By [27, Theorem 1], the Euler totient $\varphi(n)$ and the sum of divisors $\sigma(n)$ are jointly WUD modulo a fixed integer q precisely when q is coprime to 6; in fact, $\mathcal{Q}(1;\varphi,\sigma) = \{q : (q,6) = 1\}$. Theorem 2.1 shows that this joint weak equidistribution holds uniformly in $q \le (\log x)^{(1-\epsilon)\alpha(q)}$ coprime to 6, where $\alpha(q) := \prod_{\ell \mid q}(\ell-3)/(\ell-1)$ and $\epsilon > 0$ is fixed but arbitrary. In subsection § 8.1, we will show that the ranges of q in (i)–(iii) above are all essentially optimal, and that for $K \ge 2$, the range of q under condition (i) is essentially optimal, even if q is squarefree and $\{W_{i,k}\}_{1\le i\le K}$ are *all* linear, for *any* choice of (pairwise coprime) linear functions! In particular, this means that the range $(\log x)^{(1-\epsilon)\alpha(q)}$ is essentially optimal for the joint weak equidistribution of φ and σ, even if we restrict to squarefree q.

Our constructions in § 8.1 will reveal that obstructions to uniformity in q come from inputs n that are k-th powers of a prime P. Modifying these constructions, we can produce obstructions of the form mP^k with m fixed or growing slowly with x. It turns out that the problematic inputs are those with too few large prime factors. More precisely, complete uniformity in q up to a fixed but arbitrary power of $\log x$ can be restored by restricting the set of inputs n to those divisible by a sufficient number (say R) of primes exceeding q (here and below, all prime factors are counted with multiplicity unless stated otherwise). A smaller value of R suffices provided we assume that sufficiently many of these primes appear to a k-th power in n.

To make these precise, we let $P(n)$ denote the largest prime divisor of n, with the convention that $P(1) := 1$. Set $P_1(n) := P(n)$, and inductively define $P_k(n) := P_{k-1}(n/P(n))$. Thus, $P_k(n)$ is the k-th largest prime factor of n (counted with multiplicity), with $P_k(n) = 1$ if $\Omega(n) < k$. We also use n_k to denote the largest positive integer such that n_k^k is a unitary divisor of n; in other words, no prime divisor of the integer n/n_k^k appears to an exponent divisible by k. (Informally, we may call n_k the "reduced k-th power part"of n; if $k = 1$, then $n_1 = n$.) Since $D = 1$ forces $K = 1$ and $W_k = W_{1,k}$ to be linear (a case in which Theorem 2.1(i) already gives complete uniformity in $q \le (\log x)^{K_0}$), we assume in Theorems 2.2 to 2.4 below that $D \ge 2$.

Theorem 2.2. *The following formulae hold as $x \to \infty$, uniformly in coprime residues $a_1, \dots, a_K$ to moduli $q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(k; f_1, \cdots, f_K)$ and satisfying $IFH(W_{1,k}, \dots, W_{K,k}; B_0)$.*

(a)

$$\begin{aligned}(2.2)\quad &\#\{n \le x : P_R(n) > q,\ (\forall i)\ f_i(n) \equiv a_i \pmod q\}\\ &\sim \frac{1}{\varphi(q)^K}\#\{n \le x : \gcd(f(n), q) = 1\} \sim \frac{1}{\varphi(q)^K}\#\{n \le x : P_R(n) > q, \gcd(f(n), q) = 1\},\end{aligned}$$

where

$$\begin{cases} R = k(KD+1), & \text{if } k < D\\ R \text{ is the least integer exceeding } k\left(1 + (k+1)\left(K - 1/D\right)\right), & \text{if } k \ge D.\end{cases}$$

(b)

$$\begin{aligned}(2.3)\quad &\#\{n \le x : P_{KD+1}(n_k) > q,\ (\forall i)\ f_i(n) \equiv a_i \pmod q\}\\ &\sim \frac{1}{\varphi(q)^K}\#\{n \le x : \gcd(f(n), q) = 1\} \sim \frac{1}{\varphi(q)^K}\#\{n \le x : P_{KD+1}(n_k) > q, \gcd(f(n), q) = 1\}.\end{aligned}$$

We remark here that since the inputs n we work with satisfy $\gcd(f(n), q) = 1$, the k-admissibility of q guarantees that n must differ by a bounded factor from a k-full integer (see Lemma 3.3 below). This is what makes the anatomy of the reduced k-th power parts (i.e. the n_k), and hence also the kind of restriction in subpart (b), natural to consider. The two formulae (2.2) and (2.3) coincide for $k = 1$, and even in the special case $k = K = 1$, either of them improves over Theorem 1.4(a) in [37]. The value of R in Theorem 2.2(a) is optimal for the sum of divisors function $\sigma(n)$ to even moduli q; see the discussion on applications following the statement of Theorem 2.6.

