Theorem — *Cauchy-Schwarz Inequality*
Let V be an inner product space over F. Then for all $v_1, v_2 \in V$, we have

$$|\langle v_1, v_2\rangle| \leq \|v_1\| \|v_2\|.$$

Proof. If either $v_1 = 0$ or $v_2 = 0$, then the result is immediate, so we may assume that $v_1 \leq 0$, $v_2 \neq 0$.

We define v such that

$$v = v_2 - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} v_1, \qquad \text{(Projection onto } v_1\text{)}$$

and claim that $\langle v, \alpha v_1\rangle = 0$ for all $\alpha \in F$. Observe that

$$\begin{aligned}
\langle v, \alpha v_1\rangle &= \left\langle v_2 - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} v_1, \alpha v_1 \right\rangle \\
&= \langle v_2, \alpha v_1\rangle - \left\langle \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} v_1, \alpha v_1 \right\rangle \\
&= \langle v_2, \alpha v_1\rangle - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} \langle v_1, \alpha v_1\rangle \\
&= \bar{\alpha} \langle v_2, v_1\rangle - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} \bar{\alpha} \langle v_1, v_1\rangle \\
&= \bar{\alpha} \langle v_2, v_1\rangle - \bar{\alpha} \langle v_2, v_1\rangle \\
&= 0.
\end{aligned}$$

Thus the claim has been shown. We now prove the rest of the theorem. We know that inner products of vectors with themselves are always non-negative, so

$$\begin{aligned}
0 &\leq \langle v, v\rangle \\
&= \left\langle v, v_2 - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} v_1 \right\rangle \\
&= \langle v, v_2\rangle + \left\langle v, -\frac{\langle v_2, v_1\rangle}{\|v_1\|^2} v_1 \right\rangle \\
&= \langle v, v_2\rangle \\
&= \left\langle v_2 - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} v_1, v_2 \right\rangle \\
&= \langle v_2, v_2\rangle + \left\langle -\frac{\langle v_2, v_1\rangle}{\|v_1\|^2} v_1, v_2 \right\rangle \\
&= \langle v_2, v_2\rangle - \frac{\langle v_2, v_1\rangle}{\|v_1\|^2} \langle v_1, v_2\rangle \\
&= \|v_2\|^2 - \frac{\overline{\langle v_1, v_2\rangle} \langle v_1, v_2\rangle}{\|v_1\|^2} \\
&= \|v_2\|^2 - \frac{|\langle v_1, v_2\rangle|^2}{\|v_1\|^2} \\
0 &\leq \|v_1\|^2 \|v_2\|^2 - |\langle v_1, v_2\rangle|^2 \\
|\langle v_1, v_2\rangle|^2 &\leq \|v_1\|^2 \|v_2\|^2 \\
|\langle v_1, v_2\rangle| &\leq \|v_1\| \|v_2\|
\end{aligned}$$

$\square$

Theorem — *Minkowski Inequality*
Let V be an inner product space over F. Then for all $v_1, v_2 \in V$, we have

$$\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|.$$

Proof. Observe that

$$\begin{aligned}
\|v_1 + v_2\|^2 &= \langle v_1 + v_2, v_1 + v_2 \rangle \\
&= \langle v_1, v_1 \rangle + \langle v_2, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_2 \rangle \\
&= \langle v_1, v_1 \rangle + \langle v_2, v_1 \rangle + \overline{\langle v_1, v_2 \rangle} + \langle v_2, v_2 \rangle
\end{aligned}$$

Suppose $\langle v_1, v_2 \rangle = \alpha + \beta\sqrt{-1}$, so then $\langle v_1, v_2 \rangle + \overline{\langle v_1, v_2 \rangle} = \alpha + \beta\sqrt{-1} + \alpha - \beta\sqrt{-1} = 2\alpha$.

