uniformly in coprime residues $a_1, \dots, a_K$ to moduli $q \le (\log x)^{K_0}$ lying in $\mathcal{Q}(k; f_1, \cdots, f_K)$ and satisfying $IFH(W_{1,k}, \dots, W_{K,k}; B_0)$.

In this section and the next, we establish a weaker version of this result, where we reduce the congruences $f_i(n) \equiv a_i \pmod q$ to a bounded modulus.

Proposition 4.3. *Fix $K_0, B_0 > 0$ and assume that $\{W_{i,k}\}_{1 \le i \le K} \subset \mathbb{Z}[T]$ are nonconstant and multiplicatively independent. There exists a constant $\lambda := \lambda(W_{1,k}, \dots, W_{K,k}; B_0) > 0$ depending only on $\{W_{i,k}\}_{1 \le i \le K} \subset \mathbb{Z}[T]$ and B_0, such that as $x \to \infty$, we have*

$$\sum_{\substack{n \le x \text{ convenient} \\ (\forall i)\ f_i(n) \equiv a_i \pmod q}} 1 \;=\; \left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^K \sum_{\substack{n \le x:\ (f(n),q)=1 \\ (\forall i)\ f_i(n) \equiv a_i \pmod{Q_0}}} 1 \;+\; o\left(\frac{1}{\varphi(q)^K} \sum_{\substack{n \le x \\ (f(n),q)=1}} 1\right), \tag{4.6}$$

uniformly in coprime residues $a_1, \dots, a_K$ to k-admissible moduli $q \le (\log x)^{K_0}$ satisfying $IFH(W_{1,k}, \dots, W_{K,k}; B_0)$, where Q_0 is a divisor of q satisfying $Q_0 \le \lambda$.

Proof. For any $N \ge 1$ and $(w_i)_{i=1}^K \in U_q^K$, we define

$$\mathcal{V}_{N,K}^{(k)}\left(q; (w_i)_{i=1}^K\right) := \left\{(v_1, \dots, v_N) \in (U_q)^N :\ (\forall i \in [K]) \prod_{j=1}^N W_{i,k}(v_j) \equiv w_i \pmod q\right\}.$$

We write each convenient n uniquely in the form $m(P_J \cdots P_1)^k$, where $m, P_J, \dots, P_1$ satisfy (4.1). Then $f_i(n) = f_i(m) \prod_{j=1}^J W_{i,k}(P_j)$, so that the conditions $f_i(n) \equiv a_i \pmod q$ amount to $\gcd(f(m), q) = 1$ and $(P_1, \dots, P_J) \bmod q \in \mathcal{V}'_{q,m} := \mathcal{V}_{J,K}^{(k)}\left(q; (a_i f_i(m)^{-1})_{i=1}^K\right)$. Noting that the conditions $P_1 \cdots P_J \le (x/m)^{1/k}$ and $(P_1, \dots, P_J) \bmod q \in \mathcal{V}'_{q,m}$ are both independent of the ordering of $P_1, \dots, P_J$, we obtain

$$\sum_{\substack{n \le x \text{ convenient} \\ (\forall i)\ f_i(n) \equiv a_i \pmod q}} 1 \;=\; \sum_{\substack{m \le x \\ (f(m),q)=1}} \;\sum_{(v_1,\dots,v_J) \in \mathcal{V}'_{q,m}} \frac{1}{J!} \sum_{\substack{P_1,\dots,P_J > L_m \\ P_1 \cdots P_J \le (x/m)^{1/k} \\ P_1,\dots,P_J \text{ distinct} \\ (\forall j)\ P_j \equiv v_j \pmod q}} 1.$$

