Optimality under condition (iii). Fix $d > 1$ and define $W_{i,k}(T) \coloneqq (T-1)^d + i \in \mathbb{Z}[T]$, so that $\prod_{i=1}^K W_{i,k}(T+1) = \prod_{i=1}^K (T^d + i)$ is clearly separable in $\mathbb{Q}[T]$, hence so is $\prod_{i=1}^K W_{i,k}(T)$. Let $q \coloneqq Q^d$ for some $Q \le (\log x)^{K_0/d}$ satisfying $P^-(Q) = \ell_0$. Then $1 \in R_k(q)$, showing that $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$. Moreover, $i \in U_q$ for each $i \in [K]$, and any prime $P \equiv 1 \pmod{Q}$ satisfies $f_i(P^k) = W_{i,k}(P) = (P-1)^d + i \equiv i \pmod{q}$. Consequently, there are $\gg x^{1/k}/q^{1/d} \log x$ many $n \le x$ satisfying $f_i(n) \equiv i \pmod{q}$ for all i, and this last expression grows strictly faster than $\varphi(q)^{-K}\#\{n \le x : (f(n), q) = 1\}$ as soon as $q^{K-1/D_{\min}} = q^{K-1/d} \ge (\log x)^{(1+\epsilon)\alpha_k}$ for some fixed $\epsilon \in (0,1)$. This establishes that the range of q in condition (iii) of Theorem 2.1 is optimal, and concrete examples of moduli q satisfying the conditions imposed so far, are those of the form Q^d, with Q lying in $[(\log x)^{(1+\epsilon)(K-1/d)^{-1}/d}, (\log x)^{K_0/d}]$ and having least prime factor ℓ_0.

9. Restricted inputs to general moduli: Proof of Theorem 2.2

Fix $T \in \mathbb{N}_{>1}$. We first show that as $x \to \infty$ and uniformly in k-admissible $q \le (\log x)^{K_0}$,

$$\sum_{\substack{n \le x:\ P_T(n) \le q \\ \gcd(f(n),q)=1}} 1 = o\Bigg(\sum_{\substack{n \le x \\ \gcd(f(n),q)=1}} 1\Bigg), \quad \sum_{\substack{n \le x:\ P_T(n_k) \le q \\ \gcd(f(n),q)=1}} 1 = o\Bigg(\sum_{\substack{n \le x \\ \gcd(f(n),q)=1}} 1\Bigg). \tag{9.1}$$

The first asymptotic is immediate by Proposition 4.1 as $P_{Jk}(n) \le P_T(n)$. To show the second, we write any n counted in the left side uniquely in the form $n = BN^kA$, where B is k-free, A is $(k+1)$-full and the exponent of any prime in A is not a multiple of k. Then $n_k = N$, and B, N, A are pairwise coprime, so that $f(n) = f(B)f(N^k)f(A)$, and

$$\sum_{\substack{n \le x:\ P_T(n_k) \le q \\ \gcd(f(n),q)=1}} 1 \le \sum_{\substack{B \le x \\ B \text{ is } k\text{-free} \\ (f(B),q)=1}} \sum_{\substack{N,A:\ N^kA \le x/B \\ P_T(N) \le q;\ A \text{ is } (k+1)\text{-full} \\ \gcd(f(N^k)f(A),q)=1}} 1. \tag{9.2}$$

If $A > x^{1/2}$, then $N \le (x/AB)^{1/k} \le x^{1/2k}$. Since A is $(k+1)$-full, the contribution of the tuples (B, N, A) with $A > x^{1/2}$ is $\ll \sum_{B \ll 1} \sum_{N \le x^{1/2k}} (x/BN^k)^{1/(k+1)} \ll x^{1/k - 1/2k(k+1)}$, which is negligible. On the other hand, if $A \le x^{1/2}$, then given B and A, [35, Lemma 2.3] shows there are $\ll x^{1/k}(\log_2 x)^T / B^{1/k}A^{1/k} \log x$ many $N \le (x/AB)^{1/k}$ having $P_T(N) \le q$. The sum over A is $\le \prod_p (1 + \sum_{v \ge k+1} p^{-v/k}) \ll 1$, so that the total contribution of all tuples (B, N, A) with $A \le x^{1/2}$ is $O(x^{1/k}(\log_2 x)^T / \log x)$. The second formula in (9.1) now follows from (3.1).

