Theorem — *Subspace Theorem*
Let V be a vector space over F, $\varnothing \neq W \subseteq V$ a subset. Then the following are equivalent:

(a) W is a subspace of V

(b) W is closed under addition and scalar multiplication from V

(c) For all $w_1, w_2 \in W$, for all $\alpha \in F$, $\alpha w_1 + w_2 \in W$

Proof. (a $\Rightarrow$ b) This holds by definition.

(b $\Rightarrow$ a) We claim that $0_V \in W$ and $0_W = 0_V$. Because W is non-empty, there exists $w \in W$. Because W is closed under scalar multiplication, $(-1)w \in W$, so $0_V = w + (-w) \in W$. Furthermore, for all $w' \in W$, we have that $0_V + w' = w' = w' + 0_V$, so $0_V = 0_W$.

(b $\Rightarrow$ c) Because W is closed under multiplication, we have that for all $w_1 \in W$ and $\alpha \in F$, $\alpha w_1 \in W$. Furthermore, because W is closed under addition, $\alpha w_1 + w_2 \in W$ for all $w_2 \in W$.

(c $\Rightarrow$ b) If we let $\alpha = 0$, we have closure under addition. If we let $w_2 = 0$, we have closure under multiplication. $\square$

Theorem — *Toss In Theorem*
Let V be a vector space over F, $\varnothing \neq S \subseteq V$ a linearly independent subset. Suppose that $v \in V \setminus \operatorname{span}(S)$. Then $S \cup \{v\}$ is linearly independent.

Proof. Suppose towards a contradiction that $S \cup \{v\}$ is linearly dependent. In other words, there exists a subset $\{v_1, \ldots, v_n\} \subseteq S$ such that for some $\alpha_1, \ldots, \alpha_{n+1} \in F$, not all zero, such that

$$\alpha_1 v_1 + \cdots + \alpha_n v_n + \alpha_{n+1} v = 0.$$

We know that $\alpha_{n+1} \neq 0$ because otherwise we would have $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$, where not all $\alpha_i = 0$, which contradicts that S is a linearly independent subset. Since $\alpha_{n+1} \neq 0$, it has an inverse. Thus we may write

$$v = -\frac{\alpha_1}{\alpha_{n+1}} v_1 - \cdots - \frac{\alpha_n}{\alpha_{n+1}} v_n,$$

so $v \in \operatorname{span}(S)$. Thus we have arrived at the contradiction that v is both in the span of S and also not in the span of S. $\square$

Theorem — *Coordinate Theorem*
Let V be a finite-dimensional vector space over F with basis $\mathcal{B} = \{v_1, \dots, v_n\}$ and $v \in V$. Then there exists unique $\alpha_1, \dots, \alpha_n \in F$ such that

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

We call $\alpha_1, \dots, \alpha_n$ the *coordinates* of v relative to the basis $\mathcal{B}$ and call α_i the ith coordinate relative to $\mathcal{B}$.

Proof. Because $\mathcal{B}$ is a basis for V, we know that $\mathcal{B}$, so all vectors $v \in V$ may be written as a linear combination of the basis vectors. We must show that this representation is unique. Let $v \in V$ and $\alpha_i, \beta_i \in F$, for $i = 1, \dots, n$. Then

$$\beta_1 v_1 + \cdots + \beta_n v_n = v = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

Moving things to one side, we have

$$(\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_n - \beta_n) v_n = 0.$$

Because $\mathcal{B}$ is a basis and thus linearly independent, we have that $\alpha_i = \beta_i$ for all $i = 1, \dots, n$, so the representation is unique. $\square$

Theorem — *Important Exercise (General Toss Out Theorem)*
Let V be a vector space over F, with $v_1, \ldots, v_n \in V$. Then

$$\operatorname{span}(v_1, \ldots, v_n) = \operatorname{span}(v_2, \ldots, v_n)$$

if and only if $v_1 \in \operatorname{span}(v_2, \ldots, v_n)$.

