Worksheet 3

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Problem 1.

(a) Put a relation E on the integers by xEy if $x+y$ is divisible by 2. Show that E is reflexive, symmetric, and transitive. What are all the elements related to 0? What about all the elements related to 1?

(b) Now consider the relation T on the integers defined by xTy if $x+y$ is divisible by 3. Is this relation reflexive? Symmetric? Transitive?

(a) *Proof.* Observe that for any integer x, $x+x=2x$ is divisible by 2. Thus xEx and E is reflexive. Suppose xEy for some integers x and y. Then for some integer k, we have $x+y=2k=y+x$, so yEx and E is symmetric. Finally, suppose xEy and yEz. Then for some integers m and n, we have $x+y=2m$ and $y+z=2n$. Thus

$$\begin{aligned}
(x+y)+(y+z)&=2m+2n\\
x+2y+z&=2m+2n\\
x+z&=2m+2n-2y\\
x+z&=2(m+n-y).
\end{aligned}$$

Therefore xEz and E is transitive. All of the elements related to 0 are the even integers, and all of the elements related to 1 are the odd integers. $\square$

(b) *Proof.* The relation is not reflexive. Observe that $1+1=2$ is not divisible by 3. The relation is symmetric. Suppose xTy, so that there exists some integer k such that $x+y=3k$. Then $y+x=3k$, so yEx and the relation is symmetric. The relation is not transitive. Observe that $1T2$ because $1+2=3$ is divisible by 3, and $2T1$ because $2+1=3$ is divisible by 3, but $1+1=2$ is not divisible by 3. In other words, $1T2$ and $2T1$ does not imply $1T1$. $\square$

Problem 2. Suppose that X is a set and that we have a function $f\colon X \to \mathbb{R}$. Consider the relation on X defined by aRb if $f(a) \geq f(b)$.

(a) Show that this relation is reflexive and transitive.

(b) What condition on f is necessary for this relation to be antisymmetric?

(c) Describe how by choosing X and f appropriately this relation can give relations on people such as "a is related to b if a is wealthier than b", or "c is related to d if c is older than d".

(a) *Proof.* Let $x, y, z \in X$. Observe that for all x that $f(x) \geq f(x)$, so xRx and R is reflexive. Suppose xRy and yRz. Then $f(x) \geq f(y)$ and $f(y) \geq f(z)$. Thus $f(x) \geq f(z)$, so xRz and R is transitive. □

(b) We need f to be injective.

Proof. Suppose f is injective and let $x, y \in X$ such that xRy and yRx. Then $f(x) \geq f(y)$ and $f(y) \geq f(x)$, so $f(x) = f(y)$. Because f is injective, we have $x = y$ and thus R is antisymmetric.

Suppose towards a contraposition that f is not injective. Then there exists some $x, y \in X$ such that $f(x) = f(y)$ but $x \neq y$. Thus xRy and yRx and $x \neq y$, so R is not antisymmetric. □

(c) We may choose X to be the set of people and f to either give a person's wealth or their age to yield the two relations listed.

Problem 3. Put a relation Q on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by $(a, b)Q(c, d)$ if $ad = bc$. Show that Q is reflexive, symmetric, and transitive. What mathematical structure does this remind you of?

Proof. Let $x, y, z \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ such that $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$. Then xQx because $x_1x_2 = x_2x_1$ implies $x_1x_2 = x_2x_1$, so Q is reflexive. Now suppose xQy, so $x_1y_2 = x_2y_1$. Then $x_2y_1 = x_1y_2$, so Q is symmetric. Finally, suppose xQy and yQz, so $x_1y_2 = x_2y_1$ and $y_1z_2 = y_2z_1$. Multiplying the two equations, we get

$$x_1z_2(y_1y_2) = x_2z_1(y_1y_2).$$

If $y_1 \neq 0$, then we may divide both sides of the above equation to get $x_1z_2 = x_2z_1$. If $y_1 = 0$, then $x_1y_2 = 0 = y_2z_1$, so $x_1 = 0 = z_1$. Therefore $x_1z_2 = 0 = x_2z_1$. In either case, Q is transitive. $\square$

This equivalence relation is the same as the structure of the rational numbers.

Problem 4. The Fibonacci numbers are the sequence $\{F_n\}_{n=0}^{\infty}$ defined by $F_0 = 0$, $F_1 = 1$, and for $n > 1$ $F_n = F_{n-1} + F_{n-2}$. Compute the first 10 Fibonacci numbers. Show that $\sum_{i=0}^{k} F_i = F_{k+2} - 1$. Are the Fibonacci numbers increasing, decreasing, nonincreasing, nondecreasing, or none of these? What about the sequence defined by for $k \in \mathbb{N}$ by $s_k = \sum_{i=0}^{k} F_i$?

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

Proof. Observe that for $n = 0$, we have $\sum_{i=0}^{0} F_i = F_0 = 0 = F_2 - 1$. Furthermore, when $n = 1$, we have $\sum_{i=0}^{1} F_i = 0 + 1 = 2 = F_3 - 1$. Suppose that the statement holds for all non-negative integers m between 0 and k, inclusive. Then

$$\begin{aligned}\sum_{i=0}^{k+1} F_i &= \sum_{i=0}^{k} F_i + F_{k+1} \\ &= F_{k+2} - 1 + F_{k+1} \\ &= F_{k+3} - 1.\end{aligned}$$

Thus the statement holds for $n = k + 1$, so the statement is true for all non-negative integers. □

Observe that $F_2 = F_1 > F_0$, and $F_n - F_{n-1} = F_{n-2} \geq 0$ for all $n \geq 2$. Thus F_n is nondecreasing. As for s_n, observe that $s_2 > s_1 > s_0$, and $s_n - s_{n-1} = F_n > 0$ for all $n \geq 2$. Thus s_n is increasing.