1 Lecture 6

The force between long parallel wires defines the ampere. As a reminder, last class we found the formula

$$B = \frac{\mu_0 I}{2\pi r}.$$

Consider two parallel, infinitely-long wires that are carrying current. If the currents are going in the same direction, then we use the right-hand rule to find that the wires have an attractive force between them. We find that the force acting on wire 2 is

$$\begin{aligned}\vec{F} &= I_2\vec{L} \times \vec{B}_1\\ &= I_2 L \frac{\mu_0 I_1}{2\pi r} \text{ to the left,}\end{aligned}$$

so we have

$$F_2 = \frac{\mu_0}{2\pi} \cdot \frac{I_1 I_2 L}{r}.$$

Rearranging some more, we get

$$\frac{F_2}{L} = \left(2 \cdot 10^{-7}\right) \frac{I_1 I_2}{r} \text{ Newtons.}$$

What happens when we ave $I_1 = I_2$? Well, we get that

$$\frac{F}{L} = \left(2 \cdot 10^{-7}\right) \frac{I^2}{r},$$

which we can rearrange to get

$$I = \underbrace{\left(r \cdot \frac{F}{L} \cdot \frac{1}{2 \cdot 10^{-7}}\right)^{\frac{1}{2}}}_{\text{defines the Ampere}}.$$

1.1 Ampere's Law

Just like Coulomb's Law leads to Gauss' Law (which is an integral), the Biot-Savart Law leads to Ampere's Law (also an integral). Remember the Biot-Savart Law from before:

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi} \frac{I\,\mathrm{d}\vec{\ell} \times \hat{r}}{r^2}.$$

Theorem — *Ampere's Law*
The amount of magnetic field in a loop can give you the current, given by

$$\oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enc}}.$$

The analogue to the Gaussian surface is what we call an *Amperium loop*. There is no radial component to $\vec{B}$ because there are no monopoles. Thus we have that $\vec{B} \parallel \mathrm{d}\vec{\ell}$, and $\vec{B} \cdot \mathrm{d}\vec{\ell} = B\,\mathrm{d}\ell$. Then

$$\begin{aligned}\oint B\,\mathrm{d}\ell &= B \oint \mathrm{d}\ell\\ &= B \cdot 2\pi r\\ &= \mu_0 I,\end{aligned}$$

so

$$B = \frac{\mu_0 I}{2\pi r}.$$

The differential form of Ampere's Law is given by:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J},$$

where $\vec{J}$ is the current density.

1.1.1 Ampere's Law Example—Long Wire, Thin but Finite Thickness

Consider a wire with radius R that has uniform current density J. We define current density to be the current per unit area, in other words

$$J = \frac{I}{\pi R^2}, \text{ a constant.}$$

We have a few cases here:

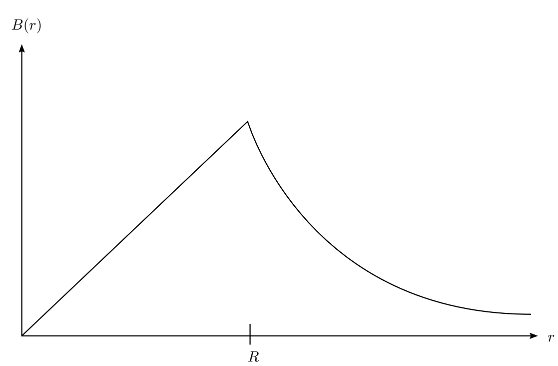
(a) Outside the wire, i.e. $r > R$.

$$\begin{aligned}\oint \vec{B} \cdot \mathrm{d}\vec{\ell} &= \mu_0 I_{\text{encl}} \\ 2\pi r B &= \mu_0 I \\ \boxed{B = \frac{\mu_0 I}{2\pi r}.}\end{aligned}$$

(b) Inside the wire, i.e. $r < R$.

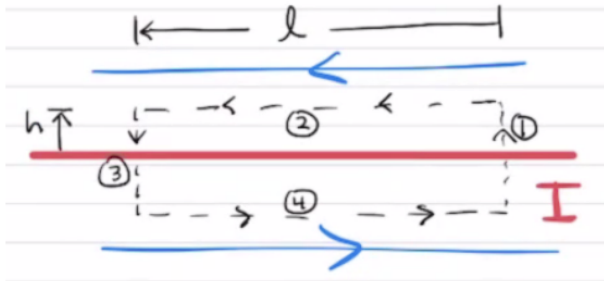
$$\begin{aligned}\oint \vec{B} \cdot \mathrm{d}\vec{\ell} &= \mu_0 I_{\text{encl}} \\ 2\pi r B &= \mu_0 I \cdot \frac{\pi r^2}{\pi R^2} \\ 2\pi r B &= \mu_0 I \cdot \frac{r^2}{R^2} \\ \boxed{B = \frac{\mu_0 I r}{2\pi R^2}.}\end{aligned}$$

Pictorially, we can draw a graph for the magnetic field as a function of the distance to the centre of the wire as follows:



1.1.2 Ampere's Law Example—Plane of Current

If we have two parallel wires next to each other with the currents moving in the same direction, by superposition of field lines we generate a space with zero field between the wires. Extending this idea to more than just two wires (read: an infinite number, creating a plane of current), we create a planar magnetic field.



Applying Ampere's Law, we have

$$\begin{aligned}\oint \vec{B}\cdot\mathrm{d}\vec{\ell} &= \mu_0 I_{\text{encl}}\\ \int_1 + \int_2 + \int_3 + \int_4 &= \mu_0 NI\\ 2\int_4 \vec{B}\cdot\mathrm{d}\vec{\ell} &= \mu_0 NI\\ 2B\ell &= \mu_0 NI\\ B &= \frac{\mu_0 nI}{2}.\end{aligned}$$

We define the winding density n to be the number of wires per unit length, or $n = \frac{N}{\ell}$.