Proof. ($\Rightarrow$) Suppose $\operatorname{span}(v_1, \ldots, v_n) = \operatorname{span}(v_2, \ldots, v_n)$. Then for all $\alpha_1, \ldots, \alpha_n \in F$, there exist $\beta_2, \ldots, \beta_n$ such that

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_2 v_2 + \cdots + \beta_n v_n.$$

Because this equality must hold for all α_i, we may assume that $\alpha_1 \neq 0$. Thus

$$\begin{aligned} \alpha_1 v_1 &= (\beta_2 - \alpha_2) v_2 + \cdots + (\beta_n - \alpha_n) v_n \\ v_1 &= \alpha_1^{-1}(\beta_2 - \alpha_2) v_2 + \cdots + \alpha_1^{-1}(\beta_n - \alpha_n) v_n, \end{aligned}$$

so $v_1 \in \operatorname{span}(v_2, \ldots, v_n)$.

($\Leftarrow$) Let $v_1 \in \operatorname{span}(v_2, \ldots, v_n)$, so $v_1 = \alpha_2 v_2 + \cdots + \alpha_n v_n$ for $\alpha_i \in F$ for $i = 2, \ldots, n$. Then

$$\begin{aligned} \operatorname{span}(v_1, \ldots, v_n) &= \beta_1 v_1 + \cdots + \beta_n v_n \\ &= \beta_1(\alpha_2 v_2 + \cdots + \alpha_n v_n) + \beta_n v_n \\ &= (\beta_1 \alpha_2 + \beta_2) v_2 + \cdots + (\beta_1 \alpha_n + \beta_n) v_n \\ &= \operatorname{span}(v_2, \ldots, v_n). \end{aligned}$$

$\square$

Theorem — *Toss Out Theorem*
Let V be a vector space over F. If V can be spanned by finitely many vectors then V is a finite-dimensional vector space over F. More precisely, if

$$V = \text{span}(v_1, \ldots, v_n),$$

then a subset of $\{v_1, \ldots, v_n\}$ is a basis for V.

Proof. If $V = 0$, there is nothing to prove, so we may assume that V is non-zero. Suppose that $V = \text{span}(v_1, \ldots, v_n)$. We prove by induction on n that a subset of $\{v_1, \ldots, v_n\}$ is a basis. When $n = 1$, we have $V = \text{span}(v_1) \neq 0$ and because $V \neq 0$, we know $v_1 \neq 0$. Thus $\{v_1\}$ is linearly independent and spans V, and so is a basis. Suppose that the statement holds for some $n = k$. We claim that a subset of $\{v_1, \ldots, v_{k+1}\}$ is a basis for V. If $\{v_1, \ldots, v_{k+1}\}$ is linearly independent, then it is a basis for V because it spans V, and we are one. If it is not linearly independent, there exists some vector that is in the span of the other vectors. Without loss of generality, say that $v_{k+1} \in \text{span}(v_1, \ldots, v_k)$. Then we have that $\text{span}(v_1, \ldots, v_k) = \text{span}(v_1, \ldots, v_{k+1})$, so we may remove v_{k+1} from the set while still spanning V. By the induction hypothesis, because $\text{span}(\{v_1, \ldots, v_k\}) = V$, a subset of $\{v_1, \ldots, v_k\}$ is a basis for V, so we are done. $\square$

Theorem — *Replacement Theorem*
Let V be a vector space over F, with $\{v_1, \dots, v_n\}$ a basis for V. Suppose that $v \in V$ satisfies

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n, \qquad (\alpha_1, \dots, \alpha_n \in F, \alpha_i \neq 0)$$

Then

$$\{v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n\}$$

is also a basis for V.

