Physics 1C Notes

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Contents

1 Lecture 1

1.1 Electromagnetism

In Physics 1B, we discussed how

$$\text{charge} \rightarrow \text{electric fields} \begin{cases} \text{Coulomb's Law} \\ \text{Gauss' Law} \end{cases}.$$

Furthermore, we know that moving charge $\rightarrow$ magnetic field (Ampere's Law).

$$\text{Gauss' Law: } \oint_{\text{surface}} \vec{E} \cdot \mathrm{d}\vec{A} = \frac{q}{\varepsilon_0} \qquad\qquad \text{Ampere's Law: } \oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I.$$

There are two other main laws for electromagnetism, and all of these laws give us magnets, motors, power generators, etc.

1.2 Electromagnetic Waves

These four laws also lead to electromagnetic waves, examples of which are radio waves, visible light, etc.

> **Definition.** *Speed of Light*
> We define the speed of light c to be
>
> $$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \cdot 10^8 \mathrm{m/s}.$$

1.3 Light and Optics

Light and matter gives us the ideas of reflection and refraction, which lead to prisms, lens, and mirrors. In 1905, Einstein noticed that space and time depend on the observer, and that energy and momentum are different from what we thought.

2   Lecture 2

> **Definition.** *Magnetic field*
> The *magnetic field* is denoted by $\vec{B}$, and its units are given in Tesla (which is a large unit!).

2.1   Demonstrations

- A current through a magnetic field experiences a force (the jumping wire). We observed that the force $\vec{F}$ was perpendicular to the magnetic field $\vec{B}$, which was also perpendicular to the current $\vec{I}$.

- The lodestone is a magnet.

- Opposite poles attract, same poles repel (bar magnets).

- We visualised the magnetic field by looking at the alignment of iron filings in a viscous fluid.

2.2   Equations

1. The force due to a magnetic field is given by

$$\vec{F}_m = q\vec{v} \times \vec{B}.$$

   This force is a part of the electromagnetic force,

$$\vec{F}_{EM} = q\vec{E} + q\vec{v} \times \vec{B}.$$

> **Note (Cross Products).**   Suppose we have two vectors $\vec{a}$ and $\vec{b}$.   Then $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$. If θ is the angle between the two vectors, then
>
> $$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right| \sin\theta = ab\sin\theta.$$
>
> The direction of the vector is given by the "right hand rule".

3   Lecture 3

3.1   Gauss' Law for Magnetic Fields

In Physics 1B, we learned Gauss' Law for electric fields:

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}.$$

We have a similar equation for magnetic fields:

$$\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0.$$

The reason why this is zero is because there are "no magnetic monopoles", i.e. every magnet has two poles.

> **Note.** This can also be interpreted to mean that for a given surface, the number of magnetic field lines coming out must equal the number of magnetic field lines going in.

3.2   The Motion of Charged Particles in a Magnetic Field

There are [some] cases:

i) When the particle is at rest, we have
$$\vec{F} = q\vec{v} \times \vec{B} = 0.$$

ii) For a particle in 2D motion with a constant magnetic field, we have

$$\vec{F} = q\vec{v} \times \vec{B},$$

which is perpendicular to both $\vec{v}$ and $\vec{B}$ (by the properties of the cross product). We use the right hand rule to get the actual direction of the force (when the charge is positive).

> **Note.** Since the magnetic force is perpendicular to the motion of the particle, we have uniform circular motion.

To get the radius of the motion, we solve:

$$F_C = F_B$$
$$\frac{mv^2}{r} = qvB$$
$$r = \frac{mv^2}{qvB}$$
$$\boxed{r = \frac{mv}{qB} = \frac{p}{qB}.}$$

iii) For a particle in 3D motion with a constant magnetic field, $\vec{v}_0$ is not necessarily perpendicular to $\vec{B}$. Thus we first decompose $\vec{v}_0$ into $\vec{v}_0 = \vec{v}_{0_\parallel} + \vec{v}_{0_\perp}$. Then we have

$$\vec{F} = q\left(\vec{v}_{0_\parallel} + \vec{v}_{0_\perp}\right) \times \vec{B}$$
$$= qv_{0_\parallel} \times B + q\vec{v}_{0_\perp} \times \vec{B}$$
$$= q\vec{v}_{0_\perp} \times \vec{B}.$$

> **Note.**   The movement of the particle is circular relative to the perpendicular plane to the magnetic field, but the velocity of the particle parallel to the magnetic field is constant. Thus the particle moves in a helix (or spiral) pattern.

3.3   Force Equation for a Current-Carrying Wire

We define $\vec{\ell}$ to be the length of the wire in the direction of $\vec{v}$. From earlier, we know that

$$\vec{F} = N \cdot q\vec{v} \times \vec{B}.$$

Furthermore, we have

$$I = \frac{Q}{t} = \frac{Nq}{t} = \frac{Nq}{\frac{\ell}{v}} = \frac{Nqv}{\ell}.$$

Thus we have

$$\begin{aligned}
\vec{F} &= N \cdot q\vec{v} \times \vec{B} \\
&= N \cdot qv \cdot \frac{\vec{\ell}}{\ell} \times \vec{B} \\
&= I\vec{\ell} \times \vec{B}.
\end{aligned}$$

