Exercises Test

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Exercise 1.B.1. Prove that $-(-v) = v$ for every $v \in V$.

Proof.

$$\begin{aligned} -(-v) &= -(-v) + 0 && \text{(Additive Identity)} \\ &= -(-v) + (-v + v) && \text{(Additive Inverse)} \\ &= (-(-v) + -v) + v && \text{(Associativity of Addition)} \\ &= 0 + v && \text{(Additive Inverse)} \\ &= v && \text{(Additive Identity)} \end{aligned}$$

□

Exercise 1.B.2. Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Proof. We begin by first proving that the statement is true for cases where $a = 0$, then by proving it true when $a \neq 0$. Suppose that $a = 0$. Then $a = 0$ and the statement is true. Suppose $a \neq 0$. Then

$$\begin{aligned} av &= 0 \\ av &= 1 \cdot 0 && \text{(Multiplicative Identity)} \\ av &= \left(a \cdot a^{-1}\right) \cdot 0 && \text{(Multiplicative Inverse)} \\ av &= a \cdot \left(a^{-1} \cdot 0\right) && \text{(Associativity of Multiplication)} \\ v &= a^{-1} \cdot 0 \\ v &= 0 \end{aligned}$$

□

Exercise 1.B.3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Proof. We will first show that there exists an $x \in V$ such that $v + 3x = w$. Consider the vector $x = \frac{1}{3}w - \frac{1}{3}v$. Then

$$\begin{aligned} v + 3x &= v + 3\left(\frac{1}{3}w - \frac{1}{3}v\right) \\ &= v + 3 \cdot \frac{1}{3}w - 3 \cdot \frac{1}{3}v \\ &= v + w - v \\ &= v + w + (-v) \\ &= v + (-v) + w \\ &= 0 + w \\ &= w \end{aligned}$$

Now we will show that such an x is unique. Suppose there exists some x' such that $v + 3x' = w$. Then

$$\begin{aligned}
v + 3x' &= w \\
v + 3x' &= 0 + w \\
v + 3x' &= (v + (-v)) + w \\
v + 3x' &= v + (-v + w) \\
3x' &= -v + w \\
3x' &= 1 \cdot (w + (-v)) \\
3x' &= 3 \cdot \frac{1}{3} \cdot (w + (-v)) \\
x' &= \frac{1}{3} \cdot (w + (-v)) \\
x' &= \frac{1}{3}w - \frac{1}{3}v \\
x' &= x
\end{aligned}$$

Thus $x' = x$ which shows that x is unique. □

Exercise 1.B.4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

The empty set does not have an additive identity 0, so it is not a vector space.

Exercise 1.B.5. Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Proof. ($\Rightarrow$) We will show that $0v = 0$ implies there exists some $w \in V$ such that $v + w = 0$.

$$\begin{aligned}
0v &= 0 \\
(1 + (-1))v &= 0 \\
v + (-1 \cdot v) &= 0
\end{aligned}$$

Thus if $0v = 0$ and $w = -1 \cdot v$, $v + w = 0$.
($\Leftarrow$) We will now show that for all $v \in V$, if there exists a $w \in V$ such that $v + w = 0$, then $0v = 0$. Suppose there exists some w such that $0v + w = 0$. Observe that

$$\begin{aligned}
0v + w &= (0 + 0)v + w \\
&= 0v + 0v + w \\
&= 0v + 0 && (0v + w = 0) \\
&= 0v
\end{aligned}$$

Thus $0v + w = 0v = 0$. □

Exercise 1.B.6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$
$$\infty + \infty = \infty, \qquad (-\infty) + (-\infty) = -\infty, \qquad \infty + (-\infty) = 0.$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbf{R}$? Explain.

Proof. It is not a vector space over $\mathbf{R}$, because addition is not associative for all vectors in $\mathbf{R}\cup\{\infty\}\cup\{-\infty\}$. Consider the following:

$$(\infty+\infty)+(-\infty)=\infty+(-\infty)=0\neq\infty=\infty+0=\infty+(\infty+(-\infty)).$$

$\square$

Exercise 1.C.1. For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$:

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$;

Yes, it is a subspace. Suppose $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are vectors in the set. Then we have the identity when $u_1 = u_2 = u_3 = 0$. Additionally, $u_1 + 2u_2 + 3u_3 = 0$ and $v_1 + 2v_2 + 3v_3 = 0$. Thus $(u_1 + v_1) + 2(u_2 + v_2) + 3(u_3 + v_3) = 0$ and we have closure under addition. Finally we have that $k(u_1) + k(2u_2) + k(3u_3) = 0$, so there is closure under scalar multiplication. Therefore the set is a subspace of $\mathbf{F}^3$.

