Lecture 3 Notes

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1 Sets and Functions

1.1 What are sets?

Definition. A *set* is a collection of objects. We usually denote sets using capital letters. Here's an example of a set:

$$X = \{a, b, c, d\} \quad a \in X.$$

We say $a \in X$ means that a is an element of X. The order of elements in a set does not matter, and neither do repetitions, so we have:

$$\{a, b, c, d\} = \{a, c, d, b\} = \{a, a, b, b, c, c, d, d\}.$$

The elements of a set don't matter either, so we may have a set like

$$\left\{\sqrt{2}, \text{James Cameron (the director)}, \text{James Cameron (the professor)}\right\}$$

Sets can have other sets as elements, for example

$$\{\{0, 1\}, 0, 1\}.$$

The three elements of the above set are: $\{0, 1\}$, $\{0\}$, and $\{1\}$.

1.2 Set-builder notation

We use set builder notation to make sets. For example,

$$\text{UCLA} = \{x : x \text{ is a person who is a student at UCLA}\}.$$

Common sets that we use in this class:

- $\mathbb{N}$ is the set of natural numbers. $\{0, 1, 2, \dots\}$
- $\mathbb{Z}$ is the set of integers. $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{Q}$ is the set of rationals (fractions)
- $\mathbb{R}$ is the set of reals
- $\mathbb{C}$ is the set of complex numbers

Consider the set $E = \{x \in \mathbb{Z} \mid x = 2k \text{ for some } k \in \mathbb{Z}\}$, which is the even integers. The empty set has no items in it, and is denoted $\varnothing = \{\}$.

Definition. Suppose X, Y are sets. We say that X is a subset of Y (we write $X \subseteq Y$) if $a \in X$ implies $a \in Y$. In other words, every element of X is also an element of Y.

Definition. For sets X, Y we have that $X = Y$ if they have the same statements.

Proposition. For sets X, Y we have $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

Proof. Suppose that $X = Y$. Let $a \in X$. Since $X = Y$, we have $a \in Y$ as well, so $X \subseteq Y$. Similarly, if $b \in Y$ then $b \in X$ because $X = Y$, so $Y \subseteq X$.

We will now show the other direction of the statement. Suppose $X \subseteq Y$ and $Y \subseteq X$. Thus if $a \in X$ then $a \in Y$ and if $b \in Y$ then $b \in X$, so X and Y have the same elements and $X = Y$. $\square$

Note. If X is any set, then $\varnothing \subseteq X$.
The empty set is not an element of every set, although it *could* be an element of a set. For example, $\varnothing \notin \mathbb{Z}$ but $\varnothing \subseteq \mathbb{Z}$.

$$\varnothing \in \{\varnothing\} = \{\{\}\} \qquad \text{(Note: } \varnothing \neq \{\varnothing\}\text{)}$$

It is raining	I got wet	If it is raining, then I got wet
T	T	T
T	F	F
F	T	T
F	F	T

The last two statements are vacuously true, because the first part is false. Think about it in terms of "lying". If it is not raining and you did not get wet, that does not mean that the person who said "If it is raining, then I got wet" is lying. Their statement was just not applicable in the situation.

1.3 Constructing sets from other sets

(i) $X \cup Y = \{a \mid a \in X \text{ or } a \in Y\}$

(ii) $X \cap Y = \{a \mid a \in X \text{ and } a \in Y\}$

(iii) $X \times Y = \{(a, b) \mid a \in X \text{ and } b \in Y\}$

(iv) $Y \setminus X = \{y \in Y \mid y \notin X\}$

Proposition. If X, Y are sets:

- $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$
- $X \cap Y \subseteq X$, $X \cap Y \subseteq Y$
- $Y \setminus X \subseteq Y$
- $X \cap (Y \setminus X) = \varnothing$

Definition. If X is a set, $\mathscr{P}(X)$ (the "power set" of X) is the set of subsets of X.

Proposition. If X has n elements, then $\mathscr{P}(X)$ has 2^n elements.