Turning to the invariant factor hypothesis, we show that the failure of this condition incurs an additional factor over the expected proportion of $n \le x$ satisfying $\gcd(f(n), q) = 1$. For certain choices of q and $\{W_{i,k}\}_{1 \le i \le K}$, this factor can be made too large, once again leading to an overrepresentation of the tuple $(a_i \bmod q)_{i=1}^K$ by the multiplicative functions $f_1, \dots, f_K$. In what follows, $P^-(q)$ denotes the smallest prime dividing q.

Theorem 2.6. *Fix $R \ge 1$ and assume that $\{W_{i,k}\}_{1 \le i \le K} \subset \mathbb{Z}[T]$ are nonconstant, monic and multiplicatively independent, so that $\beta = \beta(W_{1,k}, \dots, W_{K,k}) \in \mathbb{Z} \setminus \{0\}$. There exists a constant $C := C(W_{1,k}, \dots, W_{K,k}) > 0$ such that*

(2.4)

$$\#\{n \le x : P_{Rk}(n) > q,\ (\forall i \in [K])\ f_i(n) \equiv a_i \pmod q\} \gg \frac{2^{\#\{\ell | q:\ \gcd(\ell-1, \beta) \ne 1\}}}{\varphi(q)^K} \cdot \frac{x^{1/k} (\log\log x)^{R-2}}{\log x}$$

as $x \to \infty$, uniformly in k-admissible $q \le (\log x)^{K_0}$ having $P^-(q) > C$, and in coprime residues $(a_i)_{i=1}^K$ mod q which are all congruent to 1 modulo the largest squarefree divisor of q.

Here, the restriction on the residues a_i is imposed in order to have a positive contribution of certain character sums modulo the prime divisors of q. In subsection § 13.1, we shall construct explicit examples of $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$ and $\{W_{i,k}\}_{1 \le i \le K}$ for which the expression in the above lower bound is much larger than the expected proportion of $n \le x$ having $\gcd(f(n), q) = 1$.

We can give several applications of our main results to arithmetic functions of common interest. For instance, recall Śliwa's [47] result that $\sigma(n)$ is weakly equidistributed precisely to moduli that are not multiples of 6; in fact, his result shows that $\mathcal{Q}(1; \sigma) = \{q : \gcd(q, 2) = 1\}$ and $\mathcal{Q}(2; \sigma) = \{q : \gcd(q, 6) = 2\}$. By Theorem 2.1(i), $\sigma(n)$ is WUD uniformly to all odd moduli $q \le (\log x)^{K_0}$. Calling the members of the set $\mathcal{Q}(2; \sigma)$ "special", Theorem 2.1(ii) and (iii) show that $\sigma(n)$ is WUD uniformly to all special $q \le (\log x)^{(2-\delta)\widetilde{\alpha}(q)}$ and also to all squarefree special $q \le (\log x)^{K_0}$ satisfying $2^{\omega(q)} \le (\log x)^{(1-\epsilon)\widetilde{\alpha}(q)}$, where $\widetilde{\alpha}(q) := \alpha_2(q) = \prod_{\substack{\ell | q \\ \ell \equiv 1 \pmod 3}} (1 - 2/(\ell - 1))$. By the example constructed in [46, subsection 7.1], the latter restriction is optimal. Furthermore, by Theorems 2.2(a) or 2.4(a) (resp. by Theorem 2.3(a)), uniformity is restored to *all* (resp. to squarefree) special $q \le (\log x)^{K_0}$ by restricting to inputs n with $P_6(n) > q$ (resp. $P_4(n) > q$); here we have noted that the condition $P_3(n) > q$ forces $P_4(n) > q$ since for $\sigma(n)$ to be coprime to the even number q, it is necessary for n to be of the form m^2 or $2m^2$. By the examples constructed in [46], both of these restrictions are optimal as well. Alternatively, by Theorem 2.2(b) (resp. 2.3(b)), we may restrict to n with $P_3(n_2) > q$ (resp. $P_2(n_2) > q$) to restore complete uniformity in all (resp. squarefree) special $q \le (\log x)^{K_0}$.

For another example, we saw using Theorem 2.1 that $\varphi(n)$ and $\sigma(n)$ are jointly WUD modulo $q \le (\log x)^{(1-\epsilon)\alpha(q)}$ coprime to 6, and that these two restrictions on q are necessary and essentially optimal. By Theorem 2.2, complete uniformity is restored to all moduli $q \le (\log x)^{K_0}$ coprime to 6 by restricting to inputs n with $P_5(n) > q$.