Proceeding exactly as in [37] to remove the congruence conditions on $P_1, \dots, P_J$ by successive applications of the Siegel–Walfisz Theorem, we deduce that

$$\sum_{\substack{P_1,\dots,P_J > L_m \\ P_1 \cdots P_J \le (x/m)^{1/k} \\ P_1,\dots,P_J \text{ distinct} \\ (\forall j)\ P_j \equiv v_j \pmod q}} 1 \;=\; \frac{1}{\varphi(q)^J} \sum_{\substack{P_1,\dots,P_J > L_m \\ P_1 \cdots P_J \le (x/m)^{1/k} \\ P_1,\dots,P_J \text{ distinct}}} 1 \;+\; O\left(\frac{x^{1/k}}{m^{1/k}} \exp\left(-K_1 (\log x)^{\epsilon/4}\right)\right) \tag{4.7}$$

for some constant $K_1 := K_1(K_0) > 0$. Collecting estimates and noting that $\#\mathcal{V}'_{q,m} \le \varphi(q)^J \le (\log x)^{K_0 J}$, we obtain

(4.8)

$$\sum_{\substack{n \le x \text{ convenient} \\ (\forall i)\ f_i(n) \equiv a_i \pmod q}} 1 = \sum_{\substack{m \le x \\ (f(m),q)=1}} \frac{\#\mathcal{V}'_{q,m}}{\varphi(q)^J} \left(\frac{1}{J!} \sum_{\substack{P_1,\dots,P_J > L_m \\ P_1 \cdots P_J \le (x/m)^{1/k} \\ P_1,\dots,P_J \text{ distinct}}} 1\right) + O\left(x^{1/k} \exp\left(-\frac{K_1}{2} (\log x)^{\epsilon/4}\right)\right).$$

Here in the last step we have crudely bounded the sum $\sum_{\substack{m\le x\\(f(m),q)=1}} m^{-1/k}$ by writing each m as AB for some k-full A and k-free B satisfying $\gcd(A,B)=1$, and then noting that the sum $\sum 1/A$ is no more than $\prod_{p\le x}\left(1+1/p+O\left(1/p^{1+1/k}\right)\right)$. The following proposition is a special case of the more general Proposition 5.4 established in the next section, and will provide the needed estimate on the cardinalities of the sets $\mathcal{V}'_{q,m}$.

Proposition 4.4. *Assume that $\{W_{i,k}\}_{1\le i\le K}$ are multiplicatively independent. There exists a constant $C_0 := C_0(W_{1,k},\dots,W_{K,k};B_0) > (8D)^{2D+2}$ depending only on $\{W_{i,k}\}_{1\le i\le K}$ and B_0, such that for any constant $C>C_0$, the following estimates hold uniformly in coprime residues $(w_i)_{i=1}^K$ to moduli q satisfying $\alpha_k(q)\neq 0$ and $IFH(W_{1,k},\dots,W_{K,k};B_0)$: We have*

$$\begin{aligned}(4.9)\quad &\frac{\#\mathcal{V}^{(k)}_{N,K}\left(q;(w_i)_{i=1}^K\right)}{\varphi(q)^N}\\ &= \frac{\alpha_k(q)^N}{\alpha_k(Q_0)^N}\left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^K\left\{\frac{\#\mathcal{V}^{(k)}_{N,K}\left(Q_0;(w_i)_{i=1}^K\right)}{\varphi(Q_0)^N}+O\left(\frac{1}{C^N}\right)\right\}\prod_{\substack{\ell\mid q\\ \ell>C_0}}\left(1+O\left(\frac{(4D)^N}{\ell^{N/D-K}}\right)\right),\end{aligned}$$

uniformly for $N\ge KD+1$, where Q_0 is a C_0-smooth divisor of q of size $O_C(1)$. Moreover

$$(4.10)\qquad \frac{\#\mathcal{V}^{(k)}_{N,K}\left(q;(w_i)_{i=1}^K\right)}{\varphi(q)^N}\le\frac{\left(\prod_{\ell^e\|q}e\right)^{\mathbb{1}_{N=KD}}}{q^{N/D}}\ \exp\left(O(\omega(q))\right),\quad \textit{for each } 1\le N\le KD.$$

