Physics 1B Notes

Kyle Chui

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1 Fluids

1.1 Basic Definitions

The density of a material is a constant and is given by $\rho = \frac{m}{V}$.

Pressure is the average force per area and is given by $p = \frac{F}{A}$. The hydrostatic pressure due to the weight of fluid is $p = \rho g h$. The *gauge pressure* is basically just relative pressure. Pressure is a scalar and has no direction.

Pascal's Law states that a pressure change at any point in a confined incompressible fluid is transmitted throughout the fluid such that the same change occurs everywhere.

1.2 Archimedes' Principle

The buoyant force on an object immersed in a fluid is equal to the weight of the displaced fluid. It is given by the equation $F_{\text{buoyant}} = \rho V_{\text{disp}} g$.

> **Note (Special Cases).** If an object is floating at the surface of a fluid, not all of its volume is displacing fluid, so the buoyant force is less than you think it is. Moreover, if an object has sunken to the bottom of a fluid and is resting on the floor, don't forget to take into account the normal force on the object.

Archimedes' Principle can be explained from the existence of a pressure difference between the top and bottom of an object.

1.3 Fluid Dynamics

The continuity equation comes from the simple idea that "what comes in must go out". In other words, the volume of fluid passing through a tube is constant, regardless of where in the tube you are. Thus we have $A_1 v_1 = Q = A_2 v_2$.

Poiseuille's Law explains why there is flow in a pipe to begin with, and is given by

$$Q = \frac{\pi}{8\eta} r^4 \frac{\Delta p}{L},$$

where η is the viscosity, r is the radius of the pipe, Δp is the pressure difference over the pipe, and L is the length of the pipe.

Bernoulli's Law is basically just energy conservation, and helps us find local pressure changes in moving fluids. It is given by:

$$P_0 + \frac{1}{2}\rho v_0^2 + \rho g h_0 = P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1$$

A special case of this is Toricelli's law, which says that for an open tank, the speed of liquid coming out of the hole a distance h below the surface is equal to that acquired by an object falling freely through the same distance h. By using conservation energy, we find $v = \sqrt{2gh}$.

> **Note.** If the height of a fluid is constant, then by Bernoulli's Law, we have that faster fluids have less pressure.

1.4 Laminar and Turbulent Flows

A dimensionless parameter, called Reynold's number N_r reveals whether a flow is laminar or turbulent. We calculate this by $N_r = \frac{2\rho v r}{\eta}$. If $N_r < 2000$, then the flow is laminar. If $N_r > 3000$, then the flow is turbulent.

If it is anywhere in between, then it is unstable. Reynold's number is also the ratio of inertial forces to viscous forces (think: high ratio = more inertia, low ratio = high viscosity). A force F is needed to keep a fluid moving at a constant velocity, given by $F = \eta\frac{vA}{L}$.

2 Periodic Motion and Oscillations

2.1 Simple Harmonic Oscillators

The most basic oscillator that we study is the *simple harmonic oscillator*, often abbreviated SHO. The equation for a SHO is

$$x = A\cos\left(\frac{2\pi}{T}t + \Phi\right),$$

where x is the displacement, A is the amplitude, $\omega = \frac{2\pi}{T}$ is the angular frequency, and Φ is the phase.

Ideal springs obey Hooke's law, where their restoring forces scale linearly with the displacement from equilibruim: $F_{\text{spring}} = -kx$.

2.2 Derivation for SHO

By Newton's second law, we have

$$F = ma = m\cdot\frac{\mathrm{d}^2x}{\mathrm{d}t^2}.$$

We also know that the restoring force is $F = -kx$, so we get

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{k}{m}x = 0.$$

By Ansatz, the solution to this equation is $x = A\cdot\cos(\omega t + \phi)$, as desired.

Note. Observe that $v = \frac{\mathrm{d}x}{\mathrm{d}t} = -\omega A\sin(\omega t + \phi)$, so $v_{\max} = \omega A$. Furthermore, we have that $a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2A\cdot\cos(\omega t + \phi) = -\omega^2x$, so $a_{\max} = -\omega^2A$. Plugging these into the above differential equation, we have $\omega = \sqrt{\frac{k}{m}} = 2\pi f = \frac{2\pi}{T}$.

