

Acknowledgments and Disclosure of Funding

This work was supported by the Ministry of Science and Innovation of Spain through projects PID2019-105093GB-I00, PID2020-117971GB-C22, and PID2022-136436NB-I00; by the Ministry of Universities of Spain through the FPU contract FPU21/00968, and by ICREA under the ICREA Academia programme.

References

- Credit Card Fraud Detection: Anonymized credit card transactions labeled as fraudulent or genuine. <https://www.kaggle.com/datasets/mlg-ulb/creditcardfraud/data>. Accessed: 2023-12-01.
- H. Adams, T. Emerson, M. Kirby, R. Neville, C. Peterson, P. Shipman, S. Chepushtanova, E. Hanson, F. Motta, and L. Ziegelmeier. Persistence images: A stable vector representation of persistent homology. *Journal of Machine Learning Research*, 18(8):1–35, 2017. URL <http://jmlr.org/papers/v18/16-337.html>.
- M. Agueh and G. Carlier. Barycenters in the Wasserstein space. *SIAM Journal on Mathematical Analysis*, 43(2):904–924, 2011. doi: 10.1137/100805741. URL <https://doi.org/10.1137/100805741>.
- N. Akai, T. Hirayama, and H. Murase. Experimental stability analysis of neural networks in classification problems with confidence sets for persistence diagrams. *Neural Networks*, 143:42–51, 2021. ISSN 0893-6080. doi: <https://doi.org/10.1016/j.neunet.2021.05.007>. URL <https://www.sciencedirect.com/science/article/pii/S0893608021001994>.
- R. Andreeva, K. Limbeck, B. Rieck, and R. Sarkar. Metric space magnitude and generalisation in neural networks, 2023. URL <https://arxiv.org/abs/2305.05611>.
- M. Arjovsky, S. Chintala, and L. Bottou. Wasserstein generative adversarial networks. In D. Precup and Y. W. Teh, editors, *Proceedings of the 34th International Conference on Machine Learning*, volume 70 of *Proceedings of Machine Learning Research*, pages 214–223. PMLR, 06–11 Aug 2017. URL <https://proceedings.mlr.press/v70/arjovsky17a.html>.
- N. Atienza, R. Gonzalez-Diaz, and M. Soriano-Trigueros. On the stability of persistent entropy and new summary functions for topological data analysis. *Pattern Recognition*, 107:107509, 2020. ISSN 0031-3203. doi: <https://doi.org/10.1016/j.patcog.2020.107509>. URL <https://www.sciencedirect.com/science/article/pii/S0031320320303125>.
- Ž. Avsec, V. Agarwal, D. Visentin, J. R. Ledsam, A. Grabska-Barwinska, K. R. Taylor, Y. Assael, J. Jumper, P. Kohli, and D. R. Kelley. Effective gene expression prediction from sequence by integrating long-range interactions. *Nature Methods*, 18(10):1196–1203, Oct 2021. ISSN 1548-7105. doi: 10.1038/s41592-021-01252-x. URL <https://doi.org/10.1038/s41592-021-01252-x>.
- N. Balabin, D. Voronkova, I. Trofimov, E. Burnaev, and S. Barannikov. Disentanglement learning via topology, 2023. URL <https://arxiv.org/abs/2308.12696>.

- R. Ballester, C. Casacuberta, and S. Escalera. Decorrelating neurons using persistence. In *NeurIPS 2023 Workshop on Symmetry and Geometry in Neural Representations*, 2023a. URL <https://openreview.net/forum?id=NJS6568y79>.
- R. Ballester, X. A. Clemente, C. Casacuberta, M. Madadi, C. A. Corneanu, and S. Escalera. Predicting the generalization gap in neural networks using topological data analysis, 2023b. URL <https://arxiv.org/abs/2203.12330>.
- S. Barannikov, I. Trofimov, N. Balabin, and E. Burnaev. Representation topology divergence: A method for comparing neural network representations. In K. Chaudhuri, S. Jegelka, L. Song, C. Szepesvari, G. Niu, and S. Sabato, editors, *Proceedings of the 39th International Conference on Machine Learning*, volume 162 of *Proceedings of Machine Learning Research*, pages 1607–1626. PMLR, 17–23 Jul 2022. URL <https://proceedings.mlr.press/v162/barannikov22a.html>.
- A. Barbara, Y. Bennani, and J. Karkazan. On the use of persistent homology to control the generalization capacity of a neural network. In B. Luo, L. Cheng, Z.-G. Wu, H. Li, and C. Li, editors, *Neural Information Processing*, pages 274–286, Singapore, 2024. Springer Nature Singapore. ISBN 978-981-99-8132-8.
- U. Bauer. Ripser: Efficient computation of Vietoris–Rips persistence barcodes. *Journal of Applied and Computational Topology*, 5(3):391–423, Sep 2021. ISSN 2367-1734. doi: 10.1007/s41468-021-00071-5. URL <https://doi.org/10.1007/s41468-021-00071-5>.
- M. Bianchini and F. Scarselli. On the complexity of neural network classifiers: A comparison between shallow and deep architectures. *IEEE Transactions on Neural Networks and Learning Systems*, 25(8):1553–1565, 2014. doi: 10.1109/TNNLS.2013.2293637.
- T. Birdal, A. Lou, L. J. Guibas, and U. Simsekli. Intrinsic dimension, persistent homology and generalization in neural networks. In M. Ranzato, A. Beygelzimer, Y. Dauphin, P. Liang, and J. W. Vaughan, editors, *Advances in Neural Information Processing Systems*, volume 34, pages 6776–6789. Curran Associates, Inc., 2021. URL https://proceedings.neurips.cc/paper_files/paper/2021/file/35a12c43227f217207d4e06ffefe39d3-Paper.pdf.
- D. Blalock, J. J. Gonzalez Ortiz, J. Frankle, and J. Gutttag. What is the state of neural network pruning? In I. Dhillon, D. Papailiopoulos, and V. Sze, editors, *Proceedings of Machine Learning and Systems*, volume 2, pages 129–146, 2020. URL https://proceedings.mlsys.org/paper_files/paper/2020/file/6c44dc73014d66ba49b28d483a8f8b0d-Paper.pdf.
- T. Bonis, M. Ovsjanikov, S. Oudot, and F. Chazal. Persistence-based pooling for shape pose recognition. In A. Bac and J.-L. Mari, editors, *Computational Topology in Image Context*, pages 19–29, Cham, 2016. Springer International Publishing. ISBN 978-3-319-39441-1.
- M. Botnan and M. Lesnick. Algebraic stability of zigzag persistence modules. *Algebraic & Geometric Topology*, 18(6):3133–3204, oct 2018. doi: 10.2140/agt.2018.18.3133. URL <https://doi.org/10.2140%2Fagt.2018.18.3133>.