In all of Theorems 2.2 to 2.4, we may assume q to be sufficiently large, for otherwise these results follow directly from Theorem N and (9.1). These formulae also show the equality of the second and third expressions in (2.2) and (2.3), so it remains to show the first equality in either. Recall that for this theorem, we have $\epsilon \coloneqq 1$ and $y = \exp(\sqrt{\log x})$ in the framework developed in section 4. Now any convenient n has $P_J(n_k) > y$ and hence is counted in the left hand sides of both (2.2) and (2.3). By Theorem 4.2, it suffices to show that the contributions of the inconvenient n to the left hand sides of (2.2) and (2.3) are negligible compared to $\varphi(q)^{-K}\#\{n \le x : (f(n), q) = 1\}$. In fact, by (4.3) and (3.3), it remains to show the bounds (9.3)(i) and (ii) below to establish subparts (a) and (b) of the theorem, respectively:

(9.3)

$$\text{(i)}\ {\sum}^*_{n:\ P_R(n) > q} 1 \ll \frac{x^{1/k}}{\varphi(q)^K (\log x)^{1-2\alpha_k/3}}, \quad \text{(ii)}\ {\sum}^*_{n:\ P_{KD+1}(n_k) > q} 1 \ll \frac{x^{1/k}}{\varphi(q)^K (\log x)^{1-2\alpha_k/3}}.$$

Here and in the rest of the manuscript, any sum of the form $\sum_n^*$ denotes a sum over positive integers $n \le x$ that are not z-smooth, not divisible by the $(k+1)$-th power of a prime exceeding y, have $P_{Jk}(n) \le y$ and satisfy $f_i(n) \equiv a_i \pmod q$ for all $i \in [K]$. Other conditions imposed on this sum are additional to these.

Defining $\omega_\parallel(n) := \#\{p > q : p^k \parallel n\} = \#\{n \le x : p \parallel n_k\}$ and $\omega^*(n) := \#\{p > q : p^{k+1} \mid n\}$, we first show the following three bounds:

(9.4)
$$\sum\nolimits^*_{n:\ \omega_\parallel(n) \ge KD+1} 1,\ \sum\nolimits^*_{\substack{n:\ \omega_\parallel(n)=KD\\ \omega^*(n)\ge 1}} 1,\quad \sum_{\substack{n \le x:\ (f(n),q)=1\\ \omega^*(n)\ge Kk,\ P_{Jk}(n)\le y,\ P(n)>z\\ p>y \implies p^{k+1}\nmid n}} 1 \ \ll\ \frac{x^{1/k}}{\varphi(q)^K (\log x)^{1-2\alpha_k/3}}.$$

Any n counted in the first sum is of the form $m(P_{KD+1}\cdots P_1)^k$, where $P_{Jk}(m) \le y$, where $P_1, \dots, P_{KD+1}$ are primes exceeding q satisfying $P_1 := P(n) > z$ and $q < P_{KD+1} < \cdots < P_1$, and where $f_i(n) = f_i(m)\prod_{j=1}^{KD+1} f_i(P_j^k) = f_i(m)\prod_{j=1}^{KD+1} W_{i,k}(P_j)$. The conditions $f_i(n) \equiv a_i \pmod q$ can be rewritten as $(P_1, \dots, P_{KD+1}) \bmod q \in \mathcal{V}^{(k)}_{KD+1,K}\big(q; (a_i f_i(m)^{-1})_{i=1}^K\big)$. Given m, $(v_1, \dots, v_{KD+1}) \in \mathcal{V}^{(k)}_{KD+1,K}\big(q; (a_i f_i(m)^{-1})_{i=1}^K\big)$, and $P_2, \dots, P_{KD+1}$, the number of P_1 in $(q, x^{1/k}\big/m^{1/k}P_2\cdots P_{KD+1}]$ satisfying $P_1 \equiv v_1 \pmod q$ is $\ll x^{1/k}\log_2 x\big/m^{1/k}P_2\cdots P_{KD+1}\varphi(q)\log x$, by Brun-Titchmarsh. We sum this over all possible $P_2, \dots, P_{KD+1}$, making use of the bound $\sum_{\substack{q<p\le x\\ p\equiv v \pmod q}} 1/p \ll \log_2 x\big/\varphi(q)$ uniformly in $v \in U_q$ (this follows from Brun–Titchmarsh and partial summation). We deduce that the number of possible $(P_1, \dots, P_{KD+1})$ satisfying $P_j \equiv v_j \pmod q$ for each $j \in [KD+1]$ is no more than

$$\sum_{\substack{q<P_{KD+1}<\cdots<P_2\le x\\ (\forall j)\ P_j\equiv v_j \pmod q}} \ \sum_{\substack{z<P_1\le x^{1/k}\big/m^{1/k}P_2\cdots P_{KD+1}\\ P_1\equiv v_1 \pmod q}} 1 \ \ll\ \frac{1}{\varphi(q)^{KD+1}}\cdot\frac{x^{1/k}(\log_2 x)^{O(1)}}{m^{1/k}\log x}. \tag{9.5}$$