2.3 Energy in an SHO

The total energy in an SHO is the sum of its kinetic and elastic potential energies. As such, it follows that

$$E_{\text{total}} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2 = \text{Some constant}.$$

2.4 The Pendulum

The net force on a pendulum is $F = -mg\sin\theta = -mg\sin\left(\frac{x}{L}\right)$. Using the small angle approximation, we have $F \approx -\frac{mg}{L}\cdot x$, so the pendulum is a kind of simple harmonic oscillator. To get the angular frequency, we use $\omega = \sqrt{\frac{k}{m}}$, which yields $\omega = \sqrt{\frac{g}{L}}$.

For a physical pendulum, where mass is distributed along the rod, things get a bit more complicated. We have

$$\omega = \sqrt{\frac{mgd}{I}},$$

where m is the mass, g is the acceleration due to gravity, d is the distance from the axis of rotation to the center of gravity, and I is the moment of inertia.

Note. Observe that the simple pendulum is just a special case of the physical pendulum. The inertia of the simple pendulum is just $I = mL^2$, and so we have $\omega = \sqrt{\frac{mgL}{mL^2}} = \sqrt{\frac{g}{L}}$.

2.5 Damped Oscillations

In real life, the amplitude A will decrease with time because of friction. The damping force depends on the velocity of the mass, and the equation is given by $F_{\text{damping}} = -bv$, where b is a "damping constant" that depends on drag and friction. Taking this into account, we have a modified equation of motion:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = 0.$$

The solution is now

$$x(t) = A \cdot e^{-\frac{b}{2m}t}\cos(\omega' t + \phi)$$

Note. The amplitude of the motion is now time dependent, and we call this the envelope. The angular frequency is also different, with

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \qquad (\omega' < \omega)$$

For simplicity, the multiplier in the exponent is often denoted $\alpha = \frac{b}{2m}$. The equations them become:

$$x(t) = A \cdot e^{-\alpha t}\cos(\omega' t + \phi)$$
$$\omega' = \sqrt{\omega_0^2 - \alpha^2},$$

where ω_0 is the original frequency of the SHO without damping. If ω' is imaginary, then the oscillator is *overdamped*. If it is 0, then it is *critically damped*. If it is a positive number, then it is *underdamped*.

2.6 Driven Oscillations

Each oscillator has an intrinsic frequency ω. However, driven oscillators always oscillate at the *drive frequency*, not the fundamental frequency. If the drive frequency and intrinsic frequency are the same, then the oscillations will reach very high amplitudes. Assuming the SHO is driven sinusoidally from the outside, we have $F_{\text{ext}} = F_0\cos(\omega t)$. Then we have

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = F_0\cos(\omega t),$$

which has solution $x = A_0\sin(\omega t + \phi_0)$.

Note. The driven oscillator oscillates at the drive frequency, not the intrinsic frequency, but the amplitude and phase depend on the intrinsic frequency.

$$A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

Without any damping, A becomes infinite when ω_0 becomes ω (resonance).

3 Waves

Waves can be categorized into two different kinds: longitudinal and transverse. Solids are able to transmit both longitudinal and transverse waves, whereas fluids are only able to handle longitudinal waves.

> **Note.** Ocean waves are surface waves and live at the interface between two different media. They are neither longitudinal nor transverse!

3.1 Wave Velocity

A wave is an oscillation in both space and time. At a fixed position, the wave experienced is the same as a simple harmonic oscillator. Waves are described by the following equation:

$$y = A\sin\Big(\underbrace{\frac{2\pi}{\lambda}}_{k}x - \underbrace{\frac{2\pi}{T}}_{\omega}t\Big),$$

where k is the wave number and ω is the angular velocity. Furthermore, just like with SHOs, the energy scales with the square of the frequency and amplitude: $E \sim f^2A^2$. The wave travels one wavelength each period, so $v = \frac{\lambda}{T} = f \cdot \lambda = \frac{\omega}{k}$.

> **Note (Two different velocities).** There are *two different velocities* in a wave. It has a phase velocity (how it moves from left to right), given by $v = f\lambda = \frac{\omega}{k}$, and a transverse velocity (the medium oscillating up and down), given by $v = \omega A\sin(\omega t)$ or $v_{\text{max}} = \omega A$.

For transverse waves passing through a string/rope, we have

$$v = \sqrt{\frac{F_{\text{tension}}}{\mu}}.$$

> **Note.** When a wave travels from one medium to another, the frequency *does not change*. The wavelength and velocity will change accordingly, but the frequency is determined by the source oscillator.