- M. B. Botnan and M. Lesnick. An introduction to multiparameter persistence, 2023. URL <https://arxiv.org/abs/2203.14289>.
- R. Brüel Gabrielson and G. Carlsson. Exposition and interpretation of the topology of neural networks. In *2019 18th IEEE International Conference On Machine Learning And Applications (ICMLA)*, pages 1069–1076, 2019. doi: 10.1109/ICMLA.2019.00180.
- P. Bubenik. Statistical topological data analysis using persistence landscapes. *Journal of Machine Learning Research*, 16(3):77–102, 2015. URL <http://jmlr.org/papers/v16/bubenik15a.html>.
- G. Carlsson and R. Brüel Gabrielson. Topological approaches to deep learning. In N. A. Baas, G. E. Carlsson, G. Quick, M. Szymik, and M. Thauale, editors, *Topological Data Analysis*, pages 119–146, Cham, 2020. Springer International Publishing. ISBN 978-3-030-43408-3.
- G. Carlsson and V. de Silva. Zigzag persistence. *Foundations of Computational Mathematics*, 10(4):367–405, Aug 2010. ISSN 1615-3383. doi: 10.1007/s10208-010-9066-0. URL <https://doi.org/10.1007/s10208-010-9066-0>.
- G. Carlsson, T. Ishkhanov, V. de Silva, and A. Zomorodian. On the local behavior of spaces of natural images. *International Journal of Computer Vision*, 76(1):1–12, Jan 2008. ISSN 1573-1405. doi: 10.1007/s11263-007-0056-x. URL <https://doi.org/10.1007/s11263-007-0056-x>.
- M. Carrière and S. Oudot. Structure and stability of the one-dimensional Mapper. *Foundations of Computational Mathematics*, 18(6):1333–1396, Dec 2018. ISSN 1615-3383. doi: 10.1007/s10208-017-9370-z. URL <https://doi.org/10.1007/s10208-017-9370-z>.
- M. Carrière, M. Cuturi, and S. Oudot. Sliced Wasserstein kernel for persistence diagrams. In *Proceedings of the 34th International Conference on Machine Learning - Volume 70, ICML’17*, page 664–673. JMLR.org, 2017.
- M. Carrière, F. Chazal, M. Glisse, Y. Ike, H. Kannan, and Y. Umeda. Optimizing persistent homology based functions. In M. Meila and T. Zhang, editors, *Proceedings of the 38th International Conference on Machine Learning*, volume 139 of *Proceedings of Machine Learning Research*, pages 1294–1303. PMLR, 18–24 Jul 2021. URL <https://proceedings.mlr.press/v139/carriere21a.html>.
- J. Charlier, R. State, et al. Phom-gem: Persistent homology for generative models. In *The 6th Swiss Conference on Data Science (SDS), 2019 IEEE International Conference*. IEEE, 2019.
- F. Chazal, V. de Silva, and S. Oudot. Persistence stability for geometric complexes. *Geometriae Dedicata*, 173(1):193–214, Dec 2014. ISSN 1572-9168. doi: 10.1007/s10711-013-9937-z. URL <https://doi.org/10.1007/s10711-013-9937-z>.
- F. Chazal, V. de Silva, M. Glisse, and S. Oudot. *The Structure and Stability of Persistence Modules*, volume 10 of *SpringerBriefs in Mathematics*. Springer, 2016.