sin(kx-kvt) \\ &= A\sin(kx-\omega t). && (\omega \coloneqq kv) \end{aligned}$$

Solving for v, we find that $v = \frac{\omega}{k}$.

To generalise this to a sine wave in an arbitrary direction, we replace kx with $\vec{k}\cdot\vec{x} = |\vec{k}|\,|\vec{x}|\cdot\cos\theta$. Here we now have $\vec{k}$, which is a *wave vector*, instead of just a wave number. We now have

$$f(\vec{x},t) = A\sin\left(\vec{k}\cdot\vec{x}-\omega t\right). \qquad (v = \tfrac{\omega}{|\vec{k}|})$$

This forms a series of wave fronts that are all perpendicular to $\vec{k}$ by virtue of the dot product.

20.2 Plane Electromagnetic Waves—What Are They Like?

You have oscillating electric and magnetic waves, perpendicular to each other, and also perpendicular to the direction in which they are travelling.

Note. Because integration is a linear operation, the superposition of waves that satisfy the Maxwell equations also satisfies the Maxwell equations. In other words, the sums of solutions are solutions.

20.3 Intensity of Electromagnetic Waves, Poynting Vector

Definition. *Intensity*
The amount of power over a given area. Its units are usually given in $\mathrm{W/m^2}$.

The Poynting vector is given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}.$$

The intensity, which is also the average of the magnitude of the Poynting vector, is then

$$\begin{aligned} I &= S_{\text{avg}} \\ &= \frac{B_{\max}E_{\max}}{2\mu_0}. \end{aligned}$$

Furthermore, we have that

$$E_{\max} = cB_{\max},$$

where c is the speed of light.

21 Lecture 21

[A bunch of demos]

22 Lecture 22

Consider a square wave front moving at the speed of light in the positive x direction. Suppose that $\vec{E}$ is parallel to $\hat{\jmath}$ and $\vec{B}$ is parallel to $\hat{k}$, and that $\vec{E} = \vec{B} = 0$ when $x > ct$ (you are ahead of the wave front).

These wave fronts are consistent with Maxwell's equations, and any linear combination of wave fronts is also a valid wave front, so you can model any wave front using these square pulses.

Theorem — *Wave Equation*
For electromagnetic waves, we have
$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - c^2 \nabla^2 \mathbf{B} = 0.$$

Note. If you plug in $\vec{B} = B(\vec{x} - \vec{v}t)$ into the above equation, you see that it works out when $|\vec{v}| = c$.

22.1 Energy and Momentum in Electromagnetic Waves

Suppose we have the sine waves:
$$\begin{aligned}\vec{E} &= E_{\text{max}} \sin(kx - \omega t)\hat{\jmath}\\ \vec{B} &= B_{\text{max}} \sin(kx - \omega t)\hat{k}\end{aligned}$$

Let $E_{\text{max}} = 100$ V/m. Then
$$B_{\text{max}} = \frac{E_{\text{max}}}{c} = \frac{100}{3 \cdot 10^8}\ \text{T} = 3.33 \cdot 10^{-7}\ \text{T}.$$

Then the Energy in an $\vec{E}$-field is given by
$$\frac{U_E}{\text{vol}} = \frac{\text{energy}}{\text{volume}} = \frac{\varepsilon_0}{2}E^2 = \frac{8.85 \cdot 10^{-12}}{2}(100)^2 = 4.4 \cdot 10^{-8}\ \frac{\text{J}}{\text{m}^3}.$$

Similarly, the energy in the $\vec{B}$-field is given by
$$\frac{U_B}{\text{vol}} = \frac{B^2}{2\mu_0} = \frac{\left(\frac{E}{c}\right)}{2\mu_0} = \frac{E^2}{2\mu_0 \cdot \frac{1}{\mu_0 \varepsilon_0}} = \frac{\varepsilon_0}{2}E^2.$$

