(7.15)

$$\sum_{n\le X} \mathbb{1}_{(f(n),Q)=1} \prod_{i=1}^{K} \chi_1(f_1(n)) = \frac{H_\chi(1/k) G_{\chi,2}(1/k)}{\Gamma(\alpha_k(Q) c_{\widehat{\chi}})} \cdot \frac{X^{1/k}}{(\log X)^{1-\alpha_k(Q)c_{\widehat{\chi}}}} \Big(1+O(\exp(-\sqrt{\log X}))\Big) + O\left(\frac{X^{1/k}}{(\log X)^{2-\alpha_k(Q)\mathrm{Re}(c_{\widehat{\chi}})-(1/20+\alpha_k(Q)/5)\epsilon_1}}\right),$$

by the reflection formula for the Gamma function and as $\Gamma(z) \gg 1$ for all $z$ with $|z| \le 2$.

If $c_{\widehat{\chi}} \neq 1$, then $\mathrm{Re}(c_{\widehat{\chi}}) \le \cos(2\pi/\varphi(Q_0)) < 1-\epsilon_1$. Lemma 7.1(iii) and Proposition 7.2(i) yield

$$\sum_{n\le X} \mathbb{1}_{(f(n),Q)=1} \prod_{i=1}^{K} \chi_1(f_1(n)) \ll \frac{X^{1/k}}{(\log X)^{1-\alpha_k(Q)(\mathrm{Re}(c_{\widehat{\chi}})+\epsilon_1/5)}} \ll \frac{X^{1/k}}{(\log X)^{1-\alpha_k(Q)(1-\delta_0)}},$$

with $\delta_0 := \delta_0(\lambda) := \min\{3\epsilon_1/4, 1-\epsilon_1/2\}$. On the other hand, if $c_{\widehat{\chi}} = 1$, then since $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$, we must have $G_{\chi,2}(1/k) = 0$ (as observed before (7.12)). Hence, (7.15) yields

$$\sum_{n\le X} \chi_1(f_1(n)) \cdots \chi_K(f_K(n)) \mathbb{1}_{(f(n),Q)=1} \ll \frac{X^{1/k}}{(\log X)^{2-\alpha_k(Q)-(1/20+\alpha_k(Q)/5)\epsilon_1}} \ll \frac{X^{1/k}}{(\log X)^{1-\alpha_k(Q)(1-\delta_0)}},$$

completing the proof of Theorem 5.6 in case 1.

Finally in case 2, (7.12) and Proposition 7.4 lead to the following analogue of (7.14):

$$\sum_{n\le X} \chi_1(f_1(n)) \cdots \chi_K(f_K(n)) \mathbb{1}_{(f(n),Q)=1} = -\lim_{\delta\to 0+} \frac{I_4^* + \overline{I_4^*}}{2\pi i} + O(X^{1/k} \exp(-\kappa_0 \sqrt{\log X})). \tag{7.16}$$

An argument entirely analogous to the one given above leads to the sharper variant of (7.15) with the $\exp(-\sqrt{\log X})$ replaced by $\exp\left(-\frac{c_1 \log X}{8kK_0 \log_2 X}\right)$, completing the proof of Theorem 5.6.

This finally concludes the proof of Theorem 4.2. In order to establish Theorems 2.1 to 2.4, we thus need to appropriately bound the contributions of inconvenient $n$'s considered in the respective theorems. We take this up in the next several sections.

## 8. Equidistribution to restricted moduli: Proof of Theorem 2.1

By Theorem 4.2, it remains to show that

$$\sum_{\substack{n\le x \text{ inconvenient} \\ (\forall i)\ f_i(n)\equiv a_i \pmod q}} 1 = o\left(\frac{1}{\varphi(q)^K} \sum_{\substack{n\le x \\ (f(n),q)=1}} 1\right) \quad \text{as } x\to\infty, \tag{8.1}$$

uniformly in coprime residues $(a_i)_{i=1}^K$ to $k$-admissible moduli $q \le (\log x)^{K_0}$, under any one of the conditions (i)-(iii) of Theorem 2.1.