Alternatively, if things are always changing, then

$$\vec{F} = \int_{\text{wire}} I \, \mathrm{d}\vec{\ell} \times \vec{B}.$$

4   Lecture 4

4.1   Current Loops in Magnetic fields: Force and Torque

4.1.1   Uniform Magnetic Field Perpendicular to the Loop

Consider a fictional square loop of wire with side L, where the current flowing through the loop is I. We denote $\odot$ to be out of the page, and $\otimes$ to be into the page. We will use the equations $\vec{F} = I\vec{L} \times \vec{B}$ and $\vec{\tau} = \vec{r} \times \vec{F}$, and assume that we have a constant, uniform $\vec{B}$ field.
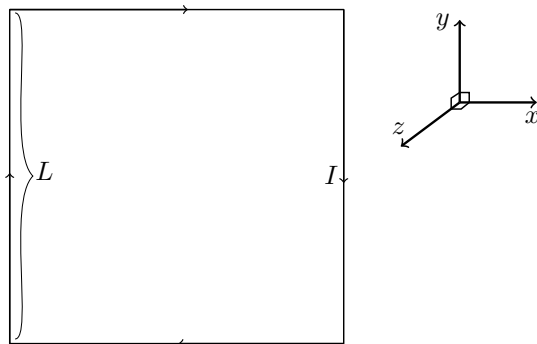
$\vec{B}$	$\vec{F}_\text{left}$	$\vec{F}_\text{top}$	$\vec{F}_\text{right}$	$\vec{F}_\text{bottom}$	$\vec{F}_\text{net}$	$\vec{\tau}_\text{net}$
$\otimes$	$\leftarrow$	$\uparrow$	$\rightarrow$	$\downarrow$	0	0
$\uparrow$	0	$\odot$	0	$\otimes$	0	$\hat{\imath}\left(\frac{L}{2}BIL \cdot 2\right) = (BIL^2)\hat{\imath}$

The magnetic moment μ is given by $\mu = IA$, where A is the area of the loop (in this case, $L = A^2$). We use this μ to find the torque, and we have $\tau = \mu B$ or $\vec{\tau} = \vec{\mu} \times \vec{B}$. The magnetic moment is defined as perpendicular to the surface of the loop.

   For the example above, the magnetic moment must be *into the page*. We can find this by using the right hand rule, where you point your thumb in the direction of the current and curl your fingers around to find the direction of the moment.

4.1.2   Generalisation to a Loop of Any Shape

We can approximate an irregularly shaped loop of wire by breaking it into smaller, rectangular loops of wire.

> **Note.** If the currents of the rectangular loops are all travelling in the same direction, the currents inside of the outer loop will cancel, leaving only the current around the perimeter of the outside loop.

5   Lecture 5

Last week, we studied the force due to a $\vec{B}$. This week, we will study how to produce a $\vec{B}$.

5.1   Biot-Savart Law

Point charge observations:

- When you have a point charge, you have an electric field $\vec{E}$ that radiates outwards from the point charge.

- This is slightly different from the magnetic field, which is generated by a *moving* point charge.

    - The closer you are to the point charge, the larger the magnetic field.

> **Theorem** — *Coulomb's Law*
> Given the charge and distance from a point charge, we may find the electric field:
> $$\vec{E} = \frac{q\hat{r}}{4\pi\varepsilon_0 r^2}.$$

Biot-Savart Law precursor:
$$\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v} \times \hat{r}}{r^2}.$$

> **Note.** In the equation for the magnetic field, we see that it is both orthogonal to the velocity vector and $\hat{r}$ (due to the cross product).

Points:

- Both are inversely proportional to the square of the distance.

- $\mu_0 \coloneqq 4\pi \cdot 10^{-7}$ Tm/A, and is used in the definition of the ampere (and so the Coulomb).

> **Note.** This means that the $\vec{E}$ is very large, but $\vec{B}$ is quite small.

> **Theorem** — *Biot-Savart Law*
> Given the current through a wire, and the displacement from a point to the wire, we can find the magnetic field:
> $$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{\ell} \times \hat{r}}{r^2}.$$
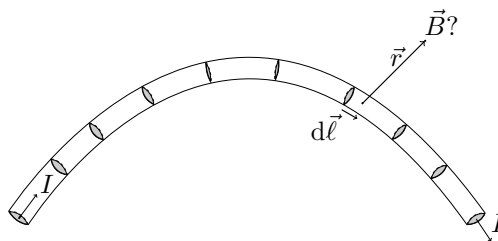
In the above diagram, you can see how the cross product of $d\vec{\ell}$ and $\vec{r}$ is in the direction of the magnetic field. Furthermore, we can integrate the Biot-Savart Law to get
$$\vec{B} = \int_{\text{wire}} d\vec{B}$$

5.2 Magnetic Field of a Straight Current-Carrying Wire

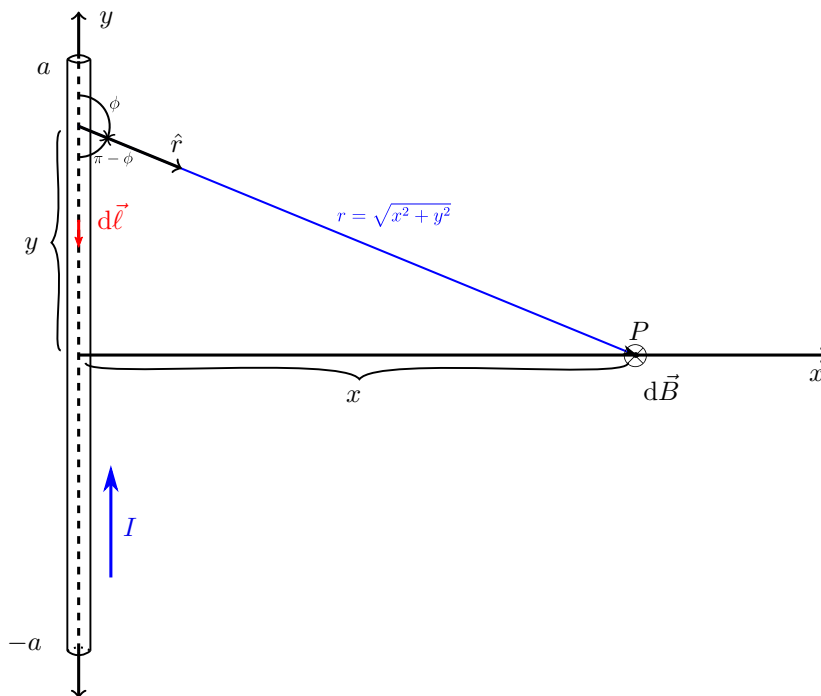For a long, straight current-carrying wire, we try to find the field on some point P from locations $-a$ to a on the wire.

