find that $h_1'(T) = 4\widetilde{G}(4T+1)\prod_{i=1}^K F_{i,j}(4T+1)^{A_{i,2}-1}$ has $t \coloneqq \mathrm{ord}_2(h_1') \le 2D_0+C_0$. Since any 2-critical point of $h_1(T) = \prod_{i=1}^K F_{i,j}(4T+1)^{A_{i,2}}$ is a root of the polynomial $2^{-\mathrm{ord}_2(\widetilde{G}(4T+1))}\widetilde{G}(4T+1)$ mod 2, it follows that the maximum multiplicity of such a 2-critical point is no more than $\deg \widetilde{G}(4T+1) - 1 \le D_0 - 1$. As such, an application of Proposition 5.2(ii) with $m \coloneqq e_2 - 2 \ge (2D_0+C_0)+3 \ge t+3$ and $\chi \coloneqq \psi$, shows that the first term in (5.24) has size at most $(12.5)\cdot 2^{2D_0+C_0}\cdot 2^{e_2(1-1/D_0)}$, proving our claim. This shows that $|Z_{2^{e_2};\ \chi_{1,2},\dots,\chi_{K,2}}(F_{1,j},\dots,F_{K,j})| \le 25\cdot 2^{2D_0+C_0}\cdot 2^{e_2(1-1/D_0)}$, completing the proof of (5.23).

Setting $C_1 \coloneqq D_0 C_0^{C_0}$ and combining (5.22) with (5.23), we find that for each $j \in [N]$,

$$|Z_{\widetilde{q};\ \chi_1,\dots,\chi_K}(F_{1,j},\dots,F_{K,j})| \le \left(\prod_{\substack{\ell^e \| \widetilde{q} \\ e_\ell \le \kappa}} \varphi(\ell^e)\right) \prod_{\substack{\ell^e \| \widetilde{q} \\ e_\ell \ge \kappa+1}} \left(\ell^{e-e_\ell}\cdot C_1 \ell^{e_\ell(1-1/D_0)}\right) \le \frac{(2C_1)^{C_0}\varphi(\widetilde{q})}{A^{1/D_0}},$$

where $A \coloneqq \prod_{\ell^e \| \widetilde{q}:\ e_\ell \ge \kappa+1} \ell^{e_\ell}$ denotes the $(\kappa+1)$-full part of $\mathrm{lcm}[\mathfrak{f}(\chi_1),\dots,\mathfrak{f}(\chi_K)]$, i.e, the largest $(\kappa+1)$-full divisor of $\mathrm{lcm}[\mathfrak{f}(\chi_1),\dots,\mathfrak{f}(\chi_K)]$. For a divisor $d$ of $\widetilde{q}$, there are $\le d^K$ tuples $(\chi_1,\dots,\chi_K)$ of characters mod $\widetilde{q}$ for which $\mathrm{lcm}[\mathfrak{f}(\chi_1),\dots,\mathfrak{f}(\chi_K)] = d$. Hence, summing the bound in the above display over all possible $(\chi_1,\dots,\chi_K)$ occurring in the sum $\mathcal{S}''$, we obtain

$$\begin{aligned} |\mathcal{S}''| &\le \frac{1}{\varphi(\widetilde{q})^K} \sum_{\substack{A | \widetilde{q}:\ A>1 \\ A \text{ is } (\kappa+1)\text{-full}}} \ \sum_{\substack{d | \widetilde{q} \\ (\kappa+1)\text{-full part of } d \text{ is } A}} d^K \cdot \frac{(2C_1)^{C_0 N}\varphi(\widetilde{q})^N}{A^{N/D_0}} \\ &\ll \frac{\varphi(\widetilde{q})^N}{\varphi(\widetilde{q})^K}\cdot (2C_1)^{C_0 N} \sum_{\substack{A | \widetilde{q}:\ A>1 \\ A \text{ is } (\kappa+1)\text{-full}}} \frac{1}{A^{N/D_0 - K}}. \end{aligned}$$

In the last step above, we have noted that for any $d$ dividing $\widetilde{q}$ whose $(\kappa+1)$-full part is $A$, we have $d \ll A$. Continuing,

$$\frac{|\mathcal{S}''|}{\varphi(\widetilde{q})^N} \ll \frac{(2C_1)^{C_0 N}}{\varphi(\widetilde{q})^K}\left\{\prod_{\ell^e \| \widetilde{q}}\left(1 + \sum_{\kappa+1 \le \nu \le e} \frac{1}{\ell^{\nu(N/D_0-K)}}\right) - 1\right\}. \tag{5.25}$$

