part of any positive integer n satisfying $\gcd(f(n), q) = 1$ must be supported on the primes in the set $\bigcup_{1\le v<k} S_v$. As a consequence, we have the following important observation.

Lemma 3.3. *If q is k-admissible, then the k-free part of any positive integer n satisfying $\gcd(f(n), q) = 1$ is bounded. More precisely, it is of size $O(1)$, where the implied constant depends only on the polynomials $\{W_{i,v}\}_{\substack{1\le i\le K\\ 1\le v\le k}}$.*

The following estimate (see [37, Lemma 2.4]) will be useful throughout the paper.

Lemma 3.4. *Let $G \in \mathbb{Z}[T]$ be a fixed nonconstant polynomial. For each positive integer q, let $\alpha_G(q) := \frac{1}{\varphi(q)}\#\{u \in U_q : G(u) \in U_q\}$. We have, uniformly in q and $x \ge 3q$,*

$$\sum_{p\le x} \frac{\mathbb{1}_{(G(p),q)=1}}{p} = \alpha_G(q)\log_2 x + O((\log_2(3q))^{O(1)}).$$

Coming to the proof of the upper bound implied in (3.1), we define $y := \exp(\sqrt{\log x})$ and start by removing those n which are divisible by the $(k+1)$-th power of a prime exceeding y. Writing any such n as AB for some k-free B and k-full A, Lemma 3.3 shows that $B \ll 1$ so that the contribution of such n to (3.1) is

(3.3)
$$\sum_{\substack{n\le x:\ (f(n),q)=1\\ \exists\ p>y:\ p^{k+1}\mid n}} 1 \ll \sum_{\substack{A\le x\\ A \text{ is } k\text{-full}\\ \exists\ p>y:\ p^{k+1}\mid n}} 1 \le \sum_{p>y}\sum_{\substack{v\ge k+1\\ p^v\le x}}\sum_{\substack{m\le x/p^v\\ m \text{ is } k\text{-full}}} 1 \ll \sum_{p>y}\sum_{v\ge k+1}\left(\frac{x}{p^v}\right)^{1/k} \ll \left(\frac{x}{y}\right)^{1/k},$$

where we have used the fact that the number of k-full integers up to X is $O(X^{1/k})$ (see [13]). The last expression above is negligible in comparison to the right hand side of (3.1), hence, it remains to bound the number of n not divisible by the $(k+1)$-th power of any prime greater than y and satisfying $(f(n), q) = 1$.

We write any such n in the form BMN, where N is y-rough, BM is y-smooth, B is k-free, M is k-full, and B, M, N are pairwise coprime. By Lemma 3.3, we see that $B = O(1)$ and that N is k-full. But since n is not divisible by the $(k+1)$-th power of any prime exceeding y, N must be the k-th power of a squarefree y-rough integer A. Consequently,

(3.4)
$$\sum_{\substack{n\le x:\ (f(n),q)=1\\ p>y \implies p^{k+1}\nmid n}} 1 \le \sum_{\substack{B\le x\\ (f(B),q)=1\\ B \text{ is } k\text{-free}}}\ \sum_{\substack{M\le x/B:\ M \text{ is } k\text{-full}\\ P(M)\le y,\ (f(M),q)=1}}\ \sum_{\substack{A\le (x/BM)^{1/k}\\ P^-(A)>y:\ (f(A^k),q)=1\\ A \text{ squarefree}}} 1.$$

We now write the right hand side of the above inequality as $\Sigma_1 + \Sigma_2$, where Σ_1 and Σ_2 count the contribution of (B, M, A) with $M \le x^{1/2}$ and $M > x^{1/2}$, respectively. Any A counted in Σ_2 satisfies $A \le (x/BM)^{1/k} \le x^{1/2k}/B^{1/k}$, so that

$$\Sigma_2 \le \sum_{\substack{B\le x\\ (f(B),q)=1\\ B \text{ is } k\text{-free}}}\ \sum_{\substack{A\le x^{1/2k}/B^{1/k}\\ P^-(A)>y:\ (f(A^k),q)=1\\ A \text{ squarefree}}}\ \sum_{\substack{M\le x/BA^k:\ P(M)\le y\\ M \text{ is } k\text{-full},\ (f(M),q)=1}} 1.$$