From the diagram, we can see that $r = (x^2 + y^2)^{\frac{1}{2}}$, and

$$\hat{r} = \frac{x\hat{\imath} - y\hat{\jmath}}{(x^2 + y^2)^{\frac{1}{2}}}. \qquad\qquad \text{(Scale down } \vec{r}\text{)}$$

Furthermore, $\mathrm{d}\vec{\ell} = \mathrm{d}y\hat{\jmath}$. The hard part is calculating the following:

$$\frac{\mathrm{d}\vec{\ell} \times \hat{r}}{r^2} = \frac{\mathrm{d}y\hat{\jmath} \times (x\hat{\imath} - y\hat{\jmath})}{(x^2 + y^2)^{\frac{3}{2}}}$$

$$= \frac{-\hat{k}x\,\mathrm{d}y}{(x^2 + y^2)^{\frac{3}{2}}}.$$

We then integrate over the length of the wire:

$$\vec{B} = -\hat{k}\frac{\mu_0 I}{4\pi} \int_{-a}^{a} \frac{x\,\mathrm{d}y}{(x^2 + y^2)^{\frac{3}{2}}}$$

Letting $y = x\tan\theta$, and going through some algebra, we get

$$= -\hat{k}\frac{\mu_0 I}{4\pi} \left(\frac{2a}{x\sqrt{a^2 + x^2}} \right).$$

For an infinitely long wire, we see what happens when $a \to \infty$. In this case, we have

$$\vec{B} = -\hat{k}\frac{\mu_0 I}{4\pi} \left(\frac{2}{x} \right) = -\hat{k}\frac{\mu_0 I}{2\pi x}.$$

We started with a special point P on the wire, which is on the perpendicular bisector of the wire. However, when $a \to \infty$, we have an infinite amount of wire on either side, so the y becomes irrelevant and all we care about is the distance x to the wire. Thus to find the magnitude of the wire, we have

$$B = \frac{\mu_0 I}{2\pi r},$$

where r is the distance to the wire. If we were to look at the wire from the top (the xz-plane), we would have that $\vec{B}$ circles the wire.

Note. Another right-hand rule is to put your thumb in the direction of the current, and your thumbs will curl to form the magnetic field.

6   Lecture 6

The force between long parallel wires defines the ampere. As a reminder, last class we found the formula

$$B = \frac{\mu_0 I}{2\pi r}.$$

Consider two parallel, infinitely-long wires that are carrying current. If the currents are going in the same direction, then we use the right-hand rule to find that the wires have an attractive force between them. We find that the force acting on wire 2 is

$$\vec{F} = I_2 \vec{L} \times \vec{B}_1$$
$$= I_2 L \frac{\mu_0 I_1}{2\pi r} \text{ to the left,}$$

so we have

$$F_2 = \frac{\mu_0}{2\pi} \cdot \frac{I_1 I_2 L}{r}.$$

Rearranging some more, we get

$$\frac{F_2}{L} = \left(2 \cdot 10^{-7}\right) \frac{I_1 I_2}{r} \text{ Newtons.}$$

What happens when we ave $I_1 = I_2$? Well, we get that

$$\frac{F}{L} = \left(2 \cdot 10^{-7}\right) \frac{I^2}{r},$$

which we can rearrange to get

$$I = \underbrace{\left(r \cdot \frac{F}{L} \cdot \frac{1}{2 \cdot 10^{-7}}\right)^{\frac{1}{2}}}_{\text{defines the Ampere}}.$$

6.1   Ampere's Law

Just like Coulomb's Law leads to Gauss' Law (which is an integral), the Biot-Savart Law leads to Ampere's Law (also an integral). Remember the Biot-Savart Law from before:

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi} \frac{I \, \mathrm{d}\vec{\ell} \times \hat{r}}{r^2}.$$

> **Theorem** — *Ampere's Law*
> The amount of magnetic field in a loop can give you the current, given by
>
> $$\oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enc}}.$$

The analogue to the Gaussian surface is what we call an *Amperium loop*. There is no radial component to $\vec{B}$ because there are no monopoles. Thus we have that $\vec{B} \parallel \mathrm{d}\vec{\ell}$, and $\vec{B} \cdot \mathrm{d}\vec{\ell} = B \, \mathrm{d}\ell$. Then

$$\oint B \, \mathrm{d}\ell = B \oint \mathrm{d}\ell$$
$$= B \cdot 2\pi r$$
$$= \mu_0 I,$$

so

$$B = \frac{\mu_0 I}{2\pi r}.$$

The differential form of Ampere's Law is given by:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J},$$

where $\vec{J}$ is the current density.

6.1.1   Ampere's Law Example—Long Wire, Thin but Finite Thickness

Consider a wire with radius R that has uniform current density J. We define current density to be the current per unit area, in other words

$$J = \frac{I}{\pi R^2}, \text{ a constant.}$$

We have a few cases here:

(a) Outside the wire, i.e. $r > R$.

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{encl}}$$

$$2\pi r B = \mu_0 I$$

$$\boxed{B = \frac{\mu_0 I}{2\pi r}.}$$
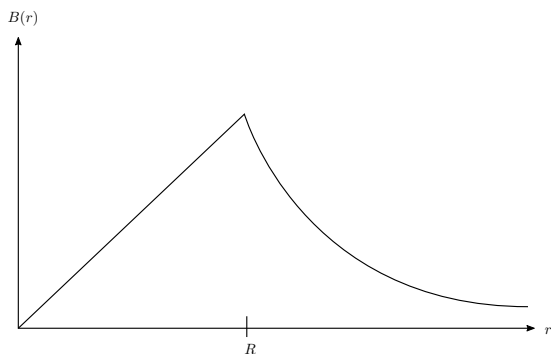
(b) Inside the wire, i.e. $r < R$.

