where $\mathcal{T}_1$ denotes the sum ignoring the distinctness condition on the $p_1, \dots, p_{k(K-1)}$, and $\mathcal{T}_2$ denotes the sum over all the tuples $(p_1, \dots, p_{k(K-1)})$ for which $p_i = p_j$ for some $i \neq j \in [k(K-1)]$. Now $\mathcal{T}_1 = \prod_{1 \le j \le k(K-1)} \left(\sum_{q < p_j \le x^{1/4Kk^2}} p_j^{-(1+1/k)} \right) \gg 1/q^{K-1} (\log q)^{k(K-1)}$ while $\mathcal{T}_2 \ll \left(\sum_{p>q} p^{-(2+2/k)} \right) \left(\sum_{p>q} p^{-(1+1/k)} \right)^{k(K-1)-2} \ll 1/q^K$. Consequently, the expression on the right hand side of (11.11) is $\gg d^{\omega(q)} x^{1/k} / \varphi(q)^K (\log_2 x)^{k(K-1)+1} \log x$, which by Proposition 3.1, grows strictly faster than $\varphi(q)^{-K} \#\{n \le x : \gcd(f(n), q) = 1\}$ as soon as $d^{\omega(q)} > (\log x)^{(1+\epsilon)\alpha_k}$. We have already constructed such q in subsection § 8.1. Hence, the condition $P_{k(Kk+K-k)+1}(n) > q$ in Theorem 2.3(a) is optimal.

Optimality in Theorem 2.3(b). We now address the optimality of the input restrictions in Theorem 2.3(b). For $K = 1$, we are assuming $P_2(n_k) > q$ when $W_k = W_{1,k}$ is not squarefull, and this is optimal for it cannot be replaced by the condition $P(n_k) > q$, as shown in (8.2). Turning to the condition $P_{2K+1}(n_k) > q$, we claim that it cannot be replaced by $P_{2K-1}(n_k) > q$ for any $K \ge 1$, even if $\prod_{i=1}^K W_{i,k}$ is assumed to be separable. Having already shown this above for $K = 1$, we assume that $K \ge 2$.

To show our claim above, we continue with the same definitions of $\{W_{i,k}\}_{1 \le i \le K}$, $\widetilde{C}_0$, ℓ_0 and q. Let $f_1, \dots, f_K \colon \mathbb{N} \to \mathbb{Z}$ be any multiplicative functions satisfying $f_i(p^v) := W_{i,v}(p)$ and $f_i(p^{2k}) := 1$ for all primes p, all $i \in [K]$ and $v \in [k]$. Consider any $n \le x$ of the form $(p_1 \cdots p_{K-1})^{2k} P^k$ with $P, p_1, \dots, p_{K-1}$ being primes satisfying $P := P(n) > x^{1/3k}$, $q < p_{K-1} < \cdots < p_1 < x^{1/4Kk}$ and $\prod_{1 \le j \le d} (P - 2j) \equiv 0 \pmod q$. Then $n_k = (p_1 \cdots p_{K-1})^2 P$, $P_{2K-1}(n_k) = p_{K-1} > q$ and $f_i(n) = W_{i,k}(P) \equiv 2(2i-1) \pmod q$ for each $i \in [K]$. Given $p_1, \dots, p_{K-1}$, the number of possible P is $\gg d^{\omega(q)} x^{1/k} / \varphi(q) (p_1 \cdots p_{K-1})^2 \log x$, since $(p_1 \cdots p_{K-1})^2 \le x^{2(K-1)/4Kk}$ $\le x^{1/2k}$. Proceeding exactly as above, we find that the number of $n \le x$ having $P_{2K-1}(n_k) > q$ and satisfying $f_i(n) \equiv 2(2i-1) \pmod q$ for all i, is $\gg d^{\omega(q)} x^{1/k} / \varphi(q)^K (\log_2 x)^K \log x$. The same q as mentioned before satisfy $d^{\omega(q)} > (\log x)^{(1+\epsilon)\alpha_k}$, making this last expression grow strictly faster than $\varphi(q)^{-K} \#\{n \le x : \gcd(f(n), q) = 1\}$. The condition $P_{2K+1}(n_k) > q$ in Theorem 2.3(b) is thus nearly optimal in that it cannot be replaced by $P_{2K-1}(n_k) > q$.

