Winter 2021 Math 61 Notes

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Contents

1 Sets and Functions

1.1 Power Sets

Definition. *Power Set*
If X is a set, the *power set* of X, denoted $\mathscr{P}(X)$, is the set of subsets of X.

Example. *Power Sets*

- $\mathscr{P}(\varnothing) = \{\varnothing\}$
- $\mathscr{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
- $\mathscr{P}(\{a, b, c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

Definition. *Cardinality of Finite Sets*
If X has finitely many elements, then $|X|$ denotes the number of elements of X.

Theorem — *Cardinality of Power Sets*
If X is finite, then $|\mathscr{P}(X)| = 2^{|X|}$.

Proof. Let us induct on the cardinality of the set X. Suppose $|X| = 0$, so that $X = \varnothing$. Then $\mathscr{P}(X) = \{\varnothing\}$, so $|\mathscr{P}(X)| = 1 = 2^0$. Thus the statement is true when $|X| = 0$.

Suppose that the statement holds for some non-negative integer k. Let Y be a set such that $|Y| = k + 1$, and $y \in Y$. Observe that we may split $\mathscr{P}(Y)$ into two groups: the subsets containing y, and the subsets that do not contain y. A subset of Y that does not contain y is exactly $Y \setminus \{y\}$, which has k elements. By the inductive hypothesis, there exist 2^k such subsets. A subset of Y that does contain y is obtained by adding y to a subset of Y which does not contain y. Again, there are 2^k such subsets. Any subset of Y either does or does not contain y (but not both), so there are $2^k + 2^k = 2^{k+1}$ subsets of Y. Therefore $\mathscr{P}(X) = 2^{|X|}$ for all finite sets $|X|$. $\square$

1.2 Functions

Definition. *Function*
If X, Y are sets, a function f from X to Y, written $f \colon X \to Y$ is a subset of $X \times Y$ satisfying two properties:

- For all $a \in X$, there exists $b \in Y$ such that $(a, b) \in f$
  - Everything in the domain must get mapped to something in the codomain
- For all $a \in X$ and $b, b' \in Y$, if $(a, b), (a, b') \in f$, then $b = b'$
  - Every element in the domain can map to at most one element in the codomain

Note (Function Notation). If $(a, b) \in f$, we write $f(a) = b$.

Example. *Functions*

- $f \colon \mathbb{Z} \to \mathbb{N}$ such that $f(x) = x^2$
- $g \colon \mathbb{N} \to \mathbb{N}$ such that $g(x) = x^2$

Note that f and g are different functions.

Definition. *Domain and Codomain of a Function*
If $f \colon X \to Y$, X is the domain of f and Y is the codomain of f.

Definition. *Range of a Function*
For $f \colon X \to Y$, the range of f is:

$$\operatorname{range} f = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

Definition. *Surjectivity*
A function $f \colon X \to Y$ is *onto* or *surjective* if $\operatorname{range} f = Y$. In other words, a function is surjective if its range is equal to its codomain.

Example. *Surjective Functions*

- $f \colon \{a, b, c\} \to \{d, e, f\}$ defined by $f = \{(a, d), (b, e), (c, f)\}$
- $f \colon \mathbb{Z} \to \mathbb{N}$ defined by $f(x) = |x|$

Definition. *Injectivity*
A function $f \colon X \to Y$ is *one-to-one* or *injective* if, for all $x, y \in X$, $f(x) = f(y)$ implies that $x = y$. In other words, different elements in the domain map to different elements in the codomain.

Example. *Injective Functions*

- $g \colon \mathbb{N} \to \mathbb{N}$ defined by $g(x) = x^2$

Note (Properties of Functions). Observe that the both the domain and codomain of a function matter when it comes to determining whether the function satisfies certain properties. For instance, $f \colon \mathbb{Z} \to \mathbb{N}$ defined by $f(x) = x^2$ is not injective, but restricting the domain to $\mathbb{N}$ would make it injective. Similarly, a function $f \colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$ is not surjective, but restricting the codomain to $\mathbb{N}$ would make it surjective.

Definition. *Composition of Functions*
If $f \colon X \to Y, g \colon Y \to Z$ are functions, then $g \circ f \colon X \to Z$ is a function defined by $(g \circ f)(x) = g(f(x))$.

Theorem — *Composition of Injective/Surjective Functions is Injective/Surjective*
Let $f\colon X \to Y$, $g\colon Y \to Z$.