Thus we see that the energy stored in both waves is the same. If we add up the waves and average them, we get
$$\begin{aligned}\left(\frac{U_{EM}}{\text{vol}}\right)_{\text{avg}} &= \frac{U_E + U_B}{\text{vol}}\\ &= \frac{1}{2}\left(\frac{\varepsilon_0}{2}E_{\text{max}}^2 + \frac{\varepsilon_0}{2}E_{\text{max}}^2\right)\\ &= \frac{\varepsilon_0}{2}E_{\text{max}}^2\\ &= 4.4 \cdot 10^{-8}\ \frac{\text{J}}{\text{m}^3}.\end{aligned}$$

To find the intensity (or power per unit area), we have
$$\begin{aligned}\frac{P}{A} &= \frac{U}{A \cdot t}\\ &= \frac{U_{EM} \cdot V}{A \cdot t}\\ &= \frac{U_{EM} \cdot A \cdot ct}{A \cdot t}\\ &= U_{EM} \cdot c.\end{aligned}$$

22.2 Electromagnetic Waves in Matter

When we have a medium, then we use ε and μ instead of ε_0 and μ_0, respectively (to take into account the constants). We have

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{K_m\mu_0K_e\varepsilon_0}} = \frac{c}{\sqrt{K_mK_e}}.$$

Definition. *Refractive Index*
We define the refractive index by

$$n = \sqrt{K_mK_e},$$

which is always greater than 1.

23 Lecture 23

23.1 The Types of Light

There are many types of light: Gamma rays, X-rays, ultraviolet light, visible light, infrared light, microwaves, and radio waves. Other descriptors include:

- Linearly, circularly, elliptically polarised, as well as unpolarised
- Coherent (like lasers) and incoherent (like sunlight)
- Incandescent, LED, fluorescent, neon, laser

Note. What we as humans perceive doesn't necessarily line up with real-world physics, because we are more sensitive to certain wavelengths ("blue", "green", and "red").

23.2 Wave Fronts, Rays, and Beams

Consider a point emitting electromagnetic waves in concentric circles. As you get further and further away, the circles' curvature keeps on decreasing until it basically becomes a plane wave.

23.3 Reflection and Refraction of Light

Reflection is exactly what you think it is. We call the angle measured from the normal of the reflective surface to the light ray the "angle of incidence" and the angle from the normal of the reflective surface to the reflected ray the "angle of reflection". For perfect reflections, we have $\theta_i = \theta_r$.

Note. This is why some mirrors are curved—they reflect light towards a common point and concentrate it. The light comes in as virtually parallel rays (because it is so far away).

Refraction is when a light ray passes from one transparent medium to another, "bending" in the process. Usually a part of the light ray is reflected when it passes through the media boundary.

Theorem — *Snell's Law*
Let the angle of incidence be denoted θ_1 and the angle of refraction θ_2. Then

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where n_1 and n_2 are the indices of refraction for the two materials, respectively.

Note. This is actually why prisms are able to separate colours, because the index of refraction for certain kinds of glass is a function of the wavelength of the light passing through it.

23.4 Huygen's Principle

In short, Huygen's principle states that we can think of each point on a wave front as sources of spherical waves. If you do this for an infinite number of points, you see that the waves overlap to form the next wave front.

24 Lecture 24

24.1 Snell's Law for Refraction

By Snell's Law, we know that $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Solving for $\sin\theta_2$, we have

$$\sin\theta_2 = \frac{n_1 \sin\theta_1}{n_2}.$$

If $n_1 < n_2$, then $\sin\theta_2 = \frac{n_1}{n_2} \cdot \sin\theta_1 < 1$ and we can find θ_2. If $n_1 > n_2$, then $\sin\theta_2 = \frac{n_1}{n_2} \cdot \sin\theta_1$ which an either be greater than 1 (no solution) or less than 1 (bends away from the normal).

Note. Total reflection is used in fibre optic cables to transmit data over thousands of kilometres.

Definition. *Polarised Light*
Light where the electric field is oscillating in only one direction.