12. Restricted inputs with higher polynomial control: Proof of Theorem 2.4

By the same initial reductions as in the proofs of Theorems 2.2 and 2.3, it suffices to show that, with the respective values of R in the two subparts, we have

$$\sum\nolimits^*_{n:\ P_R(n) > q} 1 \ll \frac{x^{1/k}}{\varphi(q)^K (\log x)^{1 - 2\alpha_k/3}}, \tag{12.1}$$

The subsequent calculations will hold for either value of R until stated explicitly. Note that (for the first time in our proofs), we will allow our implied constants to depend on V, and on the full set of polynomials $\{W_{i,v}\}_{\substack{1 \le i \le K \\ 1 \le v \le V}}$.

We will first show that in either of the two subparts of the theorem, the contribution to the left hand side of (12.1) from the n's which are divisible by the $(V+1)$-th power of a prime exceeding q can be absorbed in the right hand side. Any such n can be written in the form $mp^c P^k$, where $P := P(n) > z$, $p \in (q, P)$ is prime, $c \ge V+1$, $P_{Jk}(m) \le y$ and P mod

$q \in \mathcal{V}_{1,K}^{(k)}\big(q; (a_i f_i(mp^c)^{-1})\big)$. Proceeding as in the proof of the second bound in (9.4), we see that the contribution of such n is $\ll \frac{V'_{1,K}}{\varphi(q)q^{(V+1)/k-1}} \cdot \frac{x^{1/k}}{(\log x)^{1-2\alpha_k/3}}$. For general q, an application of (5.34) (with H being a polynomial among $\{W_{i,k}\}_{1\le i\le K}$ having least degree) shows that the expression above is $\ll x^{1/k}\big/q^K(\log x)^{1-2\alpha_k/3}$, since $(V+1)/k - 1 + 1/D_{\min} > K$ by the hypothesis of Theorem 2.4(a). On the other hand, if q is squarefree, then from $V'_{1,K} \ll D_{\min}^{\omega(q)}$ and $V \ge Kk$, it follows that the contribution of such n is once again $\ll x^{1/k}\big/q^K(\log x)^{1-2\alpha_k/3}$.

To prove (12.1), it thus only remains to consider the contribution of the n's for which $v_p(n) \le V$ for any prime $p > q$. We may further restrict to those n which have $\omega^*(n) \in [Kk-1]$ and $\omega_\parallel(n) \in [KD]$ (resp. $\omega_\parallel(n) \in [2K]$ if q is squarefree). This is because the contribution of the n having $\omega_\parallel(n) \ge KD+1$ (resp. $\omega_\parallel(n) \ge 2K+1$) has already been bounded in the first bound in (9.4) (resp. (11.4)), while the contribution of the n having $\omega^*(n) \ge Kk$ has already been bounded in the third bound of (9.4), and finally since any n for which $\omega^*(n) = 0$ must anyway have $\omega_\parallel(n) \ge KD+1$ (resp. $\omega_\parallel(n) \ge 2K+1$) as $R \ge k(KD+1)$ (resp. $R \ge k(2K+1)$). It thus remains to show that for a given $r \in [KD]$ (resp. $r \in [2K]$) and $s \in [Kk-1]$, we have

$$\widetilde{\mathcal{M}}_{r,s} \ll \frac{x^{1/k}(\log_2 x)^{O(1)}}{q^K \log x}, \tag{12.2}$$

where $\widetilde{\mathcal{M}}_{r,s}$ denotes the contribution to the left hand side of (12.1) from all the n having $\omega_\parallel(n) = r$, $\omega^*(n) = s$, and $k \le v_p(n) \le V$ for all $p > q$ dividing n. For given r and s,

$$\widetilde{\mathcal{M}}_{r,s} \le \sum_{\substack{c_1,\dots,c_s \in [k+1,V] \\ c_1+\dots+c_s \ge R-kr}} \widetilde{\mathcal{M}}_{r,s}(c_1,\dots,c_s), \tag{12.3}$$