Assuming we have a wave $y(x,t) = A\cos(kx - \omega t)$, we can derive the following by taking partial derivatives and dividing.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

3.2 Elasticity

The bulk modulus of a material is the inverse of its compressibilty. If something is easy to compress, it has a low bulk modulus. The equation is

$$\frac{\Delta V}{V_0} = -\frac{1}{B}\Delta p.$$

3.3 Wave Velocity in Fluids

- Let v be the speed of a longitudinal wave in a fluid, B the bulk modulus of the fluid, and ρ the density of the fluid. Then we have

$$v = \sqrt{\frac{B}{\rho}}.$$

- Let v be the speed of a longitudinal wave in a solid rod, Y be the Young's modulus of the material, and ρ the density of the rod. Then we have

$$v = \sqrt{\frac{Y}{\rho}}$$

Note. Both the bulk modulus and Young's modulus depend on the temperature of the material. However, most of it is due to the density of the material.

- The speed of sound in gases is related to the average speed of particles in the gas:

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}},$$

  where k is the Boltzmann constant and m is the molar mass of the gas.

- For sound in air at sea-level, the velocity of sound is

$$v_{\text{sound}} = 331\sqrt{\frac{T}{273}},$$

  where T is in units of Kelvin.

3.4 Standing Waves

A standing wave is created by two identical waves that are superimposed on one another, that are travelling in opposite directions. If the original wave is described by $y = A\sin(kx - \omega t)$, then the standing wave's equation is

$$y = 2A\sin(kx)\cos(\omega t).$$

Note. We usually denote the amplitude of the standing wave by A, because we don't care about the original travelling waves.

The *order* of a standing wave tells you the shape of the wave and gives you nice equations to find the wavelength and frequency. The equations are:

$$\lambda = \frac{2L}{n} \quad \text{and} \quad f = \frac{v}{2L}n,$$

where n is the order. Furthermore, the energy of the standing wave scales with the square of the order.

For a standing wave, the locations where the string remains at rest are called *nodes*, and the locations where the string reaches the highest displacement are called *antinodes*.

The wavelength of a standing wave depends on L only: $\lambda = 2L$(if both ends are closed), $\lambda = 4L$(if one end is open).

Thus

$$f = \frac{v}{\lambda} \quad \text{where} \quad v = \sqrt{\frac{F_T}{\mu}},$$

where F_T is the tension in the string and μ is the density of the string.

The pitch of *wind* instruments depends on the temperature of the air: As temperature goes down, so do the velocity of the sound and the frequency of the sound. However, temperature does not affect string instruments.

> **Note.** All instruments are based on standing waves. The overtones of an instrument determine the sound characteristics of it (why two instruments playing the same note sound so different).

3.5 Sound Intensity

Sound is emitted isotropically, that is to say, equally in all directions. Intensity is defined by

$$I = \frac{P}{A},$$

and is measured in $\mathrm{W/m^2}$. The power received by a sensor of area A is $P_{\text{received}} = I \cdot A$. Additionally, we also have

$$I = \frac{(p_{\max})^2}{2\rho v_w},$$

where Δp is the pressure variation, ρ is the density, and v_w is the wave velocity.

Furthermore, we also have that

$$p_{\max} = BkA,$$

where B is the bulk modulus of the medium, k is the wave number, and A is the displacement amplitude.

We use a logarithmic scale, called *sound intensity level* β to quantify loudness, which is measured in Bell. 1dB = 1 decibel = 10 bel. To convert sound intensity level in dB to intensity in $\mathrm{W/m^2}$, we have

$$I = I_0 \cdot 10^{\frac{\beta}{10}},$$

where I_0 is the threshold for human hearing, about 10^{-12} $\mathrm{W/m^2}$. The pain threshold for human hearing is $1\mathrm{W/m^2}$, or about 120dB.

3.6 Doppler Effect

The shift depends on the component of the source's velocity in your direction.

