Proof. The following general observation will play an important role in our arguments: We have $|\widetilde{H}_\chi(s)| \asymp |\widetilde{H}_\chi(w)|$ uniformly in complex numbers s and w satisfying $\mathrm{Im}(s) = \mathrm{Im}(w) =: t$, $|s-w| \ll \mathcal{L}_Q(t)^{-1}$ and $\mathrm{Re}(w) \geq \mathrm{Re}(s) \geq \frac{1}{k}\left(1 - \frac{c_1}{2\mathcal{L}_Q(t)}\right)$.

Indeed by the definitions of $\widetilde{H}_\chi(s)$ and $\widetilde{F}_\chi(s)$, we have

$$\left|\frac{\widetilde{H}'_\chi(z)}{\widetilde{H}_\chi(z)}\right| = \left|c_{\widehat{\chi}} k \left(\frac{F_1'(kz)}{F_1(kz)} + \frac{\alpha_k(Q)}{kz-1} - \frac{\alpha_k(Q)\gamma(\psi_e)}{kz-\beta_e}\right) + c_{\widehat{\chi}} k \frac{g'(kz)}{g(kz)} + \frac{G'_{\chi,1}(z)}{G_{\chi,1}(z)}\right| \ll \mathcal{L}_Q(t) \tag{7.6}$$

uniformly for complex numbers $z = u + it$ satisfying $u \geq \frac{1}{k}\left(1 - \frac{c_1}{2\mathcal{L}_Q(t)}\right)$. Here in the last step, we have applied (7.2) and [44, Lemma 15(i)], the latter with $\xi(t) := \exp(6\mathcal{L}_Q(t))$. The general observation now follows by writing $\log\left(\widetilde{H}_\chi(w)/\widetilde{H}_\chi(s)\right) = \int_{\mathrm{Re}(s)}^{\mathrm{Re}(w)} \widetilde{H}'_\chi(u+it)/\widetilde{H}_\chi(u+it)\, \mathrm{d}u$.

(i) Let $b_k(t) := \frac{1}{k}\left(1 + \frac{c_3}{\mathcal{L}_Q(t)}\right)$ for some absolute constant $c_3 > 0$. By the above observation and the definitions of $\widetilde{F}_\chi(s)$, $\widetilde{H}_\chi(s)$ and $H_\chi(s)$, it follows that

(7.7)
$$\begin{aligned}&\left|H_\chi\left(\frac{1}{k}\right)\right| \ll \left|\widetilde{H}_\chi\left(\frac{1}{k}\right)\right| (1-\beta_e)^{-\alpha_k(Q)} \ll |\widetilde{H}_\chi(b_k(0))|\ (1-\beta_e)^{-\alpha_k(Q)} \\ &\ll |\widetilde{F}_\chi(b_k(0))|\ (\log Q)(1-\beta_e)^{-2\alpha_k(Q)} \ll |F_1(kb_k(0))g(kb_k(0))|^{\mathrm{Re}(c_{\widehat{\chi}})}\ (\log Q)^2(1-\beta_e)^{-2\alpha_k(Q)}.\end{aligned}$$

Here in the last bound, we have noted that $|G_{\chi,1}(b_k(0))| \ll \log_2 Q$, as is evident from the fact that $\prod_{\substack{p|Q\\ W_k(p)\in U_Q}} (1-p^{-kb_k(0)})^{-1} \ll \exp(\sum_{p|Q} 1/p) \ll \exp(\sum_{p\leq\omega(Q)} 1/p) \ll \log\omega(Q) \ll \log_2 Q$.

Now proceeding as in [44, Lemma 8], we see that for all s with $\sigma > 1/k$, we have

$$\sum_{n\geq 1} \frac{\mathbb{1}_{(f(n^k),Q)=1}}{n^{ks}} = F_1(ks)\ g(ks)\ \widetilde{G}(s), \tag{7.8}$$

where

$$\begin{aligned}\widetilde{G}(s) = \prod_p &\left(1 + \sum_{v\geq 2} \frac{1}{p^{vks}}(\mathbb{1}_{(f(p^{kv}),Q)=1} - \mathbb{1}_{(W_k(p),Q)=1}\ \mathbb{1}_{(f(p^{k(v-1)}),Q)=1})\right)\cdot \\ &\prod_{\substack{p|Q\\ W_k(p)\in U_Q}} \left(1-\frac{1}{p^{ks}}\right)^{-1} \cdot \exp\left(\sum_{\substack{b\in U_Q\\ W_k(b)\in U_Q}} \sum_{v\geq 2}\left(\sum_{p\equiv b \pmod Q} \frac{1}{vp^{vks}} - \sum_{p:\ p^v\equiv b \pmod Q} \frac{1}{vp^{vks}}\right)\right).\end{aligned}$$

