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JOINT DISTRIBUTION IN RESIDUE CLASSES OF FAMILIES OF POLYNOMIALLY-DEFINED MULTIPLICATIVE FUNCTIONS

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Abstract. We study the distribution of families of multiplicative functions among the coprime residue classes to moduli varying uniformly in a wide range, obtaining analogues of the Siegel–Walfisz Theorem for large classes of multiplicative functions. We extend a criterion of Narkiewicz for families of multiplicative functions that can be controlled by values of polynomials at the first few prime powers, and establish results that are completely uniform in the modulus as well as optimal in most parameters and hypotheses. This also significantly generalizes and improves upon previous work done for a single such function in specialized settings. Our results have applications for most interesting multiplicative functions, such as the Euler totient function $\varphi(n)$, the sum-of-divisors function $\sigma(n)$, the coefficients of the Eisenstein series, etc., and families of these functions. For instance, an application of our results shows that for any fixed $\epsilon > 0$, the functions $\varphi(n)$ and $\sigma(n)$ are jointly asymptotically equidistributed among the reduced residue classes to moduli q coprime to 6 varying uniformly up to $(\log x)^{(1-\epsilon)\alpha(q)}$, where $\alpha(q) := \prod_{\ell \mid q}(\ell-3)/(\ell-1)$; furthermore, the coprimality restriction is necessary and the range of q is essentially optimal. One of the primary themes behind our arguments is the quantitative detection of a certain mixing (or ergodicity) phenomenon in multiplicative groups via methods belonging to the 'anatomy of integers', but we also rely heavily on more pure analytic arguments (such as a suitable modification of the Landau–Selberg–Delange method), – whilst using several tools from arithmetic and algebraic geometry, and from linear algebra over rings as well.

1. Introduction

We say that an integer-valued arithmetic function g is uniformly distributed (or equidistributed) modulo q if $\#\{n \le x : g(n) \equiv b \pmod q\} \sim x/q$ as $x \to \infty$, for each residue class b mod q. This definition generalizes naturally to families of arithmetic functions, and has been well-studied for (integral-valued) additive functions, – with work of Delange [10], [11] characterizing when a family of such functions is equidistributed to a fixed modulus q. These results have also been partially extended in [36], [1] and [45], where the modulus q itself has been allowed to vary up to a certain threshold depending on the stopping point x of inputs.

However, for multiplicative functions, there are indications that uniform distribution is not the correct notion to consider. For instance, it can be shown that the Euler totient function $\varphi(n)$ is almost always divisible by any fixed integer $q > 1$, and hence is not equidistributed to any given modulus. Motivated by this, Narkiewicz in [26] introduces the notion of weak uniform distribution: Given an integer-valued arithmetic function f and a positive integer q, we say that f is weakly uniformly distributed (or weakly equidistributed or WUD) modulo q if there are

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infinitely many positive integers n for which $\gcd(f(n), q) = 1$, and if

$$\#\{n \le x : f(n) \equiv a \pmod q\} \sim \frac{1}{\varphi(q)} \#\{n \le x : \gcd(f(n), q) = 1\}, \quad \text{as } x \to \infty,$$

for each coprime residue class a mod q. This definition extends naturally to families of arithmetic functions: we say that the integer-valued arithmetic functions $f_1, \dots, f_K$ are jointly weakly equidistributed (or jointly WUD) modulo q if there are infinitely many n for which $\gcd(f_1(n) \cdots f_K(n), q) = 1$, and if for all coprime residue classes $a_1, \dots, a_K$ mod q, we have

$$\#\{n \le x : \forall i \in [K],\ f_i(n) \equiv a_i \pmod q\} \sim \frac{1}{\varphi(q)^K} \#\{n \le x : \gcd(f_1(n) \cdots f_K(n), q) = 1\} \tag{1.1}$$

as $x \to \infty$. (Here and below, $[K]$ denotes the set $\{1, \dots, K\}$.)

The phenomenon of weak uniform distribution has captured significant interest for specific as well as for general classes of multiplicative functions. Narkiewicz [26] shows that $\varphi(n)$ is weakly equidistributed precisely to those moduli q that are coprime to 6, while Śliwa [47] shows that the sum of divisors function $\sigma(n) = \sum_{d|n} d$ is weakly equidistributed mod q exactly when q is not a multiple of 6. Generalizations of Śliwa's result to Fourier coefficients of Eisenstein series, namely the functions $\sigma_r(n) := \sum_{d|n} d^r$, has been studied in great depth by Narkiewicz, Rayner, Dobrowolski, Fomenko and others; see [14], [31], [29], [30, Theorem 6.12], [39], [40]. In fact in [26, Theorem 1], Narkiewicz gives a general criterion for deciding weak equidistribution for a single "polynomially-defined" multiplicative function f, one that can be controlled by the values of polynomials at the first few powers of all primes. While the exact statement requires some set-up, the general idea is that such a function f is weakly equidistributed modulo a fixed positive integer q precisely when for every nontrivial Dirichlet character mod q that acts trivially on a special subgroup of the unit group mod q, a certain "local factor" (or Euler factor) associated to this Dirichlet character vanishes. Narkiewicz dedicates a significant portion of his monograph [30] to give more explicit sufficient conditions that guarantee weak uniform distribution, and to obtain algorithms characterizing all the moduli to which a given "polynomially-defined" multiplicative function is weakly equidistributed.

