0.1 Equivalence Relations

Definition. *Equivalence relation*
A relation R on aset X is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Note. An equivalence relation gives us a notion of two different elements in a set being "the same".

- Reflexive: Everything is "the same" as itself
- Symmetric: If x is "the same" as y, then y is "the same" as x
- Transitive: If x is "the same" as y, and y is "the same" as z, then x is "the same" as z

Example. *Equivalence Relations*

(a) The relation E on the integers where xEy if $x - y$ is even.

- Reflexive: For all $x \in \mathbb{Z}$, $x - x = 0$, which is even, so xEx
- Symmetric: For all $x, y \in \mathbb{Z}$, if $x - y$ is even, so is $-(x - y) = y - x$. Thus if xEy, then yEx
- Transitive: For all $x, y, z \in \mathbb{Z}$, if $x - y$ is even and $y - z$ is even, then their sum, $x - z$, is also even. Thus if xEy and yEz, then xEz.

Observe that this relation relates two integers if they have the same parity.

(b) Let Y be any finite set, and $a, b \in Y^*$ (the set of all strings constructed using Y). Consider the relation L over Y^* such that aLb if a and b have the same length.

(c) Let X be the set of all animals, with animals $x, y \in X$. Consider the relation S over X such that xSy if x and y are of the same species.

(d) Let $x, y \in \mathbb{R}$. Consider the relation C over $\mathbb{R}$ such that xCy if $x - y$ is an integer.

Definition. *Equivalence Classes*
If R is an equivalence relation on a set X, then for $x \in X$, the *equivalence class* of X is the set (with respect to R), denoted by $[x] = [x]_R = \{y \in X \mid xRy\}$.

Example. *Equivalence Classes*

(a) Let E be a relation on $\mathbb{Z}$, where xEy if $x - y$ is even. The equivalence classes for E are $[0]$ (the evens) and $[1]$ (the odds). So, the set of equivalence classes $= \{[0], [1]\}$.

(b) Let $x, y \in \mathbb{R}$, with the relation C over $\mathbb{R}$ defined by xCy if $x - y$ is an integer. The set of equivalence classes $= \{[x] \mid x \in [0, 1)\}$.

If R is an equivalence relation on a set X, then:

- For all $x \in X$, if $x \in [y]$ and $x \in [z]$, then $[y] = [z]$.

  Proof. Suppose $x \in [y]$ and $x \in [z]$. Let $w \in [y]$. Because $w \in [y]$, we know that yRw. We also know that yRx because $x \in [y]$. By symmetry of R, we have wRy, and by transitivity, we have wRx. But $x \in [z]$, so zRx, and by symmetry we have xRw. By transitivity, zRw so $w \in [z]$. Thus $[y] \subseteq [z]$.

  By a similar argument, we have that $[z] \subseteq [y]$, so $[y] = [z]$. □

- For any $x \in X$, x is in some equivalence class, $x \in [x]$ by reflexivity.
  So, for every $x \in X$, x is in exactly one equivalence class. If x is in another equivalence class $[y]$, then by the above $[x] = [y]$.

Definition. *Partition*
For X a set, a *partition* $\mathcal{S}$ of X is a set of nonempty subsets of X such that every element of X is an element of exactly one of the subsets. In other words, for all $A, B \in \mathcal{S}$

- $A, B \subseteq X$
- $A, B \neq \varnothing$
- If $A \cap B \neq \varnothing$ then $A = B$
- For all $x \in X$, there exists exactly one $A \in \mathcal{S}$ such that $x \in A$

Note. We showed that if R is an equivalence relation on X then $\{[x]_R \mid x \in X\}$ is a partition of X.

Theorem — *Equivalence Relations and Partitions*
For X a set, there is a bijection $F\colon$ Set of equivalence relations on $X \to$ Set of partitions of X, defined by

$$F(E) = \{[x]_E \mid x \in X\},$$

the inverse function F^{-1} sends a partition $\mathcal{S}$ to the equivalence relation $F^{-1}(\mathcal{S})$ defined by $xF^{-1}(\mathcal{S})y$ if and only if x and y are in the same element of $\mathcal{S}$ (in the same equivalence class of E).