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 \mid x_1 + 2x_2 + 3x_3 = 4\}$;

No, this is not a subspace because there is no identity in the subset of $\mathbf{F}^3$.

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 \mid x_1 x_2 x_3 = 0\}$;

No, this is not a subspace because there is no closure under addition. Suppose $u = (0, 1, 1)$ and $v = (1, 1, 0)$. The vector $u + v = (1, 2, 1)$ is not in the subset.

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 \mid x_1 = 5x_3\}$;

Yes, this is a subspace. Suppose $u = (5u_3, u_2, u_3)$ and $v = (5v_3, v_2, v_3)$ are vectors in the set. Then we have the identity when $u_2 = u_3 = 0$. Additionally, $u + v = (5(u_3 + v_3), u_2 + v_2, u_3 + v_3)$, which is also in the set so we have closure under addition. Finally, $kv = (5(kv_3), kv_2, kv_3)$ is also in the set so we have closure under scalar multiplication.

Exercise 1.C.2. Verify all of the following:

(a) If $b \in \mathbb{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_3 = 5x_4 + b\}$$

is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

Proof. ($\Rightarrow$) Suppose $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_3 = 5x_4 + b\}$ is a subspace of $\mathbf{F}^4$. Then we have closure under addition and scalar multiplication. Thus if $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$ we have $u_3 = 5u_4 + b$ and $v_3 = 5v_4 + b$. Let $w = u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4)$. Observe that

$$\begin{aligned} w_3 &= u_3 + v_3 \\ &= (5u_4 + b) + (5v_4 + b) && (x_3 = 5x_4 + b) \\ &= 5(u_4 + v_4) + 2b. \end{aligned}$$

Because we have closure under addition, $w \in U$ so $w_3 = 5w_4 + b = 5(u_4 + v_4) + b$. Therefore $2b = b$ and $b = 0$.

($\Leftarrow$) Suppose $b = 0$. Then let $U = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 \mid x_3 = 5x_4\}$. Suppose $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2, v_3, v_4)$, and $w = u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4)$. Observe that the identity is in the set when $u_1 = u_2 = u_4 = 0$. Then

$$\begin{aligned} w_3 &= u_3 + v_3 \\ &= 5u_4 + 5v_4 && (x_3 = 5x_4) \\ &= 5(u_4 + v_4) \\ &= 5w_4. \end{aligned}$$

Thus w is in the set and we have closure under addition. We will now show that the set has closure under scalar multiplication.

$$\begin{aligned} ku &= (ku_1, ku_2, ku_3, ku_4) \\ &= (ku_1, ku_2, k(5u_4), u_4) && (x_3 = 5x_4) \\ &= (ku_1, ku_2, 5(ku_4), ku_4) \end{aligned}$$

Because $(ku)_3 = 5(ku)_4$, we have closure under scalar multiplication and thus the set is a subspace. $\square$

(b) The set of continuous real-valued functions on the interval $[0,1]$ is a subspace of $\mathbf{R}^{[0,1]}$.

Proof. Let U be the set of functions $f\colon [0,1] \to \mathbb{R}$. If $g, h \in U$, then $kg \in U$ and $g + h \in U$, for all $k \in \mathbb{R}$. Thus U is closed under addition and scalar multiplication. The identity function $f(x) = 0$ is also in U, so U is a subspace of $\mathbf{R}^{[0,1]}$. □

(c) The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

Proof. Let U be the set of differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$. Then let $g, h \in U$. If $g = 0$, then it is the identity for U because $0 + f = f + 0 = f$ for all $f \in U$. Because the sum of differentiable functions is differentiable, $f + g \in U$ and the set is closed under addition. Because the product of a scalar and a differentiable function is still differentiable, for all $k \in \mathbb{R}$, we have $kg \in U$ and thus U is closed under scalar multiplication. Therefore U is a subspace of $\mathbb{R}^{\mathbb{R}}$. □

(d) The set of differentiable real-valued functions f on the interval (0, 3) such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

