with the respective values of R.

Completing the proof of Theorem 2.3(a). Let R be any one of the two values defined in the statement (so we will only be assuming that $R \ge k(Kk+K-k)+1$ until stated otherwise). If $\omega^*(n)=0$, then $k\omega_{\|}(n) \ge R \ge k(Kk+K-k)+1$, so that $\omega_{\|}(n) \ge Kk+K-k+1 \ge 2K+1$, with the last inequality being true since *both* $K, k \ge 2$. As such, any n with $\omega^*(n)=0$ counted in (11.1)(i) is automatically counted in the first sum in (11.4). Likewise, since *both* $K, k \ge 2$, the condition $\omega_{\|}(n) = 2K$ forces $\sum_{p>q:\ p^{k+1} \mid n} v_p(n) \ge R - k\omega_{\|}(n) \ge k(Kk+K-k)+1-2Kk = k((K-1)(k-1)-1)+1 \ge 1$. Thus $\omega^*(n) \ge 1$, showing that any n with $\omega_{\|}(n)=2K$ contributing to (11.1)(i) is counted in the second sum in (11.4). Furthermore, by the third bound in (9.4), the contribution of all n having $\omega^*(n) \ge Kk$ to the left hand side of (11.1)(i) is absorbed in the right hand side. It thus suffices to show that for any $r \in [2K-1]$ and $s \in [Kk-1]$, the contribution $\Sigma_{r,s}$ of all n with $\omega_{\|}(n) = r$ and $\omega^*(n) = s$ to the left hand side of (11.1)(i) is absorbed in the right hand side.

Recall that any n counted in $\Sigma_{r,s}$ is of the form $mp_1^{c_1}\cdots p_s^{c_s}P_1^k\cdots P_r^k$ for some distinct primes $p_1,\dots,p_s$, $P_1,\dots,P_r$ and integers $m, c_1,\dots,c_s$, which satisfy the conditions (i)–(v) in the proof of Theorem 2.2(a), but with either of the current values of R. Once again, the integers $\tau_1,\dots,\tau_s$ defined by $\tau_j := \min\{c_j, R-kr\}$ satisfy $\tau_j \in [k+1, R-kr]$, $\tau_j \le c_j$ and $\tau_1+\cdots+\tau_s \ge R-kr$. (Here to have $R-kr \ge k+1$, it is important that $r \le 2K-1$ and $K,k \ge 2$.) As such,

$$\Sigma_{r,s} \le \sum_{\substack{\tau_1,\dots,\tau_s \in [k+1, R-kr] \\ \tau_1+\cdots+\tau_s \ge R-kr}} \mathcal{N}_{r,s}(\tau_1,\dots,\tau_s), \tag{11.5}$$

where $\mathcal{N}_{r,s}(\tau_1,\dots,\tau_s)$ denotes the contribution of all n counted in the left hand side of (11.1)(i) which can be written in the form $mp_1^{c_1}\cdots p_s^{c_s}\ P_1^k\cdots P_r^k$ for some distinct primes $p_1,\dots,p_s, P_1, \cdots, P_r$ and integers $m, c_1,\dots,c_s$ satisfying $c_1 \ge \tau_1,\dots,c_s \ge \tau_s$ and the conditions (i)–(v) in the proof of Theorem 2.2(a) (but with either of the current values of R). We will show that for each tuple $(\tau_1,\dots,\tau_s)$ occurring in (11.5), we have

$$\mathcal{N}_{r,s}(\tau_1,\dots,\tau_s) \ \ll\ \frac{x^{1/k}(\log_2 x)^{O(1)}}{q^K \log x} \exp\big(O(\sqrt{\log q})\big). \tag{11.6}$$

Now the bound (9.11) continues to hold, so we have

$$\mathcal{N}_{r,s}(\tau_1,\dots,\tau_s) \ll \frac{1}{q^{(\tau_1+\cdots+\tau_s)/k-s}}\ \frac{V'_{r,K}}{\varphi(q)^r} \cdot \frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x} \tag{11.7}$$

with the current values of $r, s, \tau_1,\dots,\tau_s$ and with $V'_{r,K}$ defined in the usual manner. As such, applying (5.33) with $L := 1$ and $(G_{i,1})_{i=1}^K := (W_{i,k})_{i=1}^K$, we find that

$$\mathcal{N}_{r,s}(\tau_1,\dots,\tau_s) \ll \frac{\exp\big(O(\omega(q))\big)}{q^{(\tau_1+\cdots+\tau_s)/k-s+r/2}} \cdot \frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x} \ll \frac{\exp\big(O(\omega(q))\big)}{q^{\max\{s/k+r/2,\ R/k-r/2-s\}}} \cdot \frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x}.$$