To show this, we set $z := x^{1/\log_2 x}$ and recall that, by (4.3), (3.3) and (3.1), the $n$'s that are either $z$-smooth or divisible by the $(k+1)$-th power of a prime exceeding $y$ give negligible contribution to the left hand side of (8.1) in comparison to the right hand side. The remaining $n$ can be written in the form $mP^k$, where $P := P(n) > z$, $P_{Jk}(m) \le y$, $m$ is not divisible by the $(k+1)$-th power of a prime exceeding $y$, and $\gcd(m, P) = 1$, so that $f_i(n) = f_i(m)W_{i,k}(P)$. Given $m$, the number of possible $P$ is, by the Brun-Titchmarsh inequality,

$$\ll \frac{V''_{1,q}}{\varphi(q)} \cdot \frac{(x/m)^{1/k}}{\log(z/q)} \ll \frac{V''_{1,q}}{\varphi(q)} \cdot \frac{x^{1/k}\log_2 x}{m^{1/k}\log x},$$

where $V''_{1,q} := \max\left\{\#\mathcal{V}^{(k)}_{1,K}\big(q;(w_i)_{i=1}^K\big) : \ (w_i)_{i=1}^K \in U_q^K\right\}$. Summing this over possible $m$, we get

$$\sum_{\substack{n\le x \text{ inconvenient}\\ P(n)>z;\ p>y \implies p^{k+1}\nmid n\\ (\forall i)\ f_i(n)\equiv a_i \pmod q}} 1 \ \ll\ \frac{V''_{1,q}}{\varphi(q)} \cdot \frac{x^{1/k}}{(\log x)^{1-\alpha_k\epsilon/2}} \exp\left(O((\log_3 x)^2 + (\log_2(3q))^{O(1)})\right)$$

via (4.5). By Proposition 3.1, the quantity on the right hand side above is negligible compared to the right hand side of (8.1) whenever $q^{K-1}V''_{1,q} \ll (\log x)^{(1-2\epsilon/3)\alpha_k}$. But this does hold under any one of conditions (i)-(iii) in the statement of Theorem 2.1, because:

(i) $V''_{1,q} \ll 1$ if at least of one of $\{W_{i,k}\}_{1\le i\le K}$ is linear.

(ii) $V''_{1,q} \ll D_{\min}^{\omega(q)}$ if $q$ is squarefree, since $\#\mathcal{V}^{(k)}_{1,K}(\ell;(w_i)_{i=1}^K) \le D_{\min}$ for all $\ell \gg 1$.

(iii) $V''_{1,q} \ll_\delta q^{1-1/D_{\min}+\delta}$ by (5.34). With $\delta := \frac{\epsilon}{4(1-\epsilon)}\left(K - \frac{1}{D_{\min}}\right)$, this yields $q^{K-1}V''_{1,q} \ll (\log x)^{(1-2\epsilon/3)\alpha_k}$ under condition (iii) of the theorem.

This establishes (8.1), completing the proof of Theorem 2.1. □

8.1. **Optimality in the ranges of $q$ in Theorem 2.1.** In all our examples below, $\{W_{i,k}\}_{i=1}^K \subset \mathbb{Z}[T]$ will be nonconstant with $\prod_{i=1}^K W_{i,k}$ separable over $\mathbb{Q}$. Then $\beta(W_{1,k},\dots,W_{K,k}) = 1$, guaranteeing that any integer satisfies $IFH(W_{1,k},\dots,W_{K,k};1)$. We claim that there exists a constant $\widetilde{C} := \widetilde{C}(W_{1,k},\dots,W_{K,k})$ such that any $\widetilde{C}$-rough $k$-admissible integer $q$ lies in $\mathcal{Q}(k;f_1,\cdots,f_K)$. Indeed, viewing a character of $U_q^K$ as a tuple of characters mod $q$,[7] the condition (2.1) becomes vacuously true whenever $\mathcal{T}_k(q) := \{(W_{1,k}(u),\cdots,W_{K,k}(u)) \in U_q^K : u \in U_q\}$ generates the group $U_q^K$. Now under the canonical isomorphism $U_q^K \to \prod_{\ell^e\|q} U_{\ell^e}^K$, the set $\mathcal{T}_k(q)$ maps to $\prod_{\ell^e\|q}\mathcal{T}_k(\ell^e)$. Thus by [30, Lemma 5.13], if $\mathcal{T}_k(q)$ does not generate $U_q^K$, then there is some $\ell^e \| q$ and some tuple of characters $(\psi_1,\cdots,\psi_K) \ne (\chi_{0,\ell},\dots,\chi_{0,\ell})$ mod $\ell^e$ for which $\prod_{i=1}^K \psi_i(W_{i,k}(u))$ is constant on the set $R_k(\ell^e)$. Our claim now follows from [28, Lemma 5].