Proof. Notice that $v \in \operatorname{span}(v_1, \dots, v_n)$, so $\operatorname{span}(v_1, \dots, v_n) = \operatorname{span}(v_1, \dots, v_n, v) = V$ by the General Toss out Theorem. Furthermore, because $\alpha_i \neq 0$, we may rewrite the definition of v as follows:

$$\begin{aligned} v &= \alpha_1 v_1 + \cdots + \alpha_n v_n \\ \alpha_i v_i &= v - \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j v_j \right) \\ v_i &= \alpha_i^{-1} v - \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_i^{-1} \alpha_j v_j \right) \end{aligned}$$

Thus we see that $v_i \in \operatorname{span}(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n)$ so

$$V = \operatorname{span}(v_1, \dots, v_n, v) = \operatorname{span}(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n).$$

Thus it remains to show that this set of vectors is linearly independent. Suppose towards a contradiction that they are linearly dependent, that is there exist $\beta_1, \dots, \beta_n \in F$, not all zero, such that

$$\beta_1 v_1 + \cdots + \beta_{i-1} v_{i-1} + \beta_i v + \beta_{i+1} v_{i+1} + \cdots + \beta_n v_n = 0.$$

If $\beta_i = 0$, then we reach a contradiction because $\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ is a linearly independent set so all β_j must be zero. If $\beta_i \neq 0$, then it has an inverse. Thus

$$v = -\beta_i^{-1} \left(\beta_1 v_1 + \cdots + \beta_{i-1} v_{i-1} + \beta_{i+1} v_{i+1} + \cdots + \beta_n v_n \right).$$

Moving the terms to one side and expanding, we have

$$0 = \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\beta_i^{-1} \beta_j + \alpha_j \right) v_j \right) + \alpha_i v_i.$$

Because $\{v_1, \dots, v_n\}$ is linearly independent, we know that $\alpha_i = 0$, a contradiction. Therefore we have that $\{v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n\}$ is a basis for V. $\square$

Theorem — *Main Theorem*
Suppose V is a vector space over F with $V = \operatorname{span}(v_1, \dots, v_n)$. Then any linearly independent subset of V has at most n elements.

Proof. By the Toss Out Theorem, we know that a subset of $\{v_1, \dots, v_n\}$ is a basis for V, so we may assume that $\{v_1, \dots, v_n\}$ is a basis for V. Suppose there is another linearly independent subset of V that has m elements, say $\{w_1, \dots, w_m\}$. We proceed via induction.

We claim that after changing notation, if necessary, for each $k \leq n$ that

$$\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$$

is a basis for V. Applying this claim to $k = n$, we get that $\{w_1, \dots, w_{n+1}\}$ is linearly dependent, a contradiction as $\{w_1, \dots, w_n\}$ is a basis. Thus we may assume that $k \leq n$. For $k = 1$, we have that $w_1 \in \operatorname{span}(v_1, \dots, v_n)$, so for some $\alpha_i \in F$, not all zero, $i = 1, \dots, n$,

$$w_1 = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

Rearranging if necessary, we may assume that $\alpha_1 \neq 0$. Thus by the Replacement Theorem $\{w_1, \dots, v_n\}$ is a basis for V. Suppose the statement holds for some $n = k$, so $\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$ is a basis for V. Because $w_{k+1} \in V$, for some $\alpha_i \in F$, not all zero, we have

$$w_{k+1} = \alpha_1 w_1 + \cdots + \alpha_k w_k + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n.$$

We know that at least one of the $\alpha_i \neq 0$ for $i \geq k+1$, otherwise we would have that $w_k+1 \in \operatorname{span}(w_1, \dots, w_k)$, a contradiction because the w_i's are linearly independent. Thus rearranging if necessary, we may assume that $\alpha_{k+1} \neq 0$. Therefore we may use the Replacement Theorem, and $\{w_1, \dots, w_{k+1}, v_{k+2}, \dots, v_n\}$ is a basis for V. Thus the claim has been proven by induction. $\square$

Theorem — *Extension Theorem*
Let V be a finite-dimensional vector space over F, $W \subseteq V$ a subspace. Then every linearly independent subset S in W is finite and part of a basis for W which is a finite-dimensional vector space over F.