with $\widetilde{\mathcal{M}}_{r,s}(c_1,\dots,c_s)$ denoting the count of n counted in $\widetilde{\mathcal{M}}_{r,s}$ which can be written in the form $mp_1^{c_1}\cdots p_s^{c_s}P_1^k\cdots P_r^k$, with $p_1,\dots,p_s,P_1,\dots,P_r$ being distinct primes exceeding q, $P(m) \le q$, $P_1 = P(n) > z$, $P_r < \dots < P_1$, *and* $f_i(n) = f_i(m)\prod_{l=1}^s W_{i,c_l}(p_l) \cdot \prod_{j=1}^r W_{i,k}(P_j)$. With $\mathcal{V}_{r+s,K}(q;\ (c_j)_{j=1}^s;\ (w_i)_{i=1}^K)$ being the set of tuples $(u_1,\dots,u_s,v_1,\dots,v_r) \in U_q^{s+r}$ satisfying the congruences $\prod_{l=1}^s W_{i,c_l}(u_l)\cdot\prod_{j=1}^r W_{i,k}(v_j) \equiv w_i \pmod q$ for each $i \in [K]$, the conditions $f_i(n) \equiv a_i \pmod q$ amount to $(p_1,\dots,p_s,\ P_1,\dots,P_r) \bmod q \ \in\ \mathcal{V}_{r+s,K}\big(q;\ (c_j)_{j=1}^s;\ (a_if_i(m)^{-1})_{i=1}^K\big)$.

Given m and $(u_1,\dots,u_s,v_1,\dots,v_r) \in \mathcal{V}_{r+s,K}\big(q;\ (c_j)_{j=1}^s;\ (a_if_i(m)^{-1})_{i=1}^K\big)$, we bound the number of possible $(p_1,\dots,p_s,P_1,\dots,P_r)$ satisfying $(p_1,\dots,p_s,P_1,\dots,P_r) \equiv (u_1,\dots,u_s,v_1,\dots,v_r) \bmod q$. First, given $(p_1,\dots,p_s)$, the number of possible $(P_1,\dots,P_r)$ is, by the arguments leading to (9.5), $\ll x^{1/k}(\log_2 x)^{O(1)}\big/\varphi(q)^r p_1^{c_1/k}\cdots p_s^{c_s/k} m^{1/k}\log x$. We sum this over possible $p_1,\dots,p_s > q$, making use of the observation that for fixed $\varepsilon_1 > 0$, we have $\sum_{\substack{n>q \\ n\equiv u \pmod q}} 1/n^{1+\theta} \ll_{\varepsilon_1} 1/q^{1+\theta}$, uniformly in residue classes u mod q, and uniformly in $\theta > \varepsilon_1$. We find that the number of possible $(p_1,\dots,p_s,P_1,\dots,P_r)$ is $\ll x^{1/k}(\log_2 x)^{O(1)}\big/\varphi(q)^r q^{(c_1+\dots+c_s)/k} m^{1/k}\log x$. Finally summing the above expression over all possible $(u_1,\dots,u_s,v_1,\dots,v_r)$ and then over all m via (9.10), we obtain

$$\widetilde{\mathcal{M}}_{r,s}(c_1,\dots,c_s) \ll \frac{1}{q^{(c_1+\dots+c_s)/k-s}} \cdot \frac{V'_{r+s,K}(q;\ (c_j)_{j=1}^s)}{\varphi(q)^{r+s}} \cdot \frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x}, \tag{12.4}$$

where $V'_{r+s,K}(q;\ (c_j)_{j=1}^s) := \max\big\{\#\mathcal{V}_{r+s,K}(q;\ (c_j)_{j=1}^s;\ (w_i)_{i=1}^K) : (w_i)_{i=1}^K \in U_q^K\big\}$.

Completing the proof of Theorem 2.4(a). We specialize to $R := \max\{k(KD+1), (Kk-1)D_0+2\}$, and apply Proposition 5.4(b) with $(G_{i,r})_{\substack{1\le i\le K\\ 1\le r\le L}}$ being the system $(W_{i,v})_{\substack{1\le i\le K\\ k\le v\le V}}$, so that $G_{i,r} := W_{i,k+r-1}$ and $\sum_{i=1}^K \deg G_{i,r} = D_{k+r-1}$. We also set $N := r+s$, and define $\{F_{i,j}\}_{\substack{1\le i\le K\\ 1\le j\le N}}$ by setting (for all $i \in [K]$) $F_{i,j} := W_{i,c_j}$ for $j \in [s]$ and $F_{i,j} := W_{i,k}$ for $s+1 \le j \le s+r$, so that $\widetilde{\mathcal{V}}_{N,K}\big(q; (w_i)_{i=1}^K\big) = \mathcal{V}_{r+s,K}(q;\ (c_j)_{j=1}^s;\ (w_i)_{i=1}^K)$.