Now $\max\{s/k+r/2, R/k-r/2-s\} > K$ whenever either (A_1) $k \ge 3$, $r \ge 3$, *or* (A_2) $k=2$, $r \ge 4$ holds: indeed, if $s/k+r/2 \le K$, then $s \le k(K-r/2)$, so that (as $R \ge k(Kk+K-k)+1$) we have $R/k - r/2 - s \ge K + (k-1)(r/2-1) - 1 + 1/k$. This last quantity strictly exceeds K precisely under (A_1) or (A_2), establishing (11.6) under one of these two conditions. It thus only remains to tackle the cases $r \in \{1,2\}$, and the case $k=2, r=3$.

The case $r = 1$ is dealt with easily by inserting into (11.7) the trivial bound $V'_{r,K} = V'_{1,K} \ll D_{\min}^{\omega(q)}$. It is to deal with the case $r = 2$ and the case $k = 2, r = 3$ that we need the dichotomy in the statement of the theorem.

When at least one of the $\{W_{i,k}\}_{1\le i\le K}$ *is not squarefull ...*
First assume that one of the polynomials $\{W_{i,k}\}_{1\le i\le K} \subset \mathbb{Z}[T]$ is not squarefull, say $W_{1,k}$ (this is the first time in the argument for $K \ge 2$ that we are making this assumption). If $r = 2$, then Proposition 10.1(a) yields $\#\mathcal{V}_{2,K}(q;(w_i)_{i=1}^K)/\varphi(q)^2 \le \#\mathcal{V}_{2,1}(q;w_1)/\varphi(q)^2 \ll \varphi(q)^{-1}\exp(O(\sqrt{\log q}))$, uniformly for $(w_i)_{i=1}^K \in U_q^K$. Inserting this into (11.7), we find that

$$\mathcal{N}_{2,s}(\tau_1,\dots,\tau_s) \ll \frac{1}{q^{\max\{s/k+1, R/k-1-s\}}}\cdot\frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x}\exp\big(O(\sqrt{\log q})\big). \tag{11.8}$$

Since $\max\{s/k+1, R/k-1-s\}$ is always at least K, this establishes (11.6) in the case when $r = 2$ and one of $\{W_{i,k}\}_{1\le i\le K}$ is not squarefull.

For $k = 2, r = 3$, the multiplicative independence of $\{W_{1,k}, W_{2,k}\}$ allows us to use Proposition 10.1(b) to get $\#\mathcal{V}^{(k)}_{3,K}(q;(w_i)_{i=1}^K)/\varphi(q)^3 \ll \exp\big(O(\omega(q))\big)/\varphi(q)^2$ uniformly for $(w_i)_{i=1}^K$. By (11.7), $\mathcal{N}_{3,s}(\tau_1,\dots,\tau_s) \ll \frac{\exp\left(O(\omega(q))\right)}{q^{\max\{s/2+2,\ R/2-1-s\}}}\cdot\frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x}$, and it is easily checked that $\max\{s/2+2,\ R/2-1-s\} > K$. This shows (11.6) when one of $\{W_{i,k}\}_{1\le i\le K}$ is not squarefull.

When all of the $\{W_{i,k}\}_{1\le i\le K}$ *may be squarefull ...*
In general (i.e., without any nonsquarefullness assumption on $\{W_{i,k}\}_{1\le i\le K}$), we can still use the second assertion of Corollary 5.5 for $r = 2$ and 3, in place of their improved versions in Proposition 10.1 (both of these values of r are at most $2K$ as $K \ge 2$). Coming to the case $r = 2$ (and $k \ge 2$), we invoke (5.33) to obtain $\mathcal{N}_{2,s}(\tau_1,\dots,\tau_s) \ll \frac{\exp\left(O(\omega(q)\right)}{q^{\max\{s/k+1,\ R/k-1-s\}}}\cdot\frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x}$, and we need the exponent of q in the denominator to *strictly* exceed K, in order to get a power saving of q^{-K}. (Compare this with (11.8) where owing to the absence of the factor $\exp\big(O(\omega(q))\big)$ we only needed the same exponent to be at least K.) This is where we use, for the first time (under the case $K \ge 2$) that $R = k(Kk+K-k+1)+1$. Indeed, this value of R guarantees that $\max\{s/k+1,\ R/k-1-s\}$ always exceeds K, establishing (11.6) for $r = 2$.

