

Theorem 5.6. *There exists a constant $\delta_0 := \delta_0(\lambda) > 0$ such that, uniformly in moduli $q \leq (\log x)^{K_0}$ lying in $\mathcal{Q}(k; f_1, \dots, f_K)$ and having sufficiently large radical, we have*

$$\sum_{n \leq x} \chi_1(f_1(n)) \cdots \chi_K(f_K(n)) \mathbb{1}_{(f(n), q)=1} \ll \frac{x^{1/k}}{(\log x)^{1-(1-\delta_0)\alpha_k(Q)}}$$

for all tuples of characters $(\chi_1, \dots, \chi_K) \neq (\chi_{0, Q_0}, \dots, \chi_{0, Q_0}) \bmod Q_0$.

Let $\mathcal{C}_k(Q_0)$ denote the set of tuples of characters $(\chi_1, \dots, \chi_K) \bmod Q_0$, not all trivial, such that $\prod_{i=1}^K \chi_i(W_{i,k}(u))$ is constant on the set $R_k(Q_0) = \{u \in U_{Q_0} : W_k(u) \in U_{Q_0}\}$. To prove Theorem 5.6, we separately consider the two cases when a tuple of characters mod Q_0 lies in $\mathcal{C}_k(Q_0)$ or not.

6. PROOF OF THEOREM 5.6 FOR NONTRIVIAL TUPLES OF CHARACTERS NOT IN $\mathcal{C}_k(Q_0)$

For any integer $d \geq 1$ and any nontrivial tuple (ψ_1, \dots, ψ_K) of characters mod d not lying in $\mathcal{C}_k(d)$, we have $|\sum_{u \bmod d} \chi_{0,d}(u) \psi_1(W_{1,k}(u)) \cdots \psi_K(W_{K,k}(u))| < \alpha_k(d) \varphi(d)$. With λ as in Proposition 4.3, we define the constant $\delta_1 := \delta_1(W_{1,k}, \dots, W_{K,k}; B_0) \in (0, 1)$ to be

$$\max_{\substack{d \leq \lambda \\ \alpha_k(d) \neq 0}} \max_{\substack{(\psi_1, \dots, \psi_K) \neq (\chi_{0,d}, \dots, \chi_{0,d}) \bmod d \\ (\psi_1, \dots, \psi_K) \notin \mathcal{C}_k(d)}} \frac{1}{\alpha_k(d) \varphi(d)} \left| \sum_{u \bmod d} \chi_{0,d}(u) \psi_1(W_{1,k}(u)) \cdots \psi_K(W_{K,k}(u)) \right|.$$

Then since $Q_0 \leq \lambda$, we have for any nontrivial tuple $(\chi_1, \dots, \chi_K) \notin \mathcal{C}_k(Q_0)$,

$$(6.1) \quad \left| \sum_{u \bmod Q_0} \chi_{0, Q_0}(u) \chi_1(W_{1,k}(u)) \cdots \chi_K(W_{K,k}(u)) \right| \leq \delta_1 \alpha_k(Q_0) \varphi(Q_0).$$

We set $\delta := (1 - \delta_1)/2$ and $Y := \exp((\log x)^{\delta/3})$. To establish Theorem 5.6 for all $(\chi_1, \dots, \chi_K) \notin \mathcal{C}_k(Q_0)$, it suffices to show that

$$(6.2) \quad \sum_{\substack{n \leq x \\ p > Y \Rightarrow p^{k+1} \nmid n}} \chi_1(f_1(n)) \cdots \chi_K(f_K(n)) \mathbb{1}_{(f(n), q)=1} \ll \frac{x^{1/k}}{(\log x)^{1-(\delta_1+\delta)\alpha_k}},$$

since by the arguments before (3.3), the contribution of n 's not counted above is negligible. Writing any n counted in (6.2) uniquely as BMA^k (as in (3.4)), we see that the sum equals

$$(6.3) \quad \sum_{\substack{B \leq x \\ P(B) \leq Y \\ B \text{ is } k\text{-free}}} \mathbb{1}_{(f(B), q)=1} \left(\prod_{i=1}^K \chi_i(f_i(B)) \right) \sum_{\substack{M \leq x/B \\ M \text{ is } k\text{-full} \\ P(M) \leq Y}} \mathbb{1}_{(f(M), q)=1} \left(\prod_{i=1}^K \chi_i(f_i(M)) \right) \\ \sum_{A \leq (x/BM)^{1/k}} \mathbb{1}_{P^-(A) > Y} \mathbb{1}_{(f(A^k), q)=1} \mu(A)^2 \prod_{i=1}^K \chi_i(f_i(A^k))$$