For squarefree moduli q, it suffices to have much weaker restrictions (that are also exactly or nearly optimal) on the set of inputs n so as to detect weak equidistribution.

Theorem 2.3. *The following hold as $x \to \infty$, uniformly in coprime residues $a_1, \dots, a_K$ modulo* <u>*squarefree*</u> *$q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(k; f_1, \cdots, f_K)$ and satisfying $IFH(W_{1,k}, \dots, W_{K,k}; B_0)$.*

(a) *The formulae* (2.2) *for $k \ge 2$, with*[4]

$$R := \begin{cases} k(Kk + K - k) + 1, & \textit{if at least one of } \{W_{i,k}\}_{1 \le i \le K} \subset \mathbb{Z}[T] \textit{ is not squarefull.} \\ k(Kk + K - k + 1) + 1, & \textit{in general.} \end{cases}$$

(b) *The formulae* (2.3), <u>*either*</u> *with "$KD + 1$" replaced by $2K + 1$ for any $K \ge 1$,* <u>*or*</u> *with "$KD + 1$" replaced by 2 for $K = 1$ when $W_k = W_{1,k} \in \mathbb{Z}[T]$ is not squarefull.*

Since $n_1 = n$, the case $k = 1$ missing in (a) is accounted for in (b). It is worthwhile to strive for the optimality of R since doing so ensures weak equidistribution among the largest possible set of inputs n. In subsection § 11.1, we show that the restriction on the inputs n in (a) is optimal in the sense that in order to have uniformity in $q \le (\log x)^{K_0}$, it is not possible to reduce "$k(Kk + K - k) + 1$" to "$k(Kk + K - k)$". Likewise, the restriction in (b) is nearly optimal in that it is not possible to reduce "$2K + 1$" to "$2K - 1$" for any $K \ge 2$, nor is it possible to reduce the "2" to "1" for $K = 1$. (In fact, in all these examples, $\{W_{i,k}\}_{i=1}^K$ will be pairwise coprime irreducibles, making $\prod_{i=1}^K W_{i,k}$ separable over $\mathbb{Q}$.) The restriction $k \ge 2$ and the nonsquarefullness condition in (a) are for technical reasons that will become clear from the arguments.

Our constructions demonstrating the aforementioned optimality or near-optimality of the values of R in Theorem 2.3 will come from multiplicative functions f_i for which the polynomials $\{W_{i,k}\}_{1 \le i \le K}$ are nonconstant (in fact multiplicatively independent), but for which the polynomials $\{W_{i,k+1}\}_{1 \le i \le K}$ or $\{W_{i,2k}\}_{1 \le i \le K}$ are constant. In practice however, the $W_{i,v}$ are often nonconstant for many more values of v (beyond a fixed threshold k); in fact, for many well-known arithmetic functions f (such as the Euler totient and sums of divisor-powers $\sigma_r(n) := \sum_{d \mid n} d^r$), the values $f(p^v)$ are controlled by nonconstant polynomials $W_v \in \mathbb{Z}[T]$ for *all* $v \ge 1$. Hence, it is natural to ask whether the restriction on inputs n in Theorems 2.2 and 2.3 can be weakened when such additional control on the f_i is available, or in other words, if V (the number of

[4]Here we write a polynomial $F \in \mathbb{Z}[T]$ as $F = r \prod_{j=1}^M H_j^{\nu_j}$ for some $\nu_j \in \mathbb{N}$ and pairwise coprime primitive irreducibles $H_j \in \mathbb{Z}[T]$, and we say that F is "squarefull" in $\mathbb{Z}[T]$ if $(\prod_{j=1}^M H_j)^2 \mid F$. This condition is equivalent to saying that $\prod_{\substack{\theta \in \mathbb{C} \\ F(\theta) = 0}} (T - \theta)^2 \mid F(T)$ in $\mathbb{C}[T]$, i.e., that every root of F in $\mathbb{C}$ has multiplicity at least 2.

powers of primes at which we are assuming the f_i to be controlled by nonconstant polynomials $W_{i,v}$) can be taken to be sufficiently large. It turns out that we can almost always do this for squarefree q and in several cases in general. Unlike the results stated so far, the implied constants in Theorem 2.4 below could depend on the full set of polynomials $\{W_{i,v}\}_{\substack{1\le i\le K\\ 1\le v\le V}}$.

