all the remaining tuples $(\chi_1, \dots, \chi_K)$ with $\mathrm{lcm}[\mathfrak{f}(\chi_1), \dots, \mathfrak{f}(\chi_K)] = \ell$, we may invoke Proposition 5.1 to obtain, **for all** $j \in [N]$,

$$|Z_{\ell;\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j})| = \left| \sum_{v \bmod \ell} \psi_1 \left(v^{\varphi(\ell)} \prod_{i=1}^{K} F_{i,j}(v)^{A_i} \right) \right| \le \left(\sum_{i=1}^{K} \deg F_{i,j} \right) \ell^{1/2} \le D_0 \ell^{1/2}.$$

By (5.12), we deduce that for all but $O_L(1)$ many tuples $(\chi_1, \dots, \chi_K)$ of characters mod ℓ^e satisfying $\mathrm{lcm}[\mathfrak{f}(\chi_1), \dots, \mathfrak{f}(\chi_K)] = \ell$, we have

$$|Z_{\ell^e;\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j})| \le D_0 \ell^{e-1/2} \quad \text{for every } j \in [N], \tag{5.15}$$

and when $\gcd(\ell - 1, \beta(G_{1,r}, \dots, G_{K,r})) = 1$ for all $r \in [L]$, this inequality is true for all $(\chi_1, \dots, \chi_K)$ mod ℓ^e satisfying $\mathrm{lcm}[\mathfrak{f}(\chi_1), \dots, \mathfrak{f}(\chi_K)] = \ell$.

Case 2: Now assume that $e_0 \ge 2$. Consider an arbitrary $j \in [N]$ and let $(G_{i,j'})_{i=1}^K = (F_{i,j})_{i=1}^K$ for some $j' \in [L]$. Since $\ell > C_0 > C_1(G_{1,j'}, \dots, G_{K,j'})$ and $e_0 \ge 2$, Proposition 5.3(b) and (5.13) show that $\tau(\ell) \coloneqq \mathrm{ord}_\ell \left((T^{\varphi(\ell^{e_0})} \prod_{i=1}^K F_{i,j}(T)^{A_i})' \right) = 0$. Consequently, (5.14) and Proposition 5.2(i) yield $|Z_{\ell^{e_0};\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j})| \le \left(\sum_{\theta \in \mathcal{A}_\ell} \mu_\theta(\mathcal{C}_\ell) \right) \ell^{e_0(1-1/(M_\ell+1))}$, where $\mathcal{A}_\ell \subset \mathbb{F}_\ell$ denotes the set of ℓ-critical points of the polynomial $T^{\varphi(\ell^{e_0})} \prod_{i=1}^K F_{i,j}(T)^{A_i}$, $\mathcal{C}_\ell(T) \coloneqq (T^{\varphi(\ell^{e_0})} \prod_{i=1}^K F_{i,j}(T)^{A_i})'$ and $M_\ell \coloneqq \max_{\theta \in \mathcal{A}_\ell} \mu_\theta(\mathcal{C}_\ell)$. Moreover, by the last assertion in Propositon 5.3, any $\theta \in \mathcal{A}_\ell$ is a root of the polynomial $\widetilde{F}(T) \coloneqq \sum_{i=1}^K A_i F'_{i,j}(T) \prod_{\substack{1 \le r \le K \\ r \ne i}} F_{r,j}(T)$ (a nonzero element of $\mathbb{F}_\ell[T]$), and $\mu_\theta(\mathcal{C}_\ell) = \mu_\theta(\widetilde{F})$. As such, $M_\ell \le \sum_{\theta \in \mathcal{A}_\ell} \mu_\theta(\mathcal{C}_\ell) \le D_0 - 1$, yielding $|Z_{\ell^{e_0};\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j})| \le (D_0 - 1)\ell^{e_0(1-1/D_0)}$. Thus, by (5.12),

$$|Z_{\ell^e;\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j})| \le (D_0 - 1)\ell^{e - e_0/D_0} \text{ if } \ell^{e_0} \coloneqq \mathrm{lcm}[\mathfrak{f}(\chi_1), \dots, \mathfrak{f}(\chi_K)] \in \{\ell^2, \dots, \ell^e\}. \tag{5.16}$$

