Worksheet 4

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Problem 1.

(a) Show that $\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$. You can get this result by using the binomial theorem on $(x+y)^{m+n} = (x+y)^m(x+y)^n$ or with a counting argument (think of choosing r things from $m+n$ things where you keep track of what you choose from the first m things and from the last n things).

(b) Show that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$.

(a) *Proof.* Observe that choosing r items from a set with $m+n$ items is the same as first choosing k items from a set of m, and then choosing $r-k$ items from a set of n. We need a summation in order to capture all possible different pairs of subsets of size k and $r-k$. ☐

(b) *Proof.* Observe that the left hand side is the same as $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$ because $\binom{n}{k} = \binom{n}{n-k}$. The left hand side of this formula counts the number of ways to choose a k element subset from a set of n elements, and a $n-k$ element subset from another n element subset. This is the same as choosing an n element subset from a set of $2n$ elements, which is represented on the right hand side. ☐

Problem 2. Show that $n2^{n-1} = \sum_{i=1}^{n} i\binom{n}{i}$. See if you can do this by using the binomial theorem and also with a counting argument (count the number of teams with a captain from a set n people).

Proof. Observe that the left hand side is the number of ways to choose a team with a captain from n people. We have n choices for the captain, and then 2^{n-1} different subsets of the remaining $n-1$ people. The right hand side counts how many teams with a captain can be made with 1 person, then 2 people, then 3 people, etc. There are $\binom{n}{i}$ sets of i people from the total set of n. Then we choose any of those team members to be the captain, so we multiply by i. Thus they count the same thing and are equal. □

Note (Alternate Equation). If you first chose the captain, and then the other $i-1$ team members, you would have $\sum_{i=1}^{n} n \cdot \binom{n-1}{i-1}$.