If $r+s \ge KD_0+1$, then (5.6) (applied to $N := r+s \in [KD_0+1, KD+Kk-1]$ [10]) yields $V'_{r+s,K}(q;\ (c_j)_{j=1}^s)/\varphi(q)^{r+s} \ll q^{-K}\exp\big(O(\omega(q))\big)$. Inserting this into (12.4) and using that $(c_1+\cdots+c_s)/k - s \ge s/k \ge 1/k$, we obtain $\widetilde{\mathcal{M}}_{r,s}(c_1,\dots,c_s) \ll x^{1/k}(\log_2 x)^{O(1)}/q^K \log x$. On the other hand, if $r+s \le KD_0$, then (5.6) and (12.4) lead to

$$\widetilde{\mathcal{M}}_{r,s}(c_1,\dots,c_s) \ \ll\ \frac{\big(\prod_{\ell^e \| q} e\big)\exp\big(O(\omega(q))\big)}{q^{\max\{s/k+(r+s)/D_0,\ R/k-(1-1/D_0)(r+s)\}}}\cdot\frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x},$$

where we have recalled that $(c_1+\cdots+c_s)/k - s \ge \max\{s/k, R/k-r-s\}$. Since $R > (Kk-1)D_0+1$, it is easy to check that the exponent of q above exceeds K. This proves that $\widetilde{\mathcal{M}}_{r,s}(c_1,\dots,c_s) \ll x^{1/k}(\log_2 x)^{O(1)}/q^K\log x$ for any tuple $(c_1,\dots,c_s)$ counted in the sum (12.3), and since there are $O(1)$ many such tuples, we obtain the desired bound (12.2).

Completing the proof of Theorem 2.4(b). This time we use Corollary 5.5 in place of Proposition 5.4. If $r+s \ge 2K+1$, then (5.33) yields $V'_{r+s,K}(q;\ (c_j)_{j=1}^s) \ll q^{-K}\exp\big(O(\omega(q))\big)$. Inserting this into (12.4) and again using $(c_1+\cdots+c_s)/k - s \ge s/k \ge 1/k$ shows that $\widetilde{\mathcal{M}}_{r,s}(c_1,\dots,c_s) \ll x^{1/k}(\log_2 x)^{O(1)}/q^K\log x$ in this case. On the other hand, if $r+s \le 2K$, then (5.33) yields

$$\widetilde{\mathcal{M}}_{r,s}(c_1,\dots,c_s) \ \ll\ \frac{\exp\big(O(\omega(q))\big)}{q^{\max\{s/k+(r+s)/2,\ R/k-(r+s)/2\}}}\cdot\frac{x^{1/k}(\log_2 x)^{O(1)}}{\log x},$$

and it is easy to see that the exponent of q above always exceeds K. $\square$

This finally establishes Theorems 2.1 to 2.4. As such, we shall no longer continue with the set-up for these results. In the next section, we shall prove Theorems 2.5 and 2.6, and thus shall only be assuming the hypotheses mentioned explicitly in their respective statements.

13. Necessity of the multiplicative independence and invariant factor hypotheses: Proofs of Theorems 2.5 and 2.6

We first give a lower bound that will be useful in both the theorems. Until we specialize to each theorem, we will not assume anything about $\{W_{i,k}\}_{1\le i\le K} \in \mathbb{Z}[T]$ beyond that they are nonconstant, and our estimates will be uniform in all $q \le (\log x)^{K_0}$ and $(a_i)_{i=1}^K \in U_q^K$.

Let $y := \exp(\sqrt{\log x})$ and given any fixed $R \ge 1$, we let $V'_q := \mathcal{V}^{(k)}_{R,K}\big(q; (a_i)_{i=1}^K\big) = \{(v_1,\dots,v_R) \in U_q^R : (\forall i \in [K])\ \prod_{j=1}^R W_{i,k}(v_j) \equiv a_i \pmod q\}$. Consider any $N \le x$ of the form $N = (P_1\cdots P_R)^k$, where $P_1,\dots,P_R$ are primes satisfying $y < P_R < \cdots < P_1$, and $(P_1,\dots,P_R)$ mod

[10]Here we are of course assuming that such r and s exist in the first place, which amounts to having $KD_0+1 \le KD+Kk-1$