Define $V'_{r,K} := \max\Big\{\#\mathcal{V}^{(k)}_{r,K}\big(q; (w_i)_{i=1}^K\big) : w_1, \dots, w_K \in U_q\Big\}$. Summing (9.5) over all $(v_1, \dots, v_{KD+1}) \in \mathcal{V}^{(k)}_{KD+1,K}\big(q; (a_i f_i(m)^{-1})_{i=1}^K\big)$ and then over all m via (4.5) shows that

$$\sum\nolimits^*_{n:\ \omega_\parallel(n)\ge KD+1} 1 \ \ll\ \frac{V'_{KD+1,K}}{\varphi(q)^{KD+1}}\cdot\frac{x^{1/k}}{(\log x)^{1-\alpha_k/2}}\cdot\exp\Big(O\big((\log_3 x)^2 + (\log_2(3q))^{O(1)}\big)\Big). \tag{9.6}$$

Applying (4.9) with $N := KD+1$, we get $V'_{KD+1,K}/\varphi(q)^{KD+1} \ll \varphi(q)^{-K}\prod_{\ell\mid q}(1+O(\ell^{-1/D})) \ll \varphi(q)^{-K}\exp\big(O((\log q)^{1-1/D})\big)$. This yields the first bound in (9.4).

Next, any n counted in the second sum in (9.4) can be written in the form $mp^c(P_{KD}\cdots P_1)^k$ for some m, c and distinct primes $p, P_1, \dots, P_{KD}$ exceeding q, which satisfy the conditions $P_1 = P(n) > z$, $q < P_{KD} < \cdots < P_1$, $P_{Jk}(m) \le y$, $c \ge k+1$ and $f_i(n) = f_i(m)f_i(p^c)\prod_{j=1}^{KD} W_{i,k}(P_j)$, so that $(P_1, \dots, P_{KD}) \bmod q \in \mathcal{V}^{(k)}_{KD,K}\big(q; (a_i f_i(mp^c)^{-1})_{i=1}^K\big)$. Given m, p, c and $(v_1, \dots, v_{KD}) \in \mathcal{V}^{(k)}_{KD,K}\big(q; (a_i f_i(mp^c)^{-1})_{i=1}^K\big)$, the arguments leading to (9.5) show that the number of possible $(P_1, \dots, P_{KD})$ satisfying $(P_j)_{i=1}^{KD} \equiv (v_j)_{i=1}^{KD} \pmod q$ is $\ll x^{1/k}(\log_2 x)^{O(1)}\big/\varphi(q)^{KD}m^{1/k}p^{c/k}\log x$. Summing this successively over all $(v_1, \dots, v_{KD})$, $c \ge k+1$, $p > q$ and all possible m, shows that the second of the three sums in (9.4) is $\ll \frac{V'_{KD,K}}{q^{1/k}\varphi(q)^{KD}} \cdot \frac{x^{1/k}}{(\log x)^{1-2\alpha_k/3}}$. (Here we have noted that $\sum_{p>q,\ c\ge k+1} p^{-c/k} \ll \sum_{p>q} p^{-1-1/k} \ll q^{-1/k}$.) By (4.10), we have $V'_{KD,K}\Big/ q^{1/k}\varphi(q)^{KD} \ll 1/q^K$, proving the second inequality in (9.4).

Lastly, any n counted in the third sum in (9.4) still has $P(n) > z$ and $P(n)^k \parallel q$, and thus can be written in the form $mp_1^{c_1}\cdots p_{Kk}^{c_{Kk}}P^k$ for some distinct primes $p_1, \dots, p_{Kk}, P$ exceeding q and some integers $m, c_1, \dots, c_{Kk}$, which satisfy $P = P(n) > z$, $P_{Jk}(m) \le y$, $c_j \ge k+1$ for all $j \in [Kk]$, and $\gcd(f(m), q) = 1$. Given $m, p_1, \dots, p_{Kk}, c_1, \dots, c_{Kk}$, the number of possible $P > z$ satisfying $P^k \le x/mp_1^{c_1}\cdots p_{Kk}^{c_{Kk}}$ is $\ll x^{1/k}\big/(mp_1^{c_1}\cdots p_{Kk}^{c_{Kk}})^{1/k}\log z$. Summing this over all $c_1, \dots, c_{Kk} \ge k+1$, and then over all $p_1, \dots, p_{Kk}, m$, shows the third bound in (9.4).