Now if $N \ge KD_0 + 1$, then since $\kappa > C\cdot(30D_0C_0^{C_0})^{2C_0} \ge D_0(D_0+3)$, we see that the sum on $\nu$ above is no more than $2^{-\kappa(N/D_0-K)}\left(1-2^{-1/D_0}\right)^{-1} \le \frac{2^{D_0+2}}{2^{\kappa/D_0}} \le \frac{1}{2}$. Hence $\log(1 + \sum_{\kappa+1\le\nu\le e} \ell^{-\nu(N/D_0-K)}) \ll 2^{-\kappa(N/D_0-K)} \ll 2^{-\kappa N/D_0}$. In addition, since $P(\widetilde{q}) \le C_0$, (5.25) gives

$$\frac{|\mathcal{S}''|}{\varphi(\widetilde{q})^N} \ll \frac{(2C_1)^{C_0 N}}{\varphi(\widetilde{q})^K}\left\{\exp\left(O\left(\frac{1}{2^{\kappa N/D_0}}\right)\right) - 1\right\} \ll \frac{1}{\varphi(\widetilde{q})^K}\cdot\left(\frac{(2C_1)^{C_0}}{2^{\kappa/D_0}}\right)^N \ll \frac{C^{-N}}{\varphi(\widetilde{q})^K}, \tag{5.26}$$

where in the last step, we have recalled that $\kappa/D_0 > D_0^{-1}\cdot C\cdot(30D_0C_0^{C_0})^{2C_0} > C\cdot(2C_1)^{C_0}$. Combining (5.26) with (5.21), we deduce that

$$\frac{\#\widetilde{\mathcal{V}}_{N,K}\left(\widetilde{q};(w_i)_{i=1}^K\right)}{\varphi(\widetilde{q})^N} = \frac{\mathcal{S}'+\mathcal{S}''}{\varphi(\widetilde{q})^N} = \left(\frac{\varphi(Q_0)}{\varphi(\widetilde{q})}\right)^K\left\{\frac{\#\widetilde{\mathcal{V}}_{N,K}\left(Q_0;(w_i)_{i=1}^K\right)}{\varphi(Q_0)^N} + O\left(\frac{1}{C^N}\right)\right\}, \tag{5.27}$$

uniformly for $N \ge KD_0+1$ and in coprime residues $w_1, \ldots, w_K$ to any modulus $q$. In particular, since $\widetilde{q}/\varphi(\widetilde{q}) \le 2^{\omega(\widetilde{q})} \ll 1$, we have

$$\frac{\#\widetilde{\mathcal{V}}_{N,K}\left(\widetilde{q};(w_i)_{i=1}^K\right)}{\varphi(\widetilde{q})^N} \ll \frac{1}{\varphi(\widetilde{q})^K} \ll \frac{1}{\widetilde{q}^K}. \tag{5.28}$$

Note that we did not make use of any invariant factor hypothesis to derive (5.27) or (5.28).

On the other hand, for each $N \in [KD_0]$, we have $1+\sum_{\kappa+1\le\nu\le e} \ell^{-\nu(N/D_0-K)} \le \sum_{0\le\nu\le e} \ell^{\nu(K-N/D_0)} \ll e^{\mathbb{1}_{N=KD_0}} \ \ell^{e(K-N/D_0)}$. Multiplying this over the $O(1)$ primes $\ell$ dividing $\widetilde{q}$ yields, from (5.25), $|\mathcal{S}''|/\varphi(\widetilde{q})^N \ll \left(\prod_{\ell^e\|\widetilde{q}} e\right)^{\mathbb{1}_{N=KD_0}} \Big/ \widetilde{q}^{N/D_0}$. Combining this with the trivial bound $|\mathcal{S}'|/\varphi(\widetilde{q})^N \ll \varphi(\widetilde{q})^{-K} \ll \widetilde{q}^{-K} \ll \widetilde{q}^{-N/D_0}$ coming from (5.21), we find that for each $N \in [KD_0]$, we have

$$\frac{\#\widetilde{\mathcal{V}}_{N,K}\left(\widetilde{q};(w_i)_{i=1}^K\right)}{\varphi(\widetilde{q})^N} \ll \frac{\left(\prod_{\ell^e\|\widetilde{q}} e\right)^{\mathbb{1}_{N=KD_0}}}{\widetilde{q}^{N/D_0}}, \quad \text{uniformly in } q \text{ and } (w_i)_{i=1}^K \in U_q^K. \tag{5.29}$$

