The intuitive explanation for such constraints on our inputs n comes from a certain 'mixing' phenomenon in the unit group mod q, which can be interpreted as a quantitative ergodicity phenomenon for random walks on multiplicative groups. For example, let q be an odd positive integer. From the set of units u mod q for which $u+1$ is also a unit, choose uniformly at random $u_1, u_2, u_3, \ldots$, and construct the sequence of partial products $u_1+1, (u_1+1)(u_2+1), (u_1+1)(u_2+1)(u_3+1), \ldots$. Then as we go further into the sequence, each unit mod q is roughly equally likely to appear as one of the products $(u_1+1)\cdots(u_J+1)$. This particular example lies at the core of the weak equidistribution of $\sigma(n)$ to odd moduli. The phenomenon for $\sigma(n)$ to even moduli not divisible by 3 is analogous, except that we work with units u mod q for which u^2+u+1 is also a unit mod q.

In general, for certain collections of K multivariate polynomials, the values taken by them that are coprime to q are jointly equidistributed among the unit group mod q whenever the number of variables is significantly larger compared to K: having a large number of variables amplifies the power savings in certain character sum bounds, thus ensuring that any K congruences (coming from the K polynomials) maximally "cut down" the ambient space of tuples. This is reminiscent of a very common phenomenon occurring in several applications of the circle method, such as in Waring's problem. It is to have this large number of variables that it becomes necessary to restrict our inputs n to those having sufficiently many large prime factors, so as to restore weak equidistribution in the full "Siegel-Walfisz range" $q \le (\log x)^{K_0}$.

Our arguments for the main results require ideas comprising a variety of themes. One of the central themes is the exploitation of the aforementioned mixing phenomenon in the multiplicative group via methods belonging to the 'anatomy of integers'. In addition, we crucially require several "pure analytic" ideas, where we work with certain "pretentious distances", and we also suitably modify the Landau–Selberg–Delange method to give strong estimates on the mean values of various multiplicative functions taking values in the unit disk. (To this end, we need to analyze a product of L-functions raised to complex powers.) Linear algebra over rings comes into play throughout the paper, – most prominently in combination with methods from combinatorial number theory, – in order to count solution tuples of multiple polynomial congruences in several variables. Furthermore, we need to understand the rational points of certain affine varieties over finite fields using tools from arithmetic and algebraic geometry.

2. The setting and the main results

We say that an arithmetic function f is polynomially-defined if there exists $V \ge 1$ and polynomials $\{W_v\}_{1\le v\le V}$ with integer coefficients satisfying $f(p^v) = W_v(p)$ for all primes p and all $v \in [V]$. To set up for Narkiewicz's general criterion in [28], we consider $K, V \ge 1$ and polynomially-defined multiplicative functions $f_1, \ldots, f_K \colon \mathbb{N} \to \mathbb{Z}$, with defining polynomials $\{W_{i,v}\}_{\substack{1\le i\le K\\ 1\le v\le V}} \subset \mathbb{Z}[T]$ satisfying $f_i(p^v) = W_{i,v}(p)$ for any prime p, and any $i \in [K], v \in [V]$. For any q and $v \in [V]$, define $R_v(q) := \{u \in U_q : \prod_{i=1}^K W_{i,v}(u) \in U_q\}$; here $U_q := (\mathbb{Z}/q\mathbb{Z})^\times$ denotes the multiplicative group mod q, so that saying "$r \in U_q$" for an integer r is synonymous with saying that "$\gcd(r,q)=1$". Fix $k \in [V]$ and assume that $\{W_{i,k}\}_{1\le i\le K}$ are all nonconstant. We say that a positive integer q is k-admissible (with respect to the family $(W_{i,v})_{\substack{1\le i\le K\\ 1\le v\le V}}$) if the set $R_k(q)$ is nonempty but the sets $R_v(q)$ are empty for all $v < k$. We define $\mathcal{Q}(k; f_1, \cdots, f_K)$ to be the set of all k-admissible integers q such that for every tuple $(\chi_1, \ldots, \chi_K) \ne (\chi_0, \ldots, \chi_0)$

of Dirichlet characters[1] mod q for which $\prod_{i=1}^K \chi_i(W_{i,k}(u)) = 1$ for all $u \in R_k(q)$, there exists a prime p satisfying

$$\sum_{j\geq 0} \frac{\chi_1(f_1(p^j))\cdots\chi_K(f_K(p^j))}{p^{j/k}} = 0. \tag{2.1}$$

Narkiewicz's criterion [28, Theorem 1] in this setting is then stated as follows.