25 Lecture 25

25.1 Polarisation: Malus' Law

We first consider the "front view of polarisation", where we can see the direction of travel of the light. We then put a vertical polariser in the way of the light, which filters out all non-vertical light. We then put another polariser, this time oriented at an angle ϕ relative to the vertical. We know that at any given time, the electric field that's coming from the source is given by

$$\vec{E}_0 = E_0 \cos\theta(t) \cdot \hat{\imath} + E_0 \sin\theta(t) \cdot \hat{\jmath},$$

where θ is a function of t. Similarly, because the first polariser filters out all non-vertical light, we have

$$\vec{E}_1 = E_0 \cos\theta(t) \cdot \hat{\imath}.$$

The second polariser only accepts electric fields that are parallel to its axis of polarisation, so we project onto the axis and get

$$\frac{\left|\vec{E}_2\right|}{\left|\vec{E}_1\right|} = \cos\phi.$$

Since $I \propto E^2$, we have

$$\frac{I_2}{I_1} = \cos^2\phi.$$

25.2 Dispersion

Note. Simply put, dispersion is the idea that objects don't have constant indices of refraction and so refract different wavelengths of light in different directions.

For example, the index of refraction could be defined by $n = n_0\lambda$ for some constant n_0.

26 Lecture 26

26.1 Plane Mirrors and Images

The way images work is that our brain perceives incoming light rays reflecting off of a mirror the same way as it would perceive an object on the other side of the mirror. We can find the location of the image by tracing the light rays backwards through the mirror to see where they converge. For a plane mirror, the image distance is the same as the object distance (which can be shown with basic geometry).

For multiple mirrors, we can either try finding the end result all at once via ray-tracing or we can find the "image of the image".

26.2 Concave Focusing Mirror

Let R be the radius of curvature of our spherical mirror.

1. If light comes from the centre of curvature, all the light coming from the source will reflect perfectly back to the source.

2. If light is coming from infinitely far away, then the incoming light rays would all be parallel. Through some geometry and approximations, we see that the rays converge on the line from the centre to the mirror, parallel to the incoming rays, halfway between the centre of curvature and the mirror.

> **Note.** Due to the approximations, when we have spherical mirrors (in practice) the focal distance changes. Real telescopes utilise parabolic mirrors for a constant focus.

27 Lecture 27

27.1 Focal Length Equation

Theorem — *Focal Length Equation*
If the distance from the mirror/lens to the object is given by s, and the distance to the image is denoted s', then
$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'},$$
where f is the focal length.

27.2 Ray Diagrams

For a converging lens, we draw three rays to find the image of an object. The rays all start from the tip of the object, and are as follows:

1. Parallel to the axis, then to the far focus
2. Through the centre of the lens
3. To the near focus, then parallel

We define magnification by
$$m = \frac{h'}{h} = -\frac{s'}{s}.$$
The negative takes into account the fact that the image is inverted.

27.3 Optics Sign Rules

1. When the object is on the same side of the reflecting/refracting surface as the incoming light, the object distance s is positive, otherwise negative
2. When the image is on the same side of the reflecting/refracting surface as the outgoing light, the image distance s' is positive, otherwise negative
3. When the centre of curvature C is on the same side as the outgoing light, the radius of curvature is positive, otherwise negative

28 Lecture 28

28.1 Lens-Maker's Equation

The Lens-maker's equation is given by

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

assuming that your lens is in air. We call $\frac{1}{f}$ the "bending power", whose units are metres^{-1}, or diopters.

28.2 Cameras

The focal length of the camera lens should approximately be equal to the distance from the lens to the screen, to be able to focus images onto the screen.

Note. In reality the focal distance should be slightly less because the object is not always at infinity, and should be adjustable.

Pinhole cameras don't need to be focused, but you need a very small hole to produce a sharp image, but this reduces the amount of light that the camera can get.