Applying (4.9) with $N:=J\ge KD+1$, and with C chosen to be a constant exceeding $2C_0^{C_0}$, we see that

$$\frac{\#\mathcal{V}'_{q,m}}{\varphi(q)^J}=(1+o(1))\frac{\alpha_k(q)^J}{\alpha_k(Q_0)^J}\left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^K\left\{\frac{\#\mathcal{V}'_{Q_0,m}}{\varphi(Q_0)^J}+O\left(\frac{1}{C^J}\right)\right\},$$

where $\mathcal{V}'_{Q_0,m} := \mathcal{V}^{(k)}_{J,K}\big(Q_0;(a_if_i(m)^{-1})_{i=1}^K\big)$ and we have noted that $\sum_{\substack{\ell\mid q\\ \ell>C_0}}(4D)^J/\ell^{J/D-K}\le \left(4D/C_0^{1/(2D+2)}\right)^J=o(1)$. We insert this into (4.8), and observe that since $\alpha_k(q)\neq 0$, $Q_0\mid q$ and Q_0 is C_0-smooth, we have $\alpha_k(Q_0)C\ge C\prod_{\ell\le C_0}\left(1-\frac{\ell-2}{\ell-1}\right)\ge\frac{C}{C_0^{C_0}}\ge 2$. We obtain

$$\begin{aligned}(4.11)\quad &\sum_{\substack{n\le x \text{ convenient}\\ (\forall i)\ f_i(n)\equiv a_i \pmod q}} 1\\ &=(1+o(1))\left(\frac{\varphi(Q_0)}{\varphi(q)}\right)^K\frac{\alpha_k(q)^J}{\alpha_k(Q_0)^J}\sum_{\substack{m\le x\\(f(m),q)=1}}\frac{\#\mathcal{V}'_{Q_0,m}}{\varphi(Q_0)^J}\left(\frac{1}{J!}\sum_{\substack{P_1,\dots,P_J>L_m\\ P_1\cdots P_J\le(x/m)^{1/k}\\ P_1,\dots,P_J \text{ distinct}}}1\right)+o\left(\frac{1}{\varphi(q)^K}\sum_{\substack{n\le x\\(f(n),q)=1}}1\right),\end{aligned}$$

where by the arguments leading to (4.8) and the observation $\#\{(v_1, \dots, v_J) \in U_q^J : \prod_{j=1}^J W_k(v_j) \in U_q\} = (\alpha_k(q)\varphi(q))^J$, we have noted that

(4.12)

$$\sum_{\substack{n \le x \text{ convenient} \\ \gcd(f(n),q)=1}} 1 = \alpha_k(q)^J \sum_{\substack{m \le x \\ (f(m),q)=1}} \left(\frac{1}{J!} \sum_{\substack{P_1,\dots,P_J > L_m \\ P_1 \cdots P_J \le (x/m)^{1/k} \\ P_1,\dots,P_J \text{ distinct}}} 1 \right) + O\left(x^{1/k} \exp\left(-\frac{K_1}{2} (\log x)^{\epsilon/4} \right) \right).$$

For each $(w_i)_{i=1}^K \in U_q^K$, we define $\mathcal{U}_{J,K}\big(q, Q_0; (w_i)_{i=1}^K\big)$ to be the set of tuples $(v_1, \dots, v_J) \in U_q^J$ satisfying $\prod_{j=1}^J W_{i,k}(v_j) \in U_q$ and $\prod_{j=1}^J W_{i,k}(v_j) \equiv w_i \pmod{Q_0}$ for each $i \in [K]$. Observe that any convenient n satisfying $\gcd(f(n), q) = 1$ and $f_i(n) \equiv a_i \pmod{Q_0}$ for all $i \in [K]$, can be uniquely written in the form $n = m(P_J \cdots P_1)^k$, where $P_J, \dots, P_1$ are primes satisfying (4.1), $\gcd(f(m), q) = 1$ and $(P_1, \dots, P_J) \bmod q \in \mathcal{U}_m := \mathcal{U}_{J,K}\big(q, Q_0;\, (a_i f_i(m)^{-1})_{i=1}^K\big)$. As such, by the arguments leading to (4.8), we obtain