Uniformly for s with $\sigma \geq 1/k$, we observe that the infinite product above has size at least $1 - \sum_{p,v\geq 2} 1/p^v \gg 1$ and at most $\exp(\sum_{p,v\geq 2} 1/p^v) \ll 1$. Likewise, the exponential factor has size $\asymp 1$ in the same region. Moreover, for $\sigma \geq 1/k$, the product over $p \mid Q$ is $\asymp \exp(\sum_{p|Q:\ (W_k(p),Q)=1}\ p^{-k\sigma})$, which is $\gg 1$ and $\ll \exp(\sum_{p|Q}\ p^{-1}) \ll \log_2 Q$. Putting these observations together, we find that $1 \ll \widetilde{G}(s) \ll \log_2 Q$ for $\sigma \geq 1/k$. Applying the lower bound on $s \coloneqq b_k(0)$, the equality (7.8) yields

$$|F_1(kb_k(0))\ g(kb_k(0))| \ll \sum_{n\geq 1} \frac{\mathbb{1}_{(f(n^k),Q)=1}}{n^{kb_k(0)}} \leq \zeta(kb_k(0)) = \frac{1}{kb_k(0)-1} + O(1) \ll \log Q,$$

so that from (7.7), we obtain $|H_\chi(1/k)| \ll (\log Q)^3(1-\beta_e)^{-2\alpha_k(Q)}$. Subpart (i) now follows as $Q \leq (\log x)^{K_0}$ and $1-\beta_e \gg_{\epsilon_1} Q^{-\epsilon_1/20K_0} \gg_{\epsilon_1} (\log x)^{-\epsilon_1/20}$ by Siegel's Theorem.

(ii) By the observation made at the start of the proof, we have $|\widetilde{H}_\chi(s)| \ll |\widetilde{H}_\chi(1/k)| \ll |H_\chi(1/k)|(1-\beta_e)^{-\alpha_k(Q)} \ll |H_\chi(1/k)|(\log x)^{\alpha_k(Q)\epsilon_1/20}$. The result now follows from (i).

(iii) By the aforementioned observation, we have $|\widetilde{H}_\chi(s)| \ll |\widetilde{H}_\chi(b_k(t)+it)|$, and since $|s-\theta/k| \gg 1/\mathcal{L}_Q(t)$, we have $b_k(t)+it-\theta/k \asymp s-\theta/k$ for $\theta \in \{1,\beta_e\}$. Thus $|\widetilde{F}_\chi(s)| \ll |\widetilde{F}_\chi(b_k(t)+it)|$. Using (7.5) and replicating the arguments that led to the bounds on $\widetilde{G}(s)$ above, we also obtain $(\log_2 Q)^{-1} \ll G_{\chi,1}(s) \ll \log_2 Q$ for $\sigma \geq 1/k$, so that $|\widetilde{F}_\chi(s)| \ll (\log_2 Q)\cdot|F_1(k(b_k(t)+it))g(k(b_k(t)+it))|^{\mathrm{Re}(c_{\widehat{\chi}})}$. From (7.8) and the bounds on $\widetilde{G}(s)$, we thus get $|\widetilde{F}_\chi(s)| \ll (\log_2 Q)\left|\sum_{n\geq 1}\mathbb{1}_{(f(n^k),Q)=1}/n^{k(b_k(t)+it)}\right|^{\mathrm{Re}(c_{\widehat{\chi}})} \ll (\log_2 Q)\left(\sum_{n\geq 1}\mathbb{1}_{(f(n^k),Q)=1}/n^{kb_k(t)}\right)^{\mathrm{Re}(c_{\widehat{\chi}})}$, whence $|\widetilde{F}_\chi(s)| \ll (\log_2 Q)^2|F_1(kb_k(t))g(kb_k(t))|^{\mathrm{Re}(c_{\widehat{\chi}})} \ll (\log_2 Q)^3|\widetilde{F}_\chi(b_k(t))|$. By definitions of $b_k(t)$ and $\widetilde{H}_\chi(b_k(t))$, we have $|\widetilde{F}_\chi(s)| \ll (\log_3 x)^3|\widetilde{H}_\chi(b_k(t))|\mathcal{L}_Q(t)^{\alpha_k(Q)}(1-\beta_e)^{-\alpha_k(Q)}$. Finally, recall that $|t| \leq T = \exp(\sqrt{\log x})$, that $1-\beta_e \gg_{\epsilon_1} (\log x)^{-\epsilon_1/20}$, and that $|\widetilde{H}_\chi(b_k(t))| \ll |\widetilde{H}_\chi(1/k)| \ll (\log x)^{\alpha_k(Q)\epsilon_1/4}$ (by subpart (ii) the general observation at the start of the proof). This yields $|\widetilde{F}_\chi(s)| \ll (\log x)^{\alpha_k(Q)(1/2+\epsilon_1)}$, and Lemma 7.1(iii) applies.

