if $\alpha_k(Q) = c_{\widehat{\chi}} = 1$, then the complex plane is cut along the line $\sigma \le \beta_e/k$. (In the last case, if there is no Siegel zero, then there is no cut.) Fix μ_0 satisfying $\max\{0.7, k/(k+1)\} < \mu_0 < 1$.

7.1. **Analysis of the Dirichlet series.** We start by giving a meromorphic continuation of $F_\chi(s)$ to a larger region. To do this, set $\mathcal{L}_Q(t) := \log(Q(|tk|+1))$ and recall that there exists an absolute constant $c_1 > 0$ such that the product $\prod_{\psi \bmod Q} L(s,\psi)$ has at most one zero β_e (counted with multiplicity) in the region $\sigma > 1 - c_1/\log(Q(|t|+1))$, called the "Siegel zero", which is necessarily real and simple. If β_e exists, then it is a root of $L(s,\psi_e)$ for some real character ψ_e mod Q, which we will be referring to as the "exceptional character". By reducing c_1 if necessary, we may assume that $c_1 < 1-\mu_0$, and that the conductor of ψ_e (which is squarefree) is large enough that it is not $(D+2)$-smooth.

Lemma 7.1. *The Dirichlet series $F_\chi(s)$ is absolutely convergent on the half-plane $\sigma > \frac{1}{k}$, where it satisfies*

$$F_\chi(s) = F_1(sk)^{c_{\widehat{\chi}}}\, g(sk)^{c_{\widehat{\chi}}}\, G_{\chi,1}(s)\, G_{\chi,2}(s) \tag{7.1}$$

with

$$F_1(sk) = \left(\prod_{Q_1|Q} \prod_{\substack{\psi \bmod Q_1\\ \psi \text{ primitive}}} L(sk,\psi)^{\gamma(\psi)}\right)^{\alpha_k(Q)}$$

$$g(sk) = \left(\prod_{Q_1|Q} \prod_{\substack{\psi \bmod Q_1\\ \psi \text{ primitive}}} \prod_{\ell | \frac{Q}{Q_1}} \left(1 - \frac{\psi(\ell)}{\ell^{ks}}\right)^{\gamma(\psi)}\right)^{\alpha_k(Q)}, \quad \gamma(\psi) = \frac{1}{\alpha_k(Q)\varphi(Q)} \sum_{\substack{v \in U_Q\\ W_k(v)\in U_Q}} \overline{\chi}(v).$$

Here, the functions $F_1(sk)$, $g(sk)$, $G_{\chi,1}(s)$ and $G_{\chi,2}(s)$ satisfy the following properties:

(i) *$F_1(sk)$ is holomorphic and nonvanishing in the region $\left\{s : \sigma > \frac{1}{k}\left(1 - \frac{c_1}{\mathcal{L}_Q(t)}\right)\right\} \setminus \left\{\frac{1}{k}, \frac{\beta_e}{k}\right\}$.*

(ii) *$g(sk)$ and $G_{\chi,1}(s)$ are holomorphic and nonvanishing in the half-plane $\sigma > \mu_0/k$, and we have, uniformly for all s in this region,*

$$\max\left\{\left|\frac{g'(sk)}{g(sk)}\right|, \left|\frac{G'_{\chi,1}(s)}{G_{\chi,1}(s)}\right|\right\} \ll \max\{1, (\log Q)^{1-\sigma k}\}\ \log\log Q. \tag{7.2}$$

(iii) *$G_{\chi,2}(s)$ is holomorphic in the half-plane $\sigma > \mu_0/k$, wherein $|G_{\chi,2}(s)|, |G'_{\chi,2}(s)| \ll 1$.*

Proof. For all s in the region $\sigma > 1$, we can use the Euler product of $F_\chi(s)$ to write

$$F_\chi(s) = \left(\prod_{\substack{b\in U_Q\\ W_k(b)\in U_Q}} \prod_{p \equiv b \,(\mathrm{mod}\ Q)} \left(1 - \frac{1}{p^{ks}}\right)^{-c_{\widehat{\chi}}}\right) \cdot \left(\prod_{\substack{p|Q\\ W_k(p)\in U_Q}} \left(1 - \frac{1}{p^{ks}}\right)^{-c_{\widehat{\chi}}}\right) \cdot \prod_p \left(1 + \sum_{v\ge 1} \frac{\mathbb{1}_{f(p^v),Q)=1}}{p^{vs}} \prod_{i=1}^{K} \chi_i(f_i(p^v))\right)\left(1 - \frac{\mathbb{1}_{W_k(p),Q)=1}}{p^{ks}}\right)^{c_{\widehat{\chi}}} \tag{7.3}$$