$$\begin{aligned}
&= \|v_1\|^2 + 2\alpha + \|v_2\|^2 \\
&\leq \|v_1\|^2 + 2\sqrt{\alpha^2 + \beta^2} + \|v_2\|^2 \\
&= \|v_1\|^2 + 2\left|\langle v_1, v_2 \rangle\right| + \|v_2\|^2 \\
&\leq \|v_1\|^2 + 2\|v_1\|\|v_2\| + \|v_2\|^2 && \text{(Cauchy-Schwarz)} \\
&= (\|v_1\| + \|v_2\|)^2.
\end{aligned}$$

Thus we have shown that $\|v_1 + v_2\|^2 \leq (\|v_1\| + \|v_2\|)^2$, so we have $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$. $\square$

Theorem — *Gram-Schmidt Theorem*
Let V be an inner product space over F and $\varnothing \neq S_n = \{v_1, \dots, v_n\} \subseteq V$ a linearly independent set. Then there exist $y_1, \dots, y_n \in V$ such that

(i) $y_1 = v_1$

(ii) $T_n = \{y_1, \dots, y_n\}$ is an orthogonal set and linearly independent

(iii) $\operatorname{span}(T_n) = \operatorname{span}(S_n)$

Proof. We construct T_n from S_n using the Gram-Schmidt Process, and proceed by induction. When $n = 1$, we set $y_1 = v_1$, and the statements above clearly hold. Suppose the statement is true for some $n = k$, so

(i) $y_1 = v_1$,

(ii) $T_k = \{y_1, \dots, y_k\}$ is an orthogonal set and linearly independent,

(iii) $\operatorname{span}(T_k) = \operatorname{span}(S_k)$.

We define

$$y_{k+1} = v_{k+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1}, y_i\rangle}{\|y_i\|^2} y_i. \tag{*}$$

We know that $y_{k+1} \neq 0$, because otherwise $v_{k+1} \in \operatorname{span}(T_k) = \operatorname{span}(S_k)$ and $\{v_1, \dots, v_{k+1}\}$ would not be linearly independent, a contradiction.

We claim that $\langle y_{k+1}, y_j\rangle = 0$ for all $j = 1, \dots, k$. We can see that

$$\begin{aligned}
\langle y_{k+1}, y_j\rangle &= \left\langle v_{k+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1}, y_i\rangle}{\|y_i\|^2} y_i, y_j \right\rangle \\
&= \langle v_{k+1}, y_j\rangle - \sum_{i=1}^{k} \frac{\langle v_{k+1}, y_i\rangle}{\|y_i\|^2} \langle y_i, y_j\rangle \\
&= \langle v_{k+1}, y_j\rangle - \frac{\langle v_{k+1}, y_j\rangle}{\|y_j\|^2} \langle y_j, y_j\rangle && (\langle y_i, y_j\rangle = \delta_{ij}) \\
&= \langle v_{k+1}, y_j\rangle - \langle v_{k+1}, y_j\rangle \\
&= 0.
\end{aligned}$$

Since $0 \notin T_{k+1} = \{y_1, \dots, y_{k+1}\}$ and T_{k+1} is an orthogonal set, it must be linearly independent. Furthermore, we know by rearranging (*) that $v_{k+1} \in \operatorname{span}(y_1, \dots, y_{k+1})$ and has a non-zero component for y_{k+1}, and so by the Replacement Theorem

$$\operatorname{span}(T_{k+1}) = \operatorname{span}(y_1, \dots, y_{k+1}) = \operatorname{span}(y_1, \dots, y_k, v_{k+1}) = \operatorname{span}(v_1, \dots, v_{k+1}) = S_{k+1}.$$

$\square$

Theorem — *Orthogonal Theorem*
Let V be a finite-dimensional inner product space over F. Then V has an orthogonal basis. If $F = \mathbb{R}$ or $\mathbb{C}$, then V has an orthonormal basis.