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{encl}}$$

$$2\pi r B = \mu_0 I \cdot \frac{\pi r^2}{\pi R^2}$$

$$2\pi r B = \mu_0 I \cdot \frac{r^2}{R^2}$$

$$\boxed{B = \frac{\mu_0 I r}{2\pi R^2}.}$$

Pictorially, we can draw a graph for the magnetic field as a function of the distance to the centre of the wire as follows:

6.1.2   Ampere's Law Example—Plane of Current

If we have two parallel wires next to each other with the currents moving in the same direction, by superposition of field lines we generate a space with zero field between the wires. Extending this idea to more than just two wires (read: an infinite number, creating a plane of current), we create a planar magnetic field.
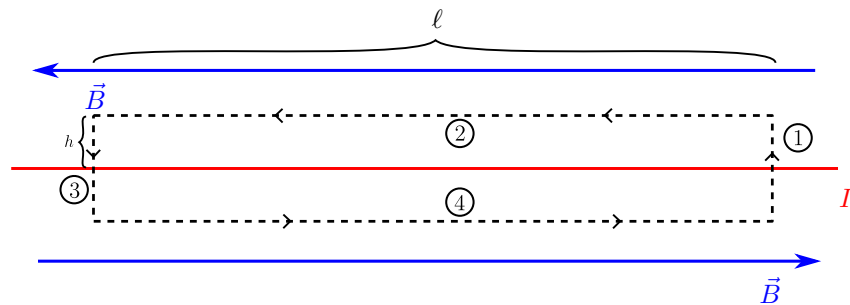
Figure 1: Plane of Current

Definition. *Winding Density*
We define the *winding density* n to be the number of wires per unit length, or

$$n = \frac{N}{\ell}.$$

Applying Ampere's Law, we have

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{encl}}$$

$$\int_1 + \int_2 + \int_3 + \int_4 = \mu_0 NI$$

$$2 \int_4 \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 NI$$

$$2B\ell = \mu_0 NI$$

$$B = \frac{\mu_0 nI}{2}.$$

7 Lecture 7

7.1 Magnetic Field from a Solenoid

> **Definition.** *Solenoid*
> A *solenoid* is the winding of wire in a helix shape.

If you take a 2D cross-section of the solenoid, you can see how the inside of the solenoid has parallel magnetic field lines due to the superposition of the field lines. This is analogous to the parallel electric field lines found between parallel, charged plates. In the infinite case, the magnetic field can be calculated with the equation
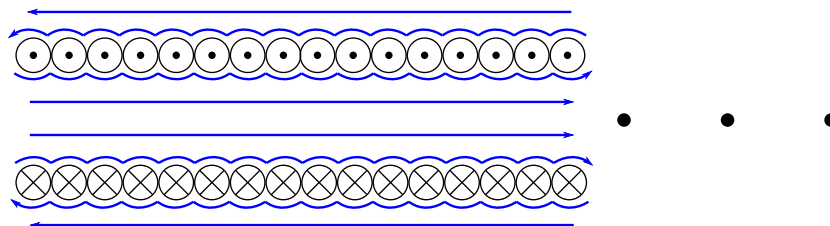$$B = \mu_0 \cdot \frac{nI}{2}.$$

Figure 2: Cross Section of a Solenoid

Using Ampere's Law, with a box around all of the currents shown, we have
$$\oint_{\text{loop}} \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{encl}}$$
$$= \int_1 + \int_3$$
$$= 0. \qquad \text{(The currents cancel)}$$

From our diagram we see that the magnetic field outside of the solenoid is in one direction, regardless where you are, so $\int_1 = \int_3 = 0$. Restricting our attention to only the top half of the currents, we have
$$\oint_{\text{loop}} \vec{B} \cdot d\vec{\ell} = \mu_0 NI$$
$$B \cdot L = \mu_0 nLI$$
$$B = \mu_0 nI.$$

The magnetic field lines outside the solenoid cancel out.

7.2   Magnetic Field from a Toroid

To get a toroid, you basically bend a solenoid into a loop. Consider a toroid with inner radius a and outer radius b. Then applying Ampere's Law, we get

$$\oint_{\text{loop}} \vec{B} \cdot \mathrm{d}\vec{\ell} = 2\pi r B$$
$$= \mu_0 N I,$$

so $B = \frac{\mu_0 N I}{2\pi r}$ (here we have $n = \frac{N}{2\pi a}$ when $a < r < b$). Outside of the toroid, the enclosed current is zero.

8   Lecture 8

8.1   Magnetism in Matter—Atoms

For a series of loops, we know that the magnetic moment is given by

$$\vec{\mu} = NIA \cdot \hat{n},$$

where N is the number of loops, I is the current, and A is the area enclosed by the loop. Furthermore, the torque on the loop(s) in an external $\vec{B}$ field is given by $\vec{\tau} = \vec{\mu} \times \vec{B}$. This is analogous to a dipole in an electric field, where we had

$$\vec{p} = q \cdot \vec{\ell} \quad \text{and} \quad \vec{\tau} = \vec{p} \times \vec{E}.$$

Question. What happens if dipoles go to minimum potential energy U?

If we call the *electric field generated by the dipole* $\vec{E}'$, then we see that it tends to point in the opposite direction to $\vec{E}_0$, the original electric field. Then we have that the net electric field is given by $\vec{E} = \vec{E}_0 + \vec{E}'$.

In contrast, the magnetic field created by the solenoid is usually in the *same* direction as $\vec{B}_0$, so the magnetic field usually gets *amplified*.

8.2   Describing Magnetic Materials

The textbook goes over the "Bohr magneton" for a single atom. We have the equations

$$\vec{M} = \frac{\vec{\mu}_{\text{total}}}{V} \quad \text{and} \quad \vec{B} = \vec{B}_0 + \mu_0 \vec{M}.$$

However, the above equations are usually not that useful, so we ignore them. More interestingly, we have

$$\mu_0 \to \mu = K_m \cdot \mu_0$$
$$\oint B \cdot \mathrm{d}\ell = \mu I$$
$$B \cdot 2\pi r = \mu I.$$

The K_m constant referenced above is called "permeability". We replace all of the μ_0's in the previous equations with just μ.

> **Note.** This is analogous to the dielectric constant for capacitors.

> **Example.** The permeability of iron is approximately 1000, so when you fill a solenoid with iron, you increase its magnetic field output by a factor of 1000. This is similar to filling the space between the plates of a capacitor with a dielectric, which increased the capacitor's capacitance.

> **Note.** Most materials have a permeability K_m that is close to 1.