We have the following four cases and equations for finding the frequency heard by the observer, using the Doppler effect:

(a) The source is coming towards you

$$f' = \frac{f}{1 - \frac{v_{\text{source}}}{v_{\text{sound}}}}$$

(b) The source is moving away from you

$$f' = \frac{f}{1 + \frac{v_{\text{source}}}{v_{\text{sound}}}}$$

(c) The observer is moving towards the source

$$f' = f\left(1 + \frac{v_{\text{observer}}}{v_{\text{sound}}}\right)$$

(d) The observer is moving away from the source

$$f' = f\left(1 - \frac{v_{\text{observer}}}{v_{\text{sound}}}\right)$$

When you have both objects moving, you calculate things sequentially by considering the frequency at a stationary point between the two objects.

3.7 Beats

Beats are created by the superposition of two waves of slightly different frequencies.

The equation of a superposition of two sound waves is

$$y = 2A \cdot \sin\left(2\pi \frac{f_A - f_B}{2} t\right) \cos\left(2\pi \frac{f_A + f_B}{2} t\right).$$

The frequency of the beat is $f_{\text{beat}} = f_1 - f_2$, because your ear cannot distinguish between the crest and the trough, so you hear it twice as often as you normally would. The frequency of the envelope is $\frac{1}{2}(f_1 + f_2)$.

In a shock wave, the pressure and temperature go up, as the speed of the wave goes down.

4 Electrostatics

4.1 Electric Charge and Electric Field

Electric charge is a fundamental property of elementary particles. You can charge an object by adding or removing electrons/ions, and the total amount of charge is always conserved.

Most static electricity is *triboelectric*, or charging by friction. The further two materials are apart on the triboelectric series, the more charge they will exchange. Electrons are transferred from the electropositive to the electronegative material.

You can measure how much charge something has by using an electroscope. Touch a charged object to the top of the electroscope (to transfer some of the charges to the electroscope). The further apart the middle piece of metal is from the vertical, the more charge the object has. The electroscope only tells you the *magnitude*, not the sign of the charge.

Conductors conduct electricity and heat well—they are usually metals. Insulators are poor conductors of electricity and heat. Semiconductors can be either conductors or insulators depending on if voltage is applied or light is shined on it.

Conductors

- Can transport electric charge freely
- Charges are distributed evenly on its surface so that they are as far apart as possible
- Charges cannot “fall off” the conductor
- When two conductors touch, charge flows from highly charged areas to areas with lower charge

Insulators

- Cannot transport electric charge freely
- Charge stays in place
- Touching two insulators does not transfer much charge

Metallic Bonding

- Metallic bonds form in elements where the valence band and the conduction band overlap
- Positive ions are fixed on a lattice structure
- Some valence electrons are shared between all ions and move freely
- Good electric conductors typically are “shiny”

The conduction band is the set of energy levels needed for the electrons to freely move (energy needed for a material to conduct electricity), and the valence band is the set of all the energy levels that the electrons in a material are at. For metals, there is overlap, so electrons can "move" and the material conducts. For semiconductors, there is a small gap between the bands, so a small initial energy is needed to conduct (voltage, light, etc.). For insulators, this *band gap* is very large, so electrons rarely move between atoms.

> **Note (Neutral objects can be attracted to charged ones).** Conductors strongly attract by macroscopic induction (the charges move in the conductor). Insulators weakly attract by microscopic induction (the insulator can be locally polarized by aligning *electric dipoles*).

An electron's charge is the smallest unit of charge—quarks have smaller charge (2/3e for up and -1/3e for down) but they cannot be separated. We denote a proton's charge as $e = 1.6 \cdot 10^{-19}$C, and an electrons charge as $-e = -1.6 \cdot 10^{-19}$C.

4.2 Electric Force

To find the electric force between two point charges, we use *Coulomb's Law*.

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 \cdot Q_2}{r^2},$$

where $\varepsilon_0 = 8.85 \cdot 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{Nm}^2}$. The constant is called the "permittivity of free space". For simplicity, we often rewrite this as

$$F = \frac{kQ_1Q_2}{r^2},$$

where $k = 9 \cdot 10^9\,\frac{\mathrm{Nm}^2}{\mathrm{C}^2}$. The net electric force on a charge can be found by summing up the individual forces (superposition).

The four fundamental forces, from weakest to strongest: gravity, weak nuclear force, electromagnetic force, strong nuclear force.

4.3 Electric Field

Electric and gravitational forces come from local interaction between an object and a field. A *field* is a variable defined at each and every point in space. The electric field is given by $\vec{E} = \frac{\vec{F}}{q}$, measured in $\frac{\mathrm{N}}{\mathrm{C}}$.