Proof. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V. By the Gram-Schmidt Theorem, we know that there exists a linearly independent and orthogonal set $\mathcal{C} = \{w_1, \dots, w_n\}$ such that $\operatorname{span}(\mathcal{C}) = \operatorname{span}(\mathcal{B}) = V$. Because $\mathcal{C}$ is both linearly independent and spans V, it must be a basis for V, and so is an orthogonal basis for V.

If $F = \mathbb{R}$, then $\left\{\frac{w}{\|w\|} \mid w \in \mathcal{C}\right\}$ is an orthonomal basis for V as $\|w\| \in \mathbb{R}$ for all $w \in \mathcal{C}$. The same reasoning applies for $F = \mathbb{C}$. $\square$

Theorem — *Orthogonal Decomposition Theorem*
Let V be an inner product space over F (not necessarily finite-dimensional), with $S \subseteq V$ a finite-dimensional subspace, and $v \in V$. Then there exist unique $s \in S$ and $s^\perp \in S^\perp$ such that $v = s + s^\perp$. In particular, $V = S + S^\perp$ and $S \cap S^\perp = 0$, so $V = S \perp S^\perp$. Moreover, if

$$v = s + s^\perp, s \in S, s^\perp \in S^\perp,$$

then

$$\|v\|^2 = \|s\|^2 + \left\|s^\perp\right\|^2. \qquad \text{(Pythagorean Theorem)}$$

If, in addition, V is a finite-dimensional inner product space over F, then

$$\dim(V) = \dim(S) + \dim(S^\perp).$$

Proof. By the Orthogonal Theorem, we know that there exists an orthogonal basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for S, the finite-dimensional inner product space over F. We will first show existence. Let $v \in V$. We define $s \in S = \operatorname{span}(\mathcal{B})$ by

$$s = \sum_{i=1}^n \frac{\langle v, v_i\rangle}{\|v_i\|^2} v_i,$$

and $s^\perp = v - s$. Then $v = s + s^\perp$ and $S \cap S^\perp = 0$, i.e. $V = S \oplus S^\perp$. We first show that $s^\perp \perp v_j$ for $j = 1, \ldots, n$. Observe that

$$\begin{aligned}
\left\langle s^\perp, v_j\right\rangle &= \langle v - s, v_j\rangle \\
&= \langle v, v_j\rangle - \langle s, v_j\rangle \\
&= \langle v, v_j\rangle - \left\langle \sum_{i=1}^n \frac{\langle v, v_i\rangle}{\|v_i\|^2} v_i, v_j\right\rangle \\
&= \langle v, v_j\rangle - \sum_{i=1}^n \frac{\langle v, v_i\rangle}{\|v_i\|^2} \langle v_i, v_j\rangle \\
&= \langle v, v_j\rangle - \frac{\langle v, v_j\rangle}{\|v_j\|^2} \langle v_j, v_j\rangle && (\langle v_i, v_j\rangle = \delta_{ij}) \\
&= \langle v, v_j\rangle - \langle v, v_j\rangle \\
&= 0.
\end{aligned}$$

Since $s = \sum_{i=1}^n \alpha_i v_i \in S$, we have

$$\begin{aligned}
\left\langle s, s^\perp\right\rangle &= \left\langle \sum_{i=1}^n \alpha_i v_i, s^\perp\right\rangle \\
&= \sum_{i=1}^n \alpha_i \left\langle v_i, s^\perp\right\rangle \\
&= 0.
\end{aligned}$$

Thus we have shown that $s^\perp \perp s$ for all $s \in S$, so $s^\perp \in S^\perp$, as needed. We must now show uniqueness. Suppose

$$s + s^\perp = v = r + r^\perp.$$

Then $s - r = r^\perp - s^\perp$. However, because $S \cap S^\perp = 0$, we have $s - r = 0 = r^\perp - s^\perp$, so $s = r$ and $s^\perp = r^\perp$. Therefore the decomposition of v is unique.