We define a new constant called "susceptibility" given by $\chi_m := K_m - 1$. This makes it such that have a "susceptibility" close to zero are the items that are not very magnetic.

If atomic loops perfectly align, then the currents inside the magnet perfectly cancel out, leaving only a net current along the boundary of the magnet. This generates the same effect as a solenoid, which just has a current looping around and around its boundary.

8.3   Three Types of Magnetic Materials

1. Iron-like: Ferromagnetic materials have $K_m \gg 1$ (much greater than one).

2. Paramagnetic materials have $K_m = 1 + \underbrace{\chi_m}_{\text{around } 10^{-4}}$.

3. Diamagnetic materials have $K_m = 1 - \underbrace{|\chi_m|}_{\text{around } 10^{-5}}$.

Ferromagnetic materials are useful:

- They have big K_m due to atoms aligning better than thermally expected (they are usually in disarray).

In the following diagram (hysteresis curve), the x-axis represents the applied magnetic field, and the y-axis the induced magnetic field. If you apply a strong magnetic field to a ferromagnetic material, it will also create its own magnetic field, amplifying the effects of the original magnetic field. What's interesting here is that when you remove the external magnetic field, the ferromagnetic field will settle into being a permanent magnet. If you apply a negative magnetic field to the object, the same happens, but in the other direction.
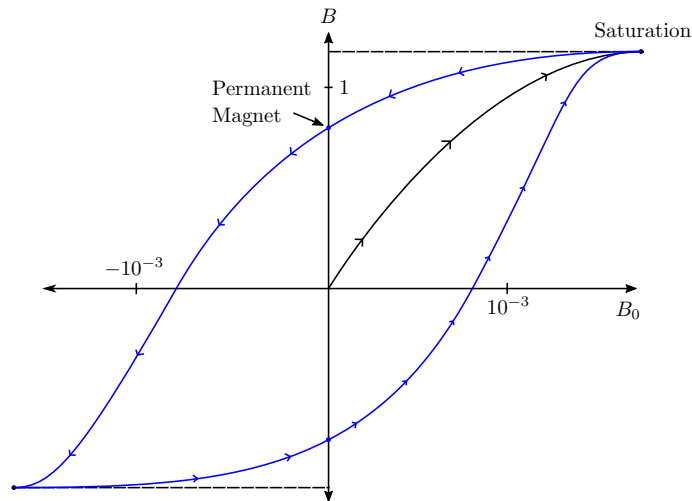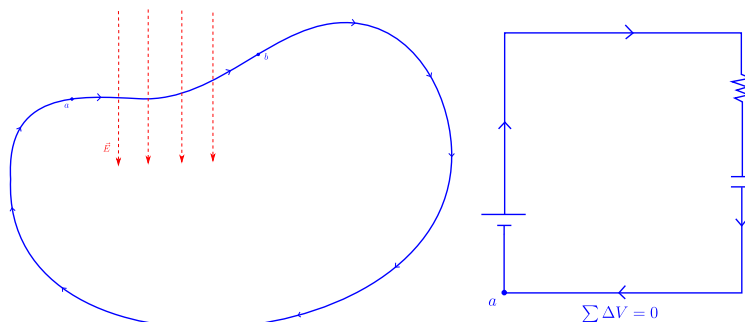

The black curve above traces the induced field as you apply an external $\vec{B}$, and the blue curves show how the induced magnetic field changes with the external field after the initial application.

9   Lecture 9

9.1   A New Fundamental Law of Nature—Faraday's Law

A changing magnetic field produces *loops* in the electric field. Recall for voltage:

$$\Delta V_{ab} = -\int_a^b \vec{E}\,\mathrm{d}\vec{\ell} \quad \text{and} \quad \oint_{\text{loop}} \vec{E}\cdot\mathrm{d}\vec{\ell} = 0.$$

Theorem — *Faraday's Law*
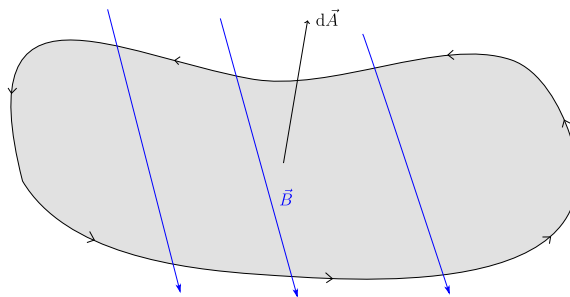If Φ_m is the magnetic flux, given by

$$\Phi_m = \oint_{\text{surface}} \vec{B}\cdot\mathrm{d}\vec{A},$$

then we have

$$\oint \vec{E}\cdot\mathrm{d}\vec{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_m = -\frac{\mathrm{d}\Phi_m}{\mathrm{d}t}.$$

Alternatively, we have the differential form of Faraday's Law, given by

$$\vec{\nabla}\times\vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}.$$

Note. Note that for Gauss' Law, we needed a potential difference in order to be able to generate an electric field, but for Faraday's Law we just need a changing magnetic flux.

9.2   Moving a Magnet Into Coil: EMF

When we move the north pole of a magnet into a coil of wire, we are increasing the magnetic field through the same area, so the magnetic flux is increasing.

Cross-section of $\mathrm{d}\vec{A}$ is out of the page

$\vec{B}$ is increasing
Φ_m is increasing
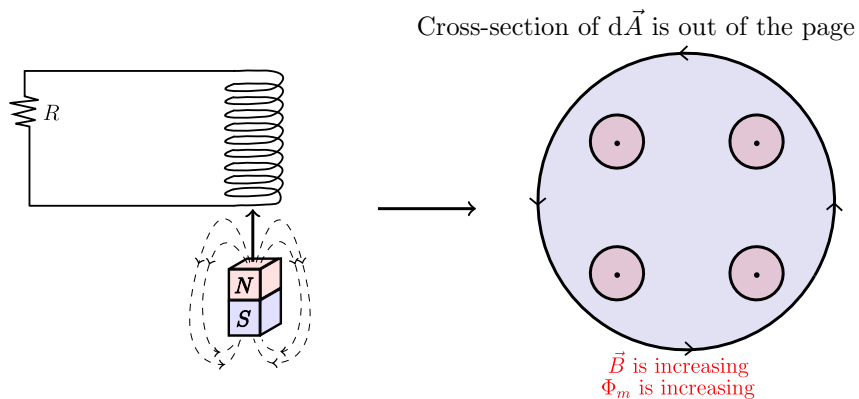
Applying Faraday's Law, we have

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} < -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_m < 0.$$

From this, we know that $\vec{E}$ is clockwise, and the current is also in the clockwise direction.