Fix any $k \in \mathbb{N}$. Let $\widetilde{C}_0 > \max\{\widetilde{C}, 4KD\}$ be any constant depending only on the polynomials $\{W_{i,k}\}_{1\le i\le K}$, which also exceeds the size of the leading coefficient and (nonzero) discriminant of $\prod_{i=1}^K W_{i,k}$. Then by Theorem N, $f_1,\dots,f_K$ are jointly weakly equidistributed modulo any (fixed) $\widetilde{C}_0$-rough $k$-admissible integer. Fix a prime $\ell_0 > \widetilde{C}_0$, and consider any nonconstant polynomials $\{W_{i,v}\}_{\substack{1\le i\le K\\ 1\le v\le k-1}} \subset \mathbb{Z}[T]$ all of whose coefficients are divisible by $\ell_0$, so that $\alpha_v(\ell_0) = 0$ for each $v < k$. Our moduli $q$ will have $P^-(q) = \ell_0$, so that $\alpha_v(q) = 0$ for all $v < k$. In each example below, we will show that $\alpha_k(q) \ne 0$, so that $q$ is $k$-admissible and lies in $\mathcal{Q}(k;f_1,\cdots,f_K)$ by definition of $\widetilde{C}_0$. The constant $K_0$ (in the assumption $q \le (\log x)^{K_0}$) is taken large enough in terms of $\{W_{i,k}\}_{i=1}^K$.

[7]Here $U_q^K$ is the direct product of $U_q$ taken $K$ times.

**Optimality under condition (i)**. We show that for any $K \geq 2$, the range of $q$ in Theorem 2.1(i) is optimal, – even if *all* of $W_{1,k}, \dots, W_{K,k}$ are assumed to be linear, for *any* choice of (pairwise coprime) linear functions. Indeed, consider $W_{i,k}(T) \coloneqq c_i T + b_i \in \mathbb{Z}[T]$ for nonzero integers $c_i$ and integers $b_i$ satisfying $b_i/c_i \neq b_j/c_j$ for all $i \neq j$. Then $\prod_{i=1}^K W_{i,k}$ is clearly separable in $\mathbb{Q}[T]$. Choose a nonzero integer $b$ such that $\prod_{i=1}^K (c_i b + b_i) \neq 0$. Let $\widetilde{C}_0 > \max\{|b|, |c_i b + b_i| : 1 \leq i \leq K\}$ be any constant satisfying the aforementioned requirements, so that any $q$ with $P^-(q) = \ell_0 > \widetilde{C}_0$ is coprime to $b$ and to $\prod_{i=1}^K W_{i,k}(b) = \prod_{i=1}^K (c_i b + b_i)$. Thus $\alpha_k(q) \neq 0$ and $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$. Now any prime $P \leq x^{1/k}$ satisfying $P \equiv b \pmod q$ also satisfies $f_i(P^k) = W_{i,k}(P) \equiv c_i b + b_i \pmod q$ for all $i \in [K]$. The Siegel–Walfisz Theorem thus shows that there are $\gg x^{1/k}/\varphi(q) \log x$ many $n \leq x$ satisfying $f_i(n) \equiv c_i b + b_i \pmod q$ for all $i \in [K]$. By Proposition 3.1, this last expression grows strictly faster than $\varphi(q)^{-K} \#\{n \leq x : (f(n), q) = 1\}$ as soon as $q \geq (\log x)^{(1+\epsilon)\alpha_k/(K-1)}$ for any fixed $\epsilon \in (0,1)$, showing that the range of $q$ in Theorem 2.1 under condition (i) is essentially optimal. Note that with $Y \in [2(1+\epsilon)\log_2 x/(K-1), (K_0/2)\log_2 x]$, the squarefree integer $q \coloneqq \prod_{\ell_0 \leq \ell \leq Y} \ell$ satisfies all desired conditions; in particular $(\log x)^{(1+\epsilon)/(K-1)} \leq q \leq (\log x)^{K_0}$ and $P^-(q) = \ell_0$.