In all these results, the modulus q has been assumed to be fixed. A natural and interesting question is whether weak equidistribution continues to hold as q varies uniformly in a suitable range depending on the stopping point x of inputs, for instance, whether it is possible to obtain analogues of the Siegel–Walfisz Theorem for primes in arithmetic progressions, but with primes replaced by values of multiplicative functions. To this end, given a constant $K_0 > 0$, we shall say that integer-valued arithmetic functions $f_1, \dots, f_K$ are jointly weakly equidistributed (or jointly WUD) mod q, uniformly for $q \le (\log x)^{K_0}$, if:

(i) For every such q, $\prod_{i=1}^K f_i(n)$ is coprime to q for infinitely many n, and

(ii) The relation (1.1) holds as $x \to \infty$, uniformly in moduli $q \le (\log x)^{K_0}$ and in coprime residue classes $a_1, \dots, a_K$ mod q. Explicitly, this means that for any $\epsilon > 0$, there exists $X(\epsilon) > 0$ such that the ratio of the left hand side of (1.1) to the right hand side lies in $(1 - \epsilon, 1 + \epsilon)$ for all $x > X(\epsilon)$, $q \le (\log x)^{K_0}$ and coprime residues $a_1, \dots, a_K$ mod q.

If $K = 1$ and $f_1 = f$, we shall simply say that f is weakly equidistributed (or WUD) mod q, uniformly for $q \le (\log x)^{K_0}$.

The question of weak equidistribution to varying moduli seems to have been first studied in [22], [35] and [37], which made some partial progress towards obtaining a uniform analogue of Narkiewicz's aforementioned criterion for a single "polynomially-defined" multiplicative function. However, the settings in these papers were highly special instances of the setting in Narkiewicz's original criterion in [26], so much so that they could not be used to obtain satisfactory uniform analogues of the aforementioned results on $\sigma_r(n)$.

As a special case of our results in this manuscript, we are able to extend Narkiewicz's criterion in its full generality to obtain results that are completely uniform in the modulus q and have optimal arithmetic restrictions on q. Certain special cases of our results also yield uniform extensions of the aforementioned results on $\sigma_r(n)$. For instance, we get all the following uniform analogues of Śliwa's result in [47]: the sum of divisors function $\sigma(n)$ is weakly equidistributed uniformly to all odd moduli $q \le (\log x)^{K_0}$ as well as to all even q not divisible by 3 that are either no more than a small power of $\log x$ or are squarefree without too many distinct prime factors. In addition, uniformity is restored to *all* (resp. to *squarefree*) even $q \le (\log x)^{K_0}$ that are not multiples of 3, provided we restrict to inputs n having *six* (resp. *four*) large prime factors counted with multiplicity. By examples constructed in [46], most of these restrictions are optimal. Applications of our main theorems also yield generalizations of these results for the functions $\sigma_r(n)$, thus obtaining complete uniform extensions of the aforementioned results of Narkiewicz, Rayner, Dobrowolski, Fomenko and others (see the discussion following the statement of Theorem 2.6).

All of these results and improvements are only for a single multiplicative function. In [28], Narkiewicz generalizes his aforementioned criterion to decide joint weak equidistribution for *families* of "polynomially defined" multiplicative functions to a fixed modulus q; he uses this generalized criterion in [27] to characterize those fixed q to which the Euler totient $\varphi(n)$ and sum of divisors $\sigma(n)$ are jointly weakly equidistributed. However, several arguments in the aforementioned papers ([22], [35], [37]) investigating uniform analogues of his previous criterion are all strictly constrained to a single multiplicative function and do not generalize to families. Our main results in this manuscript give complete uniform extensions of Narkiewicz's general criterion in [28] for families of multiplicative functions to a single modulus q, extensions that are optimal in both the range of uniformity and the arithmetic restrictions on q, as well as in various other parameters.

The qualitative summary of our main results is as follows. Under certain (provably) unavoidable conditions, a given family $f_1, \dots, f_K$ of polynomially-defined multiplicative functions is jointly weakly equidistributed *exactly* to those moduli q that satisfy Narkiewicz's criterion, and are also allowed to vary uniformly up to small powers of $\log x$, where these powers are all essentially optimal as well. In addition, weak equidistribution is restored in the full "Siegel-Walfisz range" $q \le (\log x)^{K_0}$ provided we restrict to inputs n having sufficiently many large prime factors counted with multiplicity. This threshold can be reduced and optimized (thus ensuring equidistribution among larger sample spaces of inputs) whenever q is squarefree, or whenever some reasonable additional control is available on the factorization of n or on the behavior of the functions f_i at some higher prime powers.