- If f, g are injective, so is $g \circ f$
- If f, g are surjective, so is $g \circ f$

Proof. Suppose f, g are injective functions. Let $x, x' \in X$ such that $(g \circ f)(x) = (g \circ f)(x')$. Then

$$\begin{aligned} g(f(x)) &= g(f(x')) \\ f(x) &= f(x') && \text{(Because } g \text{ is injective)} \\ x &= x' && \text{(Because } f \text{ is injective)} \end{aligned}$$

Therefore $g \circ f$ is injective. □

Proof. Suppose f, g are surjective functions. Let $z \in Z$. Because g is surjective, there exists some $y \in Y$ such that $g(y) = z$. Furthermore, because f is surjective, there exists some $x \in X$ such that $f(x) = y$. Thus, for every $z \in Z$, there exists some $x \in X$ such that $(g \circ f)(x) = g(f(x)) = g(y) = z$, so $g \circ f$ is surjective. □

Definition. *Bijectivity*
If a function is both injective and surjective, then we say that it is *bijective*.

Note (Cardinality and Bijections). If there is a bijection between two sets, they have the same number of elements.

1.3 Inverses of Functions

Definition. *Inverse of a function*
Suppose $f\colon X \to Y$, $g\colon Y \to X$ is an inverse to f (soon we'll prove that inverses are unique if they exist) if $f \circ g$ and $g \circ f$ are the identity. In other words, $(g \circ f)(x) = x$, $(f \circ g)(y) = y$ for all $x \in X$, $y \in Y$.

Theorem — *Bijective $\iff$ Inverse*
For $f\colon X \to Y$, f is a bijection if and only if f has an inverse.

Proof. Suppose f has an inverse function g. Then $f \circ g$ and $g \circ f$ are the identity. Suppose $f(a) = f(b)$. Then

$$\begin{aligned} f(a) &= f(b) \\ g(f(a)) &= g(f(b)) \\ (g \circ f)(a) &= (g \circ f)(b) \\ a &= b. \end{aligned}$$

Thus f is injective.

Suppose $b \in Y$. Since $f \circ g$ is the identity, we have that $(f \circ g)(b) = b$, so $f(g(b)) = b$. Thus f is surjective. Therefore f is a bijection.

Now suppose that f is a bijection. We define $f^{-1}(a)$ by $f^{-1}(a) = b$, where $a = f(b)$.

- Because f is surjective, we have that for all $a \in Y$, there exists some $b \in X$ such that $a = f(b)$.

- Because f is injective, any $a \in Y$ is *uniquely* mapped by some $b \in X$.

Thus f^{-1} is a function. We will now show that f^{-1} is the inverse of f. For all $x \in X$, $(f^{-1} \circ f)(x) = x$ by definition. For all $y \in Y$,

$$\begin{aligned}
(f \circ f^{-1})(y) &= f(f^{-1}(y)) \\
&= f(f^{-1}(f(x))) && \text{(Because } f \text{ is surjective)} \\
&= f(x) && ((f^{-1} \circ f)(x) = x) \\
&= y.
\end{aligned}$$

Therefore f^{-1} is the inverse of f. □

Theorem — *Uniqueness of Inverses*
Inverses of functions are unique, provided they exist.
Suppose $f\colon X \to Y$. If f has inverses $g, h\colon Y \to X$ such that $g \circ f = h \circ f = \mathrm{id}_X$, $f \circ g = f \circ h = \mathrm{id}_Y$, then $g = h$.

Proof. Let $y \in Y$. By the previous theorem we know that f is surjective, so $y = f(x)$, for some $x \in X$. Thus

$$\begin{aligned}
g(y) &= g(f(x)) \\
&= x \\
&= h(f(x)) \\
&= h(y).
\end{aligned}$$

Thus $g = h$ and the inverse is unique. □

1.4 Special Functions

Definition. *Sequence of elements*
A sequence in X is a function $s\colon D \to X$ where $D \subseteq \mathbb{Z}$.