Proof. We know that V is finite-dimensional, so it must have a finite basis that spans V, say $\{v_1, \ldots, v_n\}$. Thus by the Main Theorem, we know every linearly independent subset must have less than n vectors, and so is finite. We now show that S is a part of a basis for W. If S does not span W, then there exist vectors that are in W but not in $\mathrm{span}(S)$. Thus we may use Toss In Theorem to repeatedly add vectors to S until it spans W, at which point it becomes a basis for W. $\square$

Theorem — *Counting Theorem*
Let V be a vector space over F, with $W_1, W_2 \subseteq V$ subspaces. Suppose that both W_1 and W_2 are finite-dimensional vector spaces over F. Then

(a) $W_1 \cap W_2$ is a finite-dimensional vector space over F.

(b) $W_1 + W_2$ is a finite-dimensional vector space over F.

(c) $\dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.

Proof. We know that $W_1 \cap W_2$ is a subspace because it is closed under the same addition and scalar multiplication operations that W_1 and W_2 are closed under. In other words, for all $v, w \in W_1 \cap W_2$ and $\alpha \in F$, we have that

$$\alpha v + w \in W_1 \text{ and } \alpha v + w \in W_2,$$

so $\alpha v + w \in W_1 \cap W_2$. Similarly, if $v, w \in W_1 + W_2$, then $\alpha v + w \in W_1 + W_2$ (write each as a linear combo), so $W_1 + W_2$ is a finite-dimensional vector space over F. Let $\{w_1, \ldots, w_m\}$ be a basis for $W_1 \cap W_2$. By the Extension Theorem, we may extend it into bases for W_1 and W_2, say $\mathcal{B} = \{w_1, \ldots, w_m, u_{m+1}, \ldots, u_n\}$ and $\mathcal{C} = \{w_1, \ldots, w_m, v_{m+1}, \ldots, v_\ell\}$, respectively. Then we know that $W_1 + W_2$ is spanned by $\mathcal{B} \cup \mathcal{C}$, which has dimension $n + \ell - m$. Notice that this directly yields $\dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$. $\square$

Theorem — *Rank-Nullity (Dimension) Theorem*
Let V and W be vector spaces over a field F, and let V be finite-dimensional. Let $T\colon V \to W$ be a linear transformation. Then $\operatorname{im}(T)$ and $\ker(T)$ are finite-dimensional vector spaces over F, and $\dim(V) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$.

Proof. □

Theorem — *Monomorphism Theorem*
Let $T\colon V \to W$ be a linear transformation. Then the following are equivalent.

(a) T is injective.

(b) T takes linearly independent sets in V to linearly independent sets in W.

(c) $\ker(T) = \{0\}$.

(d) $\dim(\ker(T)) = 0$.

Theorem — *Isomorphism Theorem*
Suppose that V and W are finite-dimensional vector spaces over F with $\dim(V) = \dim(W)$. Let $T\colon V \to W$ be a linear transformation. Then the following are equivalent.

(a) T is an isomorphism.

(b) T is a monomorphism.

(c) T is an epimorphism.

(d) If $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for V, then $\{Tv_1, \ldots, Tv_n\}$ is a basis for W.

(e) There exists a basis $\mathcal{B}$ of V that maps to a basis of W.

Theorem — *Universal Property of Vector Spaces*
Let V be a finite-dimensional vector space over F, and let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V. Let W be a vector space over F, and let $w_1, \ldots, w_n \in W$ (not necessarily distinct). Then there exists a unique linear transformation T with $T\colon v_i \mapsto w_i$.

Theorem — *Classification of Finite Dimensional Vector Spaces*
Let V and W be finite-dimensional vector spaces over the field F. Then $V \cong W$ if and only if $\dim(V) = \dim(W)$.

Theorem — *Matrix Theory Theorem (MTT)*
Let V and W be finite-dimensional vector spaces of dimesnion n and m over F respectively, and let $\mathcal{B}$ and $\mathcal{C}$ be ordered bases for V and W. Then the map

$$\begin{aligned} \varphi\colon L(V, W) &\to F^{m \times n} \\ T &\mapsto [T]_{\mathcal{B},\mathcal{C}} \end{aligned}$$

is an isomorphism. In particular, $\dim(L(V, W)) = mn$.