9.3   Rotating Coil in a Magnetic Field

Suppose we were to rotate a square coil in a magnetic field, as follows:

Square coil

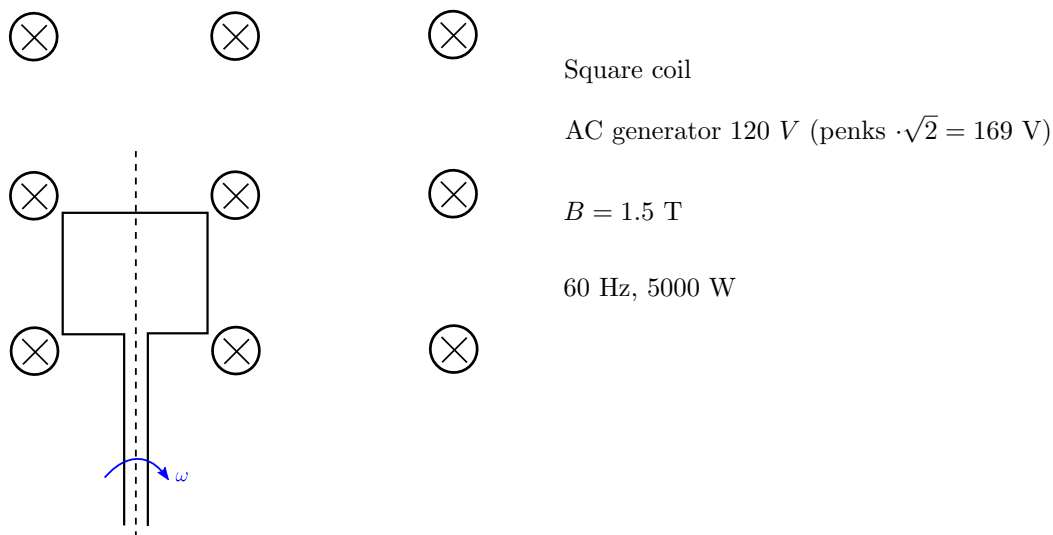AC generator 120 V (penks $\cdot\sqrt{2} = 169$ V)

$B = 1.5$ T

60 Hz, 5000 W

Figure 3: Top view of the loops of wire

Suppose that there are $N = 169$ loops of wire, so there is one loop of wire per volt needed. We can find the
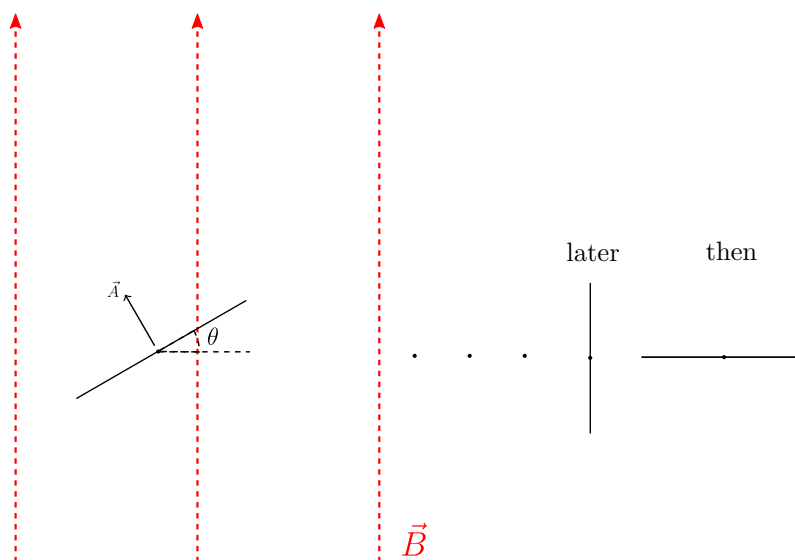

Figure 4: Side view of the loops of wire

angular velocity from the frequency, given by $\omega = 2\pi f = 377$ rad/s. We see from our diagram that

$$\Phi_m = BA \cdot \cos\theta = BA\cos(\omega t),$$

so

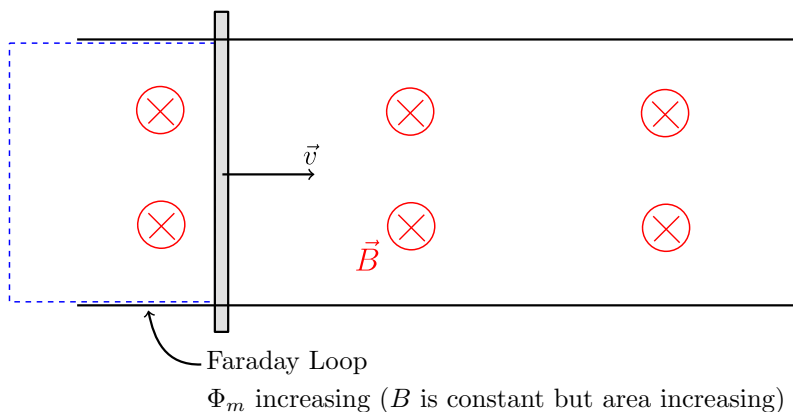$$\frac{\mathrm{d}\Phi_m}{\mathrm{d}t} = -BA\omega\sin(\omega t).$$