Proof. ($\Rightarrow$) Suppose U is the set of differentiable real-valued functions f on the interval (0, 3) such that $f'(2) = b$, and that U is a subspace of $\mathbf{R}^{(0,3)}$. Then for all $k \in \mathbb{R}$, if $g \in U$ then $kg \in U$. Thus $(kg)' = b = kb = kg'$ so $b = 0$.
($\Leftarrow$) Suppose U is the set of differentiable real-valued functions f on the interval (0, 3) such that $f'(2) = 0$. Let $g, h \in U$ so that $g'(2) = 0$ and $h'(2) = 0$. If $g = 0$, then it is the identity because $0 + f = f + 0 = f$ for all $f \in U$. Because differentiation is a linear operator, for all $k \in \mathbb{R}$, we have $(kg + h)'(2) = kg'(2) + h'(2) = 0$ so $kg + h \in U$. Therefore U is a subspace of $\mathbb{R}^{(0,3)}$. □

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^\infty$.

Proof. Let $U = \{c \mid c \text{ is a complex sequence with limit } 0\}$. Suppose $c, d \in U$ so

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0.$$

Consider the sequence $e = (0, 0, \ldots)$. Then $e + c = c + e = c$ for all sequences $c \in U$. Thus e is the identity for U. Then for all $k \in \mathbb{C}$,

$$\begin{aligned}\lim_{n\to\infty}(kc_n + d_n) &= \lim_{n\to\infty} kc_n + \lim_{n\to\infty} d_n \\ &= k\lim_{n\to\infty} c_n + \lim_{n\to\infty} d_n \\ &= k \cdot 0 + 0 \\ &= 0\end{aligned}$$

Thus $kc + d \in U$ and U is a subspace of $\mathbb{C}^\infty$. □

Exercise 1.C.3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

Proof. Let U be the set of all differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$. Suppose $g, h \in U$ so $g'(-1) = 3g(2)$ and $h'(-1) = 3h(2)$. Consider the function $e = 0$, which satisfies $e'(-1) = 0 = 3e(2)$. Then $e + g = g + e = g$ for all $g \in U$, so e is the identity for U. Then for all $k \in \mathbb{R}$,

$$\begin{aligned}(kg + h)'(-1) &= kg'(-1) + h'(-1) \\ &= k(3g(2)) + 3h(2) \\ &= 3(kg(2) + h(2)) \\ &= 3(kg + h)(2)\end{aligned}$$

Thus $kg + h \in U$ and U is a subspace of $\mathbf{R}^{(-4,4)}$. □

Exercise 1.C.4. Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

Proof. ($\Rightarrow$) Let U be the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ and U is a subspace of $\mathbf{R}^{[0,1]}$. Because U is a subspace, for all $k \in \mathbb{R}$ and all $g \in U$, we have $kg \in U$. Thus $\int_0^1 kg = \int_0^1 g = b$. However, $\int_0^1 kg = k\int_0^1 g = kb$ so we have $kb = b$ and thus $b = 0$.
($\Leftarrow$) Let U be the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$. Let $b = 0$, so $\int_0^1 f = 0$ for all functions $f \in U$. Observe that the function $f = 0$ is in U and serves as the identity for U. Let $g, h \in U$ so $\int_0^1 g = \int_0^1 h = 0$. Then for all $k \in \mathbb{R}$, we have $\int_0^1 (kg + h) = k\int_0^1 g + \int_0^h = 0$. Therefore $kg + h$ is in U and U is a subspace. $\square$

Exercise 1.C.7. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

$U = \left\{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{Z}\right\}$.

Exercise 1.C.8. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

$U = \left\{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\} \cup \left\{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\right\}$.

Exercise 1.C.9. A function $f\colon \mathbf{R} \to \mathbf{R}$ is called ***periodic*** if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Proof. No, it is not a subspace. The sum of two periodic functions is not periodic if either of the periods is irrational. Thus the set of all periodic functions is not closed under addition, and is not a subspace. $\square$