(iv) It suffices to show that uniformly for s satisfying the same conditions as in this subpart,

$$|H_\chi(s)| + |H'_\chi(s)| \ll (\log x)^{\alpha_k(Q)\epsilon_1/5}\left(\log Q + \frac{1}{1-\beta_e}\right). \tag{7.9}$$

(Here as usual, the second term on the right is omitted if there is no Siegel zero, otherwise it dominates.) Indeed once we establish (7.9), then from the bound $1-\beta_e \gg_{\epsilon_1} (\log x)^{-\epsilon_1/20}$, it follows that $|H_\chi(s)|+|H'_\chi(s)| \ll (\log x)^{(1/20+\alpha_k(Q)/5)\epsilon_1}$, which combined with Lemma 7.1(iii) and the observation $|H_\chi(1/k)G_{\chi,2}(1/k) - H_\chi(s)G_{\chi,2}(s)| = \left|\int_s^{1/k}(H_\chi(u)G_{\chi,2}(u))'\,\mathrm{d}u\right|$ completes the proof of the subpart. To show (7.9), we recall that $H_\chi(s)$ is non-vanishing for s as in the subpart. Further (7.6) applies with $z=s$ for all s considered in this subpart, yielding

$$\left|\frac{H'_\chi(s)}{H_\chi(s)}\right| = \left|\frac{\widetilde{H}'_\chi(s)}{\widetilde{H}_\chi(s)} - \frac{1}{s} + \frac{\alpha_k(Q)c_{\widehat{\chi}}\gamma(\psi_e)}{s-\beta_e/k}\right| \ll \mathcal{L}_Q(0) + 1 + \frac{1}{1-\beta_e} \ll \log Q + \frac{1}{1-\beta_e}.$$

As a consequence,

$$\left|\log\frac{H_\chi(1/k)}{H_\chi(s)}\right| = \left|\int_s^{1/k}\frac{H'_\chi(u)}{H_\chi(u)}\,\mathrm{d}u\right| \ll \left(\frac{1}{k}-s\right)\left(\log Q + \frac{1}{1-\beta_e}\right) \ll 1,$$

showing that $|H_\chi(s)| \asymp |H_\chi(1/k)|$ uniformly for all s in the statement. Collecting these bounds, we obtain for all such s,

$$|H_\chi(s)|+|H'_\chi(s)| \ll \left|H_\chi\left(\frac{1}{k}\right)\right| + \left|\frac{H'_\chi(s)}{H_\chi(s)}\right|\cdot\left|\frac{H_\chi(s)}{H_\chi(1/k)}\right|\cdot\left|H_\chi\left(\frac{1}{k}\right)\right| \ll \left|H_\chi\left(\frac{1}{k}\right)\right|\left(\log Q + \frac{1}{1-\beta_e}\right),$$

so that the desired bound (7.9) now follows from subpart (i). This concludes the proof. $\square$

7.2. **Perron's formula and the contour shifts.** We first show that there is some X sufficiently close to x for which the error term arising from an effective Perron's formula is small.