Since q and Q are supported on the same primes, Q is also k-admissible. By Lemma 3.3 and the fact that $(\chi_1, \cdots, \chi_K) \in \mathcal{C}_k(Q_0)$, we thus find that

$$\prod_{p \gg 1} \left(1 + \sum_{v \geq 1} \frac{\mathbb{1}_{f(p^v),Q)=1}}{p^{vs}} \prod_{i=1}^{K} \chi_i(f_i(p^v))\right)\left(1 - \frac{\mathbb{1}_{W_k(p),Q)=1}}{p^{ks}}\right)^{c_{\widehat{\chi}}} \tag{7.4}$$

$$= \prod_{p \gg 1} \left(1 + \frac{c_{\widehat{\chi}} \mathbb{1}_{W_k(p),Q)=1}}{p^{ks}} + O\left(\frac{1}{p^{(k+1)\sigma}}\right)\right)\left(1 - \frac{\mathbb{1}_{W_k(p),Q)=1}}{p^{ks}}\right)^{c_{\widehat{\chi}}} = \prod_{p \gg 1}\left(1 + O\left(\frac{1}{p^{(k+1)\sigma}}\right)\right),$$

which is an absolutely convergent product in the half plane $\sigma > 1/k$, showing the absolute convergence of the Dirichlet series $F_\chi(s)$ in the same half plane.

Now for $\sigma > 1/k$, the orthogonality of Dirichlet characters mod Q and the fact that $\log L(sk, \psi) = \sum_{p,v} \psi(p^v)/p^{vsk}$ show that the logarithm of the first double product in (7.3) is equal to

$$c_{\widehat{\chi}} \sum_{\substack{b \in U_Q \\ W_k(b) \in U_Q}} \left\{ \sum_p \frac{1}{\varphi(Q)} \sum_{\psi \bmod Q} \overline{\psi}(b) \frac{\psi(p)}{p^{ks}} + \sum_{v \geq 2} \sum_{p \equiv b \pmod Q} \frac{1}{vp^{vks}} \right\}$$

$$= \alpha_k(Q) c_{\widehat{\chi}} \sum_{\psi \bmod Q} \gamma(\psi) \log L(sk, \psi) + c_{\widehat{\chi}} \sum_{\substack{b \in U_Q \\ W_k(b) \in U_Q}} \sum_{v \geq 2} \left(\sum_{p \equiv b \pmod Q} \frac{1}{vp^{vks}} - \sum_{p:\ p^v \equiv b \pmod q} \frac{1}{vp^{vks}} \right).$$

We insert this into (7.3), noting that $L(sk, \psi) = L(sk, \psi^*) \prod_{\ell | \frac{Q}{Q_1}} (1 - \psi^*(\ell)/\ell^{sk})$ and that $\gamma(\psi) = \gamma(\psi^*)$ if the primitive character ψ^* mod Q_1 induces ψ mod Q. This yields (7.1), with

$$G_{\chi,2}(s) := \prod_{p \leq C_k} \left(1 + \sum_{v \geq 1} \frac{\mathbb{1}_{f(p^v),Q)=1}}{p^{vs}} \prod_{i=1}^{K} \chi_i(f_i(p^v))\right)\left(1 - \frac{\mathbb{1}_{W_k(p),Q)=1}}{p^{ks}}\right)^{c_{\widehat{\chi}}}$$

and

$$G_{\chi,1}(s) := \prod_{p > C_k} \left(1 + \sum_{v \geq 1} \frac{\mathbb{1}_{f(p^v),Q)=1}}{p^{vs}} \prod_{i=1}^{K} \chi_i(f_i(p^v))\right)\left(1 - \frac{\mathbb{1}_{W_k(p),Q)=1}}{p^{ks}}\right)^{c_{\widehat{\chi}}} \tag{7.5}$$

$$\cdot \prod_{\substack{p | Q \\ W_k(p) \in U_Q}} \left(1 - \frac{1}{p^{ks}}\right)^{-c_{\widehat{\chi}}} \cdot \exp\left(c_{\widehat{\chi}} \sum_{\substack{b \in U_Q \\ W_k(b) \in U_Q}} \sum_{v \geq 2} \left(\sum_{p \equiv b \pmod Q} \frac{1}{vp^{vks}} - \sum_{p:\ p^v \equiv b \pmod q} \frac{1}{vp^{vks}} \right)\right),$$

where $C_k > 2^{k/\mu_0}$ is a constant exceeding any k-free integer n satisfying $\gcd(f(n), q) = 1$; recall that by Lemma 3.3, C_k can be chosen to depend only on $\{W_{i,v}\}_{\substack{1 \leq i \leq K \\ 1 \leq v \leq k}}$ (and μ_0).