Example. *Sequence*

(a) $X = \{a, b, c\}$, $D = \{1, 2, 3, 4, 5\}$. We may define $s\colon D \to X$ by:

$$\begin{aligned}
1 &\mapsto a \\
2 &\mapsto b \\
3 &\mapsto c \\
4 &\mapsto b \\
5 &\mapsto a
\end{aligned}$$

(b) The Fibonacci numbers are a sequence of natural numbers. They are defined by: $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

(c) Sequence of even natural numbers: $0, 2, 4, 6, 8, \ldots$. The function $e\colon \mathbb{N} \to \mathbb{N}$ is defined by $e(n) = 2n$. Observe that the sequence of the powers of 2 is a subsequence of the even natural numbers.

Definition. *Subsequences*
A *subsequence* of $s\colon D \to X$ is a sequence obtained by restricting the domain of s. In other words, a *subsequence* is a sequence of the form $t\colon D' \to X$ where $D' \subseteq D$.

Definition. *Strings*
If X is a finite set, a *string* over X is a finite sequence of elements of X.

Example. *Strings*

(a) Let X be the English alphabet. Then c, a, t and d, o, g and m, a, t, h are all strings over X. We write strings without parentheses and commas, so c, a, t becomes *cat*.

Definition. *Special strings*
We will let X^* denote the set of strings over X. Additionally, let λ be the null string.

If α, β are strings over X, we can concatenate them to get a new string $\alpha\beta$.

Example. *Concatenation*
The string c, a, t concatenated with d, o, g becomes c, a, t, d, o, g or *catdog*.

Definition. *Substrings*
A *substring* is a string obtained by selecting some or all consecutive terms of another string. Observe that the terms must be consecutive, unlike subsequences.

2 Relations

Definition. *Relations*
A *relation* R from a set X to a set Y is a subset of $X \times Y$. We write $R(x, y)$ or xRy to denote $(x, y) \in R$. If R is a relation from X to X, we say that R is a relation on X.

Note (Relations and functions). Functions are a special type of relation.

Example. *Relations*

(a) Let X = students at UCLA, Y = Classes at UCLA in Winter '21 Quarter. Define R to be a relation between X and Y such that

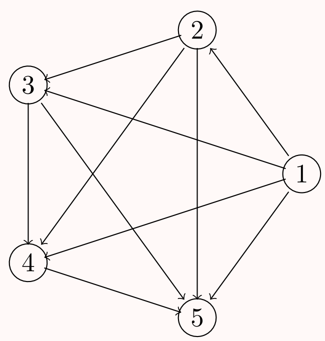
$$R = \{(x, y) \in X \times Y \mid x \text{ is a student in } y\}.$$

Is R a function? No, because a student can be taking more than one class during the Winter '21 Quarter.

(b) Let $X = \{2, 3, 4, 5\}$ and $Y = \{4, 5, 6, 7, 8\}$. Define the relation R to be: xRy if x divides y. Then

$$R = \{(2, 4), (2, 6), (2, 8), (3, 6), (4, 4), (4, 8), (5, 5)\}.$$

(c) Let $X = \{1, 2, 3, 4, 5\}$ and define a relation R on X so that xLy if $x < y$. We can visualise this by drawing an arrow $x \to y$ if $x < y$.



(d) Let $X = \{1, 2, 3, 4, 5\}$, and define a relation LE on X such that $xLEy$ if $x \leq y$. The diagram is the exact same as above, but every element is also related to itself (because $x \leq x$ for all x).

2.1 Types of Relations

(a) Reflexive: R is reflexive if for all $x \in X$, xRx (x relates to itself).

(b) Symmetric: R is symmetric if for all $x, y \in X$, $xRy \implies yRx$.

(c) Antisymmetric: R is antisymmetric if for all $x, y \in X$, xRy and yRx implies $x = y$.

(d) Transitive: R is transitive if for all $x, y, z \in X$, xRy and yRz implies xRz.

Example. *Types of relations*

(a) The relation $<$ over the reals is transitive, (vacuously) antisymmetric, not symmetric, and not reflexive.

(b) The relation $\leq$ over the reals is transitive, antisymmetric, not symmetric, and not reflexive.

(c) Let $X =$ people, and xNy if x and y have the same name. Then N is reflexive, symmetric, and transitive.

(d) Let $X =$ people, and xTy if x is taller than y. Then T is transitive, because if x is taller than y, and y is taller than z, then x is taller than z.

