Note. For the theorem above, we need to verify:

- $F^{-1}(\mathcal{S})$ is an equivalence relation,
- $F \circ F^{-1}(\mathcal{S}) = \mathcal{S}$ for all partitions $\mathcal{S}$,
- $F^{-1} \circ F(R) = R$ for all equivalence relations R.

Example. *Equivalence relations* $\iff$ *partitions*
Let

$$\begin{aligned} X &= \{1, 2, 3, 4, 5\} \\ R &= \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,5), (5,1), (1,3), (3,1), (3,5), (5,3), (2,4), (4,2)\}. \end{aligned}$$

What is the corresponding partition, i.e. what is $F(R)$?
The corresponding partition to the aforementioned relation R is $\{\{1,3,5\}, \{2,4\}\}$.

Let $\mathcal{S} = \{\{1,2,3\}, \{4,5\}\}$ what is the corresponding equivalence relation? i.e. what is $F^{-1}(\mathcal{S})$?
The corresponding equivalence relation for S is

$$\{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (4,5), (5,4)\}.$$

1 Counting

Question. If $|X| = n$, and $|Y| = m$, how many functions are there from X to Y?
Every element in the domain must get mapped to something in the codomain, but it does not matter *which* element in the codomain. Furthermore, the function maps each element in the domain to *exactly one* element in the codomain. Observe that for each $x \in X$, there are m "choices" for where x gets mapped. Thus there are $\underbrace{m \cdot m \cdots m}_{n \text{ times}} = m^n$ total functions.

Theorem — *Multiplication principle*
If a set can be enumerated/constructed in t steps (where each step is independent of the other steps) and each step has n_i choices/outcomes, then the set has $n_1 n_2 \cdots n_t$ elements.

Example. I am going to get a pizza from Vito's or D'more's:

Vito's:	D'more's:
2 crusts	1 crust
6 toppings	2 sauces
2 cheeses	3 toppings

The total number of pizzas I could order is: $2 \cdot 6 \cdot 2 + 1 \cdot 2 \cdot 3 = 30$.

Theorem — *Addition principle*
If X and Y are disjoint finite sets, then $|X \cup Y| = |X| + |Y|$.

Example. How many strings of length 5 in $\{0,1\}$ start with 10 or end with 01?

By the multiplication principle, we know there are 2^3 strings that start with 10. By similar reasoning, there are 2^3 strings that end with 01. Furthermore, there are 2 strings that satisfy both of these properties. Thus the total number of strings satisfying the statement above is $2^3 + 2^3 - 2 = 16$.

Theorem — *Inclusion/Exclusion principle*
If X, Y are finite sets, then

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$

Note. The addition principle is just a special case of the inclusion/exclusion principle where $X \cap Y = \varnothing$.