Exercise 1.C.12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. ($\Rightarrow$) Suppose V_1, V_2 are subspaces of V and $V_1 \cup V_2$ is also a subspace of V. Let $v_1 \in V_1$ and $v_2 \in V_2$. Because both v_1 and v_2 are in $V_1 \cup V_2$, their sum must be in $V_1 \cup V_2$. Without loss of generality, suppose $v_1 + v_2 \in V_1$. Then $v_1 + v_2 + (-v_1) = v_2 \in V_1$ by closure under addition. Thus every element $v_2 \in V_2$ is also in V_1, so V_2 is contained in V_1.
($\Leftarrow$) Suppose V_1, V_2 are subspaces of V such that one is contained within the other. Without loss of generality, suppose $V_1 \subseteq V_2$. Then $V_1 \cup V_2 = V_2$ so the union is also a subspace. $\square$

Exercise 1.C.16. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Proof.

$$\begin{aligned} U + W &= \{u + w \mid u \in U, w \in W\} \\ &= \{w + u \mid u \in U, w \in W\} \\ &= W + U \end{aligned}$$

$\square$

Exercise 1.C.23. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

then $U_1 = U_2$.

Proof. For every element $v \in V$, there is only one way to express it as a sum of elements from U_1 and W. Let $v = u_1 + w$, where $u_1 \in U_1$ and $w \in W$. However, we also have $v = u_2 + w$, where $u_2 \in U_2$. Because u_1, u_2, and w are unique for all v, $u_2 + w = u_1 + w$. Therefore $u_1 = u_2$ for all $v \in V$ and $U_1 = U_2$. $\square$

Exercise 2.A.1. Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Proof. Let $u_1 = v_1 - v_2$, $u_2 = v_2 - v_3$, $u_3 = v_3 - v_4$, $u_4 = v_4$. Then

$$\begin{aligned} v_1 &= u_1 + u_2 + u_3 + u_4 \\ v_2 &= u_2 + u_3 + u_4 \\ v_3 &= u_3 + u_4 \\ v_4 &= u_4 \end{aligned}$$

Thus the list also spans V. □

Exercise 2.A.2. Verify the following:

(a) A list v of one vector $v \in V$ is linearly independent if and only if $v \neq 0$.

Proof. ($\Rightarrow$, Contrapositive) Suppose $v = 0$. Then $1 \cdot v = 0$ and there exists a non-trivial solution to $av = 0$, so v is a linearly dependent list.
($\Leftarrow$, Contrapositive) Suppose v is a linearly dependent list. Then there must exist some non-zero $a \in \mathbf{F}$ such that $av = 0$. Therefore $v = 0$. □

(b) A list of two vectors in V is linearly independent if and only if neither vector is a scalar multiple of the other.

Proof. ($\Rightarrow$) Suppose that a list of two vectors v_1, v_2 is linearly independent. Without loss of generality, let $v_1 \notin \operatorname{span}(v_2)$. Thus v_1 is not a scalar multiple of v_2.
($\Leftarrow$, Contrapositive) Suppose that a list of two vectors v_1, v_2 is linearly dependent. Then there exists a non-trivial solution to the equation $a_1v_1 + a_2v_2 = 0$, so $v_2 = -\frac{a_1}{a_2}v_1$. Thus one of the vectors is a scalar multiple of the other. □

(c) $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ is linearly independent in $\mathbf{F}^4$.

Proof. Let $u = (1, 0, 0, 0)$, $v = (0, 1, 0, 0)$, and $w = (0, 0, 1, 0)$. There exists no $a_1, a_2 \in \mathbf{F}$ such that $a_1u + a_2v = w$, because w_3 is non-zero. □

(d) The list $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}(\mathbf{F})$ for each nonnegative integer m.

Proof. This problem sucks □

Exercise 2.A.4. Verify that a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1v_1 + \cdots + a_mv_m = 0$.

Proof. Suppose there exists $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1v_1 + \cdots + a_mv_m = 0$. Then the vectors are not linearly independent, so they must be linearly dependent. □

Exercise 2.A.6. Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

Proof. Suppose $a_1, a_2, a_3, a_4 \in \mathbf{F}$. Consider the equation

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4.$$

We can rewrite this as

$$a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0.$$

However, because v_1, v_2, v_3, v_4 is linearly independent, all of the coefficients must be 0. Thus $a_1 = (a_2 - a_1) = (a_3 - a_2) = (a_4 - a_3) = 0$. Therefore $a_1 = a_2 = a_3 = a_4$ and the list of vectors is linearly independent. □

Exercise 2.A.7. Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly independent.