Definition. *Inverse of a relation*
If R is a relation from X to Y, then R^{-1} is the relation from Y to X defined by:

$$R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}.$$

Definition. *Composition of relations*
If $R \subseteq X \times Y$, and $S \subseteq Y \times Z$, then $S \circ R \subseteq X \times Z$ such that

$$S \circ R = \{(x, z) \in X \times Z \mid \text{there exists } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}.$$

2.2 Equivalence Relations

Definition. *Equivalence relation*
A relation R on aset X is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Note. An equivalence relation gives us a notion of two different elements in a set being "the same".

- Reflexive: Everything is "the same" as itself
- Symmetric: If x is "the same" as y, then y is "the same" as x
- Transitive: If x is "the same" as y, and y is "the same" as z, then x is "the same" as z

Example. *Equivalence Relations*

(a) The relation E on the integers where xEy if $x - y$ is even.

- Reflexive: For all $x \in \mathbb{Z}$, $x - x = 0$, which is even, so xEx
- Symmetric: For all $x, y \in \mathbb{Z}$, if $x - y$ is even, so is $-(x - y) = y - x$. Thus if xEy, then yEx
- Transitive: For all $x, y, z \in \mathbb{Z}$, if $x - y$ is even and $y - z$ is even, then their sum, $x - z$, is also even. Thus if xEy and yEz, then xEz.

Observe that this relation relates two integers if they have the same parity.

(b) Let Y be any finite set, and $a, b \in Y^*$ (the set of all strings constructed using Y). Consider the relation L over Y^* such that aLb if a and b have the same length.

(c) Let X be the set of all animals, with animals $x, y \in X$. Consider the relation S over X such that xSy if x and y are of the same species.

(d) Let $x, y \in \mathbb{R}$. Consider the relation C over $\mathbb{R}$ such that xCy if $x - y$ is an integer.

Definition. *Equivalence Classes*
If R is an equivalence relation on a set X, then for $x \in X$, the *equivalence class* of X is the set (with respect to R), denoted by $[x] = [x]_R = \{y \in X \mid xRy\}$.

Example. *Equivalence Classes*

(a) Let E be a relation on $\mathbb{Z}$, where xEy if $x - y$ is even. The equivalence classes for E are $[0]$ (the evens) and $[1]$ (the odds). So, the set of equivalence classes $= \{[0], [1]\}$.

(b) Let $x, y \in \mathbb{R}$, with the relation C over $\mathbb{R}$ defined by xCy if $x - y$ is an integer. The set of equivalence classes $= \{[x] \mid x \in [0, 1)\}$.

If R is an equivalence relation on a set X, then:

- For all $x \in X$, if $x \in [y]$ and $x \in [z]$, then $[y] = [z]$.

  Proof. Suppose $x \in [y]$ and $x \in [z]$. Let $w \in [y]$. Because $w \in [y]$, we know that yRw. We also know that yRx because $x \in [y]$. By symmetry of R, we have wRy, and by transitivity, we have wRx. But $x \in [z]$, so zRx, and by symmetry we have xRw. By transitivity, zRw so $w \in [z]$. Thus $[y] \subseteq [z]$.

  By a similar argument, we have that $[z] \subseteq [y]$, so $[y] = [z]$. □

- For any $x \in X$, x is in some equivalence class, $x \in [x]$ by reflexivity.
  So, for every $x \in X$, x is in exactly one equivalence class. If x is in another equivalence class $[y]$, then by the above $[x] = [y]$.

Definition. *Partition*
For X a set, a *partition* $\mathcal{S}$ of X is a set of nonempty subsets of X such that every element of X is an element of exactly one of the subsets. In other words, for all $A, B \in \mathcal{S}$

- $A, B \subseteq X$
- $A, B \neq \varnothing$
- If $A \cap B \neq \varnothing$ then $A = B$
- For all $x \in X$, there exists exactly one $A \in \mathcal{S}$ such that $x \in A$

Note. We showed that if R is an equivalence relation on X then $\{[x]_R \mid x \in X\}$ is a partition of X.

Theorem — *Equivalence Relations and Partitions*
For X a set, there is a bijection $F\colon$ Set of equivalence relations on $X \to$ Set of partitions of X, defined by

$$F(E) = \{[x]_E \mid x \in X\},$$

the inverse function F^{-1} sends a partition $\mathcal{S}$ to the equivalence relation $F^{-1}(\mathcal{S})$ defined by $xF^{-1}(\mathcal{S})y$ if and only if x and y are in the same set of $\mathcal{S}$ (in the same equivalence class of E).