Theorem N. *Fix a k-admissible integer q. The functions $f_1, \ldots, f_K$ are jointly weakly equidistributed modulo q if and only if $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$.*

As mentioned in the introduction, the first steps towards obtaining uniform analogues of a special case of Theorem N for a single multiplicative function were taken in [22], [35] and [37]. However, several of the arguments in these papers cannot be generalized to families of multiplicative functions (i.e. the cases $K > 1$), and even in the special case $K = 1$, those results are far from being complete uniform analogues of Theorem N because they crucially need q to be 1-admissible and have sufficiently large prime factors, and also crucially need the defining polynomial $W_{1,1}$ to be separable.

In this work, we extend Narkiewicz's general criterion Theorem N to obtain results that are completely uniform in the modulus q varying up to a fixed but arbitrary power of $\log x$. Our results will mostly not impose any additional restrictions, beyond those that can be *proven* to be necessary and essentially optimal. These results are thus also new for a single multiplicative function as they address all the aforementioned shortcomings of [22], [35] and [37]. For concrete and provably unavoidable reasons (see Theorems 2.5 and 2.6 below), we shall need to impose two additional hypotheses to get uniform analogues of Theorem N. First, we will need the polynomials $\{W_{i,k}\}_{1\leq i\leq K}$ to be multiplicatively independent. Here, we say that the polynomials $\{F_i\}_{1\leq i\leq K} \subset \mathbb{Z}[T]$ are multiplicatively independent (over $\mathbb{Z}$) if there is no nonzero tuple of integers $(c_i)_{i=1}^K$ for which the product $\prod_{i=1}^K F_i^{c_i}$ is identically constant in $\mathbb{Q}(T)$.

For the second hypothesis, we shall need the following set-up. Given nonconstant polynomials $\{F_i\}_{i=1}^K \subset \mathbb{Z}[T]$, we factor $F_i =: r_i \prod_{j=1}^M G_j^{\mu_{ij}}$ with $r_i \in \mathbb{Z}$, $\{G_j\}_{j=1}^M \subset \mathbb{Z}[T]$ being pairwise coprime primitive[2] irreducible polynomials and with μ_{ij} being nonnegative integers, such that each G_j appears with a positive exponent μ_{ij} in some F_i. We let $\omega(F_1 \cdots F_K) \coloneqq M$ and define the exponent matrix of $(F_i)_{i=1}^K$ to be the $M \times K$ matrix

$$E_0 \coloneqq E_0(F_1, \ldots, F_K) \coloneqq \begin{pmatrix} \mu_{11} & \cdots & \mu_{K1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \mu_{1M} & \cdots & \mu_{KM} \end{pmatrix} \in \mathbb{M}_{M\times K}(\mathbb{Z}),$$

so that E_0 has a positive entry in each row. By the theory of modules over a principal ideal domain, E_0 has a Smith Normal Form given by the $M \times K$ diagonal matrix $\mathrm{diag}(\beta_1, \ldots, \beta_r)$, where $r \coloneqq \min\{M, K\}$ and $\beta_1, \ldots, \beta_r$ are integers (possibly zero) satisfying $\beta_j \mid \beta_{j+1}$ for each $1 \leq j < r$ (for the moment, we accept the convention that $0 \mid 0$). The β_j are often called the

[1]Here χ_0 or $\chi_{0,q}$ denotes, as usual, the trivial or principal character mod q.

[2]A polynomial in $\mathbb{Z}[T]$ is said to be primitive when the greatest common divisor of its coefficients is 1.

invariant factors of the matrix E_0.[3] We shall use $\beta(F_1, \dots, F_K)$ to denote the last invariant factor β_r. (Here we fixed some ordering of the G_j to define the exponent matrix $E_0(F_1, \dots, F_K)$ but the invariant factors are independent of this ordering.) We now state our second hyothesis:

Invariant Factor Hypothesis: Given $B_0 > 0$, we shall say that a positive integer q satisfies $IFH(F_1, \dots, F_K; B_0)$ if $\gcd(\ell - 1, \beta(F_1, \dots, F_K)) = 1$ for any prime $\ell \mid q$ satisfying $\ell > B_0$.

