satisfying $(A_1, \dots, A_K) \not\equiv (0, \dots, 0) \bmod \ell$. To show this, we proceed as in the proof of (a), but working with the matrix M_1 defined in (5.2) in place of the exponent matrix E_0. Observe that $\widetilde{F}(T) = \sum_{j=0}^{D-1} \left(\sum_{i=1}^K c_{i,j} A_i \right) T^j$, hence if $\kappa(\ell) := \mathrm{ord}_\ell(\widetilde{F})$, then $\ell^{\kappa(\ell)}$ divides all the entries of the matrix $M_1 (A_1 \cdots A_K)^\top$. Since M_1 has full rank and $D = \sum_{i=1}^K \deg F_i \ge K$ many rows, and since $(A_1, \dots, A_K) \not\equiv (0, \dots, 0) \bmod \ell$, an argument entirely analogous to the one leading to (5.3) shows that $\ell^{\kappa(\ell)}$ divides the last invariant factor $\widetilde{\beta}$ of M_1. Hence $\mathrm{ord}_\ell(\widetilde{F}) = \kappa(\ell) \le v_\ell(\widetilde{\beta})$ and our claim follows as $|\widetilde{\beta}| < C_1$.

As a consequence, we find that $\mathrm{ord}_\ell \left(T^{\varphi(\ell^r)} \left(\prod_{i=1}^K F_i(T)^{A_i - 1} \right) \widetilde{F}(T) \right) = \mathrm{ord}_\ell(\widetilde{F}) \le \mathbb{1}_{\ell \le C_1} C_1$ for all primes $\ell \le C_1$ satisfying $\mathrm{ord}_\ell(F_1 \cdots F_K) = 0$, and also for all primes $\ell > C_1$ (for which the condition $\mathrm{ord}_\ell(F_1 \cdots F_K) = 0$ is automatic by definition of C_1). But now since $\mathrm{ord}_\ell(\varphi(\ell^r)) \ge 1$ for $r \ge 2$ and $\mathrm{ord}_\ell(\varphi(\ell^r)) \ge C_1 + 1$ for $r \ge C_1 + 2$, (5.4) shows that $\tau(\ell) = \mathrm{ord}_\ell \left(T^{\varphi(\ell^r)} \left(\prod_{i=1}^K F_i(T)^{A_i - 1} \right) \widetilde{F}(T) \right)$, establishing subpart (b) of the proposition.

Finally, since in both the cases of (5.1), we have $\tau(\ell) < r - 1$, the identity (5.4) reveals that

$$\mathcal{C}_\ell(T) \equiv \ell^{-\tau(\ell)} \left(T^{\varphi(\ell^r)} \prod_{i=1}^K F_i(T)^{A_i} \right)' \equiv T^{\varphi(\ell^r)} \left(\prod_{i=1}^K F_i(T)^{A_i - 1} \right) \left(\ell^{-\tau(\ell)} \widetilde{F}(T) \right) \text{ in the ring } \mathbb{F}_\ell[T].$$

As such, any root of the polynomial $\theta \in \mathbb{F}_\ell$ of $\mathcal{C}_\ell(T)$ (considered as a nonzero element of $\mathbb{F}_\ell[T]$) which is not a root of $T \prod_{i=1}^K F_i(T)$, must be a root of $\ell^{-\tau(\ell)} \widetilde{F}(T)$, and θ must have the same multiplicity in $\mathcal{C}_\ell(T)$ and $\ell^{-\tau(\ell)} \widetilde{F}(T)$. This completes the proof of Proposition 5.3. □

We now come to the main result of this section: the promised generalization of Proposition 4.4. The following notation and conventions will be relevant only in the rest of the section.