We can give more applications of our main results to study the weak equidistribution of the Fourier coefficients of Eisenstein series; precisely, the functions $\sigma_r(n) := \sum_{d|n} d^r$ (for $r > 1$). An easy check shows that the polynomial $\sum_{0 \le j \le v} T^{rj} = \frac{T^{r(v+1)}-1}{T^r-1}$ shares no roots with its derivative, hence is separable. Calling the $q \in \mathcal{Q}(k; \sigma_r)$ as "k-good", Theorem 2.1 thus shows that σ_r is WUD uniformly to all k-good $q \le (\log x)^{(1-\epsilon)\alpha_k(q)(1-1/kr)^{-1}}$, and to all squarefree k-good $q \le (\log x)^{K_0}$ having $\omega(q) \le (1-\epsilon)\alpha_k(q) \log\log x/\log(kr)$. Further, by Theorems 2.2 to 2.4, weak equidistribution is restored modulo all k-good $q \le (\log x)^{K_0}$ by restricting to n with $P_{k(kr+1)}(n) > q$, whereas it is restored modulo all squarefree k-good $q \le (\log x)^{K_0}$ by restricting to n with $P_{k+1}(n) > q$ or to n with $P_2(n_k) > q$. An explicit characterization of the moduli $q \le (\log x)^{K_0}$ to which a given σ_r is weakly equidistributed thus reduces to an understanding of the possible k and of the set $\mathcal{Q}(k; \sigma_r)$ for a given (fixed) r; both of these are problems of fixed moduli that (as mentioned in the introduction) have been studied in great depth in [47], [14], [31], [29], [30], [39] and [40]. In fact, the sets $\mathcal{Q}(k; \sigma_r)$ have been explicitly characterized for all odd $r \le 200$ and all even $r \le 50$, and partial results are known for general $r \ge 4$. For example, the only two possible k's for σ_3 are $k = 1, 2$, and $\mathcal{Q}(1; \sigma_3) = \{q : \gcd(q, 14) = 1\}$ while $\mathcal{Q}(2; \sigma_3) = \{q : \gcd(q, 6) = 2\}$.

For a general family $(f_1, \dots, f_K)$, Narkiewicz [27, 30] gives algorithms to determine the sets $\mathcal{Q}(k; f_1, \cdots, f_K)$ for a fixed k. He shows (among other results) that in some of the most commonly occurring cases (which includes the cases of σ_r for all $r > 2$), the set of possible k is finite, and that for each such k, the set $\mathcal{Q}(k; f_1, \dots, f_K)$ can be characterized by certain (finitely many) coprimality conditions that can be determined effectively.

We conclude this section with the remark that although for the sake of simplicity of statements, we have been assuming that our multiplicative functions $\{f_i\}_{i=1}^K$ and polynomials $\{W_{i,v}\}_{\substack{1\le i\le K\\ 1\le v\le V}}$ are both fixed, our proofs will reveal that these results are also uniform in the $\{f_i\}_{i=1}^K$ as long as they are defined by the fixed polynomials $\{W_{i,v}\}_{\substack{1\le i\le K\\ 1\le v\le V}}$.

Notation and conventions: We do not consider the zero function as multiplicative (thus, if f is multiplicative, then $f(1) = 1$). Given $z > 0$, we say that a positive integer n is z-smooth if $P(n) \le z$, and z-rough if $P^-(n) > z$; by the z-smooth part (resp. z-rough part) of n, we shall mean the largest z-smooth (resp. z-rough) positive integer dividing n. For a ring R, we shall use $R^\times$ to denote the multiplicative group of units of R. We denote the number of primes dividing q counted with and without multiplicity by $\Omega(q)$ and $\omega(q)$ respectively, and we write $U_q := (\mathbb{Z}/q\mathbb{Z})^\times$. For a Dirichlet character χ mod q, we use $\mathfrak{f}(\chi)$ to denote the conductor of χ. When there is no danger of confusion, we shall write $(a_1, \dots, a_k)$ in place of $\gcd(a_1, \dots, a_k)$. Throughout, the letters p and ℓ are reserved for primes. For nonzero $H \in \mathbb{Z}[T]$, we use $\mathrm{ord}_\ell(H)$ to denote the highest power of ℓ dividing all the coefficients of H; for an integer $m \ne 0$, we shall sometimes use $v_\ell(m)$ in place of $\mathrm{ord}_\ell(m)$. We use $\mathbb{M}_{A\times B}(\mathbb{Z})$ to refer to the ring of $A \times B$ matrices with integer entries, while $GL_{A\times B}(\mathbb{Z})$ refers to the group of units of $\mathbb{M}_{A\times B}(\mathbb{Z})$, i.e. the matrices with determinant ± 1. Implied constants in $\ll$ and O-notation, as well as implicit constants in qualifiers like "sufficiently large", may always depend on any parameters declared as "fixed"; in particular, they will always depend on the polynomials $\{W_{i,v}\}_{\substack{1\le i\le K\\ 1\le v\le k}}$. Other dependence will be noted explicitly (for example, with parentheses or subscripts); notably, we shall use $C(F_1, \dots, F_K)$, $C'(F_1, \dots, F_K)$ and so on, to denote constants depending on the fixed polynomials $F_1, \dots, F_K$. We write $\log_k$ for the k-th iterate of the natural logarithm.