Electric field lines indicate the direction of the force acting on a positive charge, which experiences acceleration along the tangent.

- Electric field lines start on $+$ and end on $-$ charges (or continue forever)
- Electric field lines never cross
- The closer the field lines are, the stronger the field

The gravitational field g is analogous to the electric field E. Mass is analogous to charge, graviational field is to electric field, etc. Electric field lines radiate outwards from positive charges, and gravitate towards negative charges.

4.4 Normal Neutral Matter

Most matter is nearly neutral (many positives and negatives cancelling out). The field is strong when you are very close to (read: touching) the object, and non-existent otherwise. The electric forces between individual charges are what is responsible for all contact forces, including the normal force, friction, and fluid pressure.

4.5 Parallel Plate Capacitor

A parallel plate capacitor (two parallel sheets of metal that hold opposite charge) creates a homogeneous electric field. The magnitude of this field is given by

$$E = \frac{Q}{\varepsilon_0 A} = \frac{\sigma}{\varepsilon_0},$$

where $\sigma = \frac{Q}{A}$, the charge density in $\frac{\mathrm{C}}{\mathrm{m}^2}$.

4.6 Field Around Cylindrical Charges

We use the cylindrical coordinate system because it is more useful for wires. The radius is ρ, the angle is ϕ, and the height is given by z. To convert between cylindrical and Cartesian coordinates, we use

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z, \quad \rho = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right).$$

To visualize ϕ, start in the xy-plane and go counterclockwise, as you would in $\mathbb{R}^2$. To find the volume and surface element in spherical coordinates, we use

$$\mathrm{d}V = \rho \cdot \mathrm{d}\phi \cdot \mathrm{d}\rho \cdot \mathrm{d}z \quad \text{and} \quad \mathrm{d}A = \rho \cdot \mathrm{d}\phi \cdot \mathrm{d}z.$$

4.7 Electric Field of an Infinitely Long Line Charge

The total charge is infinite, and the linear charge density is defined to be $\rho_L \coloneqq \lim_{\Delta l \to 0} \frac{\Delta Q}{\Delta l} = \frac{\mathrm{d}Q}{\mathrm{d}l}$.

Let $\vec{r}$ be the location of the point of interest, and $\vec{r'}$ the location of some point on the line charge. Then $\vec{r} - \vec{r'}$ denotes the vector between the point of interest and the point on the line charge. Assume the charge is composed of infinitesimal small chunks $\mathrm{d}Q$ of charge:

$$\mathrm{d}\vec{E} = \frac{\mathrm{d}Q}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r'}}{\left|\vec{r} - \vec{r'}\right|^3} = \frac{\rho_L \cdot \mathrm{d}z'}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r'}}{\left|\vec{r} - \vec{r'}\right|^3}.$$

Let $\vec{e}_\rho, \vec{e}_z$ denote the unit vectors in the ρ and z directions. For this particular point P, we have

$$\begin{aligned} \vec{r} &= y \cdot \vec{e}_y = \rho \cdot \vec{e}_\rho \\ \vec{r'} &= z'\vec{e}_z \end{aligned}$$

We use this to find the vector and distance between the two points.

$$\begin{aligned} \vec{r} - \vec{r'} &= \rho \cdot \vec{e}_\rho - z'\vec{e}_z \\ \left|\vec{r} - \vec{r'}\right| &= \sqrt{\rho^2 + z'^2} \end{aligned}$$

Thus for a line charge, we have

$$\mathrm{d}\vec{E} = \frac{\rho_L \mathrm{d}z'}{4\pi\varepsilon_0} \frac{\rho\vec{e}_\rho - z'\vec{e}_z}{(\rho^2 + z'^2)^{\frac{3}{2}}}.$$

However, each point $\vec{r'}$ has a corresponding point $-\vec{r'}$ which cancels out the z-component of the field, so

$$\mathrm{d}E_\rho = \frac{\rho_L \rho \mathrm{d}z'}{4\pi\varepsilon_0 (\rho^2 + z'^2)^{\frac{3}{2}}}.$$

Summing over all of the line charge elements, we have

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \cdot \rho}{4\pi\varepsilon_0 (\rho^2 + z'^2)^{\frac{3}{2}}} \, \mathrm{d}z'$$

We use an integral table to evaluate this, yielding

$$\boxed{E_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho}.}$$

Note. The electric field around a line charge scales with $\frac{1}{\rho}$, which is slower than the field drop-off for a point charge.