To bound the innermost sum, we invoke Lemma 3.2; here $U = \frac{\log(x/BA^k)}{\log y} \ge \frac{1}{2}\sqrt{\log x}$. This yields

$$\Sigma_2 \ll \sum_{\substack{B \le x\\ (f(B),q)=1\\ B \text{ is } k\text{-free}}} \sum_{\substack{A \le x^{1/2k}/B^{1/k}\\ P^-(A)>y:\ (f(A^k),q)=1\\ A \text{ squarefree}}} \frac{x^{1/k}}{B^{1/k}A} \exp\left(-\frac{1}{6k}\sqrt{\log x}\cdot \log_2 x\right).$$

Recalling that $B = O(1)$ and bounding the sum on A trivially by $2\log x$, we deduce that $\Sigma_2 \ll x^{1/k}\exp\left(-\sqrt{\log x}\right)$, which is negligible compared to the right hand side of (3.1).

To estimate Σ_1, we invoke [17, Theorem 01, p. 2] on the multiplicative function $g(A) := \mu(A)^2 \mathbb{1}_{P^-(A)>y}\mathbb{1}_{(f(A^k),q)=1}$, with μ denoting the Möbius function. Since $M \le x^{1/2}$ and $B \ll 1$,

$$\Sigma_1 \ll \frac{x^{1/k}}{\log x}\exp\left(\sum_{y<p\le x}\frac{\mathbb{1}_{(W_k(p),q)=1}}{p}\right) \sum_{\substack{M \le x^{1/2}:\ M \text{ is } k\text{-full}\\ P(M)\le y,\ (f(M),q)=1}} \frac{1}{M^{1/k}}.$$

But since the sum on M above is no more than

(3.5)
$$\sum_{\substack{M \text{ is } k\text{-full}\\ P(M)\le y,\ (f(M),q)=1}} \frac{1}{M^{1/k}} \le \prod_{p\le y}\left(1+\frac{\mathbb{1}_{(f(p^k),q)=1}}{p}+O\left(\frac{1}{p^{1+1/k}}\right)\right) \ll \exp\left(\sum_{p\le y}\frac{\mathbb{1}_{(W_k(p),q)=1}}{p}\right),$$

it follows by an application of Lemma 3.4 to estimate the sum $\sum_{p\le x}\mathbb{1}_{(W_k(p),q)=1}/p$, that Σ_1 is absorbed in the right hand side of (3.1). This establishes Proposition 3.1.

4. The main term in Theorems 2.1 to 2.4: Contribution of "convenient" n

In what follows, we define

$$J := \lfloor \log_3 x \rfloor \text{ and } y := \exp((\log x)^{\epsilon/2}),$$

where ϵ is as in the statement of Theorem 2.1 and $\epsilon := 1$ for Theorems 2.2 to 2.4. We call $n \le x$ convenient if the largest J *distinct* prime divisors of n exceed y and each appear to exactly the k-th power in n. In other words, n is convenient iff it can be uniquely written in the form $n = m(P_J\cdots P_1)^k$ for $m \le x$ and primes $P_1, \dots, P_J$ satisfying

$$L_m := \max\{y, P(m)\} < P_J < \cdots < P_1. \tag{4.1}$$

Note that any n having $P_{Jk}(n) \le y$ must be inconvenient; on the other hand, if n is inconvenient and satisfies $\gcd(f(n),q)=1$ then either $P_{Jk}(n)\le y$ or n is divisible by the $(k+1)$-th power of a prime exceeding y. We start by showing that there are a negligible number of inconvenient $n \le x$ satisfying $\gcd(f(n),q)=1$.

Proposition 4.1. *We have as $x \to \infty$,*

$$\sum_{\substack{n\le x:\ (f(n),q)=1\\ n \text{ inconvenient}}} 1 = o\Bigg(\sum_{\substack{n\le x\\ (f(n),q)=1}} 1\Bigg), \tag{4.2}$$

uniformly in k-admissible $q \le (\log x)^{K_0}$.

