$q \in V'_q$. Then $P_{Rk}(N) > y > q$ and $f_i(N) = \prod_{j=1}^{R} W_{i,k}(P_j) \equiv a_i \pmod q$, so that estimating the count of such N by the arguments leading to (4.8), we obtain for some constant $K_1 > 0$,

$$\sum_{\substack{n \le x:\ P_{Rk}(n) > q \\ (\forall i)\ f_i(n) \equiv a_i \pmod q}} 1 \ \ge\ \frac{V'_q}{\varphi(q)^R} \cdot \frac{1}{R!} \sum_{\substack{P_1, \dots, P_R > y \\ P_1 \cdots P_R \le x^{1/k} \\ P_1, \dots, P_R \text{ distinct}}} 1 \ -\ x^{1/k} \exp(-K_1 (\log x)^{1/4}).$$

The sum in the main term is exactly the count of squarefree y-rough integers $m \le x^{1/k}$ having $\Omega(m) = R$. Ignoring this squarefreeness condition with a negligible error of $O(x^{1/k}/y)$, we thus find that the main term equals $\#\{m \le x^{1/k} : P^-(m) > y,\ \Omega(m) = R\}$, which is $\gg x^{1/k} (\log_2 x)^{R-1} / \log x$ by a straightforward induction on R (via Chebyshev's estimates). So

$$\sum_{\substack{n \le x:\ P_{Rk}(n) > q \\ (\forall i)\ f_i(n) \equiv a_i \pmod q}} 1 \ \gg\ \frac{V'_q}{\varphi(q)^R} \cdot \frac{x^{1/k} (\log_2 x)^{R-1}}{\log x} \ -\ x^{1/k} \exp(-K_1 (\log x)^{1/4}). \tag{13.1}$$

Completing the proof of Theorem 2.5. We now restrict to the $\{W_{i,k}\}_{1 \le i \le K}$ and $(a_i)_{i=1}^K$ considered in Theorem 2.5, so $K \ge 2$, $\{W_{i,k}\}_{1 \le i \le K-1} \subset \mathbb{Z}[T]$ are multiplicatively independent, $W_{K,k} = \prod_{i=1}^{K-1} W_{i,k}^{\lambda_i}$ for some tuple $(\lambda_i)_{i=1}^{K-1} \ne (0, \dots, 0)$ of nonnegative integers, and $(a_i)_{i=1}^K \in U_q^K$ satisfy $a_K \equiv \prod_{i=1}^{K-1} a_i^{\lambda_i} \pmod q$. The key observation is that relations assumed between the $\{W_{i,k}\}_{1 \le i \le K}$ and $(a_i)_{i=1}^K$ guarantee that $V'_q = \mathcal{V}_{R,K}^{(k)}\big(q; (a_i)_{i=1}^K\big) = \mathcal{V}_{R,K-1}^{(k)}\big(q; (a_i)_{i=1}^{K-1}\big)$, with the set $\mathcal{V}_{R,K-1}^{(k)}\big(q; (a_i)_{i=1}^{K-1}\big)$ defined by the congruences $\prod_{j=1}^R W_{i,k}(v_j) \equiv a_i \pmod q$, $i \in [K-1]$.

Define $D_1 := \sum_{i=1}^{K-1} \deg W_{i,k} > 1$ and let "C" in the statement of the theorem be any constant $C^* := C^*(W_{1,k}, \cdots, W_{K-1,k})$ exceeding $(32 D_1)^{2D_1+2}$, the sizes of the leading and constant coefficients of $\{W_{i,k}\}_{i=1}^K$, and the constant $C_1^* := C_1(W_{1,k}, \dots, W_{K-1,k})$ coming from an application of Proposition 5.3 to the family $\{W_{i,k}\}_{i=1}^{K-1}$ of nonconstant multiplicatively independent polynomials. To show the lower bound in Theorem 2.5, we may assume that $R > 4 K D_1 (D_1 + 1)$. We shall carry out some of the arguments of Proposition 5.4; note that $\alpha_k(q) = \frac{1}{\varphi(q)} \#\{u \in U_q : \prod_{i=1}^{K-1} W_{i,k}(u) \in U_q\} \ne 0$. For each prime $\ell \mid q$, we have $\gcd(\ell - 1, \beta(W_{1,k}, \cdots, W_{K-1,k})) = 1$ and $\ell > C^* > C_1^*$. Thus the hypothesis $IFH(W_{1,k}, \dots, W_{K-1,k}; 1)$ holds true, and so does the corresponding analogue of the inequality (5.18). We find that