Let $\{G_{i,r}\}_{\substack{1 \le i \le K \\ 1 \le r \le L}} \subset \mathbb{Z}[T]$ be a fixed collection of nonconstant polynomials such that for each $r \in [L]$, the polynomials $\{G_{i,r}\}_{1 \le i \le K} \subset \mathbb{Z}[T]$ are multiplicatively independent. Define $D_0 := \max_{1 \le r \le L} \sum_{i=1}^K \deg G_{i,r}$. Let $N \ge 1$ and $\{F_{i,j}\}_{\substack{1 \le i \le K \\ 1 \le j \le N}} \subset \mathbb{Z}[T]$ be a family of polynomials such that for each $j \in [N]$, the vector $(F_{i,j})_{i=1}^K$ coincides with one of the vectors $(G_{i,j'})_{i=1}^K$ for some $j' \in [L]$ (possibly depending on j). In this case we define, for any integer q,

$$\widetilde{\alpha}_j(q) := \frac{1}{\varphi(q)} \#\{u \in U_q : \prod_{i=1}^K F_{i,j}(u) \in U_q\} = \frac{1}{\varphi(q)} \#\{u \in U_q : \prod_{i=1}^K G_{i,j'}(u) \in U_q\},$$

and let $\alpha_N^*(q) := \prod_{j=1}^N \widetilde{\alpha}_j(q)$. For any $(w_i)_{i=1}^K \in U_q^K$, define

$$\widetilde{\mathcal{V}}_{N,K}\left(q; (w_i)_{i=1}^K\right) := \left\{ (v_1, \dots, v_N) \in U_q^N : \ (\forall i \in [K]) \ \prod_{j=1}^N F_{i,j}(v_j) \equiv w_i \pmod q \right\}.$$

Fix $B_0 > 0$. In the next three results, the implied constants may depend only on B_0 and on the fixed collection of polynomials $\{G_{i,r}\}_{\substack{1 \le i \le K \\ 1 \le r \le L}}$ (besides other parameters declared explicitly).

Proposition 5.4. *There exists a constant* $C_0 := C_0\left(\{G_{i,r}\}_{\substack{1\le i\le K\\ 1\le r\le L}}; B_0\right) > (8D_0)^{2D_0+2}$ *depending only on* $\{G_{i,r}\}_{\substack{1\le i\le K\\ 1\le r\le L}}$ *and* B_0*, such that for any constant* $C > C_0$*, the following hold.*

(a) *Uniformly for* $N \ge KD_0+1$ *and in coprime residues* $w_1, \dots, w_K$ *to moduli* q *satisfying* $\alpha_N^*(q) \neq 0$ *and* $IFH(G_{1,r}, \dots, G_{K,r}; B_0)$ *for each* $r \in [L]$*, we have*

$$\frac{\#\widetilde{\mathcal{V}}_{N,K}\left(q; (w_i)_{i=1}^K\right)}{\varphi(q)^N} \tag{5.5}$$
$$= \frac{\alpha_N^*(q)}{\alpha_N^*(Q_0)}\left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^K\left\{\frac{\#\widetilde{\mathcal{V}}_{N,K}\left(Q_0; (w_i)_{i=1}^K\right)}{\varphi(Q_0)^N} + O\left(\frac{1}{C^N}\right)\right\}\prod_{\substack{\ell\mid q\\ \ell>C_0}}\left(1+O\left(\frac{(4D_0)^N}{\ell^{N/D_0-K}}\right)\right),$$

where Q_0 *is a* C_0*-smooth divisor of* q *of size* $O_C(1)$*.*

(b) *For any* ***fixed*** $N \ge 1$ *and uniformly in coprime residues* $w_1, \dots, w_K$ *mod* q*, we have*

$$\frac{\#\widetilde{\mathcal{V}}_{N,K}\left(q; (w_i)_{i=1}^K\right)}{\varphi(q)^N} \le \frac{\left(\prod_{\ell^e\| q} e\right)^{\mathbb{1}_{N=KD_0}}}{q^{\min\{K, N/D_0\}}}\ \exp\big(O(\omega(q))\big). \tag{5.6}$$

Proof. In what follows, q is an arbitrary positive integer (unless stated otherwise). We may assume that $\alpha_N^*(q) \neq 0$, for both the assertions of the proposition are vacuous or tautological otherwise. In particular, this means that $\mathrm{ord}_\ell(\prod_{i=1}^K\prod_{j=1}^N F_{i,j}) = 0$ for each prime $\ell \mid q$. Fix $C_0 := C_0\left(\{G_{i,r}\}_{\substack{1\le i\le K\\ 1\le r\le L}}; B_0\right)$ to be any constant exceeding B_0, $(32D_0)^{2D_0+2}$, the sizes of the leading and constant coefficients of $\{G_{i,r}\}_{\substack{1\le i\le K\\ 1\le r\le L}}$, as well as the constants $C_1(G_{1,r}, \dots, G_{K,r})$ coming from applications of Proposition 5.3 to each of the families $\{G_{i,r}\}_{1\le i\le K}$ of multiplicatively independent polynomials. We will show that any such choice of C_0 suffices.