The electromotive force $\varepsilon = BA\omega$ should be equal to 1 volt (because each of the 169 loops contributes 1 volt to the net potential). Thus we can solve for the area:

$$A = \frac{\varepsilon}{B\omega} \approx 26.5 \text{ cm}^2.$$

9.4   Motional EMF

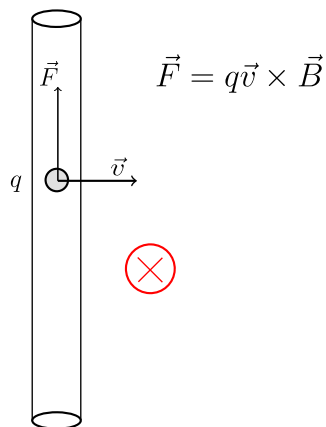Consider a wire laying on rails, with magnetic field between the rails, as shown:

Faraday Loop
Φ_m increasing (B is constant but area increasing)

If the width of the Faraday loop is x, then the area is

$$A = xL, \text{ and } \Phi_m = B \cdot A = B \times L.$$

Furthermore, we have

$$\varepsilon = -\frac{\mathrm{d}\Phi_m}{\mathrm{d}t} = -BL \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = -BLv$$

across the end of the slider. From another point of view, we see

$$\vec{F} = q\vec{v} \times \vec{B}$$

The magnetic force on the charges inside the rod move the positive charges upwards, and the negative charges downwards. This creates an electric field in the rod from top to bottom, which counteracts the magnetic force. From either perspective, we reach the same conclusion that there is a potential difference across the rod.

10   Lecture 10

10.1   Lenz's Law, Eddy Currents

This is useful for determining the sign and direction of things in a system involving magnetic fields/forces.

> **Note.** When changing a magnetic system, forces and currents act in the direction to *oppose* the change.
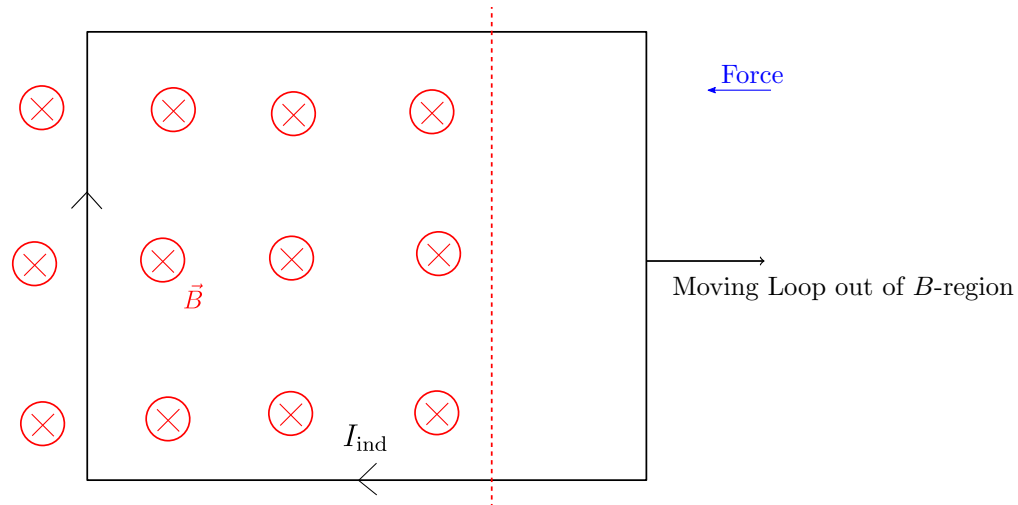

Figure 5: Pulling a Loop of Wire out of a Magnetic Field

As you pull the loop of wire out of the magnetic field that is into the page, the area of the loop that is in the constant magnetic field will decrease. Thus the loop of wire will have an induced current such that it will generate more field into the page. By the right-hand rule, we can see that the induced current should run clockwise. Furthermore, the magnetic forces on the top and bottom portions of the wire are equal and in opposite directions, and so cancel out. There is a force on the left part of the loop that is pointing to the left, which opposes the force that is pulling the loop out of the magnetic field.

11   Lecture 11

11.1   Motional EMF—Conductors Moving in Magnetic Fields

See the diagram in section 9.4. From the Lorentz force equation, we have

$$\vec{F} = q\vec{v} \times \vec{B}.$$

The electric field generated by the separation of charges (positives to the top, negatives to the bottom) in the conductor will create a force equal to and opposing the magnetic force. In other words,

$$\vec{F}_E = q\vec{E} = -q\vec{v} \times \vec{B} = -\vec{F}_M,$$

so $\vec{E} = -\vec{v} \times \vec{B}$. If the conductor is connected to metal rails, we see that this electric field will actually send the charges through the loop. The magnitude of this EMF is given by $\varepsilon = BLv$, which agrees with what we got in section 9.4.

   We can generalise this to an arbitrary loop of wire with segments all moving in random directions, by cutting the loop up into small segments of length $\mathrm{d}\vec{\ell}$ and integrating:

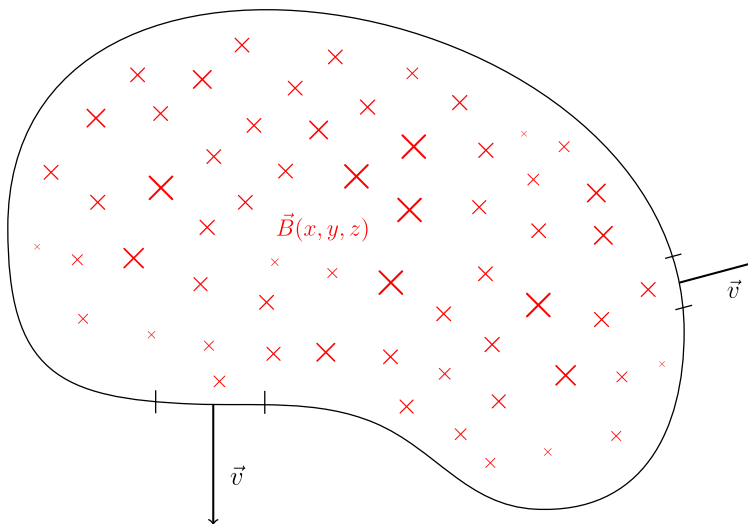$$\varepsilon = \oint (\vec{v} \times \vec{B}) \, \mathrm{d}\vec{\ell}.$$