$$\frac{1}{\big(\alpha_k(\ell) \varphi(\ell^e)\big)^R} \sum_{(\chi_1, \dots, \chi_{K-1}) \ne (\chi_{0,\ell}, \dots, \chi_{0,\ell}) \bmod \ell^e} \big| Z_{\ell^e;\ \chi_1, \dots, \chi_{K-1}}(W_{1,k}, \dots, W_{K-1,k}) \big|^R \ \le\ \frac{2(4D_1)^R}{\ell^{R/D_1 - K}}, \tag{13.2}$$

where as usual $Z_{\ell^e;\ \chi_1, \dots, \chi_{K-1}}(W_{1,k}, \dots, W_{K-1,k}) = \sum_{u \bmod \ell^e} \chi_{0,\ell}(u) \prod_{i=1}^{K-1} \chi_i(W_{i,k}(u))$. Now since $R \ge 4 K D_1 (D_1 + 1)$ and $\ell > C^* > (32 D_1)^{2D_1+2}$, we see that $\ell^{R/D_1 - K} \ge \ell^{R/(D_1+1)} \ge \ell^{R/(2D_1+2)} \cdot (C^*)^{R/(2D_1+2)} \ge \ell^2 (32 D_1)^R$, showing that the right hand expression in (13.2) is at most $1/4\ell^2$. Invoking the corresponding analogue of (5.11), we see for each prime power $\ell^e \parallel q$ that $\#\mathcal{V}_{R,K-1}^{(k)}(\ell^e; (a_i)_{i=1}^{K-1}) / \varphi(\ell^e)^R \ \ge\ (\alpha_k(\ell)^R / \varphi(\ell^e)^{K-1}) \cdot (1 - 1/2\ell^2)$. But since $\prod_{\ell \mid q} (1 - 1/2\ell^2) \ge 1 - \frac{1}{2} \sum_{\ell \ge 2} 1/\ell^2 \ge 1/2$, we obtain $V'_q / \varphi(q)^R = \mathcal{V}_{R,K-1}^{(k)}\big(q; (a_i)_{i=1}^{K-1}\big) / \varphi(q)^R \ge \alpha_k(q)^R / 2\varphi(q)^{K-1}$, which holds true uniformly in q having $P^-(q) > C^*$. Inserting this bound into (13.1) and recalling that $\alpha_k(q) \gg 1/(\log_2(3q))^D$, we are done. $\square$

Completing the proof of Theorem 2.6. Again, it suffices to consider the case $R > 18KD(D+1)$ to prove (2.4). We start by choosing "C" in the statement of the theorem to be a constant $C_2 := C_2(W_{1,k}, \ldots, W_{K,k})$ exceeding $(32D)^{6D+6}$, the sizes of the leading and constant coefficients of $\{W_{i,k}\}_{i=1}^K$, and the constant $C_1(W_{1,k}, \ldots, W_{K,k})$ obtained by applying Proposition 5.3 to the family $\{W_{i,k}\}_{1 \le i \le K}$ of multiplicatively independent polynomials. The analogue of (5.16) continues to hold for each $\ell \mid q$, and the computation leading to (5.18) yields

(13.3)
$$\frac{1}{(\alpha_k(\ell)\varphi(\ell^e))^R} \sum_{\substack{(\chi_1,\ldots,\chi_K) \bmod \ell^e \\ \mathrm{lcm}[\mathfrak{f}(\chi_1),\ldots,\mathfrak{f}(\chi_K)] \in \{\ell^2,\ldots,\ell^e\}}} |Z_{\ell^e;\, \chi_1,\ldots,\chi_K}(W_{1,k}, \ldots, W_{K,k})|^R \le \frac{2(4D)^R}{\ell^{R/D-K}} \le \frac{1}{4\ell^2},$$

where in the last inequality, we have recalled that $R > 4KD(D+1)$ and $\ell > C_2 \ge (32D)^{6D+6}$.