We first consider the case $D_0 > 1$; we will deal with the $D_0 = 1$ case towards the end of this argument. For an arbitrary positive integer Q and coprime residues $w_1, \dots, w_K$ mod Q, we apply the orthogonality of Dirichlet characters to detect the congruences defining $\widetilde{\mathcal{V}}_{N,K}\left(Q; (w_i)_{i=1}^K\right)$. This yields

$$\#\widetilde{\mathcal{V}}_{N,K}\left(Q; (w_i)_{i=1}^K\right) = \frac{1}{\varphi(Q)^K}\sum_{\chi_1,\dots,\chi_K \bmod Q}\overline{\chi}_1(w_1)\cdots\overline{\chi}_K(w_K)\prod_{j=1}^N Z_{Q;\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j}), \tag{5.7}$$

where $Z_{Q;\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j}) := \sum_{v \bmod Q}\chi_{0,Q}(v)\prod_{i=1}^K\chi_i(F_{i,j}(v))$ and $\chi_{0,Q}$ denotes (as usual) the trivial character mod Q.

We show the following estimates, both uniform in residues $w_1, \dots, w_K \in U_{\ell^e}$ for primes $\ell > C_0$:

(i) If $\alpha_N^*(\ell) \neq 0$ and $\gcd(\ell-1, \beta(G_{1,r}, \dots, G_{K,r})) = 1$ for each $r \in [L]$, then

$$\frac{\#\widetilde{\mathcal{V}}_{N,K}(\ell^e; (w_i)_{i=1}^K)}{\varphi(\ell^e)^N} = \frac{\alpha_N^*(\ell)}{\varphi(\ell^e)^K}\left(1+O\left(\frac{(4D_0)^N}{\ell^{N/D_0-K}}\right)\right), \tag{5.8}$$

uniformly in $N \geq KD_0 + 1$.

(ii) For each fixed $N \geq 1$, there is a constant K' depending at most on N and $\{G_{i,r}\}_{\substack{1 \leq i \leq K \\ 1 \leq r \leq L}}$ such that

$$\frac{\#\widetilde{\mathcal{V}}_{N,K}(\ell^e; (w_i)_{i=1}^K)}{\varphi(\ell^e)^N} \leq K' \, \frac{e^{\mathbb{1}_{N=KD_0}}}{(\ell^e)^{\min\{K, N/D_0\}}}. \tag{5.9}$$

To show these, we start by applying (5.7) with $Q \coloneqq \ell^e$ to get

$$\begin{aligned} &\frac{\#\widetilde{\mathcal{V}}_{N,K}(\ell^e; (w_i)_{i=1}^K)}{\varphi(\ell^e)^N} \\ &\quad \leq \frac{1}{\varphi(\ell^e)^K} \left\{ 1 + \frac{1}{\varphi(\ell^e)^N} \sum_{(\chi_1, \dots, \chi_K) \neq (\chi_{0,\ell}, \dots, \chi_{0,\ell}) \bmod \ell^e} \prod_{j=1}^N |Z_{\ell^e;\ \chi_1, \dots, \chi_K}(F_{1,j}, \dots, F_{K,j})| \right\}; \end{aligned} \tag{5.10}$$

in addition, if $\alpha_N^*(\ell) \neq 0$, then from $Z_{\ell^e;\ \chi_{0,\ell}, \dots, \chi_{0,\ell}}(F_{1,j}, \dots,\ F_{K,j}) = = \widetilde{\alpha}_j(\ell)\varphi(\ell^e)$, we have