Moreover, the arguments leading to the bound for Σ_2 in section 3 show that the tuples (B, M, A) having $M > x^{1/2}$ give negligible contribution to the above sum. It thus remains to

consider the contribution of tuples (B, M, A) with $M \leq x^{1/2}$. To deal with such tuples, we will establish the following general upper bound uniformly for $X \geq \exp((\log Y)^2)$:

$$(6.4) \quad \sum_{A \leq X} \mathbb{1}_{P^-(A) > Y} \mathbb{1}_{(f(A^k), q)=1} \mu(A)^2 \prod_{i=1}^K \chi_i(f_i(A^k)) \ll \frac{X}{(\log X)^{1-\alpha_k(\delta_1+\delta/2)}}.$$

We apply a quantitative version of Halász's Theorem [48, Corollary III.4.12] on the multiplicative function $F(A) := \mathbb{1}_{P^-(A) > Y} \mathbb{1}_{(f(A^k), q)=1} \mu(A)^2 \prod_{i=1}^K \chi_i(f_i(A^k))$, taking $T := \log X$. This requires us to put, for each $t \in [-T, T]$, a lower bound on the sum below (which is the square of a certain "pretentious distance"):

$$(6.5) \quad \begin{aligned} \mathcal{D}(X; t) &:= \sum_{p \leq X} \frac{1}{p} \left(1 - \operatorname{Re} \left(\mathbb{1}_{p > Y} \mathbb{1}_{(f(p^k), q)=1} \mu(p)^2 p^{-it} \prod_{i=1}^K \chi_i(f_i(p^k)) \right) \right) \\ &= (1 - \alpha_k) \log_2 X + \alpha_k \log_2 Y + \sum_{\substack{Y < p \leq X \\ (W_k(p), q)=1}} \frac{1}{p} \left(1 - \operatorname{Re} \left(p^{-it} \prod_{i=1}^K \chi_i(W_{i,k}(p)) \right) \right) \\ &\quad + O((\log_2(3q))^{O(1)}); \end{aligned}$$

here the second line uses Lemma 3.4. To get this lower bound, we proceed analogously to the proof of [36, Lemma 3.3]. The key idea is to split the range of the above sum into blocks of small multiplicative width, so that the complex number p^{-it} is essentially constant for all p lying in a given block. More precisely, we cover the interval $(Y, X]$ with finitely many disjoint intervals $\mathcal{I} := (\eta, \eta(1 + 1/\log^2 X)]$ for certain choices of $\eta \in (Y, X]$, choosing the smallest η to be Y and allowing the rightmost endpoint of such an interval to jut out slightly past X but no more than $X(1 + 1/\log^2 X)$. Then the last sum in (6.5) equals

$$(6.6) \quad \sum_{\mathcal{I}} \sum_{\substack{p \in \mathcal{I} \\ (W_k(p), q)=1}} \frac{1}{p} \left(1 - \operatorname{Re} \left(p^{-it} \prod_{i=1}^K \chi_i(W_{i,k}(p)) \right) \right) + O\left(\frac{1}{\log^3 X}\right)$$

Consider any \mathcal{I} occurring in the sum above. For each $p \in \mathcal{I}$, we have

$$|p^{-it} - \eta^{-it}| \leq \left| \int_{t \log \eta}^{t \log p} \exp(-i\varrho) d\varrho \right| \leq |t \log p - t \log \eta| \leq \frac{|t|}{\log^2 X} \leq \frac{1}{\log X}.$$

This shows that each inner sum in (6.6) is equal to

$$(6.7) \quad \sum_{\substack{u \in U_q \\ (W_k(u), q)=1}} \left(1 - \operatorname{Re} \left(\eta^{-it} \prod_{i=1}^K \chi_i(W_{i,k}(u)) \right) \right) \sum_{\substack{p \in \mathcal{I} \\ p \equiv u \pmod{q}}} \frac{1}{p} + O\left(\frac{1}{\log X} \sum_{p \in \mathcal{I}} \frac{1}{p}\right)$$

Note that $p = (1 + o(1))\eta$ for all $p \in \mathcal{I}$. (Here and in what follows, the asymptotic notation refers to the behavior as $x \rightarrow \infty$, and is uniform in the choice of \mathcal{I} .) For parameters Z, W depending on X , we write $Z \gtrsim W$ to mean $Z \geq (1 + o(1))W$. By the Siegel Walfisz Theorem,

$$\sum_{\substack{p \in \mathcal{I} \\ p \equiv u \pmod{q}}} \frac{1}{p} \gtrsim \frac{1}{\eta} \sum_{\substack{p \in \mathcal{I} \\ p \equiv u \pmod{q}}} 1 \gtrsim \frac{1}{\varphi(q)} \cdot \frac{1}{\eta} \sum_{p \in \mathcal{I}} 1 \gtrsim \frac{1}{\varphi(q)} \sum_{p \in \mathcal{I}} \frac{1}{p}.$$