If $(\chi_1, \ldots, \chi_K)$ is a tuple of characters mod ℓ^e having $\mathrm{lcm}[\mathfrak{f}(\chi_1), \ldots, \mathfrak{f}(\chi_K)] = \ell$, then with ψ_ℓ being a generator of the character group mod ℓ, we have $\chi_i = \psi_\ell^{A_i}$ for some unique $(A_1, \ldots, A_K) \in [\ell-1]^K$ satisfying $(A_1, \ldots, A_K) \not\equiv (0, \ldots, 0) \pmod{\ell-1}$. Recall from the arguments leading to (5.15) that if $\prod_{i=1}^K W_{i,k}^{A_i}$ is *not* of the form $c \cdot G^{\ell-1}$ in $\mathbb{F}_\ell[T]$, then $|Z_{\ell^e;\, \chi_1,\ldots,\chi_K}(W_{1,k}, \ldots, W_{K,k})| \le D\ell^{e-1/2}$. On the other hand, if $\prod_{i=1}^K W_{i,k}^{A_i}$ *is* of that form (with G monic, say), then since each $W_{i,k}$ is monic, we must have $\prod_{i=1}^K W_{i,k}^{A_i} = G^{\ell-1}$. Since $G(v)$ is a unit mod ℓ iff $\prod_{i=1}^K W_{i,k}(v)$ is, it follows that $Z_{\ell^e;\, \chi_1,\ldots,\chi_K}(W_{1,k}, \ldots, W_{K,k}) = \ell^{e-1} \sum_{v \bmod \ell} \psi_\ell\left((vG(v))^{\ell-1}\right) = \alpha_k(\ell)\varphi(\ell^e)$. Combining these observations with (13.3) and using that $\prod_{i=1}^K \overline{\chi}_i(a_i) = 1$ for any characters $(\chi_1, \ldots, \chi_K)$ mod ℓ^e with $\mathrm{lcm}[\mathfrak{f}(\chi_1), \ldots, \mathfrak{f}(\chi_K)] = \ell$ (as $a_i \equiv 1 \bmod \ell$), we get

$$\frac{\#\mathcal{V}_{R,K}^{(k)}\left(\ell^e; (a_i)_{i=1}^K\right)}{\varphi(\ell^e)^R} \ge \frac{\alpha_k(\ell)^R}{\varphi(\ell^e)^K}\left(1 + \mathcal{B}_\ell - \frac{1}{2\ell^2}\right), \tag{13.4}$$

where $\mathcal{B}_\ell$ denotes the number of tuples $(A_1, \ldots, A_K) \in [\ell-1]^K \setminus \{(0, \ldots, 0)\}$ for which $\prod_{i=1}^K W_{i,k}^{A_i}$ is a perfect $(\ell-1)$-th power in $\mathbb{F}_\ell[T]$.

Now recalling the definition of the constant $C_1 = C_1(W_{1,k}, \ldots, W_{K,k})$ from the proof of Proposition 5.3, we know that for any $\ell > C_1$, the pairwise coprime irreducible factors of the product $\prod_{i=1}^K W_{i,k}$ in $\mathbb{Z}[T]$ continue to be separable and pairwise coprime in the ring $\mathbb{F}_\ell[T]$. By the arguments given in the proof of Proposition 5.3(a), $\prod_{i=1}^K W_{i,k}^{A_i}$ is a perfect $(\ell-1)$-th power in $\mathbb{F}_\ell[T]$ precisely when $E_0(A_1 \cdots A_K)^\top \equiv (0 \cdot \cdots \cdot 0)^\top \pmod{\ell-1}$, where $E_0 = E_0(W_{1,k}, \ldots, W_{K,k})$ is the exponent matrix. Thus, $\mathcal{B}_\ell$ is exactly the number of nonzero vectors $X \in (\mathbb{Z}/(\ell-1)\mathbb{Z})^K$ satisfying the matrix equality $E_0X = 0$ over the ring $\mathbb{Z}/(\ell-1)\mathbb{Z}$.

