0.1 Special Functions

Definition. *Sequence of elements*
A sequence in X is a function $s\colon D \to X$ where $D \subseteq \mathbb{Z}$.

Example. *Sequence*

(a) $X = \{a, b, c\}$, $D = \{1, 2, 3, 4, 5\}$. We may define $s\colon D \to X$ by:

$$\begin{aligned} 1 &\mapsto a \\ 2 &\mapsto b \\ 3 &\mapsto c \\ 4 &\mapsto b \\ 5 &\mapsto a \end{aligned}$$

(b) The Fibonacci numbers are a sequence of natural numbers. They are defined by: $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

(c) Sequence of even natural numbers: $0, 2, 4, 6, 8, \dots$. The function $e\colon \mathbb{N} \to \mathbb{N}$ is defined by $e(n) = 2n$. Observe that the sequence of the powers of 2 is a subsequence of the even natural numbers.

Definition. *Subsequences*
A *subsequence* of $s\colon D \to X$ is a sequence obtained by restricting the domain of s. In other words, a *subsequence* is a sequence of the form $t\colon D' \to X$ where $D' \subseteq D$.

Definition. *Strings*
If X is a finite set, a *string* over X is a finite sequence of elements of X.

Example. *Strings*

(a) Let X be the English alphabet. Then c, a, t and d, o, g and m, a, t, h are all strings over X. We write strings without parentheses and commas, so c, a, t becomes *cat*.

Definition. *Special strings*
We will let X^* denote the set of strings over X. Additionally, let λ be the null string.

If α, β are strings over X, we can concatenate them to get a new string $\alpha\beta$.

Example. *Concatenation*
The string c, a, t concatenated with d, o, g becomes c, a, t, d, o, g or *catdog*.

Definition. *Substrings*
A *substring* is a string obtained by selecting some or all consecutive terms of another string. Observe that the terms must be consecutive, unlike subsequences.

1 Relations

Definition. *Relations*
A *relation* R from a set X to a set Y is a subset of $X \times Y$. We write $R(x, y)$ or xRy to denote $(x, y) \in R$. If R is a relation from X to X, we say that R is a relation on X.

Note (Relations and functions). Functions are a special type of relation.

Example. *Relations*

(a) Let X = students at UCLA, Y = Classes at UCLA in Winter '21 Quarter. Define R to be a relation between X and Y such that

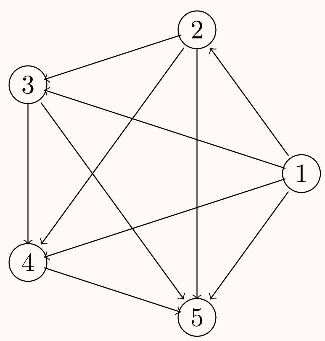
$$R = \{(x, y) \in X \times Y \mid x \text{ is a student in } y\}.$$

Is R a function? No, because a student can be taking more than one class during the Winter '21 Quarter.

(b) Let $X = \{2, 3, 4, 5\}$ and $Y = \{4, 5, 6, 7, 8\}$. Define the relation R to be: xRy if x divides y. Then

$$R = \{(2, 4), (2, 6), (2, 8), (3, 6), (4, 4), (4, 8), (5, 5)\}.$$

(c) Let $X = \{1, 2, 3, 4, 5\}$ and define a relation R on X so that xLy if $x < y$. We can visualise this by drawing an arrow $x \to y$ if $x < y$.



(d) Let $X = \{1, 2, 3, 4, 5\}$, and define a relation LE on X such that $xLEy$ if $x \leq y$. The diagram is the exact same as above, but every element is also related to itself (because $x \leq x$ for all x).

1.1 Types of Relations

(a) Reflexive: R is reflexive if for all $x \in X$, xRx (x relates to itself).

(b) Symmetric: R is symmetric if for all $x, y \in X$, $xRy \implies yRx$.

(c) Antisymmetric: R is antisymmetric if for all $x, y \in X$, xRy and yRx implies $x = y$.

(d) Transitive: R is transitive if for all $x, y, z \in X$, xRy and yRz implies xRz.

Example. *Types of relations*

(a) The relation $<$ over the reals is transitive, (vacuously) antisymmetric, not symmetric, and not reflexive.

(b) The relation $\leq$ over the reals is transitive, antisymmetric, not symmetric, and not reflexive.

(c) Let $X =$ people, and xNy if x and y have the same name. Then N is reflexive, symmetric, and transitive.

(d) Let $X =$ people, and xTy if x is taller than y. Then T is transitive, because if x is taller than y, and y is taller than z, then x is taller than z.

Definition. *Inverse of a relation*
If R is a relation from X to Y, then R^{-1} is the relation from Y to X defined by:

$$R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}.$$

Definition. *Composition of relations*
If $R \subseteq X \times Y$, and $S \subseteq Y \times Z$, then $S \circ R \subseteq X \times Z$ such that

$$S \circ R = \{(x, z) \in X \times Z \mid \text{there exists } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}.$$

Definition. *Equivalence relation*
A relation is an *equivalence relation* if it is reflexive, symmetric, and transitive.