

## 0.1 Equivalence Relations

### Definition. *Equivalence relation*

A relation  $R$  on a set  $X$  is an *equivalence relation* if it is reflexive, symmetric, and transitive.

**Note.** An equivalence relation gives us a notion of two different elements in a set being “the same”.

- Reflexive: Everything is “the same” as itself
- Symmetric: If  $x$  is “the same” as  $y$ , then  $y$  is “the same” as  $x$
- Transitive: If  $x$  is “the same” as  $y$ , and  $y$  is “the same” as  $z$ , then  $x$  is “the same” as  $z$

### Example. *Equivalence Relations*

- (a) The relation  $E$  on the integers where  $xEy$  if  $x - y$  is even.
- Reflexive: For all  $x \in \mathbb{Z}$ ,  $x - x = 0$ , which is even, so  $xEx$
  - Symmetric: For all  $x, y \in \mathbb{Z}$ , if  $x - y$  is even, so is  $-(x - y) = y - x$ . Thus if  $xEy$ , then  $yEx$
  - Transitive: For all  $x, y, z \in \mathbb{Z}$ , if  $x - y$  is even and  $y - z$  is even, then their sum,  $x - z$ , is also even. Thus if  $xEy$  and  $yEz$ , then  $xEz$ .

Observe that this relation relates two integers if they have the same parity.

- (b) Let  $Y$  be any finite set, and  $a, b \in Y^*$  (the set of all strings constructed using  $Y$ ). Consider the relation  $L$  over  $Y^*$  such that  $aLb$  if  $a$  and  $b$  have the same length.
- (c) Let  $X$  be the set of all animals, with animals  $x, y \in X$ . Consider the relation  $S$  over  $X$  such that  $xSy$  if  $x$  and  $y$  are of the same species.
- (d) Let  $x, y \in \mathbb{R}$ . Consider the relation  $C$  over  $\mathbb{R}$  such that  $xCy$  if  $x - y$  is an integer.

### Definition. *Equivalence Classes*

If  $R$  is an equivalence relation on a set  $X$ , then for  $x \in X$ , the *equivalence class* of  $x$  is the set (with respect to  $R$ ), denoted by  $[x] = [x]_R = \{y \in X \mid xRy\}$ .

### Example. *Equivalence Classes*

- (a) Let  $E$  be a relation on  $\mathbb{Z}$ , where  $xEy$  if  $x - y$  is even. The equivalence classes for  $E$  are  $[0]$  (the evens) and  $[1]$  (the odds). So, the set of equivalence classes  $= \{[0], [1]\}$ .
- (b) Let  $x, y \in \mathbb{R}$ , with the relation  $C$  over  $\mathbb{R}$  defined by  $xCy$  if  $x - y$  is an integer. The set of equivalence classes  $= \{[x] \mid x \in [0, 1)\}$ .

If  $R$  is an equivalence relation on a set  $X$ , then:

- For all  $x \in X$ , if  $x \in [y]$  and  $x \in [z]$ , then  $[y] = [z]$ .

*Proof.* Suppose  $x \in [y]$  and  $x \in [z]$ . Let  $w \in [y]$ . Because  $w \in [y]$ , we know that  $yRw$ . We also know that  $yRx$  because  $x \in [y]$ . By symmetry of  $R$ , we have  $wRy$ , and by transitivity, we have  $wRx$ . But  $x \in [z]$ , so  $zRx$ , and by symmetry we have  $xRw$ . By transitivity,  $zRw$  so  $w \in [z]$ . Thus  $[y] \subseteq [z]$ .

By a similar argument, we have that  $[z] \subseteq [y]$ , so  $[y] = [z]$ .  $\square$

- For any  $x \in X$ ,  $x$  is in some equivalence class,  $x \in [x]$  by reflexivity.  
So, for every  $x \in X$ ,  $x$  is in exactly one equivalence class. If  $x$  is in another equivalence class  $[y]$ , then by the above  $[x] = [y]$ .

**Definition.** *Partition*

For  $X$  a set, a *partition*  $\mathcal{S}$  of  $X$  is a set of nonempty subsets of  $X$  such that every element of  $X$  is an element of exactly one of the subsets. In other words, for all  $A, B \in \mathcal{S}$

- $A, B \subseteq X$
- $A, B \neq \emptyset$
- If  $A \cap B \neq \emptyset$  then  $A = B$
- For all  $x \in X$ , there exists exactly one  $A \in \mathcal{S}$  such that  $x \in A$

**Note.** We showed that if  $R$  is an equivalence relation on  $X$  then  $\{[x]_R \mid x \in X\}$  is a partition of  $X$ .

**Theorem —** *Equivalence Relations and Partitions*

For  $X$  a set, there is a bijection  $F$ : Set of equivalence relations on  $X \rightarrow$  Set of partitions of  $X$ , defined by

$$F(E) = \{[x]_E \mid x \in X\},$$

the inverse function  $F^{-1}$  sends a partition  $\mathcal{S}$  to the equivalence relation  $F^{-1}(\mathcal{S})$  defined by  $xF^{-1}(\mathcal{S})y$  if and only if  $x$  and  $y$  are in the same set of  $\mathcal{S}$  (in the same equivalence class of  $E$ ).