Proof. Suppose $a_1, a_2, \ldots, a_m \in \mathbf{F}$. Consider the equation

$$a_1(5v_1 - 4v_2) + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m = 0,$$

which can be rewritten to

$$(5a_1)v_1 + (a_2 - 4a_1)v_2 + a_3 v_3 + \cdots + a_m v_m = 0.$$

Then, because $v_1, v_2, \ldots, v_m$ is linearly independent, $(5a_1) = (a_2 - 4a_1) = a_3 = \ldots = a_m = 0$. Thus $a_1 = a_2 = \ldots = a_m = 0$ so the list is linearly independent. □

Exercise 2.A.8. Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

Proof. Because $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors, we know that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0$$

is true only if $a_1 = a_2 = \ldots = a_m = 0$. Multiplying both sides by λ, we get

$$a_1 \lambda v_1 + a_2 \lambda v_2 + \cdots + a_m \lambda v_m = 0,$$

which is also only true when all of the coefficients are 0. Thus the list of vectors is linearly independent. □

Exercise 2.A.9. Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

Proof. Consider the vectors $v = (1, 0)$ and $w = (-1, 0)$. Then $v + w = (0, 0)$ is not a linearly independent list. The statement is false. □

Exercise 2.A.10. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Proof. Suppose $b_1, \ldots, b_m \in \mathbf{F}$. Because $v_1 + w, \ldots, v_m + m$ is linearly dependent, there exists some $b_1, \ldots, b_n$ not all zero such that

$$b_1(v_1 + w) + \cdots + b_m(v_m + w) = 0.$$

Observe that

$$\begin{aligned} b_1(v_1 + w) + \cdots + b_m(v_m + w) &= 0 \\ b_1 v_1 + \cdots + b_m v_m &= -(b_1 + \cdots + b_m)w \\ -\frac{1}{b_1 + \cdots + b_m}(b_1 v_1 + \cdots b_m v_m) &= w && \text{(Not all } b \text{ are 0)} \end{aligned}$$

Thus w can be written as a linear combination of $v_1, \ldots, v_m$ so $w \in \operatorname{span}(v_1, \ldots, v_m)$. □

Exercise 2.A.14. Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Proof. ($\Rightarrow$) Suppose V is infinite-dimensional. Then there exists no list of vectors that spans V. Thus we may keep adding linearly independent vectors to a sequence indefinitely.
($\Leftarrow$) Suppose that for every integer m, there exists a sequence of vectors $v_1, \ldots, v_m$ in V such that the list is linearly independent. Suppose, for the sake of contradiction, that V is finite-dimensional with dimension k. However, when $m > k$, $v_1, \ldots, v_m$ is not linearly independent because it is longer than the spanning list of vectors. Thus V must be infinite-dimensional. $\square$

Exercise 2.B.1. Find all vector spaces that have exactly one basis.

Proof. The $\{0\}$ vector space is spanned by the empty set and is the only vector space to have exactly one basis. We will now show that all other vector spaces have more than one basis.

Let v be a vector in a finite-dimensional vector space V, and $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis for V. Consider the set of vectors $\mathcal{C} = \{2b_1, \dots, 2b_n\}$. Because $\mathcal{B}$ is a basis for V, there exists a unique $a_1, \dots, a_n \in \mathbf{F}$ such that

$$v = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

However, we may express any v as

$$v = \frac{a_1}{2}(2b_1) + \frac{a_2}{2}(2b_2) + \cdots + \frac{a_n}{2}(2b_n),$$

so $\mathcal{C}$ spans V. We also know that $\mathcal{C}$ is linearly independent because if the only solution to $a_1b_1+\cdots+a_nb_n = 0$ is the trivial solution, then the only solution to $a_1(2b_1) + \cdots + a_n(2b_n) = 0$ is also the trivial solution. Thus $\mathcal{C}$ is a basis for V. Because $\mathcal{B}$ and $\mathcal{C}$ are both bases of V, there is no vector space with exactly one basis besides $\{0\}$. □

Exercise 2.B.5. Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Proof. Let $p_0 = 1, p_1 = x, p_2 = x^3 - x^2, p_3 = x^3$. Because $p_3 - p_2 = x^2$, we may replace p_2 with x^2 without changing the span of the polynomials. Thus p_0, p_1, p_2, p_3 span $\mathcal{P}(\mathbf{F})$ and the statement is false. □

