with the last bound above being a consequence of Mertens' Theorem along with the fact that

$$\sum_{p>X}\frac{1}{p^{1+1/\log X}}\le\sum_{j\ge0}\sum_{X^{2^j}<p\le X^{2^{j+1}}}\frac{1}{p^{1+1/\log X}}\le\sum_{j\ge0}\exp(-2^j)\sum_{X^{2^j}<p\le X^{2^{j+1}}}\frac{1}{p}\ll 1.$$

(Recall that $T=\exp(\sqrt{\log x})\ge\exp\left(\frac{1}{2}\sqrt{\log X}\right)$.) As such, [48, Theorem II.2.3] yields

$$\sum_{n\le X}\chi_1(f_1(n))\cdots\chi_K(f_K(n))\mathbb{1}_{(f(n),Q)=1}=\frac{1}{2\pi i}\int_{\frac{1}{k}\left(1+\frac{1}{\log X}\right)-iT}^{\frac{1}{k}\left(1+\frac{1}{\log X}\right)+iT}\frac{F_\chi(s)X^s}{s}\,\mathrm{d}s+O\left(\frac{X^{1/k}\log X}{T}\right). \tag{7.11}$$

Our arguments will be divided into three possibilities:
Case 1: When $(\alpha_k(Q),c_{\widehat{\chi}})\ne(1,1)$ and there is a Seigel zero β_e mod Q.
Case 2: When $(\alpha_k(Q),c_{\widehat{\chi}})\ne(1,1)$ and there is no Seigel zero mod Q.
Case 3: When $(\alpha_k(Q),c_{\widehat{\chi}})=(1,1)$.
In Case 1, we will be assuming henceforth that $\beta_e>1-\frac{5c_1}{24\log Q}$; otherwise decreasing c_1 reduces to Case 2. Let $\beta^*:=\frac{2}{3}+\frac{\beta_e}{3}$ and $\sigma_k(t):=\frac{1}{k}\left(1-\frac{c_1}{4\mathcal{L}_Q(t)}\right)$, so that $\frac{\beta_e}{k}>\sigma_k(0)$. Let $\delta,\delta_1\in(0,\beta_e/10k)$ satisfy $\sigma_k(0)<\frac{\beta_e}{k}-2\delta_1<\frac{\beta_e}{k}+2\delta_1<\frac{\beta^*}{k}<\frac{1}{k}-2\delta$. Consider the contours

- Γ_2, the horizontal segment traversed from $\frac{1}{k}\left(1+\frac{1}{\log X}\right)+iT$ to $\sigma_k(T)+iT$.
- Γ_3, the part of the curve $\sigma_k(t)+it$ traversed from $t=T$ to $t=0$.
- $\Gamma_4:=\Gamma_4(\delta_1)$, the segment traversed from $\sigma_k(0)$ to $\beta_e/k-\delta_1$ **above** the branch cut.
- $\Gamma_5:=\Gamma_5(\delta_1)$, the semicircle of radius δ_1 centered at β_e/k, lying in the upper half plane and traversed clockwise.
- $\Gamma_6:=\Gamma_6(\delta_1)$, the segment traversed from $\beta_e/k+\delta_1$ to β^*/k **above** the branch cut.
- $\Gamma_7:=\Gamma_7(\delta)$, the segment traversed from β^*/k to $1/k-\delta$ **above** the branch cut.
- $\Gamma_8:=\Gamma_8(\delta)$, the circle of radius δ centered at $1/k$, traversed clockwise from the point $1/k-\delta$ above the branch cut to its reflection below the branch cut.
- $\Gamma_4^*:=\Gamma_4^*(\delta)$, the segment traversed from $\sigma_k(0)$ to $1/k-\delta$ **above** the branch cut.
- $\Gamma_5^*:=\Gamma_5^*(\delta_1)$, the circle of radius δ_1 centered at β_e/k, traversed clockwise from the point $\beta_e/k-\delta_1$ above the branch cut to its reflection below the branch cut.