4.8 Electric Field for a Uniform Disk of Charge

We denote the surface charge density by $\sigma = \frac{\mathrm{d}Q}{\mathrm{d}A}$, which has units of $\frac{\mathrm{C}}{\mathrm{m}^2}$. Then we have $\mathrm{d}Q = \sigma \cdot \mathrm{d}A$, where $\mathrm{d}A = 2\pi r \cdot \mathrm{d}r$. Let r be the distance to some point on a disk, R the radius of the disk, and x the distance from the origin to some point of interest. Using rectangular coordinates, we have

$$\begin{aligned}
\mathrm{d}E_x &= \frac{1}{4\pi\varepsilon_0}\frac{2\pi\sigma r x\,\mathrm{d}r}{(x^2+r^2)^{\frac{3}{2}}} \\
E_x &= \int_0^R \frac{1}{4\pi\varepsilon_0}\frac{2\pi\sigma r x}{(x^2+r^2)^{\frac{3}{2}}}\,\mathrm{d}r \\
&= \frac{\sigma x}{4\varepsilon_0}\int_0^R \frac{2r}{(x^2+r^2)^{\frac{3}{2}}}\,\mathrm{d}r && (u\text{-substitution}) \\
&= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{1}{\sqrt{\left(\frac{R}{x}\right)^2+1}}\right).
\end{aligned}$$

If the disk is very large or close ($R >> x$) then the second term goes to zero. So for an infinite sheet of charge, we have

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}.}$$

Electric field due to various charge distributions

Shape	Spatial variation	Source
Point charge	$E(r) \sim \frac{1}{r^2}$	Q
Line charge	$E(\rho) \sim \frac{1}{\rho}$	$\rho_L = \frac{\mathrm{d}Q}{\mathrm{d}l}$
Sheet of charge	E is independent of r	$\sigma = \frac{\mathrm{d}Q}{\mathrm{d}A}$

If you have two infinite charged sheets of opposite polarity (read: infinite capacitor), the electric field between the two sheets is doubled and the field outside of two sheets is zero (because of superposition).

4.9 Turning Insulators Into Conductors

Air is an insulator, but can be a conductor if a strong enough electric field is applied to it. A free electron in the field accelerates fast enough to hit air molecules and knock free more electrons, creating an "avalanche" of electrons (read: lightning). The threshold field needed to start such a reaction is called the *dielectric strength*.

4.10 Electric Field and Conductors

- The electric field inside a conductor is zero (charges are at rest)
- There is no "net charge" inside a conductor
- The voltage inside a conductor is constant
- Any net charge distributes itself on a conductor's surface
- The electric field is always normal to the surface of a conductor

Note. The last bullet is true because if it were not the case, the tangential component of the field would move the charges around on the surface of the conductor until it is normal.

If we place a conductor inside an electric field, the field inside the conductor gets reduced to nothing. If the conductor is hollow, the field inside is still zero, and this is called a *Faraday cage*. However, the Faraday cage only shields from fields produced by outside charges, and does not shield the outside from charges inside the cage.

4.11 Dipoles and Insulators

The *electric dipole* has zero net charge, but still produces a field, which drops off rapidly ($E \sim \frac{1}{r^3}$). The *dipole moment* $p = qd$ points from $-$ to $+$ (opposite of the field). Most insulators are composed of electric dipoles and are called *dielectrics*. A dipole will keep moving/rotating in an electric field until its dipole moment aligns with the external electric field. When you apply an external electric field to an insulator, the charges can't flow throughout the insulator, but you will align the dipole moments inside it. The electric field created by the dipoles E_{dipole} is always smaller than the external electric field E_0. The total field in the insulator decreases to

$$E = \frac{E_0}{K},$$

where K is a unit-less constant called the "dielectric constant" (always greater than 1). Energy is now stored in the aligned dipoles.

5 Gauss's Law

Recall that electric field changes with the inverse square of distance for point charges, inverse of distance for line charge, and does not change for sheet charges. The idea behind Gauss's Law is to take advantage of symmetry in objects to calculate things in a more efficient way. To get an idea for the field strength we can count how many field lines go through a particular area.