Definition. *F-stop*
The F-stop is defined by the distance to the back of the box over the diameter of the lens, or

$$\text{f-stop} \coloneqq \frac{d}{d_{\text{lens}}}.$$

Note. A smaller f-stop is generally better because it implies a larger lens, but the small-angle approximation become worse and worse. Real cameras use multiple lenses to get a better image.

28.3 Magnifying Lenses

The goal of a magnifying lens is not to make the image larger, but rather allow you to focus on something that is closer. The book derives that

$$M \coloneqq \frac{\theta'}{\theta} = \frac{25\text{ cm}}{f}. \qquad \text{(object at } \infty\text{)}$$

29 Lecture 29

29.1 Microscopes

In a microscope, the first lens magnifies the real object, and the second lens magnifies the image produced by lens 1. We know that the magnification of the first lens is going to be

$$M_1 = -\frac{s'}{s} \approx \frac{s'}{f_1},$$

so make s small and s' large. We know from before that the second lens acts as a magnifying glass, looking at the image from lens 1 as input. Thus we have

$$M_2 = \frac{25\text{ cm}}{f_2},$$

and so the total magnification due to both lenses is

$$M = M_1 M_2 = \frac{s'}{f_1} \cdot \frac{25\text{ cm}}{f_2}.$$

29.2 Telescopes

For the first lens for a telescope, we know that $s' = f$ (because $s = \infty$). Furthermore, we know

$$\theta' \approx \tan\theta' = \frac{h}{f_2} \quad \text{and} \quad \theta \approx \tan\theta = \frac{h}{f_1}.$$

Thus

$$M \coloneqq \frac{\theta'}{\theta} = \frac{f_1}{f_2}.$$

30 Lecture 30

30.1 Double-slit Interference—Maxima and Minima

Suppose we have a barrier with two slits in it, to allow for diffraction. If $\vec{E}_1$ denotes the electric field from the first slit, and $\vec{E}_2$ the electric field from the second slit, then at a point P away from the slits we have

$$\vec{E} = \vec{E}_1 + \vec{E}_2.$$

If $\vec{E}_1 = E_0 \sin(\omega t)\hat{k}$, at point P, then $\vec{E}_2 = E_0 \sin(\omega t + \phi)\hat{k}$, where

$$\phi = \frac{2\pi}{\lambda}(r_2 - r_1).$$

When $\phi = 2k\pi, k \in \mathbb{N}$, then we have *constructive interference*. When $\phi = (2k+1)\pi, k \in \mathbb{N}$, then we have *destructive interference*. However,

$$\phi = \frac{2\pi}{\lambda}(r_2 - r_1),$$

so we have constructive interference when $\Delta r = 2\pi\phi\lambda = m\lambda$. We have destructive interference when $\Delta r = \left(m + \frac{1}{2}\right)\lambda$.

31 Lecture 31

31.1 Maxima and Minima (Continued)

Using the same situation as last lecture, we want to find Δr as a function of y. Via some trigonometry nonsense, we have $\Delta r = d\sin\theta$. Thus for constructive interference, we have the equation

$$d\sin\theta = m\lambda,$$

and for destructive interference we have

$$d\sin\theta = \left(m + \frac{1}{2}\right)\lambda.$$

Example. Suppose we have a He/Ne laser with wavelength $\lambda = 632.8$ nm, which passes through two slits spaced 0.5 mm apart. Furthermore, let R be the distance from the slits to the screen. Then we know that $\sin\theta$ is an integral multiple of $\frac{\lambda}{d}$, which is

$$\frac{\lambda}{d} = \frac{632.8 \cdot 10^{-9}\text{ m}}{5 \cdot 10^{-4}\text{ m}} = 1.26 \cdot 10^{-3}\text{ m}.$$

If $R = 1$ meter, then the separation between maxima is

$$\begin{aligned}
\Delta y &= R(\tan\theta_{m+1} - \tan\theta_m) \\
&\approx R(\theta_{m+1} - \theta_m) && \text{(Small angle)} \\
&= R \cdot 1.26 \cdot 10^{-3} \\
&= 1.26\text{ mm}.
\end{aligned}$$

