0.1 Inverses of Functions

Definition. *Inverse of a function*
Suppose $f\colon X \to Y$, $g\colon Y \to X$ is an inverse to f (soon we'll prove that inverses are unique if they exist) if $f \circ g$ and $g \circ f$ are the identity. In other words, $(g \circ f)(x) = x$, $(f \circ g)(y) = y$ for all $x \in X$, $y \in Y$.

Theorem — *Bijective $\iff$ Inverse*
For $f\colon X \to Y$, f is a bijection if and only if f has an inverse.

Proof. Suppose f has an inverse function g. Then $f \circ g$ and $g \circ f$ are the identity. Suppose $f(a) = f(b)$. Then

$$\begin{aligned} f(a) &= f(b) \\ g(f(a)) &= g(f(b)) \\ (g \circ f)(a) &= (g \circ f)(b) \\ a &= b. \end{aligned}$$

Thus f is injective.

Suppose $b \in Y$. Since $f \circ g$ is the identity, we have that $(f \circ g)(b) = b$, so $f(g(b)) = b$. Thus f is surjective. Therefore f is a bijection.

Now suppose that f is a bijection. We define $f^{-1}(a)$ by $f^{-1}(a) = b$, where $a = f(b)$.

- Because f is surjective, we have that for all $a \in Y$, there exists some $b \in X$ such that $a = f(b)$.
- Because f is injective, any $a \in Y$ is *uniquely* mapped by some $b \in X$.

Thus f^{-1} is a function. We will now show that f^{-1} is the inverse of f. For all $x \in X$, $(f^{-1} \circ f)(x) = x$ by definition. For all $y \in Y$,

$$\begin{aligned} (f \circ f^{-1})(y) &= f(f^{-1}(y)) \\ &= f(f^{-1}(f(x))) && \text{(Because } f \text{ is surjective)} \\ &= f(x) && ((f^{-1} \circ f)(x) = x) \\ &= y. \end{aligned}$$

Therefore f^{-1} is the inverse of f. $\square$

Theorem — *Uniqueness of Inverses*
Inverses of functions are unique, provided they exist.
Suppose $f\colon X \to Y$. If f has inverses $g, h\colon Y \to X$ such that $g \circ f = h \circ f = \mathrm{id}_X$, $f \circ g = f \circ h = \mathrm{id}_Y$, then $g = h$.

Proof. Let $y \in Y$. By the previous theorem we know that f is surjective, so $y = f(x)$, for some $x \in X$. Thus

$$\begin{aligned} g(y) &= g(f(x)) \\ &= x \\ &= h(f(x)) \\ &= h(y). \end{aligned}$$

Thus $g = h$ and the inverse is unique. $\square$