5.1 Electric Flux

We define *flux* to be the number of field lines crossing a certain area. It is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = vA\cos\phi = A_\perp v = \vec{v} \cdot \vec{A}.$$

Notice that when $\phi = 0$ we have the maximum flux, which makes sense because ϕ is measured from the perpendicular to the field. For a uniform electric field, we have

$$\Phi_E = \vec{E} \cdot \vec{A} = E \cdot A \cdot \cos\phi.$$

If the surface is not flat and/or the electric field is not homogeneous, the flux is the integral over the area:

$$\Phi_E = \int \vec{E}\,\mathrm{d}\vec{A} = \int E\cos\theta\,\mathrm{d}A.$$

5.2 Spherical Coordinates

Gauss's Law only really works well in a spherical coordinate system. In this system, r is the radius, ϕ is the angle in the xy-plane measured counterclockwise from the positive x-axis, and θ is the angle measured vertically from the positive z-axis. Thus we have:

$$\begin{aligned} x &= r\sin\theta\cos\phi & r &= \sqrt{x^2+y^2+z^2} \\ y &= r\sin\theta\sin\phi & \theta &= \cos^{-1}\left(\frac{z}{r}\right) \\ z &= r\cos\theta & \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

The volume and surface elements in spherical coordinates are given by:

Surface element	$\mathrm{d}A = r^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta$ $\mathrm{d}\vec{A} = r^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\vec{a}_r$
Volume element	$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r$

Example. *Spherical Charge*
For a point charge, the electric field everywhere on a concentric sphere of radius r is constant and perpendicular to the sphere, so the flux is given by

$$\Phi_E = \int \vec{E}\,\mathrm{d}\vec{A} = \int E\,\mathrm{d}A = E\int \mathrm{d}A = E\cdot A = \frac{q}{4\pi\varepsilon_0 r^2}4\pi r^2 = \frac{q}{\varepsilon_0}.$$

Note (Flux is independent of radius). As you go further away from the charge the field decreases, but the area increases at the same rate. Thus the flux through the sphere surrounding the charge is a measure of the charge itself.

Theorem — *Gauss's Law*
The total electric flux through *any* closed surface is proportional to the enclosed charge.

$$\oint \vec{E}\,\mathrm{d}\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \quad \text{and} \quad \vec{\nabla}\vec{E} = \frac{\rho_V}{\varepsilon_0}$$

Note that the surface must be closed, and limits must be chosen properly.

Gauss's Law is true for any closed surface but only works well for symmetric closed surfaces (you will need to take into account angle to the normal of the surface).

5.3 Using Gauss's Law to Calculate Fields

Example. *Conducting Sphere of Charge Q*
Just like for a point charge, we choose a concentric sphere to be our Gaussian surface. Then

$$\oint \vec{E}\,\mathrm{d}\vec{A} = E\cdot 4\pi r^2 = \frac{Q_{\text{enclosed}}}{\varepsilon_0}.$$

The interesting thing is evaluating Q_{enclosed}. Observe that

$$Q_{\text{enclosed}} = \begin{cases} 0 & \text{if } r < R, \\ Q & \text{if } r \text{ is much greater than } R. \end{cases}$$

Example. *Insulating Sphere of Charge Q*
Suppose we have an insulating sphere of charge Q, with homogeneous charge density ρ_V. Just like for a point charge, we choose a concentric sphere to be our Gaussian surface. Then

$$\oint \vec{E}\,\mathrm{d}\vec{A} = E \cdot 4\pi r^2 = \frac{Q_{\text{enclosed}}}{\varepsilon_0}.$$

The interesting thing is evaluating Q_{enclosed}. Observe that

$$Q_{\text{enclosed}} = \begin{cases} \int \rho_V \,\mathrm{d}V & \text{if } r < R, \\ Q & \text{if } r \text{ is much greater than } R. \end{cases}$$

Evaluating the integral for a sphere with radius $r < R$, we have

$$Q_{\text{enclosed}} = \rho_V \cdot \frac{4}{3}\pi r^3 = \frac{Q}{\frac{4}{3}\pi R^3} \cdot \frac{4}{3}\pi r^3 = Q\frac{r^3}{R^3}.$$

Plugging this into Gauss's Law, we get

$$E \cdot 4\pi r^2 = \frac{Qr^3}{\varepsilon_0 R^3},$$

so $E = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R^3}r$. So we have that $E \sim r$ when inside the homogeneous insulator.