Finally, we turn to the case $k = 2$, $r = 3$. Here we only need that $R = k(Kk+K-k+1)+1 = 6K-1 \ge 6K-2$. Inserting the bound coming from (5.33) into (11.7), we get $\mathcal{N}_{3,s}(\tau_1,\dots,\tau_s) \ll \frac{\exp\left(O(\omega(q))\right)}{q^{\max\{s/2+3/2, 3K-5/2-s\}}}\cdot\frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x}$, and it is again easily seen that $\max\{s/2+3/2, 3K-5/2-s\} > K$. This completes the proof of (11.6), summing which over all the $O(1)$ tuples $(\tau_1,\dots,\tau_s)$ occurring in (11.5) establishes Theorem 2.3(a).

Completing the proof of Theorem 2.3(b). By arguments analogous to those given for subpart (a), it suffices to show that for any $r \in [2K-1]$ and $s \in [Kk-1]$, the contribution $\widetilde{\Sigma}_{r,s}$ of all n with $\omega_\|(n) = r$ and $\omega_k(n) = s$ to the left hand side of (11.1)(ii) satisfies

$$\widetilde{\Sigma}_{r,s} \ll \frac{x^{1/k}}{\varphi(q)^K(\log x)^{1-2\alpha_k/3}}. \tag{11.9}$$

Any n counted in $\widetilde{\Sigma}_{r,s}$ has n_k of the form $m'p_1^{c_1}\cdots p_s^{c_s}P_1\cdots P_r$ for some distinct primes $p_1,\dots,p_s$, $P_1,\dots,P_r$ and integers $m', c_1,\dots,c_s$, which satisfy conditions (i)–(v) in the proof of Theorem

2.2(b), but with "$KD+1-r$" replaced by "$2K+1-r$". Hence again n is of the form $mp_1^{c_1k}\cdots p_s^{c_sk}P_1^k\cdots P_r^k$, where $p_1,\dots,p_s,P_1,\dots,P_r$ are as above, $P_{Jk}(m)\le y$, and $f_i(n)=f_i(m)f_i(p_1^{c_1k})\cdots f_i(p_s^{c_sk})\prod_{j=1}^r W_{i,k}(P_j)$ for each $i\in[K]$. Defining $\tau_j:=\min\{c_j,2K+1-r\}$ for all $j\in[s]$, we see that $\tau_j\ge 2$ (since $r\le 2K-1$) and that $\tau_1+\cdots+\tau_s\ge 2K+1-r$. Thus

$$\widetilde{\Sigma}_{r,s}\le \sum_{\substack{\tau_1,\dots,\tau_s\in[2,2K+1-r]\\ \tau_1+\cdots+\tau_s\ge 2K+1-r}} \widetilde{\mathcal{N}}_{r,s}(\tau_1,\dots,\tau_s), \tag{11.10}$$

where (exactly as in the proof of Theorem 2.2(b)), $\widetilde{\mathcal{N}}_{r,s}(\tau_1,\dots,\tau_s)$ denotes the number of n counted in (11.1)(ii) that can be written in the form $mp_1^{c_1k}\cdots p_s^{c_sk}P_1^k\cdots P_r^k$ with $m,p_1,\dots,p_s,c_1,\dots,c_s,P_1,\dots,P_r$ being pairwise coprime and satisfying $P_1>z$; $q<P_r<\cdots<P_1$; $p_1,\dots,p_s>q$; $P_{Jk}(m)\le y$; $c_1\ge\tau_1,\dots,c_s\ge\tau_s$. Combining (9.15), (5.33), and the fact that $\max\{s+r/2,\ 2K+1-(s+r/2)\}>K$, we get $\widetilde{\mathcal{N}}_{r,s}(\tau_1,\dots,\tau_s)\ll x^{1/k}/\varphi(q)^K(\log x)^{1-2\alpha_k/3}$, for each $\tau_1,\dots,\tau_s$ counted in (11.10). This yields (11.9), concluding the proof of Theorem 2.3. ☐