Recall that E_0 has $\mathbb{Q}$-linearly independent columns and non-zero last invariant factor $\beta = \beta(W_{1,k}, \ldots, W_{K,k}) \in \mathbb{Z}$. By [33, Theorem 6.4.17], the matrix equation $E_0X = 0$ has a nontrivial solution in the ring $\mathbb{Z}/(\ell-1)\mathbb{Z}$ precisely when some nonzero element of $\mathbb{Z}/(\ell-1)\mathbb{Z}$ annihilates all the $K \times K$ minors of the matrix E_0. But if $\gcd(\ell-1, \beta) \ne 1$, then the canonical image of $d := (\ell-1)/\gcd(\ell-1, \beta)$ in $\mathbb{Z}/(\ell-1)\mathbb{Z}$ clearly does this, since $d\beta \equiv 0 \pmod{\ell-1}$ and since β divides the gcd of the $K \times K$ minors of E_0 (in $\mathbb{Z}$). We thus obtain $\mathcal{B}_\ell \ge 1$ for each prime prime $\ell \mid q$ satisfying $\gcd(\ell-1, \beta) \ne 1$, which from (13.4) yields $V_q'/\varphi(q)^R \ge 2^{\#\{\ell \mid q:\ (\ell-1,\beta)\ne 1\}}\alpha_k(q)^R/2\varphi(q)^K$. Inserting this into (13.1) establishes (2.4). $\square$

Remark: If $K = 1$ and $W_{1,k}$ is a constant c, then the k-admissibility of q forces $\gcd(q, c) = 1$, which by (13.1) gives $\#\{n \le x : P_{Rk}(n) > q, f(n) \equiv c^R \pmod q\} \gg x^{1/k}(\log_2 x)^{R-1}/\log x$.

13.1. **Explicit Examples.** We now construct examples where the lower bounds in Theorems 2.5 and 2.6 grow strictly faster than the expected quantity $\varphi(q)^{-K}\#\{n \le x : (f(n), q) = 1\}$.

Failure of joint weak equidistribution upon violation of multiplicative independence hypothesis (example for Theorem 2.5). By Proposition 3.1, it is clear that the lower bound in Theorem 2.5 grows strictly faster once q grows fast enough compared to $\log x$. For a concrete example, we start with any $\{W_{i,k}\}_{1\le i\le K-1} \subset \mathbb{Z}[T]$ for which $\beta^* = \beta(W_{1,k}, \dots, W_{K-1,k})$ is odd (for instance, $W_{i,k} := H_i^{b_i}$ for some pairwise coprime irreducibles $H_1, \dots, H_{K-1} \in \mathbb{Z}[T]$ and odd integers $b_i > 1$ satisfying $b_i \mid b_{i+1}$ for each $i < K-1$). Fix nonnegative integers $(\lambda_i)_{i=1}^{K-1} \ne (0, \dots, 0)$ and nonzero integers $(a_i)_{i=1}^{K}$ satisfying $a_K = \prod_{i-1}^{K-1} a_i^{\lambda_i}$ (in $\mathbb{Z}$), and let $W_{K,k} = \prod_{i=1}^{K-1} W_{i,k}^{\lambda_i}$. Consider a constant $\widetilde{C} > \max\{C^*, \prod_{i=1}^{K} |a_i|\}$, such that any $\widetilde{C}$-rough k-admissible integer lies in $\mathcal{Q}(k; f_1, \cdots, f_K)$. Here C^* as in the proof of Theorem 2.5, so that $\widetilde{C} > D_1 + 1 = \sum_{i=1}^{K-1} \deg W_{i,k} + 1$. Let ℓ_0 be the least prime exceeding $\widetilde{C}$ and satisfying $\ell_0 \equiv -1 \bmod \beta^*$. [11] Let $\{W_{i,v}\}_{\substack{1\le i\le K\\ 1\le v\le k-1}} \subset \mathbb{Z}[T]$ be nonconstant polynomials with all coefficients divisible by ℓ_0, and let $q := \prod_{\substack{\ell_0\le \ell\le Y\\ \ell\equiv -1 \pmod{\beta^*}}} \ell$, with Y any parameter lying in $(4|\beta^*|\log_2 x, (K_0/2)\log_2 x)$. Since $\alpha_k(\ell) \ge 1 - D_1/(\ell-1) > 0$ for $\ell > \widetilde{C}$, we see that $q \le (\log x)^{K_0}$ is k-admissible and hence lies in $\mathcal{Q}(k; f_1, \cdots, f_K)$. As β^* is odd and $\ell \equiv -1 \pmod{\beta^*}$ for all $\ell \mid q$, we have $\gcd(\ell - 1, \beta^*) = 1$ for all such ℓ. Further, $q = \exp\left(\sum_{\substack{\ell_0\le \ell\le Y\\ \ell\equiv -1 \pmod{\beta^*}}} \log \ell\right) \ge \exp(Y/2|\beta^*|) \ge \log^2 x$, so the lower bound in Theorem 2.5 grows strictly faster than $\varphi(q)^{-K}\#\{n \le x : (f(n), q) = 1\}$.

