Worksheet 2

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Problem 1. Let $\{0,1\}^X$ denote the set of functions $X \to 2$ for some set X.
Define a function F from $\mathcal{P}(X)$ to $\{0,1\}^X$ by for $A \in \mathcal{P}(X)$, $F(A)$ is the function $X \to \{0,1\}^X$ defined by $F(A)(x) = \begin{cases} 0 & x \in A \\ 1 & x \notin A \end{cases}$.

(a) List the elements of the set $\{0,1\}^{\{a,b\}}$.

(b) Let $X = \{a,b,c\}$. Compute $F(\{b,c\})$, $F(\{a\})$, and $F(\{a,b,c\})$. (Remember all the outputs are functions $X \to \{0,1\}$).

(c) Again let $X = \{a,b,c\}$. Let $g\colon X \to \{0,1\}$ be the function defined by $g(a) = 1$, $g(b) = 1$, $g(c) = 1$. Find a subset A of X so that $F(A) = g$.

(d) Show that for any set X the function $F\colon \mathcal{P}(X) \to \{0,1\}^X$ is a bijection (is injective and surjective).

(e) Use this bijection to give another proof that if X is a finite set then $|\mathcal{P}(X)| = 2^{|X|}$.

(a) The elements of the set $\{0,1\}^{\{a,b\}}$ are:

$$\{(a,0),(b,0)\}, \{(a,1),(b,0)\}, \{(a,0),(b,1)\}, \{(a,1),(b,1)\}.$$

(b)

$$\begin{aligned} F(\{b,c\}) &= \{(a,1),(b,0),(c,0)\} \\ F(\{a\}) &= \{(a,0),(b,1),(c,1)\} \\ F(\{a,b,c\}) &= \{(a,0),(b,0),(c,0)\} \end{aligned}$$

(c) Because all elements of X map to 1, we know that none of the elements of X are in A. Thus $A = \varnothing$.

(d) *Proof.* We will first show that F is injective. Let $A \neq B$. Without loss of generality, there exists some $x \in A$ such that $x \notin B$. Because F is a function, every element in the domain must get mapped to something, so either $(x,0)$ or $(x,1)$ is in $F(A)$, but not in $F(B)$. Therefore $F(A) \neq F(B)$ and F is injective.

We will now show that F is surjective. Observe that for every function $f \in \{0,1\}^X$, the set

$$S = \{x \in X \mid (x,0) \in f\}$$

maps to f. Thus F is a bijection. □

(e) *Proof.* Observe that $\left|\{0,1\}^X\right| = 2^X$, because every element in X can map to one of two elements, 0 or 1. □