11.1. **Optimality in the conditions** $P_{k(Kk+K-k)+1}(n)>q$ **and** $P_{2K+1}(n_k)>q$**.** We will now show that the smaller value of R given in Theorem 2.3(a) is optimal and that the value "$2K+1$" in (b) is nearly optimal. We retain the setting in subsection § 8.1 we had used to show optimality in Theorem 2.1(ii). To recall: fix an arbitrary $k\in\mathbb{N}$ and $d>1$, and define $W_{i,k}(T):=\prod_{j=1}^d(T-2j)+2(2i-1)$, so that $\prod_{i=1}^K W_{i,k}$ is separable (over $\mathbb{Q}$). Let $\widetilde{C}_0>4KD$ be any constant (depending only on $\{W_{i,k}\}_{1\le i\le K}$) exceeding the size of the (nonzero) discriminant of $\prod_{i=1}^K W_{i,k}$, and such that any $\widetilde{C}_0$-rough k-admissible integer lies in $\mathcal{Q}(k;f_1,\cdots,f_K)$. Fix a prime $\ell_0>C_0$ and nonconstant polynomials $\{W_{i,v}\}_{\substack{1\le i\le K\\1\le v<k}}\subset\mathbb{Z}[T]$ with all coefficients divisible by ℓ_0. Let $q\le(\log x)^{K_0}$ be any squarefree integer having $P^-(q)=\ell_0$, so that as before $q\in\mathcal{Q}(k;f_1,\cdots,f_K)$. Recall also that $(2(2i-1))_{i=1}^K\in U_q^K$, that any prime P satisfying $\prod_{j=1}^d(P-2j)\equiv 0\pmod q$ also satisfies $f_i(P^k)\equiv 2(2i-1)\pmod q$, and that the congruence $\prod_{j=1}^d(v-2j)\equiv 0\pmod q$ has exactly $d^{\omega(q)}$ distinct solutions $v\in U_q$.

Optimality in Theorem 2.3(a). First, we show that the condition "$R=k(Kk+K-k)+1$" in Theorem 2.3(a) cannot be weakened to "$R=k(Kk+K-k)$". To this end, let $f_1,\dots,f_K\colon\mathbb{N}\to\mathbb{Z}$ be any multiplicative functions such that $f_i(p^v):=W_{i,v}(p)$ and $f_i(p^{k+1}):=1$ for all primes p, all $i\in[K]$ and $v\in[k]$. Consider n of the form $(p_1\cdots p_{k(K-1)})^{k+1}P^k\le x$ where $P,p_1,\dots,p_{k(K-1)}$ are primes satisfying the conditions $P:=P(n)>x^{1/3k}$, $q<p_{k(K-1)}<\cdots<p_1<x^{1/4Kk^2}$, and $\prod_{1\le j\le d}(P-2j)\equiv 0\pmod q$. Then $P_{k(Kk+K-k)}(n)=p_{k(K-1)}>q$ and $f_i(n)=f_i(P^k)\prod_{j=1}^{k(K-1)}f_i(p_j^{k+1})\equiv 2(2i-1)\pmod q$ for each $i\in[K]$. Given $p_1,\dots,p_{k(K-1)}$, the number of primes P satisfying $x^{1/3k}<P\le x^{1/k}/(p_1\cdots p_{k(K-1)})^{1+1/k}$ is, by the Siegel–Walfisz Theorem, $\gg d^{\omega(q)}x^{1/k}/\varphi(q)(p_1\cdots p_{k(K-1)})^{1+1/k}\log x$, where we have noted that $(p_1\cdots p_{k(K-1)})^{1+1/k}\le x^{(K-1)(k+1)/4Kk^2}\le x^{1/2k}$. Dividing by $k!$ allows us to replace the condition $p_{k(K-1)}<\cdots<p_1$ by a distinctness condition, giving us

$$\sum_{\substack{n\le x\\ P_{k(Kk+K-k)}(n)>q\\ (\forall i)f_i(n)\equiv 2(2i-1)\ (\mathrm{mod}\ q)}} 1\ \gg\ \frac{d^{\omega(q)}x^{1/k}}{\varphi(q)\log x}\left(\mathcal{T}_1-\mathcal{T}_2\right), \tag{11.11}$$