

**Theorem 5.6.** *There exists a constant  $\delta_0 := \delta_0(\lambda) > 0$  such that, uniformly in moduli  $q \leq (\log x)^{K_0}$  lying in  $\mathcal{Q}(k; f_1, \dots, f_K)$  and having sufficiently large radical, we have*

$$\sum_{n \leq x} \chi_1(f_1(n)) \cdots \chi_K(f_K(n)) \mathbb{1}_{(f(n), q)=1} \ll \frac{x^{1/k}}{(\log x)^{1-(1-\delta_0)\alpha_k(Q)}}$$

for all tuples of characters  $(\chi_1, \dots, \chi_K) \neq (\chi_{0,Q_0}, \dots, \chi_{0,Q_0}) \pmod{Q_0}$ .

Let  $\mathcal{C}_k(Q_0)$  denote the set of tuples of characters  $(\chi_1, \dots, \chi_K) \pmod{Q_0}$ , not all trivial, such that  $\prod_{i=1}^K \chi_i(W_{i,k}(u))$  is constant on the set  $R_k(Q_0) = \{u \in U_{Q_0} : W_k(u) \in U_{Q_0}\}$ . To prove Theorem 5.6, we separately consider the two cases when a tuple of characters mod  $Q_0$  lies in  $\mathcal{C}_k(Q_0)$  or not.

## 6. PROOF OF THEOREM 5.6 FOR NONTRIVIAL TUPLES OF CHARACTERS NOT IN $\mathcal{C}_k(Q_0)$

For any integer  $d \geq 1$  and any nontrivial tuple  $(\psi_1, \dots, \psi_K)$  of characters mod  $d$  not lying in  $\mathcal{C}_k(d)$ , we have  $|\sum_{u \pmod{d}} \chi_{0,d}(u) \psi_1(W_{1,k}(u)) \cdots \psi_K(W_{K,k}(u))| < \alpha_k(d) \varphi(d)$ . With  $\lambda$  as in Proposition 4.3, we define the constant  $\delta_1 := \delta_1(W_{1,k}, \dots, W_{K,k}; B_0) \in (0, 1)$  to be

$$\max_{\substack{d \leq \lambda \\ \alpha_k(d) \neq 0}} \max_{\substack{(\psi_1, \dots, \psi_K) \neq (\chi_{0,d}, \dots, \chi_{0,d}) \pmod{d \\ (\psi_1, \dots, \psi_K) \notin \mathcal{C}_k(d)}}} \frac{1}{\alpha_k(d) \varphi(d)} \left| \sum_{u \pmod{d}} \chi_{0,d}(u) \psi_1(W_{1,k}(u)) \cdots \psi_K(W_{K,k}(u)) \right|.$$

Then since  $Q_0 \leq \lambda$ , we have for any nontrivial tuple  $(\chi_1, \dots, \chi_K) \notin \mathcal{C}_k(Q_0)$ ,

$$(6.1) \quad \left| \sum_{u \pmod{Q_0}} \chi_{0,Q_0}(u) \chi_1(W_{1,k}(u)) \cdots \chi_K(W_{K,k}(u)) \right| \leq \delta_1 \alpha_k(Q_0) \varphi(Q_0).$$

We set  $\delta := (1 - \delta_1)/2$  and  $Y := \exp((\log x)^{\delta/3})$ . To establish Theorem 5.6 for all  $(\chi_1, \dots, \chi_K) \notin \mathcal{C}_k(Q_0)$ , it suffices to show that

$$(6.2) \quad \sum_{\substack{n \leq x \\ p > Y \implies p^{k+1} \nmid n}} \chi_1(f_1(n)) \cdots \chi_K(f_K(n)) \mathbb{1}_{(f(n), q)=1} \ll \frac{x^{1/k}}{(\log x)^{1-(\delta_1+\delta)\alpha_k}},$$

since by the arguments before (3.3), the contribution of  $n$ 's not counted above is negligible. Writing any  $n$  counted in (6.2) uniquely as  $BMA^k$  (as in (3.4)), we see that the sum equals

$$(6.3) \quad \begin{aligned} & \sum_{\substack{B \leq x \\ P(B) \leq Y \\ B \text{ is } k\text{-free}}} \mathbb{1}_{(f(B), q)=1} \left( \prod_{i=1}^K \chi_i(f_i(B)) \right) \sum_{\substack{M \leq x/B \\ M \text{ is } k\text{-full} \\ P(M) \leq Y}} \mathbb{1}_{(f(M), q)=1} \left( \prod_{i=1}^K \chi_i(f_i(M)) \right) \\ & \quad \sum_{A \leq (x/BM)^{1/k}} \mathbb{1}_{P^-(A) > Y} \mathbb{1}_{(f(A^k), q)=1} \mu(A)^2 \prod_{i=1}^K \chi_i(f_i(A^k)) \end{aligned}$$