Exercise 2.B.6. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Proof. Let u be a vector in the vector space V. Because v_1, v_2, v_3, v_4 is a basis for V, there exist unique a_1, a_2, a_3, a_4 such that

$$u = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4.$$

However, the same vector can be rewritten as

$$\begin{aligned} u &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\ &= a_1(v_1 + v_2) + (a_2 - a_1)(v_2 + v_3) + (a_3 - a_2 + a_1)(v_3 + v_4) + (a_4 - a_3 + a_2 - a_1)v_4. \end{aligned}$$

Thus the set of vectors spans V. We will now show that they are linearly independent. Consider the following equation:

$$\begin{gathered} \alpha_1(v_1 + v_2) + \alpha_2(v_2 + v_3) + \alpha_3(v_3 + v_4) + \alpha_4v_4 = 0 \\ \alpha_1v_1 + (\alpha_1 + \alpha_2)v_2 + (\alpha_2 + \alpha_3)v_3 + (\alpha_3 + \alpha_4)v_4 = 0. \end{gathered}$$

Because v_1, v_2, v_3, v_4 is a basis for V we have that $\alpha_1 = \alpha_1 + \alpha_2 = \alpha_2 + \alpha_3 = \alpha_3 + \alpha_4 = 0$. Thus $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and the given set of vectors is linearly independent. Therefore it is a basis of V. □

Exercise 2.B.7. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Proof. The statement is false. Let $V = \mathbb{R}^4$, $U = \{(a, b, c, 0) \mid a, b, c \in \mathbf{F}\}$, and

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}.$$

□

Exercise 2.B.8. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Proof. Because $V = U \oplus W$, there is a unique way to describe every vector in V using a linear combination of U and W and thus $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V. Additionally, because the linear combination is unique, $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent, otherwise there would be more than one way to sum to each vector $v \in V$. Thus the set is a basis of V. $\square$

Exercise 2.C.1. Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Proof. Because the basis of U is linearly independent and has length $\dim V$, it is also a basis for V. Then U and V share a basis, so they must be the same vector space. $\square$

Exercise 2.C.2. Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, $\mathbf{R}^2$, and all lines in $\mathbf{R}^2$ through the origin.

Proof. We will begin by showing that the above are all subspaces of $\mathbb{R}^2$. The first is trivially true. The second is also trivially true because $\mathbb{R}^2$ is a vector space. Finally, we will show that all lines through the origin are subspaces of $\mathbb{R}$. Observe that all lines through the origin are in the set $A = \{(x, y) \mid ax + by = 0\}$ for some $a, b \in \mathbb{R}$. Then for some $k \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2) \in A$, we have

$$\begin{aligned} k(x_1, y_1) + (x_2, y_2) &= (kx_1 + ky_1) + (x_2, y_2) \\ &= (kx_1 + x_2, ky_1 + y_2). \end{aligned}$$

Then

$$\begin{aligned} a(kx_1 + x_2) + b(ky_1y_2) &= k(ax_1) + ax_2 + k(by_1) + by_2 \\ &= k(ax_1 + by_1) + (ax_2 + by_2) \\ &= k \cdot 0 + 0 \\ &= 0. \end{aligned}$$

Thus $k(x_1, y_1) + (x_2, y_2) \in A$, so all lines through the origin form subspaces of $\mathbb{R}^2$.

We must now show that there exist no other subspaces of $\mathbb{R}^2$. The only subspace of $\mathbb{R}^2$ with dimension 0 is the trivial subspace. The only subspace of $\mathbb{R}^2$ with dimension 2 is $\mathbb{R}^2$ itself. We must now show that the only subspaces of $\mathbb{R}^2$ with dimension 1 are lines through the origin. Consider the set $\{(x, y) \mid ax + by = c\}$ where a, b, c are nonzero. Then the set does not contain the zero vector, so it is not a subspace. Therefore the only subspaces of $\mathbb{R}^2$ are $\{0\}$, $\mathbb{R}^2$, and all lines going through the origin. $\square$

Exercise 2.C.9. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Proof. $\square$

Exercise 2.C.10. Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_j has degree j. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Proof. Observe that for every p_j, we may write

$$p_j = a_j z^j + \sum_{n=0}^{j-1} a_{j,n} z^n, \quad \text{with} \quad p_0 = a_0 z^0.$$

Thus the $p_0, \ldots, p_m$ is a linearly independent set. Additionally, it has length $m + 1$, so it is a basis of $\mathcal{P}(\mathbf{F})$. $\square$

Exercise 3.A.1. Hello world