Therefore, our multiplicative independence and invariant factor hypotheses are both necessary for achieving uniformity in $q \leq (\log x)^{K_0}$ in our main results, and neither of them can be bypassed by restricting to inputs n with sufficiently many prime factors exceeding q.

14. Concluding Remarks

It is interesting to note that despite the extensive amount of 'multiplicative machinery' known in analytic number theory, there does not seem to be any estimate in the literature, a direct application of which can replace our arguments in section 7. For instance, Halász's Theorem only yields an upper bound on the character sums that is not precise enough, while a direct application of the (known forms of) the Landau-Selberg-Delange method, – one of the most precise estimates on the mean values of multiplicative functions known in literature, – seems to give an extremely small range of uniformity in q.

Theorem 2.3 suggests a few directions of improvement. First, we are still "one step away" from optimality in the $K \geq 2$, $k = 1$ case in subpart (b): we proved that "$2K+1$" is sufficient while "$2K-1$" is not, so the question is whether the optimal value is "$2K$" or "$2K+1$". If it is the former, then we will need a sharper bound on $V'_{2K,K}$ than what comes from our methods in section 11. One can also ask whether it is possible to weaken the nonsquarefullness condition in subpart (a). Theorem 2.4 also suggests other avenues for improvement, for instance, by optimizing the values of R and V. We hope to return to these questions in future papers.

Acknowledgements

This work was done in partial fulfillment of my PhD at the University of Georgia. As such, I would like to thank my advisor, Prof. Paul Pollack, for the past joint research and fruitful discussions that have led me to think about this question, as well as for his continued support and encouragement. I would also like to thank the Department of Mathematics at UGA for their support and hospitality.

Data Availability The manuscript has no associated data.

Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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