In what follows, note that R as in the statement of the theorem is the least integer exceeding

$$\max\left\{k(KD+1)-1, k\left(1+(k+1)\left(K-\frac{1}{D}\right)\right)\right\} = \begin{cases} k(KD+1)-1, & \text{if } k < D \\ k\left(1+(k+1)\left(K-1/D\right)\right) & \text{if } k \ge D. \end{cases}$$

Completing the proof of Theorem 2.2(a). Since q is sufficiently large, the q-rough part of any n satisfying $\gcd(f(n), q) = 1$ is k-full (by Lemma 3.3). As such, any n with $\omega^*(n) = 0$ counted in (9.3)(i) must have $\omega_\parallel(n) \ge \lfloor R/k \rfloor \ge KD+1$, and hence is counted in the first sum in (9.4). Moreover, any n with $\omega_\parallel(n) = KD$ counted in (9.3)(i) must also have $\omega^*(n) \ge R - k\omega_\parallel(n) \ge k(KD+1) - kKD \ge 1$, and hence is counted in the second sum in (9.4). By (9.4), it thus remains to show that the contribution of n having $\omega_\parallel(n) \in [KD-1]$ and $\omega^*(n) \in [Kk-1]$ to the left hand side of (9.3) is absorbed in the right hand side. This would follow once we show that for any fixed $r \in [KD-1]$ and $s \in [Kk-1]$, the contribution $\Sigma_{r,s}$ of all n with $\omega_\parallel(n) = r$ and $\omega^*(n) = s$ to the left hand side of (9.3)(i) is absorbed in the right hand side.

Now any n counted in $\Sigma_{r,s}$ is of the form $mp_1^{c_1}\cdots p_s^{c_s}P_1^k\cdots P_r^k$ for some distinct primes $p_1, \dots, p_s$, $P_1, \dots, P_r$ and integers $m, c_1, \dots, c_s$, which satisfy the following conditions: **(i)** $P(m) \le q$; **(ii)** $P_1 := P(n) > z$; $q < P_r < \cdots < P_1$; **(iii)** $p_1, \dots, p_s > q$; **(iv)** $c_1, \dots, c_s \ge k+1$ and $c_1 + \cdots + c_s \ge R - kr$; **(v)** $m, p_1, \dots, p_s, P_1, \dots, P_r$ are all pairwise coprime, so that $f_i(n) = f_i(m)f(p_1^{c_1})\cdots f(p_s^{c_s})\prod_{j=1}^r W_{i,k}(P_j)$ for each $i \in [K]$. Here, property (i) holds because the q-rough part of any n satisfying $\gcd(f(n), q) = 1$ is k-full, whereas $\omega_\parallel(n) = r$, $\omega^*(n) = s$.

With $\tau_i := \min\{c_i, R - kr\}$, it is easy to see that the integers $\tau_1, \dots, \tau_s \in [k+1, R-kr]$ satisfy $\tau_1 \le c_1, \dots, \tau_s \le c_s$ and $\tau_1 + \cdots + \tau_s \ge R - kr$. (Here it is important that $R \ge k(KD+1)$, $r \le KD-1$ and $c_1 + \cdots + c_s \ge R - kr$.) Turning this around, we find that

$$\Sigma_{r,s} \le \sum_{\substack{\tau_1,\dots,\tau_s \in [k+1, R-kr] \\ \tau_1+\cdots+\tau_s \ge R-kr}} \mathcal{N}_{r,s}(\tau_1, \dots, \tau_s), \tag{9.7}$$

where $\mathcal{N}_{r,s}(\tau_1, \dots, \tau_s)$ denotes the contribution of all n counted in (9.3)(i) which can be written in the form $mp_1^{c_1}\cdots p_s^{c_s}P_1^k\cdots P_r^k$ for some distinct primes $p_1, \dots, p_s, P_1, \cdots, P_r$ and integers $m, c_1, \dots, c_s$ satisfying the conditions (i)-(v) above, along with the condition $c_1 \ge \tau_1, \dots, c_s \ge \tau_s$. We will show that for each tuple $(\tau_1, \dots, \tau_s)$ occurring in (9.7), we have

$$\mathcal{N}_{r,s}(\tau_1, \dots, \tau_s) \ll \frac{x^{1/k}(\log_2 x)^{O(1)}}{q^K \log x}. \tag{9.8}$$