Now (i) follows by the result quoted before the statement of the Lemma and (iii) is immediate by a mechanical calculation. It is also clear that $g(sk)$ is holomorphic and nonvanishing in the half-plane $\sigma > 0$ and the assertion of (7.2) relevant to $g(sk)$ is an immediate consequence of [44, Lemma 9(ii)]. To show the assertions for $G_{\chi,1}(s)$, we recall that for each prime $p > C_k$, the first local factor defining $G_{\chi,1}(s)$ in (7.5) is $1 + c_{\widehat{\chi}} \mathbb{1}_{W_k(p),Q)=1}/p^{ks} + O(p^{-(k+1)\sigma})$, whereupon a computation analogous to (7.4) shows that the first product (over primes $p > C_k$) in (7.5) is absolutely convergent and defines a holomorphic function in the half plane $\sigma > \mu_0/k$. (Here is it important that $\mu_0/k > 1/(k+1)$.) Likewise the exponential factor in (7.5) defines a holomorphic function in the same half plane, hence so does $G_{\chi,1}(s)$. To see that $G_{\chi,1}(s)$ is also nonvanishing in this region, we need only see that the condition $p > C_k > 2^{k/\mu_0}$ guarantees the nonvanishing of each of the factors in the (absolutely convergent) product over $p > C_k$. Finally, a straightforward computation using (7.5) shows that for $\sigma > \mu_0/k$, we have

$$\frac{G'_{\chi,1}(s)}{G_{\chi,1}(s)} = -c_{\widehat{\chi}} k \sum_{\substack{p \mid Q \\ W_k(p) \in U_Q}} \frac{\log p}{p^{ks}} + O(1) \ll \sum_{p \mid Q} \frac{\log p}{p^{k\sigma}},$$

completing the proof of (7.2) via [44, Lemma 3(i)(a)]. □

Our objective is to relate the sum in Theorem 5.6 to the Dirichlet series $F_\chi(s)$ by an effective version of Perron's formula, and shift the contour to the left of the line $\sigma = 1/k$. As such, we will need the following proposition in order to estimate the resulting integrals.

To set up, we choose $\epsilon_1 := \epsilon_1(\lambda)$ to be a constant (depending only on λ) satisfying $0 < \epsilon_1 < 1 - \cos(2\pi/d)$ for any positive integer $d \le \lambda$. Consider the functions

$$\widetilde{F}_\chi(s) := F_1(sk)^{c_{\widehat{\chi}}}\, g(sk)^{c_{\widehat{\chi}}}\, G_{\chi,1}(s)$$

$$\widetilde{H}_\chi(s) := \widetilde{F}_\chi(s) \left(s - \frac{1}{k}\right)^{\alpha_k(Q) c_{\widehat{\chi}}} \left(s - \frac{\beta_e}{k}\right)^{-\alpha_k(Q) c_{\widehat{\chi}} \gamma(\psi_e)}, \quad H_\chi(s) := \frac{\widetilde{F}_\chi(s)}{s} \left(s - \frac{1}{k}\right)^{\alpha_k(Q) c_{\widehat{\chi}}},$$

where here and in what follows, any term or factor involving β_e is to be understood as omitted if the Siegel zero doesn't exist. By assertions (i) and (ii) of the previous lemma, we see that:

- $\widetilde{F}_\chi(s)$ is holomorphic and nonvanishing in the region $\left\{s : \sigma > \frac{1}{k}\left(1 - \frac{c_1}{\mathcal{L}_Q(t)}\right),\ s \ne \frac{1}{k}, \frac{\beta_e}{k}\right\}$,
- $H_\chi(s)$ is holomorphic and nonvanishing in the region $\left\{s : \sigma > \frac{1}{k}\left(1 - \frac{c_1}{\mathcal{L}_Q(t)}\right),\ s \ne \frac{\beta_e}{k}\right\}$,
- $\widetilde{H}_\chi(s)$ is holomorphic and nonvanishing in the region $\left\{s : \sigma > \frac{1}{k}\left(1 - \frac{c_1}{\mathcal{L}_Q(t)}\right)\right\}$

(Recall our branch cut conventions elucidated at the start of the section.) Let $T := \exp(\sqrt{\log x})$.

Proposition 7.2. *We have the following bounds:*

(i) $|H_\chi(1/k)| \ll (\log x)^{\alpha_k(Q)\epsilon_1/5}$.

(ii) $|\widetilde{H}_\chi(s)| \ll (\log x)^{\alpha_k(Q)\epsilon_1/4}$ *uniformly for real s satisfying* $\frac{1}{k}\left(1 - \frac{c_1}{4\log Q}\right) \le s \le \frac{1}{k}$.

(iii) $|F_\chi(s)| \ll (\log x)^{(1/2+\epsilon_1)\alpha_k(Q)}$ *uniformly for complex numbers s satisfying* $\sigma \ge \frac{1}{k}\left(1 - \frac{c_1}{2\mathcal{L}_Q(t)}\right)$, $|t| \le T$ *and* $|s - \theta/k| \gg 1/\mathcal{L}_Q(t)$ *for* $\theta \in \{1, \beta_e\}$.

(iv) *Uniformly in real $s \le 1/k$ satisfying* $s \ge \frac{1}{k}\left(\frac{2}{3} + \frac{\beta_e}{3}\right)$ *(if the Siegel zero exists) or* $s \ge \frac{1}{k}\left(1 - \frac{c_1}{4\log Q}\right)$ *(otherwise), we have*

$$\left| H_\chi\left(\frac{1}{k}\right) G_{\chi,2}\left(\frac{1}{k}\right) - H_\chi(s) G_{\chi,2}(s) \right| \ll (\log x)^{(1/20+\alpha_k(Q)/5)\epsilon_1} \left(\frac{1}{k} - s\right).$$