To show the Pythagorean Theorem, we have

$$\|v\|^2 = \left\|s + s^\perp\right\|^2 = \left\langle s + s^\perp, s + s^\perp\right\rangle = \langle s, s\rangle + \left\langle s, s^\perp\right\rangle + \left\langle s^\perp, s\right\rangle + \left\langle s^\perp, s^\perp\right\rangle = \|s\|^2 + \left\|s^\perp\right\|^2.$$

If V is finite-dimensional, by the Counting Theorem, we have that

$$\dim(V) = \dim(S + S^{\perp}) = \dim(S) + \dim(S^{\perp}) - \dim(S \cap S^{\perp}) = \dim(S) + \dim(S^{\perp}).$$

$\square$

Theorem — *Approximation Theorem*
Let V be an inner product space over F, $S \subseteq V$ a finite-dimensional subspace, and $v \in V$. Then v_S is *closer* to v than any other vector in S, i.e.,

$$d(v, v_S) = \|v - v_S\| \leq \|v - r\| = d(v, r)$$

in $\mathbb{R}$ for all $r \in S$. Equivalently, we may write

$$d(v, S) = d(v, v_S).$$

Moreover, if $r \in S$, then

$$\|v - v_S\| = \|v - r\| \text{ in } \mathbb{R} \text{ if and only if } r = v_S.$$

We say v_S gives the *best approximation* to v in S.

Proof. Let $v = s + s^\perp$ with $s = v_S$, $s^\perp = v - s = v - v_S$ and $s^\perp \in S^\perp$. Let $r \in S$. Then

$$v - r = (v - v_S) + (v_S - r) = s^\perp + (v_S - r).$$

Because $v_S - r \in S$, we have $\|v - r\|^2 = \|v - v_S\|^2 + \|v_S - r\|^2 \geq \|v - v_S\|^2$ with equality if and only if $\|v_S - r\|^2 = 0$, or $v_S = r$. □

Theorem — *Hermitian Corollary*
Let V be an inner product space over F, $T\colon V \to V$ linear. Suppose that T is hermitian. Then any eigenvalue of T (if any) is real, i.e. lies in $F \cap \mathbb{R}$.

Proof. Because T is hermitian, we have $\langle Tv, v\rangle = \langle v, Tv\rangle$ for all $v \in V$. If λ is an eigenvalue of T, then

$$\lambda = \frac{\lambda \langle v, v\rangle}{\|v\|^2} = \frac{\langle Tv, v\rangle}{\|v\|^2} = \frac{\langle v, Tv\rangle}{\|v\|^2} = \frac{\bar{\lambda} \langle v, v\rangle}{\|v\|^2} = \bar{\lambda},$$

so $\lambda = \bar{\lambda}$ and λ must be real. $\square$

Theorem — *Key Lemma (for Hermitian Operators)*
Let V be an inner product space over F, $T\colon V \to V$ hermitian, $S \subseteq V$ a T-invariant subspace. Then

(i) $S^\perp$ is T-invariant, i.e. $T(S^\perp) \subseteq S^\perp$

(ii) $T|_{S^\perp}\colon S^\perp \to S^\perp$ is hermitian

Proof. (i) Let $s^\perp \in S^\perp$. Then it suffices to show that $Ts^\perp \in S^\perp$. Observe that for all $s \in S$, we have

$$\begin{aligned}\left\langle s, Ts^\perp \right\rangle &= \left\langle Ts, s^\perp \right\rangle && (T \text{ is hermitian})\\ &= 0. && (S \text{ is a } T\text{-invariant subspace})\end{aligned}$$

Therefore $Ts^\perp \in S^\perp$ so $S^\perp$ is T-invariant.

(ii) By (i) we know that $T|_{S^\perp}$ is linear, and since T is hermitian in V, it must be hermitian in $S^\perp$. In other words, because $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in V$, it must also hold for all $v, w \in S^\perp$. $\square$

Theorem — *Spectral Theorem (for Hermitian Operators) (Refined Version)*
Let $F = \mathbb{R}$ or $\mathbb{C}$, V a finite-dimensional inner product space over F, $T\colon V \to V$ hermitian. Then there exists an ordered orthonormal basis $\mathcal{C}$ of eigenvectors for V of T and every eigenvalue of T is real. Moreover, if $\mathcal{B}$ is any ordered orthonormal basis for V, then

$$[T]_{\mathcal{C}} = C[T]_{\mathcal{B}}C^*$$

for some invertible matrix $C \in \mathbb{M}_n F$, i.e. $C = [1_V]_{\mathcal{B},\mathcal{C}}$.