Moreover, the arguments leading to the bound for  $\Sigma_2$  in section 3 show that the tuples  $(B, M, A)$  having  $M > x^{1/2}$  give negligible contribution to the above sum. It thus remains to

consider the contribution of tuples  $(B, M, A)$  with  $M \leq x^{1/2}$ . To deal with such tuples, we will establish the following general upper bound uniformly for  $X \geq \exp((\log Y)^2)$ :

$$(6.4) \quad \sum_{A \leq X} \mathbb{1}_{P-(A)>Y} \mathbb{1}_{(f(A^k), q)=1} \mu(A)^2 \prod_{i=1}^K \chi_i(f_i(A^k)) \ll \frac{X}{(\log X)^{1-\alpha_k(\delta_1+\delta/2)}}.$$

We apply a quantitative version of Halász's Theorem [48, Corollary III.4.12] on the multiplicative function  $F(A) := \mathbb{1}_{P-(A)>Y} \mathbb{1}_{(f(A^k), q)=1} \mu(A)^2 \prod_{i=1}^K \chi_i(f_i(A^k))$ , taking  $T := \log X$ . This requires us to put, for each  $t \in [-T, T]$ , a lower bound on the sum below (which is the square of a certain “pretentious distance”):

$$(6.5) \quad \begin{aligned} \mathcal{D}(X; t) &:= \sum_{p \leq X} \frac{1}{p} \left( 1 - \operatorname{Re} \left( \mathbb{1}_{p>Y} \mathbb{1}_{(f(p^k), q)=1} \mu(p)^2 p^{-it} \prod_{i=1}^K \chi_i(f_i(p^k)) \right) \right) \\ &= (1 - \alpha_k) \log_2 X + \alpha_k \log_2 Y + \sum_{\substack{Y < p \leq X \\ (W_k(p), q)=1}} \frac{1}{p} \left( 1 - \operatorname{Re} \left( p^{-it} \prod_{i=1}^K \chi_i(W_{i,k}(p)) \right) \right) \\ &\quad + O((\log_2(3q))^{O(1)}); \end{aligned}$$

here the second line uses Lemma 3.4. To get this lower bound, we proceed analogously to the proof of [36, Lemma 3.3]. The key idea is to split the range of the above sum into blocks of small multiplicative width, so that the complex number  $p^{-it}$  is essentially constant for all  $p$  lying in a given block. More precisely, we cover the interval  $(Y, X]$  with finitely many disjoint intervals  $\mathcal{I} := (\eta, \eta(1 + 1/\log^2 X)]$  for certain choices of  $\eta \in (Y, X]$ , choosing the smallest  $\eta$  to be  $Y$  and allowing the rightmost endpoint of such an interval to jut out slightly past  $X$  but no more than  $X(1 + 1/\log^2 X)$ . Then the last sum in (6.5) equals

$$(6.6) \quad \sum_{\mathcal{I}} \sum_{\substack{p \in \mathcal{I} \\ (W_k(p), q)=1}} \frac{1}{p} \left( 1 - \operatorname{Re} \left( p^{-it} \prod_{i=1}^K \chi_i(W_{i,k}(p)) \right) \right) + O\left(\frac{1}{\log^3 X}\right)$$

Consider any  $\mathcal{I}$  occurring in the sum above. For each  $p \in \mathcal{I}$ , we have

$$|p^{-it} - \eta^{-it}| \leq \left| \int_{t \log \eta}^{t \log p} \exp(-i\varrho) d\varrho \right| \leq |t \log p - t \log \eta| \leq \frac{|t|}{\log^2 X} \leq \frac{1}{\log X}.$$

This shows that each inner sum in (6.6) is equal to

$$(6.7) \quad \sum_{\substack{u \in U_q \\ (W_k(u), q)=1}} \left( 1 - \operatorname{Re} \left( \eta^{-it} \prod_{i=1}^K \chi_i(W_{i,k}(u)) \right) \right) \sum_{\substack{p \in \mathcal{I} \\ p \equiv u \pmod{q}}} \frac{1}{p} + O\left(\frac{1}{\log X} \sum_{p \in \mathcal{I}} \frac{1}{p}\right)$$

Note that  $p = (1 + o(1))\eta$  for all  $p \in \mathcal{I}$ . (Here and in what follows, the asymptotic notation refers to the behavior as  $x \rightarrow \infty$ , and is uniform in the choice of  $\mathcal{I}$ .) For parameters  $Z, W$  depending on  $X$ , we write  $Z \gtrsim W$  to mean  $Z \geq (1 + o(1))W$ . By the Siegel-Walfisz Theorem,

$$\sum_{\substack{p \in \mathcal{I} \\ p \equiv u \pmod{q}}} \frac{1}{p} \gtrsim \frac{1}{\eta} \sum_{\substack{p \in \mathcal{I} \\ p \equiv u \pmod{q}}} 1 \gtrsim \frac{1}{\varphi(q)} \cdot \frac{1}{\eta} \sum_{p \in \mathcal{I}} 1 \gtrsim \frac{1}{\varphi(q)} \sum_{p \in \mathcal{I}} \frac{1}{p}.$$