For any $e_0 \in \{1, \dots, e\}$ there are at most $\ell^{e_0 K}$ tuples $(\chi_1, \dots, \chi_K)$ of characters mod ℓ^e having $\mathrm{lcm}[\mathfrak{f}(\chi_1), \dots, \mathfrak{f}(\chi_K)] = \ell^{e_0}$. Combining (5.16) with the respective assertion in (5.15), we get

$$\sum_{(\chi_1,\dots,\chi_K) \ne (\chi_{0,\ell},\dots,\chi_{0,\ell}) \bmod \ell^e} \left| \prod_{j=1}^{N} Z_{\ell^e;\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j}) \right| \le D_0^N \ell^{eN} \sum_{1 \le e_0 \le e} \ell^{e_0(K - N/D_0)}, \tag{5.17}$$

for any prime power ℓ^e with $\ell > C_0$ satisfying $\alpha_N^*(\ell) \ne 0$ and $\gcd(\ell - 1, \beta(G_{1,r}, \dots, G_{K,r})) = 1$ for each $r \in [L]$. (In the last inequality above, we have used the fact that $D_0 \ge 2$.) Now for each $j \in [N]$, $\widetilde{\alpha}_j(\ell) \ge 1 - D_0/(\ell - 1) > 1 - D_0/(C_0 - 1) > 1/2$, so that $\alpha_N^*(\ell) \ge 1/2^N$. If $N \ge KD_0 + 1$, then $\ell^{K - N/D_0} \le \ell^{-1/D_0} \le C_0^{-1/D_0} \le 1/2$, and (5.17) yields

$$\frac{1}{\alpha_N^*(\ell)\varphi(\ell^e)^N} \sum_{(\chi_1,\dots,\chi_K) \ne (\chi_{0,\ell},\dots,\chi_{0,\ell}) \bmod \ell^e} \left| \prod_{j=1}^{N} Z_{\ell^e;\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j}) \right| \le \frac{2(4D_0)^N}{\ell^{N/D_0 - K}}. \tag{5.18}$$

Inserting this bound into (5.11) shows the assertion (5.8). On the other hand for *any* prime power ℓ^e with $\ell > C_0$, (5.15) and (5.16) show that for each fixed $N \ge 1$,

$$\sum_{(\chi_1,\dots,\chi_K) \ne (\chi_{0,\ell},\dots,\chi_{0,\ell}) \bmod \ell^e} \left| \prod_{j=1}^{N} Z_{\ell^e;\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j}) \right| \ll \varphi(\ell^e)^N + D_0^N \ell^{eN} \sum_{1 \le e_0 \le e} \ell^{e_0(K - N/D_0)}. \tag{5.19}$$

Thus for a fixed $N \geq KD_0 + 1$, a calculation analogous to (5.18) yields

$$\frac{1}{\varphi(\ell^e)^N} \sum_{(\chi_1,\dots,\chi_K)\neq(\chi_{0,\ell},\dots,\chi_{0,\ell}) \bmod \ell^e} \left| \prod_{j=1}^{N} Z_{\ell^e;\ \chi_1,\dots,\chi_K}(F_{1,j},\dots,F_{K,j}) \right| \ll 1.$$

On the other hand if $N \in \{1,\dots,KD_0\}$, the expression in (5.19) leads to

$$\frac{1}{\varphi(\ell^e)^N} \sum_{(\chi_1,\dots,\chi_K)\neq(\chi_{0,\ell},\dots,\chi_{0,\ell}) \bmod \ell^e} \left| \prod_{j=1}^{N} Z_{\ell^e;\ \chi_1,\dots,\chi_K}(F_{1,j},\dots,F_{K,j}) \right| \ll e^{\mathbb{1}_{N=KD_0}} \ell^{e(K-N/D_0)}.$$

Inserting the last two bounds displays into (5.10) yields (5.9).