Here $\Gamma_5^*(\delta_1)$ is relevant only when our branch cut is along $\sigma\le\beta_e/k$ (i.e., when $\alpha_k(Q)=c_{\widehat{\chi}}=1$ and β_e exists), while the rest of the contours are defined irrespective of the branch cut. For a contour Ω, let $-\overline{\Omega}$ denote the contour given by the complex conjugate of Ω traversed in the opposite direction and **below** the respective branch cuts. (Note that $-\overline{\Gamma_5}$ is still traversed **clockwise** but below the branch cut.) We define the contour Γ_1 by

$$\Gamma_1:=\begin{cases}\sum_{j=2}^{8}\Gamma_j+\sum_{j=2}^{7}(-\overline{\Gamma_j}), & \text{under Case 1}\\ \Gamma_2+\Gamma_3+\Gamma_4^*+\Gamma_8+(-\overline{\Gamma_4^*})+(-\overline{\Gamma_3})+(-\overline{\Gamma_2}), & \text{under Case 2}\\ \sum_{j=2}^{4}\Gamma_j+\Gamma_5^*+\sum_{j=2}^{4}(-\overline{\Gamma_j}), & \text{under Case 3.}\end{cases}$$

In Case 3, if β_e doesn't exist, then there is no branch cut and Γ_4, $\overline{\Gamma}_4$ and Γ_5^* are excluded from Γ_1. In all three cases, the integrand in (7.11) is analytic in the region enclosed by Γ_1 and the segment joining $\frac{1}{k}\left(1+\frac{1}{\log X}\right)-iT$ and $\frac{1}{k}\left(1+\frac{1}{\log X}\right)+iT$. (Note that if $c_{\widehat{\chi}}=1$, the definitions of $\mathcal{Q}(k;f_1,\cdots,f_K)$ and $G_{\chi,1}$, $G_{\chi,2}$ in Lemma 7.1 give $G_{\chi,2}(1/k)=0$, canceling the simple pole of $F_1(sk)$ at $s=1/k$. In particular, this happens in Case 3.) So

$$\sum_{n\le X}\chi_1(f_1(n))\cdots\chi_K(f_K(n))\mathbb{1}_{(f(n),Q)=1}=-\frac{1}{2\pi i}\int_{\Gamma_1}\frac{F_\chi(s)X^s}{s}\,\mathrm{d}s+O\left(\frac{X^{1/k}\log X}{T}\right). \tag{7.12}$$

We now proceed to estimate the integrals occurring on the right hand side above. In the following proposition, any result about an integral is valid whenever the corresponding contour is a part of Γ_1: so for instance, the assertion on Γ_8 (resp. Γ_5^*) holds under Cases 1 or 2 (resp. Case 3), those on Γ_5 and Γ_6 hold under Case 1, and the bound involving Γ_4 holds under Cases 1 and 3. Let I_j (resp. $\overline{I_j}$, I_j^*) denote the corresponding integral along Γ_j (resp. $\overline{\Gamma_j}$, Γ_j^*).

Proposition 7.4. *We have the following bounds:*

(i) $|I_2|+|\overline{I_2}|+|I_3|+|\overline{I_3}|\ll X^{1/k}\exp(-\kappa_0\sqrt{\log X})$ *for some constant* $\kappa_0:=\kappa_0(c_1,k)>0$.

(ii) $\max\{|I_4+\overline{I_4}|,|I_6+\overline{I_6}|\}\ll X^{1/k}\exp(-\sqrt{\log X})$ *uniformly in* δ,δ_1 *as above.*

(iii) $\lim_{\delta_1\to0+}|I_5|=\lim_{\delta_1\to0+}|\overline{I_5}|=\lim_{\delta_1\to0+}|I_5^*|=\lim_{\delta\to0+}|I_8|=0.$

Proof. To show subpart (i), we use the fact that since $\beta_e>1-5c_1/24\log Q$, any s lying on Γ_2, Γ_3 or their conjugates satisfies the requirements of Proposition 7.2(iii). As such, (i) follows immediately from Proposition 7.2(iii) and the fact that $|s|\gg|t|+1$ for all s.