Failure of joint weak equidistribution upon violation of Invariant Factor Hypothesis (example for Theorem 2.6). Define $W_{i,k}(T) := T - i$ for each $i \in [K-1]$ and $W_{K,k}(T) := (T-K)^d$, for some fixed $d \in \{2, \dots, K\}$. Then $\{W_{i,k}\}_{1\le i\le K}$ are nonconstant, monic and pairwise coprime (hence multiplicatively independent); also $E_0(W_{1,k}, \dots, W_{K,k}) = \mathrm{diag}(1, \dots, 1, d)$ so $\beta := \beta(W_{1,k}, \dots, W_{K,k}) = d$. Note that $\alpha_k(\ell) = 1 - K/(\ell-1) > 0$ for any prime $\ell > K+1$. Let $C_3 := C_3(W_{1,k}, \dots, W_{K,k})$ be a constant exceeding the constant C_2 in the proof of Theorem 2.6, such that any k-admissible C_3-rough integer lies in $\mathcal{Q}(k; f_1, \cdots, f_K)$; note that $C_3 > D + 1 \ge K + 2$. Let ℓ_0 be the least prime exceeding C_3 and satisfying $\ell_0 \equiv 1 \pmod d$, let $\{W_{i,v}\}_{\substack{1\le i\le K\\ 1\le v<k}} \subset \mathbb{Z}[T]$ be nonconstant polynomials all of whose coefficients are divisible by ℓ_0, and let $q := \prod_{\substack{\ell_0\le \ell\le Y\\ \ell\equiv 1 \pmod d}} \ell$, with $Y \le (K_0/2)\log_2 x$ a parameter to be chosen later.

Then $q \le (\log x)^{K_0}$, $P^-(q) > C_3$ and $q \in \mathcal{Q}(k; f_1, \cdots, f_K)$. By Theorem 2.6 and Proposition 3.1, it follows that the residues $a_i \equiv 1 \pmod q$ are overrepresented if $\#\{\ell \mid q : (\ell - 1, \beta) \ne 1\} \ge 4\alpha_k \log_2 x$. But $\#\{\ell \mid q : (\ell-1, \beta) \ne 1\} = \sum_{\substack{\ell_0\le \ell\le Y\\ \ell\equiv 1 \pmod d}} 1 \ge Y/2\varphi(d)\log Y$, whereas (since $K \ge \varphi(d)$), we have $\alpha_k \le K_3/\log Y$ for some constant $K_3 > 0$ depending at most on C_3, K and d, so we only need $8K_3\varphi(d)\log_2 x < Y < (K_0/2)\log_2 x$.

[11] Our arguments go through for any $c^* \in U_{\beta^*}$ for which $c^* - 1 \in U_{\beta^*}$, in place of the residue $-1 \bmod \beta^*$.