Now for an arbitrary q, we let $\widetilde{q} \coloneqq \prod_{\substack{\ell^e \| q \\ \ell \leq C_0}} \ell^e$ denote the C_0-smooth part of q. By (5.7),

$$\#\widetilde{\mathcal{V}}_{N,K}\left(\widetilde{q}; (w_i)_{i=1}^K\right) = \frac{1}{\varphi(\widetilde{q})^K} \sum_{\chi_1,\dots,\chi_K \bmod \widetilde{q}} \overline{\chi}_1(w_1)\cdots\overline{\chi}_K(w_K) \prod_{j=1}^{N} Z_{\widetilde{q};\ \chi_1,\dots,\chi_K}(F_{1,j},\dots,F_{K,j}). \tag{5.20}$$

Given a constant $C > C_0$, we fix κ to be any integer constant exceeding $C \cdot (30 D_0 C_0^{C_0})^{2C_0}$, and let $Q_0 \coloneqq \prod_{\ell^e \| \widetilde{q}} \ell^{\min\{e,\kappa\}} = \prod_{\ell \leq C_0} \ell^{\min\{v_\ell(q),\kappa\}}$ denote the largest $(\kappa+1)$-free divisor of $\widetilde{q}$. Write the expression on the right hand side of (5.20) as $\mathcal{S}' + \mathcal{S}''$, where $\mathcal{S}'$ denotes the contribution of those tuples $(\chi_1,\dots,\chi_K) \bmod \widetilde{q}$ for which $\operatorname{lcm}[\mathfrak{f}(\chi_1),\dots,\mathfrak{f}(\chi_K)]$ is $(\kappa+1)$-free, or equivalently, those $(\chi_1,\dots,\chi_K)$ for which $\operatorname{lcm}[\mathfrak{f}(\chi_1),\dots,\mathfrak{f}(\chi_K)]$ divides Q_0.

For each tuple $(\chi_1,\dots,\chi_K)$ counted in $\mathcal{S}'$, there exists a unique tuple $(\psi_1,\dots,\psi_K)$ of characters mod Q_0 inducing $(\chi_1,\dots,\chi_K)$ mod $\widetilde{q}$, respectively. Noting that $\widetilde{\alpha}_j(\widetilde{q}) = \widetilde{\alpha}_j(Q_0)$, a straightforward calculation using (4.14) shows that

$$Z_{\widetilde{q};\ \chi_1,\dots,\chi_K}(F_{1,j},\dots,F_{K,j}) = \sum_{v \bmod \widetilde{q}} \chi_{0,\widetilde{q}}(v) \prod_{i=1}^{K} \chi_i(F_{i,j}(v)) = \frac{\varphi(\widetilde{q})}{\varphi(Q_0)} Z_{Q_0;\ \psi_1,\dots,\psi_K}(F_{1,j},\dots,F_{K,j})$$

for each $j \in [N]$. Consequently,

$$\mathcal{S}' = \frac{1}{\varphi(\widetilde{q})^K} \left(\frac{\varphi(\widetilde{q})}{\varphi(Q_0)}\right)^N \sum_{\psi_1,\dots,\psi_K \bmod Q_0} \overline{\psi}_1(w_1)\cdots\overline{\psi}_K(w_K) \prod_{j=1}^{N} Z_{Q_0;\ \psi_1,\dots,\psi_K}(F_{1,j},\dots,F_{K,j}),$$

so that invoking (5.7) with $Q \coloneqq Q_0$, we obtain

$$\frac{\mathcal{S}'}{\varphi(\widetilde{q})^N} = \left(\frac{\varphi(Q_0)}{\varphi(\widetilde{q})}\right)^K \frac{\#\widetilde{\mathcal{V}}_{N,K}\left(Q_0; (w_i)_{i=1}^K\right)}{\varphi(Q_0)^N}. \tag{5.21}$$