Hence the main term in (6.7) is

$$\gtrsim \frac{1}{\varphi(q)} \sum_{p \in \mathcal{I}} \frac{1}{p} \sum_{\substack{u \in U_q \\ (W_k(u), q)=1}} \left(1 - \operatorname{Re} \left(\eta^{-it} \prod_{i=1}^K \chi_i(W_{i,k}(u)) \right) \right) \gtrsim (\alpha_k - \alpha_k \delta_1) \left(\sum_{p \in \mathcal{I}} \frac{1}{p} \right),$$

where in the last step, we have used (4.14) and (6.1) to see that

$$\frac{1}{\varphi(q)} \left| \sum_{\substack{u \in U_q \\ (W_k(u), q)=1}} \prod_{i=1}^K \chi_i(W_{i,k}(u)) \right| = \frac{\alpha_k(q)}{\alpha_k(Q_0) \varphi(Q_0)} \left| \sum_{r \bmod Q_0} \chi_{0, Q_0}(r) \prod_{i=1}^K \chi_i(W_{i,k}(r)) \right| \leq \alpha_k \delta_1.$$

Inserting the bound obtained in the previous display into (6.7), we find that each inner sum in (6.6) is $\gtrsim \alpha_k(1 - \delta_1) \sum_{p \in \mathcal{I}} 1/p + O((\log X)^{-1} \sum_{p \in \mathcal{I}} 1/p)$. The O -term when summed over all \mathcal{I} is $\ll (\log X)^{-1} \sum_{p \leq 2X} p^{-1} \ll \log_2 X / \log X$. Thus, the main term in (6.6) is at least $\alpha_k(1 - \delta_1 - \frac{\delta}{2})(\log_2 X - \log_2 Y)$. Inserting this into (6.5) yields

$$\mathcal{D}(X; t) \geq \left(1 - \alpha_k \left(\delta_1 + \frac{\delta}{2} \right) \right) \log_2 X + \alpha_k \left(\delta_1 + \frac{\delta}{2} \right) \log_2 Y + O((\log_2(3q))^{O(1)}),$$

uniformly for $t \in [-T, T]$. As such, Corollary [48, III.4.12] establishes the claimed bound (6.4).

Now for each $M \leq x^{1/2}$, we have $(x/BM)^{1/k} \gg x^{1/2k}$. Applying (6.4) to each of the innermost sums in (6.3), we see that the total contribution of all tuples (B, M, A) with $M \leq x^{1/2}$ is

$$\ll \sum_{B \ll 1} \sum_{\substack{M \leq x^{1/2}: M \text{ is } k\text{-full} \\ P(M) \leq Y, (f(M), q)=1}} \frac{(x/BM)^{1/k}}{(\log x)^{1-\alpha_k(\delta_1+\delta/2)}} \ll \frac{x^{1/k}}{(\log x)^{1-\alpha_k(\delta_1+\delta)}},$$

where we have used (3.5) (with Y in place of y) and Lemma 3.4. This proves (6.2), and hence also Theorem 5.6 for all nontrivial tuples of characters $(\chi_1, \dots, \chi_K) \bmod Q_0$ not in $\mathcal{C}_k(Q_0)$. \square

7. PROOF OF THEOREM 5.6 FOR TUPLES OF CHARACTERS IN $\mathcal{C}_k(Q_0)$

It suffices to consider the case when x is an integer, and we will do so in the rest of the section. Our argument consists of suitably modifying the Landau–Selberg–Delange method for mean values of multiplicative functions (see for instance [48, Chapter II.5]), and to study the behavior of a product of L -functions raised to complex powers by accounting for the presence of Siegel zeros modulo q . This is partly inspired from work of Scourfield [44] and will also need some results from her paper. We will denote complex numbers in the standard notation $s = \sigma + it$.

⁶ To begin with, we consider the Dirichlet series

$$F_\chi(s) := \sum_{n \geq 1} \frac{\mathbb{1}_{(f(n), q)=1}}{n^s} \prod_{i=1}^K \chi_i(f_i(n)) = \sum_{n \geq 1} \frac{\mathbb{1}_{(f(n), Q)=1}}{n^s} \prod_{i=1}^K \chi_i(f_i(n))$$

which is absolutely convergent in the half-plane $\sigma > 1$. Let $c_{\widehat{\chi}}$ denote the constant value of $\prod_{i=1}^K \chi_i(W_{i,k}(u))$ on the set $R_k(Q_0) = W_k^{-1}(U_{Q_0}) \cap U_{Q_0}$. In the rest of the section, we assume that the complex plane has been cut along the line $\sigma \leq 1/k$ if $\alpha_k(Q)$ and $c_{\widehat{\chi}}$ are not both 1, while

⁶The parameters σ and σ_k (to be defined later) in this section have nothing to do with the divisor functions $\sigma_r(n) = \sum_{d|n} d^r$ mentioned in the introduction. We are not working with the divisor functions in this section.