For subpart (ii), we note that for all $s\in\Gamma_4$, we have $(s-1/k)^{-\alpha_k(Q)c_{\widehat{\chi}}}=(1/k-s)^{-\alpha_k(Q)c_{\widehat{\chi}}}\,e^{-i\pi\alpha_k(Q)c_{\widehat{\chi}}}$ and $(s-\beta_e/k)^{\alpha_k(Q)c_{\widehat{\chi}}\gamma(\psi_e)}=(\beta_e/k-s)^{\alpha_k(Q)c_{\widehat{\chi}}\gamma(\psi_e)}\,e^{i\pi\alpha_k(Q)c_{\widehat{\chi}}\gamma(\psi_e)}$. (This is clear if the branch cut is along $\sigma\le1/k$, and also if the branch cut is along $\sigma\le\beta_e/k$ which is when $(\alpha_k(Q),c_{\widehat{\chi}})=(1,1)$.) Likewise, for all $s\in\overline{\Gamma_4}$, we have $(s-1/k)^{-\alpha_k(Q)c_{\widehat{\chi}}}=(1/k-s)^{-\alpha_k(Q)c_{\widehat{\chi}}}\,e^{i\pi\alpha_k(Q)c_{\widehat{\chi}}}$ and $(s-\beta_e/k)^{\alpha_k(Q)c_{\widehat{\chi}}\gamma(\psi_e)}=(\beta_e/k-s)^{\alpha_k(Q)c_{\widehat{\chi}}\gamma(\psi_e)}\,e^{-i\pi\alpha_k(Q)c_{\widehat{\chi}}\gamma(\psi_e)}$. Since $e^{\pm i\pi\alpha_k(Q)c_{\widehat{\chi}}(\gamma(\psi_e)-1)}\ll1$, the definitions of $\widetilde{F}_\chi(s)$ and $\widetilde{H}_\chi(s)$ show that

$$|I_4+\overline{I_4}|\ll\left|\int_{\sigma_k(0)}^{\beta_e/k-\delta_1}\frac{\widetilde{H}_\chi(s)G_{\chi,2}(s)X^s}{s}\left(\frac{1}{k}-s\right)^{-\alpha_k(Q)c_{\widehat{\chi}}}\left(\frac{\beta_e}{k}-s\right)^{\alpha_k(Q)c_{\widehat{\chi}}\gamma(\psi_e)}\mathrm{d}s\right|.$$

But now by Lemma 7.1(iii) and Proposition 7.2(ii), we see that

$$\begin{aligned}|I_4+\overline{I_4}|&\ll X^{\beta_e/k}(\log X)^{\alpha_k(Q)\epsilon_1/4}(1-\beta_e)^{-\alpha_k(Q)}\int_{\sigma_k(0)}^{\beta_e/k-\delta_1}\left(\frac{\beta_e}{k}-s\right)^{\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}}\gamma(\psi_e))}\mathrm{d}s\\&\ll X^{\beta_e/k}(\log X)^{3\alpha_k(Q)\epsilon_1/10}\cdot\left(\frac{\beta_e}{k}-\sigma_k(0)\right)^{1+\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}}\gamma(\psi_e))}\ll X^{1/k}\exp(-\sqrt{\log X}).\end{aligned}$$

Here we have recalled that $\beta_e\le1-c(\epsilon_1)/Q^{\epsilon_1/20K_0}\le1-c(\epsilon_1)/(\log X)^{\epsilon_1/20}$ for some constant $c(\epsilon_1)>0$, and (as argued before Lemma 7.1) that $Q_e:=\mathfrak{f}(\psi_e)$ has a prime factor $\ell_e>D+2$,

which upon factoring $\psi_e = \prod_{\ell|Q} \psi_{e,\ell}$ with $\psi_{e,\ell}$ being a character mod ℓ, led to
(7.13)

$$\alpha_k(Q)|\gamma(\psi_e)| \le \alpha_k(Q) \prod_{\ell|Q_e} \left| \frac{\sum_{v:\ vW_k(v)\in U_\ell} \overline{\psi}_{e,\ell}(v)}{\alpha_k(\ell)(\ell-1)} \right| \le \frac{1}{\ell_e - 1} \left| \sum_{\substack{v \bmod \ell_e \\ W_k(v)\equiv 0 \ (\mathrm{mod}\ \ell_e)}} \overline{\psi}_{e,\ell}(v) \right| \le \frac{D}{D+1}.$$

This shows the desired bound on I_4 in (ii), and the assertion for I_6 is entirely analogous.

Coming to subpart (iii), we parametrize the points of Γ_5 by $s = \beta_e/k + \delta_1 e^{i\theta}$ where $\pi \ge \theta \ge 0$. Since $\widetilde{M} := \sup_{|s-\frac{\beta_e}{k}|\le\frac{1}{2}\left(\frac{\beta_e}{k}-\sigma_k(0)\right)} |\widetilde{H}_\chi(s)|$ is finite, we have for all sufficiently small $\delta_1 > 0$,

$$|I_5| \ll \widetilde{M} \int_0^\pi X^{\beta_e/k+\delta_1} \left(\frac{1-\beta_e}{k} - \delta_1 \right)^{-\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}})} \delta_1^{1+\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}}\gamma(\psi_e))} \,\mathrm{d}\theta \ll \frac{\widetilde{M} X^{\beta_e/k+\delta_1} \delta_1^{1/(D+1)}}{\left(\frac{1-\beta_e}{k} - \delta_1\right)^{\alpha_k(Q)}},$$