Lemma 7.3. *Let $h := x/\log^2 x$. There exists a positive integer $X \in (x, x+h]$ satisfying*

$$\sum_{\substack{3X/4<n<5X/4\\ n\neq X}} \frac{\mathbb{1}_{(f(n),Q)=1}}{|\log(X/n)|} \ll X^{1/k}\log X.$$

Proof. This would follow once we show that

$$\sum_{x<X\le x+h}\ \sum_{\substack{3X/4<n<5X/4\\ n\neq X}} \frac{\mathbb{1}_{(f(n),Q)=1}}{|\log(X/n)|} \ll x^{1/k} h\log x, \tag{7.10}$$

with the outer sum being over integers $X \in (x, x+h]$. (Recall that $x \in \mathbb{Z}^+$ in this entire section.) To show this, we write the sum on the left hand side as $S_1 + S_2$, where S_1 denotes the contribution of the case $3X/4 < n \le X-1$. Writing any n contributing to S_1 as $X - v$ for some integer $v \in [1, X/4)$, we see that $|\log(X/n)| = -\log(1-v/X) \gg v/X \gg v/x$. Recalling that $n = Bm$ for some k-free B of size $O(1)$ and some k-full m, we thus have

$$\begin{aligned} S_1 &\le \sum_{3x/4<n<x+h}\ \sum_{\substack{x<X\le x+h\\ n+1\le X<4n/3}} \frac{\mathbb{1}_{(f(n),Q)=1}}{|\log(X/n)|} \ll x\sum_{B\ll 1}\ \sum_{\substack{\frac{3x}{4B}<m<\frac{x+h}{B}\\ m \text{ is } k\text{-full}}}\ \sum_{\substack{1\le v<\frac{x+h}{4}\\ x<v+Bm\le x+h}} \frac{1}{v} \\ &\ll x\sum_{1\le v\le\frac{x+h}{4}} \frac{1}{v}\sum_{B\ll 1}\ \sum_{\substack{\frac{x-v}{B}<m\le\frac{x-v+h}{B}\\ m\text{ is } k\text{-full}}} 1 \ll x\log x\left(x^{1/k}\frac{h}{x} + x^{1/(k+1)}\right) \ll x^{1/k}h\log x, \end{aligned}$$

where we have bounded the last inner sum on m using the Erdös-Szekeres estimate on the count of k-full integers (see [13]). This shows that the sum S_1 is bounded by the right hand expression in (7.10), and similarly so is the sum S_2, establishing (7.10). $\square$

To complete the proof of Theorem 5.6, it suffices to establish the bound therein for X in place of x, for once we do so, we may simply note that

$$\Big|\sum_{x<n\le X} \chi_1(f_1(n))\cdots\chi_K(f_K(n))\mathbb{1}_{(f(n),q)=1}\Big| \le \sum_{x<n\le X}\mathbb{1}_{(f(n),Q)=1} \le \sum_{B\ll 1}\ \sum_{\substack{\frac{x}{B}<m\le\frac{X}{B}\\ m\text{ is } k\text{-full}}} 1 \ll \frac{x^{1/k}}{\log^2 x}.$$

To show the bound in Theorem 5.6 for X, we start by applying an effective version of Perron's formula [48, Theorem II.2.3]. To bound the resulting error, we use Lemma 7.3 and note that

$$\begin{aligned} &X^{\frac{1}{k}\left(1+\frac{1}{\log X}\right)}\left(\sum_{n\le 3X/4} + \sum_{n\ge 5X/4}\right)\frac{\mathbb{1}_{(f(n),Q)=1}}{T|\log(X/n)|n^{\frac{1}{k}\left(1+\frac{1}{\log X}\right)}} \ll \frac{X^{1/k}}{T}\sum_{B\ll 1}\ \sum_{\substack{m\ge 1\\ m\text{ is } k\text{-full}}} \frac{1}{m^{\frac{1}{k}\left(1+\frac{1}{\log X}\right)}} \\ &\ll \frac{X^{1/k}}{T}\prod_p\left(1+\frac{1}{p^{1+1/\log X}} + O\left(\frac{1}{p^{1+1/k}}\right)\right) \ll \frac{X^{1/k}}{T}\exp\left(\sum_p \frac{1}{p^{1+1/\log X}}\right) \ll \frac{X^{1/k}\log X}{T}, \end{aligned}$$