We now deal with the remaining sum

$$\mathcal{S}'' = \frac{1}{\varphi(\widetilde{q})^K} \sum_{\substack{\chi_1,\dots,\chi_K \bmod \widetilde{q} \\ \operatorname{lcm}[\mathfrak{f}(\chi_1),\dots,\mathfrak{f}(\chi_K)] \text{ is not } (\kappa+1)\text{-free}}} \overline{\chi}_1(w_1)\cdots\overline{\chi}_K(w_K) \prod_{j=1}^{N} Z_{\widetilde{q};\ \chi_1,\dots,\chi_K}(F_{1,j},\dots,F_{K,j}).$$

For each tuple $(\chi_1,\dots,\chi_K)$ of characters mod $\widetilde{q}$ considered in the sum above, we factor $\chi_i \eqqcolon \prod_{\ell^e \| \widetilde{q}} \chi_{i,\ell}$, where $\chi_{i,\ell}$ is a character mod ℓ^e. With $e_\ell \coloneqq v_\ell\left(\operatorname{lcm}[\mathfrak{f}(\chi_1),\dots,\mathfrak{f}(\chi_K)]\right)$, we observe that since $\mathfrak{f}(\chi_i) = \prod_{\ell^e \| q} \mathfrak{f}(\chi_{i,\ell})$ and each $\mathfrak{f}(\chi_{i,\ell})$ is a power of ℓ, we must have

$\mathrm{lcm}[\mathfrak{f}(\chi_{1,\ell}), \dots, \mathfrak{f}(\chi_{K,\ell})] = \ell^{e_\ell}$. Letting $\chi_{1,\ell}, \dots, \chi_{K,\ell}$ also denote the characters mod ℓ^{e_ℓ} inducing $\chi_{1,\ell}, \dots, \chi_{K,\ell}$ mod ℓ^e respectively (for each $\ell^e \parallel \widetilde{q}$), we see that at least one of $\chi_{1,\ell}, \dots, \chi_{K,\ell}$ must be primitive mod ℓ^{e_ℓ}. Furthermore for each $j \in [N]$, $Z_{\widetilde{q};\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j}) = \prod_{\ell^e \parallel \widetilde{q}} Z_{\ell^e;\ \chi_{1,\ell},\dots,\chi_{K,\ell}}(F_{1,j}, \dots, F_{K,j})$, so that

(5.22)
$$|Z_{\widetilde{q};\ \chi_1,\dots,\chi_K}(F_{1,j}, \dots, F_{K,j})| \le \left(\prod_{\substack{\ell^e \parallel \widetilde{q} \\ e_\ell \le \kappa}} \varphi(\ell^e) \right) \prod_{\substack{\ell^e \parallel \widetilde{q} \\ e_\ell \ge \kappa+1}} \left(\ell^{e-e_\ell} |Z_{\ell^{e_\ell};\ \chi_{1,\ell},\dots,\chi_{K,\ell}}(F_{1,j}, \dots, F_{K,j})| \right).$$

We will show that prime powers for all $\ell^e \parallel \widetilde{q}$ with $e_\ell \ge \kappa + 1$, we have

$$|Z_{\ell^{e_\ell};\ \chi_{1,\ell},\dots,\chi_{K,\ell}}(F_{1,j}, \dots, F_{K,j})| \le (D_0 C_0^{C_0})\ \ell^{e_\ell(1-1/D_0)}. \tag{5.23}$$

This follows for odd ℓ, by essentially the same argument as that given for (5.16), the only difference is that this time we use *both* the assertions in (5.1) since $e_\ell \ge \kappa + 1 > (30 D_0 C_0)^{2C_0} + 1 > C_0 + 2$. So assume that $\ell = 2$, i.e. $e_2 = v_2(\mathrm{lcm}[\mathfrak{f}(\chi_1), \dots, \mathfrak{f}(\chi_K)]) \ge \kappa + 1 \ge 31$.