3. Technical preparation: The number of $n \le x$ for which $\gcd(f(n), q) = 1$

In this section, we shall provide a rough estimate on the count of $n \le x$ for which $f(n) = \prod_{i=1}^{K} f_i(n)$ is coprime to the modulus q, uniformly in $q \le (\log x)^{K_0}$. We aim to show the following estimate, which generalizes Proposition 2.1 in [37]. In the rest of the paper, we abbreviate $\alpha_v(q)$ to α_v for each $v \in [V]$.

Proposition 3.1. *For all sufficiently large x and uniformly in k-admissible $q \le (\log x)^{K_0}$,*

$$\sum_{\substack{n \le x \\ (f(n),q)=1}} 1 \; = \sum_{\substack{n \le x \\ \text{each } (f_i(n),q)=1}} 1 = \frac{x^{1/k}}{(\log x)^{1-\alpha_k}} \exp(O((\log_2(3q))^{O(1)})). \tag{3.1}$$

3.1. **Proof of the lower bound.** Any $m \le x^{1/k}$ satisfying $\gcd(f(m^k), q) = 1$ is certainly counted in the left hand side of (3.1). To estimate the number of such m, we apply Proposition [37, Proposition 2.1], with $f(n^k)$ and $x^{1/k}$ playing the roles of "$f(n)$" and "x" in the quoted proposition. This shows that the sum in (3.1) is bounded below by the right hand side.

3.2. **Proof of the upper bound.** We start by giving an upper bound on the count of r-full smooth numbers; here we consider any $n \in \mathbb{N}$ to be 1-full (and we consider 1 as being r-full for any $r \ge 1$). The case $r = 1$ of the lemma below is a known estimate on smooth numbers.

Lemma 3.2. *Fix $r \in \mathbb{N}$. We have as $X, Z \to \infty$,*

$$\#\{n \le X : P(n) \le Z,\ n \text{ is } r\text{-full}\} \ll X^{1/r}(\log Z) \exp\left(-\frac{U}{r}\log U + O(U \log_2(3U))\right),$$

uniformly for $(\log X)^{\max\{3,2r\}} \le Z \le X^{1/2}$, where $U := \log X/\log Z$.

Proof of Lemma 3.2. The lemma is a classical application of Rankin's trick. We start by letting $\eta \le \min\{1/3, 1/2r\}$ be a positive parameter to be chosen later, and observe that

$$\sum_{\substack{n \le X:\ P(n) \le Z \\ n \text{ is } r\text{-full}}} 1 \le \sum_{\substack{n \text{ is } r\text{-full} \\ P(n) \le Z}} \left(\frac{X}{n}\right)^{(1-\eta)/r} \ll X^{(1-\eta)/r} \exp\left(\sum_{p \le Z} \frac{1}{p^{1-\eta}}\right), \tag{3.2}$$

where we have used the Euler product and noted that $\sum_p \sum_{v \ge r+1} p^{-v(1-\eta)/r} \ll \sum_p p^{-(1-\eta)(1+1/r)} \ll_r 1$ since $(1-\eta)(1+1/r) \ge (1+1/r)(1 - \min\{1/3, 1/2r\}) > 1$.

Let $\eta := \frac{\log U}{\log Z} \le \min\left\{\frac{1}{3}, \frac{1}{2r}\right\}$. We write $\sum_{p \le Z} 1/p^{1-\eta} = \log_2 Z + \sum_{p \le Z}(\exp(\eta \log p) - 1)/p + O(1)$. Since $\eta \log p \le \log 2 \ll 1$ for all $p \le 2^{1/\eta}$, we find that the contribution of $p \le 2^{1/\eta}$ to the last sum above is $\ll \eta \sum_{p \le 2^{1/\eta}} \log p/p \ll 1$, while that of $p \in (2^{1/\eta}, Z]$ is at most $(\exp(\eta \log Z) - 1) \sum_{2^{1/\eta} < p \le Z} 1/p \le U(\log_2 U + O(1))$. Collecting estimates, we obtain $\sum_{p \le Z} 1/p^{1-\eta} = \log_2 Z + O(U \log_2(3U))$, which from (3.2) completes the proof of the lemma. ∎

Since $\alpha_v(\ell) > 0$ for all $\ell > D_v + 1$, it follows that for each $1 \le v < k$, the set $S_v := \{\ell \text{ prime} : \alpha_v(\ell) = 0\}$ consists only of primes of size $O(1)$, with the implied constant depending only on the polynomials $W_{1,v}, \ldots, W_{K,v}$. It is easy to show that if q is k-admissible, then the k-free