Theorem — *Spectral Theorem (for Hermitian Operators) (Refined Version)*
Let $F = \mathbb{R}$ or $\mathbb{C}$, V a finite-dimensional inner product space over F, $T\colon V \to V$ hermitian. Then there exists an orthonormal basis $\mathcal{B} = \{v_1, \dots, v_n\}$ for V with each v_i, $i = 1, \dots, n$ an eigenvector for some eigenvalue $\lambda_i \in \mathbb{R}$, $i = 1, \dots, n$ (not necessarily distinct). In particular, T is diagonalizable.

Proof. We prove $\mathcal{B}$ exists by induction on $\dim(V) = n$.

Observe that for the $n = 1$ case, we have that $V = \operatorname{span}(v)$, for any non-zero $v \in V$. As $Tv \in \operatorname{span}(v)$, there exists an $\alpha \in F$ such that $Tv = \alpha v$, so $v \in E_T(\alpha)$. As T is hermitian, $\alpha \in \mathbb{R}$ by the Hermitian Corollary, even if $F = \mathbb{C}$. Thus $\mathcal{B} = \left\{\frac{v}{\|v\|}\right\}$ is a valid orthonormal basis for V.

Suppose the statement holds for some n, that is if W is a finite-dimensional inner product space over F, $\dim(W) = n - 1$, $T_0\colon W \to W$ hermitian, then there exists an orthonormal basis for W of eigenvectors of T_0 and every eigenvalue of T_0 is real. Let $\mathcal{C}$ be an ordered orthonormal basis for n-dimensional V, which exists as $F = \mathbb{R}$ or $\mathbb{C}$. Let $A = [T]_{\mathcal{C}} \in \mathbb{M}_n F \subseteq \mathbb{M}_n\mathbb{C}$. Then for all $x, y \in \mathbb{C}^{n\times 1}$, we have that $Ax \cdot y = x \cdot Ay$ (because T is hermitian). In other words, $A\colon \mathbb{C}^{n\times 1} \to \mathbb{C}^{n\times 1}$ is hermitian, where $\mathbb{C}^{n\times 1}$ is an inner product space over $\mathbb{C}$ via the dot product. By the Fundamental Theorem of Algebra, f_A has a root $\lambda \in \mathbb{C}$, hence λ is an eigenvalue of $A\colon \mathbb{C}^{n\times 1} \to \mathbb{C}^{n\times 1}$, which is hermitian. Thus by the Hermitian Corollary, we know that $\lambda \in \mathbb{R}$. But

$$f_T = f_{[T]_{\mathcal{C}}} = f_A,$$

so f_T has a real root $\lambda \in \mathbb{R}$, if $F = \mathbb{R}$ or $\mathbb{C}$. Thus there exists a non-zero vector $v \in E_T(\lambda) \subseteq V$ that is an eigenvector of T. Let $Fv = \operatorname{span}(v) \subseteq E_T(\lambda)$. Then Fv is T-invariant. By the Orthogonal Decomposition Theorem

$$v = Fv \perp (Fv)^{\perp}$$

and

$$\dim(V) = \dim(Fv) + \dim(Fv)^{\perp} = 1 + \dim(Fv)^{\perp}.$$

Hence $\dim(Fv)^{\perp} = n - 1$. By the Key Lemma, since Fv is T-invariant and $T\colon V \to V$ is hermitian, $(Fv)^{\perp}$ is T-invariant. □

Theorem — *New Key Lemma*
Let V be a finite-dimensional inner product space over F, $T\colon V \to V$ linear. Suppose that V has an orthonormal basis and $W \subseteq V$ is a T-invariant subspace. Then $W^\perp \subseteq V$ is T-invariant. In particular,

$$T^*|_{W^\perp} : W^\perp \to W^\perp$$

is linear.