Hence the main term in (6.7) is

$$\gtrsim \frac{1}{\varphi(q)} \sum_{p \in \mathcal{I}} \frac{1}{p} \sum_{\substack{u \in U_q \\ (W_k(u), q) = 1}} \left( 1 - \operatorname{Re} \left( \eta^{-it} \prod_{i=1}^K \chi_i(W_{i,k}(u)) \right) \right) \gtrsim (\alpha_k - \alpha_k \delta_1) \left( \sum_{p \in \mathcal{I}} \frac{1}{p} \right),$$

where in the last step, we have used (4.14) and (6.1) to see that

$$\frac{1}{\varphi(q)} \left| \sum_{\substack{u \in U_q \\ (W_k(u), q) = 1}} \prod_{i=1}^K \chi_i(W_{i,k}(u)) \right| = \frac{\alpha_k(q)}{\alpha_k(Q_0) \varphi(Q_0)} \left| \sum_{r \bmod Q_0} \chi_{0,Q_0}(r) \prod_{i=1}^K \chi_i(W_{i,k}(r)) \right| \leq \alpha_k \delta_1.$$

Inserting the bound obtained in the previous display into (6.7), we find that each inner sum in (6.6) is  $\gtrsim \alpha_k(1 - \delta_1) \sum_{p \in \mathcal{I}} 1/p + O((\log X)^{-1} \sum_{p \in \mathcal{I}} 1/p)$ . The  $O$ -term when summed over all  $\mathcal{I}$  is  $\ll (\log X)^{-1} \sum_{p \leq 2X} p^{-1} \ll \log_2 X / \log X$ . Thus, the main term in (6.6) is at least  $\alpha_k(1 - \delta_1 - \frac{\delta}{2})(\log_2 X - \log_2 Y)$ . Inserting this into (6.5) yields

$$\mathcal{D}(X; t) \geq \left( 1 - \alpha_k \left( \delta_1 + \frac{\delta}{2} \right) \right) \log_2 X + \alpha_k \left( \delta_1 + \frac{\delta}{2} \right) \log_2 Y + O((\log_2(3q))^{O(1)}),$$

uniformly for  $t \in [-T, T]$ . As such, Corollary [48, III.4.12] establishes the claimed bound (6.4).

Now for each  $M \leq x^{1/2}$ , we have  $(x/BM)^{1/k} \gg x^{1/2k}$ . Applying (6.4) to each of the innermost sums in (6.3), we see that the total contribution of all tuples  $(B, M, A)$  with  $M \leq x^{1/2}$  is

$$\ll \sum_{B \ll 1} \sum_{\substack{M \leq x^{1/2}: M \text{ is } k\text{-full} \\ P(M) \leq Y, (f(M), q) = 1}} \frac{(x/BM)^{1/k}}{(\log x)^{1-\alpha_k(\delta_1+\delta/2)}} \ll \frac{x^{1/k}}{(\log x)^{1-\alpha_k(\delta_1+\delta)}},$$

where we have used (3.5) (with  $Y$  in place of  $y$ ) and Lemma 3.4. This proves (6.2), and hence also Theorem 5.6 for all nontrivial tuples of characters  $(\chi_1, \dots, \chi_K) \bmod Q_0$  not in  $\mathcal{C}_k(Q_0)$ .  $\square$

## 7. PROOF OF THEOREM 5.6 FOR TUPLES OF CHARACTERS IN $\mathcal{C}_k(Q_0)$

It suffices to consider the case when  $x$  is an integer, and we will do so in the rest of the section. Our argument consists of suitably modifying the Landau–Selberg–Delange method for mean values of multiplicative functions (see for instance [48, Chapter II.5]), and to study the behavior of a product of  $L$ -functions raised to complex powers by accounting for the presence of Siegel zeros modulo  $q$ . This is partly inspired from work of Scourfield [44] and will also need some results from her paper. We will denote complex numbers in the standard notation  $s = \sigma + it$ .

<sup>6</sup> To begin with, we consider the Dirichlet series

$$F_\chi(s) := \sum_{n \geq 1} \frac{\mathbb{1}_{(f(n), q)=1}}{n^s} \prod_{i=1}^K \chi_i(f_i(n)) = \sum_{n \geq 1} \frac{\mathbb{1}_{(f(n), Q)=1}}{n^s} \prod_{i=1}^K \chi_i(f_i(n))$$

which is absolutely convergent in the half-plane  $\sigma > 1$ . Let  $c_{\tilde{\chi}}$  denote the constant value of  $\prod_{i=1}^K \chi_i(W_{i,k}(u))$  on the set  $R_k(Q_0) = W_k^{-1}(U_{Q_0}) \cap U_{Q_0}$ . In the rest of the section, we assume that the complex plane has been cut along the line  $\sigma \leq 1/k$  if  $\alpha_k(Q)$  and  $c_{\tilde{\chi}}$  are not both 1, while

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<sup>6</sup>The parameters  $\sigma$  and  $\sigma_k$  (to be defined later) in this section have nothing to do with the divisor functions  $\sigma_r(n) = \sum_{d|n} d^r$  mentioned in the introduction. We are not working with the divisor functions in this section.