$$\begin{aligned} \frac{\#\widetilde{\mathcal{V}}_{N,K}(\ell^e; (w_i)_{i=1}^K)}{\varphi(\ell^e)^N} = \frac{\alpha_N^*(\ell)}{\varphi(\ell^e)^K} \Bigg\{ 1 + \\ \frac{1}{\alpha_N^*(\ell)\varphi(\ell^e)^N} \sum_{(\chi_1, \dots, \chi_K) \neq (\chi_{0,\ell}, \dots, \chi_{0,\ell}) \bmod \ell^e} \overline{\chi}_1(w_1) \cdots \overline{\chi}_K(w_K) \prod_{j=1}^N Z_{\ell^e;\ \chi_1, \dots, \chi_K}(F_{1,j}, \dots, F_{K,j}) \Bigg\}. \end{aligned} \tag{5.11}$$

Now consider any tuple $(\chi_1, \dots, \chi_K) \neq (\chi_{0,\ell}, \dots, \chi_{0,\ell})$ mod ℓ^e and any $j \in [N]$. Let $\ell^{e_0} \coloneqq \operatorname{lcm}[\mathfrak{f}(\chi_1), \dots, \mathfrak{f}(\chi_K)] \in \{\ell, \dots, \ell^e\}$. Using $\chi_1, \dots, \chi_K$ to also denote the characters mod ℓ^{e_0} inducing $\chi_1, \dots, \chi_K$ respectively, we get

$$Z_{\ell^e;\ \chi_1, \dots, \chi_K}(F_{1,j}, \dots, F_{K,j}) = \ell^{e-e_0}\ Z_{\ell^{e_0};\ \chi_1, \dots, \chi_K}(F_{1,j}, \dots, F_{K,j}) \tag{5.12}$$

Since $\ell > C_0 > 2$, the character group mod ℓ^{e_0} is generated by the character ψ_{e_0} given by $\psi_{e_0}(\gamma) \coloneqq \exp(2\pi i/\varphi(\ell^{e_0}))$, for some generator γ of $U_{\ell^{e_0}}$. As such, there exists a tuple $(A_1, \dots, A_K) \in [\varphi(\ell^{e_0})]$ satisfying $\chi_i = \psi_{e_0}^{A_i}$ for each i, and

$$(A_1, \dots, A_K) \not\equiv \begin{cases} (0, \dots, 0) \pmod{\ell}, & \text{if } e_0 > 1, \\ (0, \dots, 0) \pmod{\ell - 1}, & \text{if } e_0 = 1, \end{cases} \tag{5.13}$$

since at least one of $\chi_1, \dots, \chi_K$ is primitive mod ℓ^{e_0}. This gives

$$Z_{\ell^{e_0};\ \chi_1, \dots, \chi_K}(F_{1,j}, \dots, F_{K,j}) = \sum_{v \bmod \ell^{e_0}} \psi_{e_0}\left(v^{\varphi(\ell^{e_0})} \prod_{i=1}^K F_{i,j}(v)^{A_i} \right). \tag{5.14}$$

We now consider two possibilities, namely when $e_0 = 1$ or $e_0 \geq 2$.

Case 1*:* Suppose $e_0 = 1$. For each $j \in [N]$, consider $j' \in [L]$ satisfying $(G_{i,j'})_{i=1}^K = (F_{i,j})_{i=1}^K$. By Proposition 5.3(a), we see there are $O_L(1)$ many possible tuples $(\chi_1, \dots, \chi_K)$ of characters mod ℓ^e having $\operatorname{lcm}[\mathfrak{f}(\chi_1), \dots, \mathfrak{f}(\chi_K)] = \ell$, for which $T^{\varphi(\ell)} \prod_{i=1}^K F_{i,j}(T)^{A_i} = T^{\varphi(\ell)} \prod_{i=1}^K G_{i,j'}(T)^{A_i}$ is of the form $c \cdot G(T)^{\ell-1}$ in $\mathbb{F}_\ell[T]$ for some $j \in [N]$ (here A_i are as above). Moreover if $\gcd(\ell - 1, \beta(G_{1,r}, \dots, G_{K,r})) = 1$ for all $r \in [L]$, then there is no such tuple $(\chi_1, \dots, \chi_K)$. For