To establish Proposition 5.4(a), we multiply (5.27) with the relations (5.8) for all $\ell^e \parallel q$ with $\ell > C_0$, noting that any such $\ell$ exceeds $B_0$ and hence automatically satisfies $\gcd(\ell - 1, \beta(G_{1,r}, \ldots, G_{K,r})) = 1$ for each $r \in [L]$ as $q$ satisfies $IFH(G_{1,r}, \ldots, G_{K,r}; B_0)$. (Note here that $\prod_{\ell|q:\ \ell>C_0} \alpha_N^*(\ell) = \alpha_N^*(q)/\alpha_N^*(Q_0)$.) On the other hand, (b) follows by multiplying the relations (5.9) over all $\ell > C_0$, with (5.29) (resp. (5.28)) for each $N \in [KD_0]$ (resp. each $N \ge KD_0+1$). This establishes Proposition 5.4 for $D_0 > 1$.

Now we consider the case $D_0 = 1$, so that $K = 1$ and $G_{1,r}(T) \coloneqq G_r(T) \coloneqq R_rT + S_r$ for some integers $R_r$ and $S_r$ with $R_r \ne 0$. We set $F_j \coloneqq F_{1,j}$ for each $j \in [N]$ and show that

$$\frac{\#\widetilde{\mathcal{V}}_{N,1}(\ell^e; w)}{\varphi(\ell^e)^N} = \frac{\alpha_N^*(\ell)}{\varphi(\ell^e)}\left(1 + O\left(\left(\frac{2}{\ell-1}\right)^{N-1}\right)\right), \tag{5.30}$$

uniformly for $N \ge 1$ and for $\ell^e \parallel q$ with $\ell > C_0$. To this end, we start by using (5.11) to write

$$\frac{\#\widetilde{\mathcal{V}}_{N,1}(\ell^e; w)}{\varphi(\ell^e)^N} = \frac{\alpha_N^*(\ell)}{\varphi(\ell^e)}\left(1 + \frac{1}{\alpha_N^*(\ell)\varphi(\ell^e)^N} \sum_{\chi\ne\chi_{0,\ell} \bmod \ell^e} \overline{\chi}(w) \prod_{j=1}^N Z_{\ell^e;\chi}(F_j)\right). \tag{5.31}$$

If $\mathfrak{f}(\chi) = \ell^{e_0}$ for some $e_0 \in \{2, \ldots, e\}$, then it is easy to see that $Z_{\ell^e;\chi}(F_j) = 0$ for any $j \in [N]$: this is immediate by orthogonality if $S_{j'} = 0$, and follows from Proposition 5.2 otherwise, since the polynomial $T^{\varphi(\ell^{e_0})}(R_{j'}T + S_{j'})$ has no $\ell$-critical points. On the other hand, if $\mathfrak{f}(\chi) = \ell$, then $|Z_{\ell^e;\chi}(F_j)| = \ell^{e-1} \left|\sum_{v \bmod \ell} \chi(R_{j'}v + S_{j'}) - \chi(S_{j'})\right| = \ell^{e-1} \left|\sum_{u \bmod \ell} \chi(u) - \chi(S_{j'})\right| \le \ell^{e-1}$; here we have recalled that $\ell \nmid R_{j'}$ (by choice of $C_0$). Now (5.30) follows by combining these observations with the fact that there are $\ell - 2$ many characters mod $\ell^e$ with conductor $\ell$.

Letting $\widetilde{q} \coloneqq \prod_{\substack{\ell^e\|q \\ \ell\le C_0}} \ell^e$ as before, we fix an integer $\kappa > C_0 + 3$, and write $\#\widetilde{\mathcal{V}}_{N,1}(\widetilde{q}; w) = \frac{1}{\varphi(\widetilde{q})} \sum_{\chi \bmod \widetilde{q}} \overline{\chi}(w) \prod_{j=1}^N Z_{\widetilde{q};\ \chi}(F_j) = \mathcal{S}' + \mathcal{S}''$, where $\mathcal{S}'$ again denotes the sum over those $\chi$ mod $\widetilde{q}$ for which $\mathfrak{f}(\chi)$ is $(\kappa+1)$-free. Then (5.21) continues to hold, and $\mathcal{S}'' = 0$, once again since there are no critical points. This yields $\widetilde{\mathcal{V}}_{N,1}(\widetilde{q}; w)/\varphi(\widetilde{q})^N = (\varphi(Q_0)/\varphi(\widetilde{q})) \cdot (\#\widetilde{\mathcal{V}}_{N,1}(Q_0; w)/\varphi(Q_0)^N)$, which along with (5.30) establishes Proposition 5.4 in the remaining case $D_0 = 1$. $\square$

While proving Theorem 2.3, we will also need the following variant of the Proposition 5.4, whose argument in a simpler version of that given for (5.9). Indeed applying (5.10) with $e := 1$, and recalling the two assertions around (5.15), we obtain the following corollary.