$$\sum_{\substack{n \le x \text{ convenient} \\ \gcd(f(n),q)=1 \\ (\forall i)\ f_i(n) \equiv a_i \pmod{Q_0}}} 1 = \sum_{\substack{m \le x \\ (f(m),q)=1}} \frac{\#\mathcal{U}_m}{\varphi(q)^J} \left(\frac{1}{J!} \sum_{\substack{P_1,\dots,P_J > L_m \\ P_1 \cdots P_J \le (x/m)^{1/k} \\ P_1,\dots,P_J \text{ distinct}}} 1 \right) + o\left(\frac{1}{\varphi(q)^K} \sum_{\substack{n \le x \\ (f(n),q)=1}} 1 \right). \tag{4.13}$$

Now, a simple counting argument shows the following general observation: let $F \in \mathbb{Z}[T]$ be a nonconstant polynomial, and let Q, d be positive integers such that $d \mid Q$ and $\alpha_F(Q) := \frac{1}{\varphi(Q)} \#\{u \in U_Q : F(u) \in U_Q\}$ is nonzero (hence so is $\alpha_F(d)$). Then for any $u \in U_d$ for which $F(u) \in U_d$, we have

$$\#\{U \in U_Q :\ U \equiv u \pmod{d},\ F(U) \in U_Q\} = \frac{\alpha_F(Q)\varphi(Q)}{\alpha_F(d)\varphi(d)}. \tag{4.14}$$

Using this for $F := W_k = \prod_{i=1}^K W_{i,k}$ (so that $\alpha_F = \alpha_k$), we immediately obtain

$$\#\mathcal{U}_{J,K}\big(q, Q_0; (w_i)_{i=1}^K\big) = \left(\frac{\alpha_k(q)\varphi(q)}{\alpha_k(Q_0)\varphi(Q_0)} \right)^J \#\mathcal{V}_{J,K}^{(k)}\big(Q_0, (w_i)_{i=1}^K\big)$$

for all $(w_i)_{i=1}^K \in U_q^K$. Applying this with $w_i := a_i f_i(m)^{-1}$ and recalling that $\mathcal{V}'_{Q_0,m} = \mathcal{V}_{J,K}^{(k)}\big(Q_0; (a_i f_i(m)^{-1})_{i=1}^K\big)$, we get from (4.13),

$$\sum_{\substack{n \le x \text{ convenient} \\ \gcd(f(n),q)=1 \\ (\forall i)\ f_i(n) \equiv a_i \pmod{Q_0}}} 1 = \frac{\alpha_k(q)^J}{\alpha_k(Q_0)^J} \sum_{\substack{m \le x \\ (f(m),q)=1}} \frac{\mathcal{V}'_{Q_0,m}}{\varphi(Q_0)^J} \left(\frac{1}{J!} \sum_{\substack{P_1,\dots,P_J > L_m \\ P_1 \cdots P_J \le (x/m)^{1/k} \\ P_1,\dots,P_J \text{ distinct}}} 1 \right) + o\left(\frac{1}{\varphi(q)^K} \sum_{\substack{n \le x \\ (f(n),q)=1}} 1 \right).$$

Comparing this with (4.11), we obtain

$$\sum_{\substack{n \le x \text{ convenient} \\ (\forall i)\ f_i(n) \equiv a_i \pmod{q}}} 1 = (1 + o(1)) \left(\frac{\varphi(Q_0)}{\varphi(q)} \right)^K \sum_{\substack{n \le x \text{ convenient} \\ \gcd(f(n),q)=1 \\ (\forall i)\ f_i(n) \equiv a_i \pmod{Q_0}}} 1 + o\left(\frac{1}{\varphi(q)^K} \sum_{\substack{n \le x \\ (f(n),q)=1}} 1 \right).$$