Proof. Note that because the zero map is linear, it is in $L(V, W)$ and thus $L(V, W)$ is non-empty. We claim that φ is linear. Let $\mathcal{B} = (v_1, \ldots, v_n)$ and $\mathcal{C} = (w_1, \ldots, w_m)$ be ordered bases for V and W, respectively. Let T_1, T_2 be linear maps from V to W, and $\alpha \in F$. Let the element in the ith row and jth column of

$\varphi(\alpha T_1 + T_2)$ be $\lambda_{i,j}$, the element in the ith row and jth column of $\varphi(T_1)$ be $\eta_{i,j}$, and the element in the ith row and jth column of $\varphi(T_2)$ be $\varepsilon_{i,j}$. Then

$$\begin{aligned}(\alpha T_1 + T_2)v_j &= \lambda_{1,j}w_1 + \cdots + \lambda_{m,j}w_m \\ T_1 v_j &= \eta_{1,j}w_1 + \cdots + \eta_{m,j}w_m \\ T_2 v_j &= \varepsilon_{1,j}w_1 + \cdots + \varepsilon_{m,j}w_m.\end{aligned}$$

From these three equations, we have

$$\begin{aligned}\lambda_{1,j}w_1 + \cdots + \lambda_{m,j}w_m &= (\alpha T_1 + T_2)v_j \\ &= \alpha T_1 v_j + T_2 v_j \\ &= \alpha(\eta_{1,j}w_1 + \cdots + \eta_{m,j}w_m) + \varepsilon_{1,j}w_1 + \cdots + \varepsilon_{m,j}w_m \\ &= (\alpha\eta_{1,j} + \varepsilon_{1,j})w_1 + \cdots + (\alpha\eta_{m,j} + \varepsilon_{m,j})w_m.\end{aligned}$$

Because $(w_1, \ldots, w_m)$ is an ordered basis for W, and every vector has a unique representation according to any given basis, we have that $\lambda_{i,j} = \alpha\eta_{i,j} + \varepsilon_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus $\varphi(\alpha T_1 + T_2) = \alpha\varphi(T_1) + \varphi(T_2)$ and φ is linear.

We will now show that φ is a bijection, first showing that it is injective. Suppose $\varphi(T) = 0$. Then $T(v_i) = 0$ for all $v_i \in \mathcal{B}$, so $Tv = 0$ for all $v \in V$. Thus T is the zero transformation and $\ker(\varphi) = 0$. Therefore φ is injective. We will now show that φ is surjective. For every matrix in $F^{m\times n}$, consider mapping $v_i \in \mathcal{B}$ to the ith column of the matrix. Thus for ever matrix in $F^{m\times n}$, we have a linear transformation that maps to it, so φ is surjective. Therefore φ is a bijection and so an isomorphism. $\square$

Theorem — *12.2*
Let V, W, U be finite-dimensional vector spaces over F with ordered bases $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ respectively. If $T\colon V \to W$ and $S\colon W \to U$ are linear, then

$$[S \circ T]_{\mathcal{B},\mathcal{D}} = [S]_{\mathcal{C},\mathcal{D}} \cdot [T]_{\mathcal{B},\mathcal{C}}.$$

Theorem — *Change of Basis Theorem*
Let V and W be finite-dimensional vector spaces over F with ordered bases $\mathcal{B}, \mathcal{B}'$ for V and $\mathcal{C}$, $\mathcal{C}'$ for W. Let $T\colon V \to W$ be linear. Then

$$\begin{aligned}[T]_{\mathcal{B},\mathcal{C}} &= [1_W]_{\mathcal{C}',\mathcal{C}}[T]_{\mathcal{B}',\mathcal{C}'}[1_V]_{\mathcal{B},\mathcal{B}'} \\ &= [1_W]^{-1}_{\mathcal{C},\mathcal{C}'}[T]_{\mathcal{B}',\mathcal{C}'}[1_V]_{\mathcal{B},\mathcal{B}'} \\ &= [1_W]_{\mathcal{C}',\mathcal{C}}[T]_{\mathcal{B}',\mathcal{C}'}[1_V]^{-1}_{\mathcal{B}',\mathcal{B}}\end{aligned}$$