Proof. By (3.3) and (3.1), the contribution of the n's that are divisible by the $(k+1)$-th power of a prime exceeding y is negligible. Letting $z := x^{1/\log_2 x}$, we show that the same is true for the contribution of z-smooth n to the left hand side of (4.2). Indeed, writing any such n in the form AB for some k-free B and k-full A, we have $P(A) \le z$ whereas (by Lemma 3.3) $B = O(1)$. Hence the contribution of z-smooth n is, by Lemma 3.2,

$$\sum_{\substack{n\le x:\ P(n)\le z\\ (f(n),q)=1}} 1 \ll \sum_{\substack{A\le x:\ P(A)\le z\\ A \text{ is } k\text{-full}}} 1 \ll x^{1/k}\exp\left(-\left(\frac{1}{k}+o(1)\right)\log_2 x\log_3 x\right), \tag{4.3}$$

which is negligible compared to the right hand side of (4.2).

It remains to consider the contribution of those n which are not z-smooth and are not divisible by the $(k+1)$-th power of a prime exceeding y. Since n is inconvenient, we have $P_{Jk}(n) \le y$. Hence, n can be written in the form mP^k where $P := P(n) > z$ and $m = n/P^k$, so that $P_{Jk}(m) \le y$, $\gcd(m,P) = 1$ and $f(n) = f(m)f(P^k)$. Given m, there are at most $\sum_{z<P\le(x/m)^{1/k}} 1 \ll x^{1/k}/m^{1/k}\log z$ many possibilities for P. Consequently,

$$\sum_{\substack{n\le x \text{ inconvenient}\\ P(n)>z,\ (f(n),q)=1\\ p>y \implies p^{k+1}\nmid n}} 1 \le \sum_{\substack{n\le x:\ P_{Jk}(n)\le y\\ P(n)>z,\ (f(n),q)=1\\ p>y\implies p^{k+1}\nmid n}} 1 \ll \frac{x^{1/k}\log_2 x}{\log x}\sum_{\substack{m\le x\\ P_{Jk}(m)\le y,\ (f(m),q)=1\\ p>y\implies p^{k+1}\nmid m}} \frac{1}{m^{1/k}}. \tag{4.4}$$

As in the argument preceding (3.4), we write any m occurring in the above sum (uniquely) in the form BMA^k, where B is k-free, M is k-full, A is squarefree, $P(BM) \le y < P^-(A)$, and $\Omega(A) \le J$ (since $P_{Jk}(n) \le y$). Since $B = O(1)$, we deduce that

$$\sum_{\substack{m\le x\\ P_{Jk}(m)\le y,\ (f(m),q)=1\\ p>y\implies p^{k+1}\nmid m}} \frac{1}{m^{1/k}} \ll \sum_{\substack{M\ k\text{-full}\\ P(M)\le y,\ (f(M),q)=1}} \frac{1}{M^{1/k}} \sum_{\substack{A\le x\\ \Omega(A)\le J}} \frac{1}{A}.$$

The sum on A is no more than $(1+\sum_{p\le x} 1/p)^J \le (2\log_2 x)^J \le \exp(O((\log_3 x)^2))$, while the sum on M is $\ll \exp(\alpha_k \log_2 y + O((\log_2(3q))^{O(1)}))$ by (3.5) and Lemma 3.4. Altogether,

$$\sum_{\substack{m\le x\\ P_{Jk}(m)\le y,\ (f(m),q)=1\\ p>y\implies p^{k+1}\nmid m}} \frac{1}{m^{1/k}} \ll (\log x)^{\alpha_k\epsilon/2}\exp\left(O((\log_3 x)^2 + (\log_2(3q))^{O(1)})\right), \tag{4.5}$$

and inserting this into (4.4) completes the proof via Proposition 3.1. $\square$

It is the convenient n which give rise to the main term in the count of $n \le x$ satisfying the congruences $f_i(n) \equiv a_i \pmod q$. We shall spend the next few sections proving this.

Theorem 4.2. *Fix $K_0, B_0 > 0$ and assume that $\{W_{i,k}\}_{1\le i\le K} \subset \mathbb{Z}[T]$ are nonconstant and multiplicatively independent. As $x \to \infty$, we have*

$$\sum_{\substack{n\le x \text{ convenient}\\ (\forall i)\ f_i(n)\equiv a_i \pmod q}} 1 \sim \frac{1}{\varphi(q)^K}\sum_{\substack{n\le x\\ (f(n),q)=1}} 1,$$