Example. *Infinitely Long Cylindrical Charge*
For a line charge, we choose a cylindrical surface to be our Gaussian surface. The way that we do this is split up the cylinder into its left surface, right surface, and side. Thus we have

$$\oint \vec{E}\,\mathrm{d}\vec{A} = \int_{\text{side}} \vec{E}\,\mathrm{d}\vec{A} + \int_{\text{left}} \vec{E}\,\mathrm{d}\vec{A} + \int_{\text{right}} \vec{E}\,\mathrm{d}\vec{A}.$$

Note that the $\mathrm{d}\vec{A}$ is different for the side surface in comparison to the left and right surfaces. For the side surface, it is given by $\mathrm{d}\vec{A} = \rho \cdot \mathrm{d}\phi \cdot \mathrm{d}z \cdot \vec{a}_\rho$, whereas for the left and right surfaces, it is given by $\mathrm{d}\vec{A} = \rho \cdot \mathrm{d}\phi \cdot \mathrm{d}\rho \cdot \vec{a}_z$. However, the left and right surfaces' fields cancel out, so we are left with

$$\begin{aligned} \oint \vec{E}\,\mathrm{d}\vec{A} &= \int_{\text{side}} \vec{E}\,\mathrm{d}\vec{A} \\ \frac{Q_{\text{enclosed}}}{\varepsilon_0} &= \int_0^{2\pi} \mathrm{d}\phi \int_0^L \rho \cdot E\,\mathrm{d}z \\ \frac{\rho_L \cdot L}{\varepsilon_0} &= 2\pi\rho L \cdot E \\ E &= \frac{\rho_L}{2\pi\varepsilon_0}\frac{1}{\rho}. \end{aligned}$$

We may also write this as $\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0}\frac{1}{\rho}\vec{a}_\rho$.

Example. *Infinite Plane Sheet of Charge*
For an infinite sheet, we choose a box to be our Gaussian surface. Similar to the cylinder, we create the box by "stitching together" the six different sides of the box. Thus we have the sum of six integrals, but the left, right, front, and back fields cancel out with each other, so we are left with

$$\oint \vec{E}\,\mathrm{d}\vec{A} = \int_{\text{bottom}} \vec{E}\,\mathrm{d}\vec{A} + \int_{\text{top}} \vec{E}\,\mathrm{d}\vec{A}.$$

The electric field is constant on the top and bottom, so we have

$$\frac{Q_{\text{enclosed}}}{\varepsilon_0} = EA + EA = 2EA.$$

Thus we have that

$$E = \frac{Q_{\text{enclosed}}}{2A\varepsilon_0} = \frac{\sigma}{2\varepsilon_0}. \qquad (A\sigma = Q_{\text{enclosed}})$$

Example. *Asymmetrical Shapes*
For asymmetrical shapes and fields, we choose a Gaussian surface of a box and add the six sides. The opposing sides happen add together nicely, and we get

$$\begin{aligned}
\oint_A \vec{E}\,\mathrm{d}\vec{A} &= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \cdot \Delta x \cdot \Delta y \cdot \Delta z \\
\frac{Q_{\text{enclosed}}}{\varepsilon_0} &= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \cdot \Delta V \\
\frac{Q_{\text{enclosed}}}{\varepsilon_0 \Delta V} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\
\frac{\rho_V}{\varepsilon_0} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\
\operatorname{div}(\vec{E}) &= \lim_{\Delta V \to 0} \frac{\oint_A \vec{E}\,\mathrm{d}\vec{A}}{\Delta V} \\
\operatorname{div}(\vec{E}) &= \frac{\rho_V}{\varepsilon_0}
\end{aligned}$$

The divergence of a vector flux density is the outflow from a small closed surface per unit volume as the volume shrinks to zero.

The divergence of a vector can be expressed in many ways:

Rectangular	$\operatorname{div}(\vec{E}) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$
Cylindrical	$\operatorname{div}(\vec{E}) = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \cdot E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$
Spherical	$\operatorname{div}(\vec{E}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial \phi}$

The two forms of Gauss's Law are useful in different scenarios. The integral form is useful for finding the field when given the charge density, and the differential form is useful for finding the charge density when given the field.

Theorem — *Divergence Theorem*
If we take one form of Gauss's Law and plug it into the other, we get

$$\oint_A \vec{E}\,\mathrm{d}\vec{A} = \int_V \vec{\nabla}\vec{E}\,\mathrm{d}V.$$