Figure 6: Arbitrary Loop of Wire in Magnetic Field
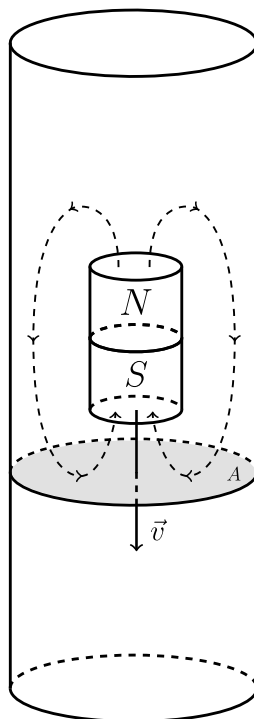
11.2   Eddy Current Demonstrations

Figure 7: Magnet Falling Through a Conducting Tube

Notice that as the magnet falls, the magnetic flux through the area A is increasing upwards. By Lenz's Law, there will be an induced magnetic field pointing downwards, opposing the change in magnetic flux. This induced magnetic field exerts a force on the magnet, causing it to slow down and fall through the tube more slowly. As the tube gets thicker and more conductive, the eddy currents and induced magnetic field also get larger, causing the magnet to fall even more slowly.

12   Lecture 12

12.1   Magnetic Induction in Circuits

Take $\vec{B}$ uniform, increasing with $B = \alpha \cdot t$ into the page. We choose the area vector $\vec{A}$ to be out of the page, resulting in

$$\Phi = \int \vec{B} \cdot d\vec{A} = \vec{B} \cdot \vec{A} = -BA < 0,$$

and

$$\frac{d\Phi}{dt} = -A \cdot \frac{dB}{dt} = -A\alpha < 0.$$

Then $\varepsilon = -\frac{d\Phi}{dt} = A \cdot \alpha > 0$. We know that ε is defined by

$$\varepsilon = \oint \vec{E} \cdot d\vec{\ell} > 0,$$

and the direction of the loop is set by the direction of $\vec{A}$. By the right hand rule, this direction must be counter-clockwise. Thus having a positive ε means $\vec{E}$ is in the same direction as $d\vec{\ell}$, so $\vec{E}$ is also counter-clockwise.

> **Note.** Just like motional EMF: charges will accumulate almost instantaneously since $\vec{E} = 0$ inside the conductor.

Let's make this a circuit by attaching a resistor: If we look at the direction of this current, it will induce a magnetic field B' inside of the loop, that is pointing out of the page. This opposes the B that is going into the page.

12.2   Concept of Self-inductance L, Solenoid Example

We know that the magnetic field inside a solenoid is

$$B_{\text{in}} = \mu \frac{N}{\ell} I = \mu n I, \qquad\qquad\qquad (n := \tfrac{N}{\ell})$$

where N is the number of turns, ℓ is the length of the solenoid, and I is the current. The flux inside the solenoid is given by

$$\Phi_B = B_{\text{in}} \cdot A \cdot N = (\mu n I) A N.$$

We differentiate this to get the electromotive force (and assume the current is the only one that changes),

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\mu n A N \cdot \frac{dI}{dt}.$$

We define the self-inductance L by $L := \mu n A N$, which simplifies the above into

$$\boxed{\varepsilon = -L \cdot \frac{dI}{dt}.}$$

For other geometries:

- $B = (\text{constant of geometry}) \cdot I$

- $\Phi_B = (\text{another constant of geometry}) \cdot I$

- $\varepsilon = -\frac{d\Phi_B}{dt} = -(\text{some constant}) \cdot \frac{dI}{dt} = -L \cdot \frac{dI}{dt}$

12.3   Magnetic Field Energy

Consider a voltage V sending a current I through an inductor L. To find the power dispersed across the inductor, we use

$$P = |I\varepsilon| = IL \cdot \frac{\mathrm{d}I}{\mathrm{d}t} \qquad \text{(into/out of the inductor)}$$

Suppose V starts at zero, then increases. The back ε opposes the increase of current—power needs to be supplied. We use the equation

$$\text{Energy} = \int_0^t \text{Power} \cdot \mathrm{d}t' \qquad (t' \text{ is dummy variable})$$

to find the energy stored in the inductor (U). Thus we have

$$
\begin{aligned}
U &= \int_0^t P \, \mathrm{d}t' \\
&= \int_0^t IL \cdot \left(\frac{\mathrm{d}I}{\mathrm{d}t'}\right) \mathrm{d}t' \\
&= \int_0^I I'L \, \mathrm{d}I' \qquad \text{(Change dummy variable)} \\
&= \frac{1}{2} I^2 L.
\end{aligned}
$$

> **Note.** This is similar to the equation for the energy of a capacitor, which was $U = \frac{1}{2}CV^2$.

> **Note.** Just as how we think of energy stored in a capacitor as being stored in the electric field between the plates, we may think of energy stored in an inductor as being stored in the magnetic field generated by the inductor.

12.4   Magnetic Field Energy Density—Use Solenoid Calculation

We use the three equations we derived earlier:

$$\varepsilon = -L \cdot \frac{\mathrm{d}I}{\mathrm{d}t}$$

$$U = \frac{1}{2}LI^2$$

$$B_{\text{in}} = \mu n I$$

Plugging in the definition of L, we have

$$
\begin{aligned}
U &= \frac{1}{2}\mu n N A I^2 \\
&= \frac{1}{2}\mu n N A \cdot \frac{B^2}{\mu^2 n^2} \\
&= \frac{1}{2}NA \cdot \frac{B^2}{\mu n} \\
&= \frac{\ell A B^2}{2\mu}. \qquad \left(n = \frac{N}{\ell}\right)
\end{aligned}
$$

Since the volume of the solenoid is $A \cdot \ell$, we have

$$\frac{U}{\text{volume}} = \frac{B^2}{2\mu}.$$

Note. This is analogous to the capacitor's energy density, given by

$$\frac{U}{\text{volume}} = \frac{\varepsilon}{2} E^2.$$