**Corollary 5.5.** *In the setting preceding Proposition 5.4, the following estimates hold uniformly in coprime residues $w_1, \dots, w_K$ to squarefree moduli $q$.*

**(a)** *For each fixed $N \geq 2K+1$,*

$$\frac{\#\widetilde{\mathcal{V}}_{N,K}\left(q; (w_i)_{i=1}^K\right)}{\varphi(q)^N} \ll \frac{1}{\varphi(q)^K} \exp\left(O(\sqrt{\log q})\right), \tag{5.32}$$

*if $q$ satisfies $IFH(G_{1,r}, \dots, G_{K,r}; B_0)$ for each $r \in [L]$.*

**(b)** *For each fixed $N \geq 1$,*

$$\frac{\#\widetilde{\mathcal{V}}_{N,K}\left(q; (w_i)_{i=1}^K\right)}{\varphi(q)^N} \ll \frac{1}{q^{\min\{K, N/2\}}} \exp\left(O(\omega(q))\right). \tag{5.33}$$

Proposition 4.4 is a special case of Proposition 5.4, with $L := 1$ and $(G_{i,1})_{i=1}^K := (W_{i,k})_{i=1}^K$, so that $D_0 = D = \sum_{i=1}^K \deg W_{i,k}$, $(F_{i,j})_{i=1}^K = (W_{i,k})_{i=1}^K$, $\widetilde{\alpha}_j(q) = \alpha_k(q)$, $\alpha_N^*(q) = \alpha_k(q)^N$, and $\widetilde{\mathcal{V}}_{N,K}\left(q; (w_i)_{i=1}^K\right) = \mathcal{V}_{N,K}^{(k)}\left(q; (w_i)_{i=1}^K\right)$. (Here of course, all the quantities on the right hand side are in accordance with notation until the previous section.) This also completes the proof of Proposition 4.3.

**Remark.** Taking $K = L = N = 1$ and $G_{1,1} = H \in \mathbb{Z}[T]$ with $\deg H = d > 1$ in (5.6), we get

$$\frac{1}{\varphi(q)} \#\{v \in U_q : H(v) \equiv w \pmod{q}\} \ll \frac{\exp\left(O(\omega(q))\right)}{q^{1/d}} \ll_\delta \frac{1}{q^{1/d - \delta}} \tag{5.34}$$

for any fixed $\delta > 0$. This is only slightly weaker than the results of Konyagin in [18, 19].

In order to deduce Theorem 4.2 from Proposition 4.3, we apply the orthogonality of Dirichlet characters to see that the main term in the right hand side of (4.6) is equal to

$$\frac{1}{\varphi(q)^K} \sum_{\substack{n \leq x \\ (f(n), q) = 1}} 1 + \frac{1}{\varphi(q)^K} \sum_{(\chi_1, \dots, \chi_K) \neq (\chi_{0,Q_0}, \dots, \chi_{0,Q_0}) \bmod Q_0} \left(\prod_{i=1}^K \overline{\chi}_i(a_i)\right) \sum_{n \leq x} \mathbb{1}_{(f(n),q)=1} \prod_{i=1}^K \chi_i(f_i(n)).$$

Let $Q := \prod_{\ell \mid q} \ell$ denote the radical of $q$. To obtain Theorem 4.2, it remains to prove that each inner sum above is $o\left(\sum_{\substack{n \leq x \\ (f(n),q)=1}} 1\right)$. For $Q \ll 1$, this follows by applying Theorem N to the divisor $Q^* := \operatorname{lcm}[Q, Q_0] \ll 1$ of $q$. (Note that as $q$ lies in $\mathcal{Q}(k; f_1, \cdots, f_K)$, so does $Q^*$, since $q$ and $Q^*$ have the same prime factors). So we may assume that $Q$ is sufficiently large. Theorem 4.2 would follow once we show the result below. Here $\lambda$ and $Q_0$ are as in Proposition 4.3.