**Optimality under condition (ii).** To show that the range of squarefree $q$ in Theorem 2.1(ii) is optimal, we define $W_{i,k}(T) \coloneqq \prod_{1 \leq j \leq d}(T - 2j) + 2(2i-1) \in \mathbb{Z}[T]$ for some fixed $d > 1$. Eisenstein's criterion at the prime 2 shows that each $W_{i,k}$ is irreducible in $\mathbb{Q}[T]$, and the distinct $W_{i,k}$'s differ by a constant, making $\prod_{i=1}^K W_{i,k}$ separable over $\mathbb{Q}$. Now $2 \in U_q$, and $W_{i,k}(2) = 2(2i-1) \leq 2(2K-1) < 4KD < \widetilde{C}_0 < P^-(q)$ for each $i \in [K]$. Thus, $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$ and $(2(2i-1))_{i=1}^K \in U_q^K$. Further, any prime $P$ satisfying $\prod_{1 \leq j \leq d}(P - 2j) \equiv 0 \pmod q$ also satisfies $f_i(P^k) = W_{i,k}(P) \equiv 2(2i-1) \pmod q$ for each $i$. Since $2d = 2 \deg W_{i,k} < 4KD < P^-(q)$, we see that $2, 4, \dots, 2d$ are all distinct coprime residues modulo each prime dividing $q$, whereupon it follows that the congruence $\prod_{1 \leq j \leq d}(v - 2j) \equiv 0 \pmod q$ has exactly $d^{\omega(q)}$ distinct solutions $v \in U_q$ for squarefree $q$. Hence, there are $\gg \frac{d^{\omega(q)}}{\varphi(q)} \cdot \frac{x^{1/k}}{\log x}$ many primes $P \leq x^{1/k}$ satisfying $f_i(P^k) \equiv 2(2i-1) \pmod q$ for all $i$, so there are also at least as many $n \leq x$ for which all $f_i(n) \equiv 2(2i-1) \pmod q$. The last expression grows strictly faster than $\varphi(q)^{-K} \#\{n \leq x : (f(n), q) = 1\}$ as soon as $q^{K-1} D_{\min}^{\omega(q)} = q^{K-1} d^{\omega(q)} > (\log x)^{(1+\epsilon)\alpha_k}$ for any fixed $\epsilon > 0$, showing that the range of $q$ in Theorem 2.1(ii) is essentially optimal.

Note that it is possible to construct squarefree $q \leq (\log x)^{K_0}$ satisfying the much stronger requirement that $d^{\omega(q)} > (\log x)^{(1+\epsilon)\alpha_k}$ (and $P^-(q) = \ell_0$). Indeed, let $q \coloneqq \prod_{\ell_0 \leq \ell \leq Y} \ell$ for some $Y \leq (K_0/2)\log_2 x$. Then $\omega(q) = \sum_{\ell_0 \leq \ell \leq Y} 1 \geq Y/2 \log Y$, while by the Chinese Remainder Theorem and the Prime Ideal Theorem, $\alpha_k(q) \leq \kappa'/\log Y$ for some constant $\kappa' \coloneqq \kappa'(W_{1,k}, \dots, W_{K,k}; \ell_0)$. So we need only choose $Y \in (4\kappa' \log_2 x/\log d, (K_0/2)\log_2 x)$ to have $q \leq (\log x)^{K_0}$ and $d^{\omega(q)} > (\log x)^{(1+\epsilon)\alpha_k}$.

For future reference, we observe that any $n$ of the form $P^k$ with $P$ a prime exceeding $q$ satisfies $P(n_k) > q$. Hence in the above setting, we have shown the stronger lower bound

$$\sum_{\substack{n \leq x:\ P(n_k) > q \\ (\forall i)\ f_i(n) \equiv 2(2i-1) \pmod q}} 1 \geq \sum_{\substack{q < P \leq x^{1/k} \\ \prod_{1 \leq j \leq d}(P - 2j) \equiv 0 \pmod q}} 1 \gg \frac{d^{\omega(q)}}{\varphi(q)} \cdot \frac{x^{1/k}}{\log x}. \tag{8.2}$$