Proof. Let $w^\perp \in W^\perp$ and $x \in W$ be arbitrary. Then

$$\left\langle x, T^* w^\perp \right\rangle = \left\langle Tx, w^\perp \right\rangle = 0,$$

as $Tx \in W$ (because W is T-invariant). Thus $T^* w^\perp \in W^\perp$, and so $W^\perp$ is also T-invariant. $\square$

Theorem — *Schur's Theorem*
Let V be a finite-dimensional inner product space over $\mathbb{C}$, $T\colon V \to V$ linear. Then T is triangularizable. Moreover, there exists an ordered orthonormal basis $\mathcal{B}$ for T such that $[T]_{\mathcal{B}}$ is upper triangular.

Proof. We induct on $n = \dim(V)$. Let $n = 1$. Then for some non-zero $v \in V$, we have that $\mathcal{B} = \left\{\frac{v}{\|v\|}\right\}$ is a valid orthonormal basis. Let $n > 1$. By the Fundamental Theorem of Algebra, the characteristic polynomial f_{T^*} for T^* has a root $\lambda \in \mathbb{C}$, hence λ is an eigenvalue of T^*. Let $0 \neq v \in E_{T^*}(\lambda)$. By the Orthogonal Decomposition Theorem,

$$V = (\mathbb{C}v) + (\mathbb{C}v)^\perp$$

and

$$\begin{aligned} n &= \dim(V) \\ &= \dim(\mathbb{C}v) + \dim(\mathbb{C}v)^\perp \\ &= 1 + \dim(\mathbb{C}v)^\perp, \end{aligned}$$

so $\dim(\mathbb{C}v)^\perp = n - 1$. We know that $(\mathbb{C}v)$ is T^*-invariant as $v \in E_{T^*}(\lambda)$, so $(\mathbb{C}v)^\perp$ is $(T^*)^* = T$-invariant by the New Key Lemma. So we may view

$$T|_{(\mathbb{C}v)^\perp} \colon (\mathbb{C}v)^\perp \to (\mathbb{C}v)^\perp \text{ as linear.}$$

By induction, there exists an ordered orthonormal basis $\mathcal{B}_0 = \{v_1, \ldots, v_{n-1}\}$ for $(\mathbb{C}v)^\perp$ such that $[T|_{(\mathbb{C}v)^\perp}]_{\mathcal{B}_0}$ is upper triangular. Let $\mathcal{B} = \left\{v_1, \ldots, v_{n-1}, \frac{v}{\|v\|}\right\}$, an ordered orthonormal basis for V. Then by (*), we have

$$\begin{bmatrix} [T|_{(\mathbb{C}v)^\perp}]_{\mathcal{B}_0} & \cdots & * \\ \vdots & \ddots & * \\ 0 & \cdots & * \end{bmatrix} \in \mathbb{M}_n\mathbb{C},$$

where the last column (denoted by asterisks) is $\left[T\left(\dfrac{v}{\|v\|}\right)\right]_{\mathcal{B}}$. □

Theorem — *Spectral Theorem (for Normal Operators)*
Let V be a finite-dimensional inner product space over $\mathbb{C}$, $T\colon V \to V$ normal. Then there exists an ordered orthonormal basis $\mathcal{C}$ for V consisting of eigenvectors of T. In particular, T is diagonalizable. Moreover, if $\mathcal{B}$ is an ordered orthonormal basis for V, then

$$T_{\mathcal{C}} = [1_V]_{\mathcal{B},\mathcal{C}}[T]_{\mathcal{B}}[1_V]^*_{\mathcal{B},\mathcal{C}}.$$