where we have again seen that $1+\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}}\gamma(\psi_e)) \ge 1/(D+1)$ by (7.13). The last expression shows that $\lim_{\delta_1\to 0+} |I_5| = 0$, and the assertions on $|\overline{I_5}|$ and $|I_5^*|$ are proved similarly. The same argument also shows that $|I_8| \ll M^* X^{1/k+\delta} \delta^{1-\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}})} \left(\frac{1-\beta_e}{k} - \delta\right)^{-\alpha_k(Q)}$ for all sufficiently small $\delta > 0$, where $M^* = \sup_{|s-\frac{1}{k}|\le\frac{1-\beta^*}{k}} |\widetilde{H}_\chi(s)|$. This yields $\lim_{\delta\to 0+} |I_8| = 0$, because $\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}}) < 1$ whenever $(\alpha_k(Q), c_{\widehat{\chi}}) \ne (1,1)$. $\square$

Now in case 3, we let $\delta_1 \downarrow 0$ in (7.12) and invoke the relevant assertions of Proposition 7.4 to obtain $\sum_{n\le X} \chi_1(f_1(n)) \cdots \chi_K(f_K(n)) \mathbb{1}_{(f(n),Q)=1} \ll X^{1/k} \exp(-\kappa_1\sqrt{\log X})$ for some constant $\kappa_1 > 0$. Hence to complete the proof of Theorem 5.6, it suffices to assume that $(\alpha_k(Q), c_{\widehat{\chi}}) \ne (1,1)$. In case 1, we obtain, by letting $\delta \downarrow 0$ and $\delta_1 \downarrow 0$ in (7.12),

$$\sum_{n\le X} \chi_1(f_1(n)) \cdots \chi_K(f_K(n)) \mathbb{1}_{(f(n),Q)=1} = -\lim_{\delta\to 0+} \frac{I_7 + \overline{I_7}}{2\pi i} + O(X^{1/k} \exp(-\kappa_1\sqrt{\log X})). \tag{7.14}$$

By an argument analogous to that given for Proposition 7.4(ii), it is easy to see that the above limit exists. Furthermore, writing $(s-1/k)^{-\alpha_k(Q)c_{\widehat{\chi}}} = (1/k-s)^{-\alpha_k(Q)c_{\widehat{\chi}}} e^{\pm i\pi\alpha_k(Q)c_{\widehat{\chi}}}$ as before, we see that the limit in (7.14) is equal to

$$\frac{\sin(\pi\alpha_k(Q)c_{\widehat{\chi}})}{\pi} \int_{\beta^*/k}^{1/k} H_\chi(s) G_{\chi,2}(s) X^s \left(\frac{1}{k} - s\right)^{-\alpha_k(Q)c_{\widehat{\chi}}} \mathrm{d}s,$$

We write the above integral as $H_\chi(1/k)G_{\chi,2}(1/k)I_1 - I_2$, where $I_1 := \int_{\beta^*/k}^{1/k} X^s (1/k-s)^{-\alpha_k(Q)c_{\widehat{\chi}}} \mathrm{d}s$. Letting $s = 1/k - u/\log X$, and using $\beta^* = 2/3 + \beta_e/3 \le 1 - c(\epsilon_1)/3(\log X)^{\epsilon_1/20}$ along with a standard bound on the tail of the integral defining a Gamma function [44, Lemma 7], we get

$$I_1 = \frac{X^{1/k}}{(\log X)^{1-\alpha_k(Q)c_{\widehat{\chi}}}} \left\{ \Gamma(1-\alpha_k(Q)c_{\widehat{\chi}}) + O(\exp(-\sqrt{\log X})) \right\}.$$

Now using Proposition 7.2(iv) and making the same change of variable, we find that

$$I_2 \ll (\log X)^{\left(\frac{1}{20}+\frac{\alpha_k(Q)}{5}\right)\epsilon_1} \int_{\beta^*/k}^{1/k} X^s \left(\frac{1}{k} - s\right)^{1-\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}})} \mathrm{d}s \ll \frac{X^{1/k}}{(\log X)^{2-\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}})-(1/20+\alpha_k(Q)/5)\epsilon_1}}$$

as $\Gamma(2-\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}})) \ll 1$. Collecting estimates, we obtain from (7.14),