We shall use Proposition 5.2(ii) to bound the sum $Z_{2^{e_2};\ \chi_{1,2},\dots,\chi_{K,2}}(F_{1,j}, \dots, F_{K,j})$. To do this, we observe that since $e_2 \ge 4$, the characters ψ, η mod 2^{e_2} defined by

$$\psi(5) := \exp(2\pi i/2^{e_2-2}),\ \psi(-1) := 1 \quad \text{and} \quad \eta(5) := 1, \eta(-1) := -1$$

generate the character group mod 2^{e_2}. Hence for each $i \in [K]$, there exist $A_{i,2} \in [2^{e_2-2}]$ and $B_{i,2} \in [2]$ satisfying $\chi_{i,2} = \psi^{A_{i,2}} \eta^{B_{i,2}}$ and $(A_{1,2}, \dots, A_{K,2}) \not\equiv (0, \dots, 0) \bmod 2$ (since $e_2 \ge 4$ and at least one of $\chi_{1,2}, \dots, \chi_{K,2}$ is primitive mod 2^{e_2}). This allows us to write

$$Z_{2^{e_2};\ \chi_{1,2},\dots,\chi_{K,2}}(F_{1,j}, \dots, F_{K,j}) = \sum_{v \bmod 2^{e_2}} \psi\left(g(v)\right)\ \eta\left(v^2 \prod_{i=1}^K F_{i,j}(v)^{B_{i,2}} \right),$$

where $g(T) := \prod_{i=1}^K F_{i,j}(T)^{A_{i,2}}$. Now η is induced by the nontrivial character mod 4 and $v^2 \prod_{i=1}^K F_{i,j}(v)^{B_{i,2}} \equiv \prod_{i=1}^K F_{i,j}(\iota)^{B_{i,2}}$ if $v \equiv \iota \pmod 4$ $(\iota = \pm 1)$. Thus, writing $v := 4u + \iota$ gives

$$\begin{aligned} &Z_{2^{e_2};\ \chi_{1,2},\dots,\chi_{K,2}}(F_{1,j}, \dots, F_{K,j}) \\ &\quad = \eta\left(\prod_{i=1}^K F_{i,j}(1)^{B_{i,2}} \right) \sum_{u \bmod 2^{e_2-2}} \psi\left(h_1(u)\right) + \eta\left(\prod_{i=1}^K F_{i,j}(-1)^{B_{i,2}} \right) \sum_{u \bmod 2^{e_2-2}} \psi\left(h_{-1}(u)\right), \end{aligned}$$

where $h_\iota(T) := g(4T + \iota)$. Note that

$$\begin{aligned} &Z_{2^{e_2};\ \chi_{1,2},\dots,\chi_{K,2}}(F_{1,j}, \dots, F_{K,j}) \\ &\quad = \frac{1}{4} \cdot \eta\left(\prod_{i=1}^K F_{i,j}(1)^{B_{i,2}} \right) \sum_{u \bmod 2^{e_2}} \psi\left(h_1(u)\right) + \frac{1}{4} \cdot \eta\left(\prod_{i=1}^K F_{i,j}(-1)^{B_{i,2}} \right) \sum_{u \bmod 2^{e_2}} \psi\left(h_{-1}(u)\right). \end{aligned} \tag{5.24}$$

We will now show that the first of the two terms must have size no more than $(12.5) \cdot 2^{2D_0 + C_0} \cdot 2^{e_2(1-1/D_0)}$; by an analogous argument, so does the second. If the first term is nonzero, then $\prod_{i=1}^K F_{i,j}(1)^{B_{i,2}} \equiv 1 \pmod 2$, so that $\mathrm{ord}_2\left(\prod_{i=1}^K F_{i,j}(4T+1)^{A_{i,2}-1} \right) = 0$. On the other hand, (5.1) shows that with $\widetilde{G}(T) := \sum_{i=1}^K A_{i,2} F'_{i,j}(T) \prod_{\substack{1 \le r \le K \\ r \ne i}} F_{r,j}(T)$, we have $\mathrm{ord}_2\left(\widetilde{G}(T)\right) \le C_0$, so that $\mathrm{ord}_2(\widetilde{G}(4T+1)) \le C_0 + 2 \deg \widetilde{G}(T) \le C_0 + 2(D_0 - 1)$. Combining these observations, we