Proof. We induct on $n = \dim(V)$. When $n = 1$, the statement is immediate. Let $n > 1$. By the Fundamental Theorem of Algebra, there exists $\bar{\lambda} \in \mathbb{C}$ a root of $f_{T^*} \in \mathbb{C}[t]$, and thus is an eigenvalue of T^*. Let $0 \neq v \in E_{T^*}\left(\bar{\lambda}\right)$. By the lemma, $v \in E_T(\lambda)$. Thus $(\mathbb{C}v)$ is both T-invariant and T^*-invariant. Hence by the New Key Lemma we know that $(\mathbb{C}v)^\perp$ is also both T-invariant and T^*-invariant. In particular, for all $x, y \in (\mathbb{C}v)^\perp$, we have

$$\langle x, T^*y\rangle = \langle Tx, y\rangle$$

and $\left(T|_{(\mathbb{C}v)^\perp}\right)^*$ is the unique linear map

$$\left(T|_{(\mathbb{C}v)^\perp}\right)^* : (\mathbb{C}v)^\perp \to (\mathbb{C}v)^\perp$$

such that for all $x, y \in (\mathbb{C}v)^\perp$,

$$\begin{aligned}\left\langle x, \left(T|_{(\mathbb{C}v)^\perp}\right)^* y\right\rangle_{(\mathbb{C}v)^\perp} &= \left\langle T|_{(\mathbb{C}v)^\perp} x, y\right\rangle_{(\mathbb{C}v)^\perp} \\ &= \langle Tx, y\rangle_V \\ &= \langle x, T^*y\rangle_V \\ &= \left\langle x, T^*|_{(\mathbb{C}v)^\perp} y\right\rangle_{(\mathbb{C}v)^\perp}.\end{aligned}$$

It follows by the uniqueness of the adjoint that $T^*|_{(\mathbb{C}v)^\perp} = \left(T|_{(\mathbb{C}v)^\perp}\right)^*$. Hence we have

$$T|_{(\mathbb{C}v)^\perp} \colon (\mathbb{C}v)^\perp \to (\mathbb{C}v)^\perp \text{ is also normal.}$$

Since

$$\dim(V) = \dim(\mathbb{C}v) + \dim((\mathbb{C}v)^\perp) = 1 + \dim((\mathbb{C}v)^\perp),$$

by the Orthogonal Decomposition Theorem, by induction there exists an orthonormal basis $\mathcal{C}_0 = \{v_2, \ldots, v_n\}$ for $(\mathbb{C}v)^\perp$ of eigenvectors of $T|_{(\mathbb{C}v)^\perp}$ hence of eigenvectors of T. It follows that

$$\mathcal{C} = \left\{\frac{v}{\|v\|}, v_2, \ldots, v_n\right\}$$

is an orthonormal basis for V consisting of eigenvectors of T. If $\mathcal{B}$ is an orthonormal basis for V, then $[1_V]_{\mathcal{B},\mathcal{C}}* = [1_V]^{-1}_{\mathcal{B},\mathcal{C}}$, so

$$T_{\mathcal{C}} = [1_V]_{\mathcal{B},\mathcal{C}}[T]_{\mathcal{B}}[1_V]^*_{\mathcal{B},\mathcal{C}}$$

by the Change of Basis Theorem. □

Theorem — *Singular Value Theorem*
Let $F = \mathbb{R}$ or $\mathbb{C}$, $A \in F^{m\times n}$. Then there exists $U \in U_n(F) \coloneqq \{B \in \mathbb{M}_n F \mid BB^* = I\}$, $X \in U_m F$ such that

$$X^* AU = D \coloneqq \begin{bmatrix} u_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in F^{m\times n}$$

is "diagonal" (i.e., $D_{ij} = 0$ for all $i \neq j$) with $D_{ii} = 0$ for all $i > r$, $D_{ii} = \mu_i, i \leq r$ with

$$u_1 \geq \cdots \geq u_r > 0 \quad \text{and} \quad r = \operatorname{rank}(A).$$