Lecture 4 Notes

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1 Subspaces

Note (Subset $\neq$ Subspace). Most subsets of vector spaces are not subspaces!

Example. *Subspaces*

1. Let F be a field. Then $F = F[t]_0 < F[t]_1 < \cdots < F[t]_n < \cdots < F[t]$ are vector spaces over F, each a subspace of the vector space that contains them.
2. If $W_1 \subseteq W_2 \subseteq V$, where W_1, W_2 are subspaces of V, then $W_1 \subseteq W_2$ is a subspace.
3. If $W_1 \subseteq W_2$ is a subspace and $W_2 \subseteq V$ is a subspace, then $W \subseteq V$ is a subspace.
4. Let $W \coloneqq \{(0, \alpha_1, \ldots, \alpha_n) \mid \alpha_i \in F, 2 \leq i \leq n\} \subseteq \mathbb{F}^n$.
5. Every line or plane through the origin in $\mathbb{R}^3$ is a subspace.

1.1 Linear Combinations and Span

Definition. *Linear Combination*
Let V be a vector space over a field F, $v_1, v_2, \ldots, v_n \in V$. We say $v \in V$ is a *linear combination* of $v_1, \ldots, v_n$ if there exists $\alpha_1, \ldots, \alpha_n \in F$ such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$.

Let $\operatorname{span}(v_1, \ldots, v_n) \coloneqq \{\text{All linear combinations of } v_1, v_2, \ldots, v_n\}$. We may also write

$$\operatorname{span}(v_1, \ldots, v_n) = \left\{\sum_{i=1}^{n} \alpha_i v_i \mid \alpha_1, \ldots, \alpha_n \in F\right\}.$$

This is a subspace of V (by Subspace Theorem) called the *span* of $v_1, \ldots, v_n$. It is the (unique) "smallest" subspace of V containing $v_1, \ldots, v_n$. i.e. if $W \subseteq V$ is a subspace and $v_1, \ldots, v_n \in W$, then $\operatorname{span}(v_1, \ldots, v_n) \subseteq W$. We also let $\operatorname{span}(\varnothing) \coloneqq \{0_V\} = 0$. The smallest vector space containing no vectors.

Question. If we view $\mathbb{C}$ as a vector space over $\mathbb{R}$, then $\mathbb{R}$ is a subspace of $\mathbb{C}$, but if we view $\mathbb{C}$ as a vector space over $\mathbb{C}$, then $\mathbb{R}$ is *not* a subspace of $\mathbb{C}$ (why?). What is going on?

Definition. *Span of a Set of Vectors*
Let V be a vector space over a field F, $\varnothing \neq S \subseteq V$ a subset. Then

$$\operatorname{span}(S) \coloneqq \text{the set of all } \textit{finite} \text{ linear combinations of vectors in } S.$$

i.e. if $v \in \operatorname{span}(S)$, then

$$\exists v_1, \ldots, v_n \in S, \alpha_1, \ldots, \alpha_n \in F \text{ such that } v = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

Thus $\operatorname{span}(S) \subseteq V$ is a subspace.

Question. What is the smallest set of vectors in V such that $\operatorname{span}(S) = V$?

Note (Vector spaces). There exist sets that are not vector spaces over one field, but are vector spaces over other fields.

Definition. *Kernel of a Matrix*
The *kernel* of a matrix A is the set of all vectors that get mapped to the zero vector. In other words,

$$\ker A \coloneqq \left\{x \in F^{n \times 1} \mid Ax = 0\right\}.$$

Theorem — *Intersection of Subspaces is a Subspace*
Let $W_i \in V$, $i \in I$ be subspaces. Then

$$\bigcap_I W = \bigcap_{i \in I} W_i = \{x \in V \mid x \in W_i, \forall i \in I\}$$

is a subspace of V.

Note (Union of Subspaces). The union of subspaces is not necessarily a subspace. For instance, consider two lines going through the origin in $\mathbb{R}^2$ that are not the same line.

Definition. *Sum of Subspaces*
Let $W_1, W_2 \subseteq V$ be subspaces. Define

$$\begin{aligned} W_1 + W_2 &:= \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\} \\ &= \operatorname{span}(W_1 \cup W_2). \end{aligned}$$

So $W_1 + W_2 \subseteq V$ is a subspace and the smallest subspace of V containing W_1 and W_2.

1.2 Linear Independence

We know that $\mathbb{R}^n$ is an n-dimensional vector space over $\mathbb{R}$. Since we need n coordinates to describe all vectors in $\mathbb{R}^n$ but no fewer will do. We want something like the following:

Let V be a vector space over a field F with $V \neq 0$. Can we find distinct vectors $v_1, \dots, v_n$ in V, some n with the following properties:

1. $V = \operatorname{span}(v_1, \dots, v_n)$
2. No v_i is a linear combination of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$

Then we want to call V an n-dimensional vector space over F.

Lemma. Let V be a vector space over F, and $n > 1$. Suppose $v_1, \dots, v_n$ are distinct. Then (2) is equivalent to if

$$\alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n, \alpha_i, \beta_i \in F, \forall i,$$

then $\alpha_i = \beta_i, \forall i$. In other words, the "coordinates" are unique.

Proof. (Contradiction) Without loss of generality, suppose $\alpha_1 \neq \beta_1$. Then $\alpha_1 - \beta_1 \neq 0$, so it has a multiplicative inverse. Thus

$$\begin{aligned} (\alpha_1 - \beta_1) v_1 &= \sum_{i=2}^n (\beta_i - \alpha_i) v_i \\ v_1 &= \frac{1}{\alpha_1 - \beta_1} \cdot \sum_{i=2}^n (\beta_i - \alpha_i) v_i && (\alpha_1 - \beta_1 \neq 0) \end{aligned}$$

so $v_1 \in \operatorname{span}(v_2, \dots, v_n)$, which is a contradiction. $\square$