For instance, if $\prod_{i=1}^K F_i$ is separable over $\mathbb{Q}$ (or more generally, if the exponent matrix $E_0(F_1, \dots, F_K)$ is equivalent to the diagonal matrix $\mathrm{diag}(1, \dots, 1)$), then $\beta(F_1, \dots, F_K) = 1$, so any q satisfies $IFH(F_1, \dots, F_K; B_0)$ for any $B_0 > 0$. Note that the polynomials $\{F_i\}_{i=1}^K \subset \mathbb{Z}[T]$ are multiplicatively independent if and only if the columns of $E_0(F_1, \dots, F_K)$ are $\mathbb{Q}$-linearly independent. In this case, $\omega(F_1 \cdots F_K) = M \geq K$ and $\beta(F_1, \dots, F_K) = \beta_K \neq 0$ as the computation of the Smith normal form is a base-change over $\mathbb{Z}$.

We now state the main results of this manuscript, uniform analogues of Theorem N. The following set-up will be assumed in the main results below: Fix $K, V \geq 1$ and $K_0, B_0 > 0$.

- Consider multiplicative functions $f_1, \dots, f_K \colon \mathbb{N} \to \mathbb{Z}$ and polynomials $\{W_{i,v}\}_{\substack{1 \leq i \leq K \\ 1 \leq v \leq V}} \subset \mathbb{Z}[T]$ satisfying $f_i(p^v) = W_{i,v}(p)$ for any prime p, any $i \in [K]$ and $v \in [V]$.
- Consider the multiplicative function $f := \prod_{i=1}^K f_i$ and the polynomials $\{W_v\}_{1 \leq v \leq V} \subset \mathbb{Z}[T]$ given by $W_v := \prod_{i=1}^K W_{i,v}$, so that $f(p^v) = W_v(p)$ for all primes p and all $v \in [V]$.
- For any q and $v \in [V]$, define $R_v(q)$ as before the statement of Theorem N so that $R_v(q) = \{u \in U_q : W_v(u) \in U_q\}$. Let $\alpha_v(q) := \frac{1}{\varphi(q)} \# R_v(q)$. Also fix $k \in [V]$ and define k-admissibility and the set $\mathcal{Q}(k; f_1, \cdots, f_K)$ as before Theorem N.
- For each $v \in [V]$, let $D_v := \deg W_v = \sum_{i=1}^K \deg W_{i,v}$, $D := D_k = \sum_{i=1}^K \deg W_{i,k}$, and $D_{\min} := \min_{1 \leq i \leq K} \deg W_{i,k}$. Note that if q is k-admissible, then $\alpha_v(q) = 0$ for $1 \leq v < k$, while $\alpha_k(q) \gg_{W_k} (\log\log(3q))^{-D}$ by the Chinese Remainder Theorem and a standard argument using Mertens' Theorem.
- Assume that $\{W_{i,k}\}_{1 \leq i \leq K}$ are multiplicatively independent.

In Theorems 2.1 to 2.3 below, our implied constants depend only on B_0 and on the polynomials $\{W_{i,v}\}_{\substack{1 \leq i \leq K \\ 1 \leq v \leq k}}$, and are in particular independent of V and of the polynomials $\{W_{i,v}\}_{\substack{1 \leq i \leq K \\ k < v \leq V}}$.

Theorem 2.1. *Fix $\epsilon \in (0,1)$. The functions $f_1, \dots, f_K$ are jointly weakly equidistributed, uniformly to all moduli $q \leq (\log x)^{K_0}$ lying in $\mathcal{Q}(k; f_1, \cdots, f_K)$ and satisfying $IFH(W_{1,k}, \dots, W_{K,k}; B_0)$, provided any one of the following holds.*

(i) *Either $K = 1$ and $W_{1,k} = W_k$ is linear, or $K \geq 2$, $q \leq (\log x)^{(1-\epsilon)\alpha_k(q)/(K-1)}$ and at least one of $\{W_{i,k}\}_{1 \leq i \leq K}$ is linear (i.e., $D_{\min} = 1$).*

(ii) *q is squarefree and $q^{K-1} D_{\min}^{\omega(q)} \leq (\log x)^{(1-\epsilon)\alpha_k(q)}$.*

(iii) *$D_{\min} > 1$ and $q \leq (\log x)^{(1-\epsilon)\alpha_k(q)(K - 1/D_{\min})^{-1}}$.*

[3] In practice, it is usually the nonzero β_j that are called the invariant factors but this terminology will be more convenient for us (and the possibility of any β_j being zero shall soon become obsolete anyway).