31.2 Double-slit Intensity

For an arbitrary point P,

$$E_P = 2 \cdot \cos\left(\frac{\phi}{2}\right) = 2E_0\cos\left(\frac{\pi}{\lambda}(r_2 - r_1)\right).$$

Since intensity scales with the square of the electric field,

$$\frac{I_2}{I_1} = \frac{4E_0^2\cos^2(\frac{\phi}{2})}{E_0^2} = 4\cos^2\left(\frac{\phi}{2}\right).$$

32 Lecture 32

32.1 Triple-slit Phasors

If each vector has magnitude E_0 and is an angle ϕ relative to the previous vector, then we have a few special cases:

- $\phi = 0$
  The vectors are all in the same direction, so
  $$E_p = 3E_0. \qquad \text{(maximum)}$$
- $\phi = \frac{2\pi}{3}$
  The vectors rotate to form an equilateral triangle, so
  $$E_p = 0.$$
- $\phi = \pi$
  The first two vectors cancel, leaving the last one, so
  $$E_p = E_0. \qquad (2^{\text{nd}} \text{ maximum})$$

32.2 Diffraction Gratings

As a function of ϕ, you have constructive interference where $\phi = \pm m \cdot 2\pi$, and destructive interference where $\phi = \frac{2\pi}{N}$. If you only have a single slit, you can still have destructive interference when $d \sin\theta = \frac{\lambda}{2}$ (which can be shown using a picture).

33 Lecture 33

33.1 Galilean Relativity

Inertial frames are frames in which if $\vec{F} = m\vec{a} = 0$, then $\vec{a} = 0$. Consider two different coordinate systems, one called S which is the "stationary" coordinate system, and another that is shifted over from S labelled S', which is a "moving" coordinate system that is moving to the right at speed v. Then we have

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}$$

Differentiating with respect to t, we have

$$\begin{aligned} v_x &= v_{x'} + v \\ v_y &= v_{y'} \\ v_z &= v_{z'} \end{aligned}$$

Assuming v is constant, then the accelerations for any object is the same in either coordinate system.

33.2 Einstein Relativity

We know that the speed of light is

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \approx 3 \cdot 10^8 \ \frac{\text{m}}{\text{s}}.$$

What is the inertial frame in which this applies?

33.2.1 Michelson–Morley Experiment

The Michelson–Morley experiment measured the speed of light from a distant star from two different points in Earth's orbit (6 months apart), and measured the same speed of light.

Einstein proposed that there is no "preferred" inertial frame, i.e. that Galilean relativity is wrong. Let's assume that electromagnetism laws are correct, and that the speed of light is constant.

33.2.2 The "Light Clock"

Consider a container that holds a light emitter, a detector, and a mirror, where the emitter emits a beam of light that reflects off of the mirror and into the detector. If the distance from the emitter/detector to the mirror is D, then the time is takes for the light to be detected is $\Delta\tau = \frac{2D}{c}$.

Now consider the situation in which the container is moving to the right extremely quickly at a speed v. The light beam must be angled forwards in order to reflect back and hit the detector. Denoting the total distance travelled by the light by s, we have

$$\begin{aligned} s^2 &= D^2 + \left(\frac{v\Delta t}{2}\right)^2 \\ \left(c \cdot \frac{\Delta t}{2}\right)^2 &= \left(\frac{c\Delta\tau}{2}\right)^2 + \left(\frac{v\Delta t}{2}\right)^2 \\ c^2(\Delta t)^2 &= c^2(\Delta\tau)^2 + v^2(\Delta t)^2 \\ (\Delta\tau)^2 &= \frac{c^2 - v^2}{c^2} \cdot (\Delta t)^2 \\ \Delta t &= \frac{c}{\sqrt{c^2 - v^2}} \cdot \Delta\tau. \end{aligned}$$

From this we can see that assuming that the speed of light is constant, that faster objects experience time slower than slower or stationary objects.