Theorem 2.4. *Assume that the polynomials $\{W_{i,v}\}_{1\le i\le K} \subset \mathbb{Z}[T]$ are multiplicatively independent for each v satisfying $k \le v \le V$. Let $D_0 := \max_{k\le v\le V} D_v = \max_{k\le v\le V} \sum_{i=1}^K \deg W_{i,v}$.*

- **(a)** *If either $V > k(K + 1 - 1/D_{\min}) - 1$ and $R := \max\{k(KD + 1), (Kk - 1)D_0 + 2\}$, or*
- **(b)** *If q is squarefree, $V \ge Kk$, and $R := k(2K + 1)$,*

then the relations (2.2) *hold, uniformly in coprime residues $a_1, \dots, a_K$ modulo $q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(k; f_1, \cdots, f_K)$ and satisfying $IFH(W_{1,k}, \dots, W_{K,k}; B_0)$.*

Notice that for any $K > 2$, the result under (b) unconditionally improves over Theorem 2.3(a) in terms of weakening the restriction on inputs n. On the other hand, the result under condition (a) improves over Theorem 2.2(a) whenever k or D is large enough compared to D_0.

We now explain the necessity of the two key additional hypotheses that we have been assuming in our main results so far, namely the multiplicative independence of $\{W_{i,k}\}_{1\le i\le K} \subset \mathbb{Z}[T]$ and the invariant factor hypothesis. It turns out that without the former condition, the K congruences $f_i(n) \equiv a_i \pmod q$ (for $1 \le i \le K$) may degenerate to fewer congruences for sufficiently many inputs n, making weak equidistribution fail uniformly to *all* sufficiently large $q \le (\log x)^{K_0}$. In this situation, weak equidistribution *cannot* be restored *no matter* how much we restrict the set of inputs n to those having sufficiently many large prime factors. We make this explicit in the next result.

Theorem 2.5. *Fix $R \ge 1$, $K > 1$ and assume that $\{W_{i,k}\}_{1\le i\le K-1} \subset \mathbb{Z}[T]$ are multiplicatively independent, with $\sum_{i=1}^{K-1} \deg W_{i,k} > 1$. Suppose $W_{K,k} = \prod_{i=1}^{K-1} W_{i,k}^{\lambda_i}$ for some nonnegative integers $(\lambda_i)_{i=1}^{K-1} \ne (0, \dots, 0)$. There exists a constant $C := C(W_{1,k}, \dots, W_{K-1,k}) > 0$ such that*

$$\#\{n \le x : P_{Rk}(n) > q,\ (\forall i \in [K])\ f_i(n) \equiv a_i \pmod q\} \gg \frac{1}{\varphi(q)^{K-1}} \cdot \frac{x^{1/k}(\log\log x)^{R-2}}{\log x}$$

as $x \to \infty$, uniformly in k-admissible $q \le (\log x)^{K_0}$ supported on primes $\ell > C$ satisfying $\gcd(\ell - 1, \beta(W_{1,k}, \dots, W_{K-1,k})) = 1$, and in $a_i \in U_q$ with $a_K \equiv \prod_{i=1}^{K-1} a_i^{\lambda_i} \pmod q$.

The compatibility of the relations involving $W_{K.k}$ and a_K suggests the aforementioned degeneracy from K to $K-1$ congruences. Note that the above lower bound will in fact come from the n which are supported on primes much larger than q. A similar lower bound holds for $K = 1$ when $W_k = W_{1,k}$ is constant (see the remark preceding subsection § 13.1). Using the above theorem, we shall construct (in § 13.1) explicit examples of polynomials $\{W_{i,k}\}_{1\le i\le K-1}$ and moduli $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$ where the above lower bound grows strictly faster than the expected proportion of $n \le x$ having $\gcd(f(n), q) = 1$. This would demonstrate an overrepresentation of the coprime residues $(a_i \bmod q)_{i=1}^K$ by the multiplicative functions $f_1, \dots, f_K$, coming from inputs n that have at least Rk many prime factors exceeding q, showing the necessity of our hypothesis on the multiplicative independence of $\{W_{i,k}\}_{1\le i\le K}$.