# Global solution and optimal control of an epidemic propagation with a heterogeneous diffusion

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#### **Abstract**

In this paper, we explore the solvability and the optimal control problem for a compartmental model based on reaction-diffusion partial differential equations describing a transmissible disease. The nonlinear model takes into account the disease spreading due to the human social diffusion, under a dynamic heterogeneity in infection risk. The analysis of the resulting system provides the existence proof for a global solution and determines the conditions of optimality to reduce the concentration of the infected population in certain spatial areas.

**Keywords:** existence and uniqueness of solutions, optimal control in coefficients, partial differential equations, reaction-diffusion system, epidemic models, COVID-19.

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# 1 Introduction

The great impact on the development of human society that infectious diseases can have requires prevention and control policies significant for public health. The recent outbreak of infectious diseases, in particular the COVID-19 pandemic, has highlighted an important

role played by global public surveillance systems and response which can lead to reducing their effects on the socioeconomic activities and human health. In the past decades, a number of mathematical models were developed to investigate infectious disease evolution and the control of their spreading (an overview can be found in [13,20], see the references therein). In most of the cases, the models present in the literature are based on systems of ordinary differential equations (ODEs) in time, describing the compartmental evolution of various types of populations possibly evolving during an epidemic.

From the scientific contributions there is evidence showing that environmental heterogeneity and human mobility have a significant impact on the spread of infectious diseases (cf., e.g., [17,19]); in addition, let us mention that epidemic models accounting for spatial diffusion have been proposed and investigated since long time ago (one may see [4,7,8,24]). An SIS functional partial differential model cooperated with spatial heterogeneity and lag effect of media impact was studied in [20]. A family of epidemiological models, that extend the classic Susceptible-Infectious-Recovered/Removed (SIR) model to account for dynamic heterogeneity in infection risk, was proposed in [2]. The family of models takes the form of a system of reaction-diffusion equations for a given population structured by heterogeneous susceptibility to infection. Recently, a new epidemic diffusion model with nonlinear transmission rates and diffusion coefficients was introduced and tested in [21,22], while in [1] the authors proved well-posedness for an initial-boundary value problem associated to a variant of the compartmental model for COVID-19 studied in [21,22]. Global existence and large time behavior of the solution to a system describing a spatio-temporal spread of an infectious disease are deduced in [6, Section 3] as an application of the results obtained in the same work [6], for a more general model of semilinear reactiondiffusion-advection systems, whose coefficients satisfy general assumptions. In fact, in [6] the authors deal with unique, globally defined uniformly bounded weak solutions and provide a main issue showing that the quasi-positive systems that satisfy an intermediate sum condition automatically give rise to a new class of  $L^p$ -energy type functionals that allow to derive uniform a priori bounds.

Since control measures are essential for a disease mitigation, optimal control studies have been proposed in the literature, considering generally the vaccine as a control variable (cf., e.g., [18, 23, 25]. However, other precursory policies imposed to lower the contagion focus on the reduction of the reproduction rate, in particular of the transmission rates and of the population movement. Identification of such coefficients have been done in [9,14,15]. Another approach of COVID-19 control has been introduced in the literature via the meanfield control model (see [10,11]), where the aim is controlling the propagation of epidemics on a spatial domain. In the paper [11] the control variable, the spatial velocity, was introduced for the classical disease models, such as the SIR model, and some numerical algorithms based on proximal primal-dual methods were provided. The same method was used in [10] by choosing two controls for the pandemic: relocation of populations and distribution of vaccines. Optimization solutions in view of reducing the number of infective individuals by using as controls the transmission rates have been proposed in [16] and [3]. Also, we quote the paper [5] for the study of a more general problem of optimally controlling an epidemic outbreak of a disease structured by age since exposure, with the aid of two types of control instruments, namely social distancing and vaccination. In [5] the aim is minimizing the direct health cost of the epidemic arising from the overall epidemic incidence, as well as the indirect epidemic cost, namely the broader societal and

economic cost due to social distancing and its impact on the labor force and production, on overall social and relational activities.

In this paper, we explore an optimal control problem for a compartmental model based on partial differential equations (PDEs), to account for the disease spreading due to the human social diffusion under a dynamic heterogeneity in infection risk and provide the conditions of optimality to reduce the concentration of the infected population in certain spatial areas.

The model we consider describes the evolution of an epidemic spreading in a nonsedentary population

$$\partial_t s + \beta_i s \, i + \beta_e s \, e - \operatorname{div}(\kappa_s \nabla s) - \gamma r = 0 \qquad \text{in } Q \tag{1.1}$$

$$\partial_t e - \beta_i s \, i - \beta_e s \, e + \sigma e + \phi_e e - \operatorname{div}(\kappa_e \nabla e) = 0 \qquad \text{in } Q \tag{1.2}$$

$$\partial_t i + \phi_r i - \operatorname{div}(\kappa_i \nabla i) - \sigma e = 0 \qquad \text{in } Q \tag{1.3}$$

$$\partial_t r - \phi_r i - \phi_e e - \operatorname{div}(\kappa_r \nabla r) + \gamma r = 0 \qquad \text{in } Q \qquad (1.4)$$

$$\partial_{\mathbf{n}}s = \partial_{\mathbf{n}}e = \partial_{\mathbf{n}}i = \partial_{\mathbf{n}}r = 0$$
 on  $\Sigma$  (1.5)

$$(s, e, i, r)(0) = (s_0, e_0, i_0, r_0)$$
 in  $\Omega$  (1.6)

where

$$Q := \Omega \times (0, T) \quad \text{and} \quad \Sigma := \Gamma \times (0, T)$$
 (1.7)

 $\Omega$  being a bounded domain of  $\mathbb{R}^d$  (with d=1, 2 or 3) within which the population is located; here,  $\Gamma$  denotes the boundary of  $\Omega$  and T>0 is a fixed final time. Moreover, in (1.5),  $\partial_n$  denotes the derivative in the direction of the outward unit normal field n on the boundary.

Let us describe the meaning of the physical variables and the coefficients that enter the system (1.1)–(1.6). The functions s, e, i and r represent the susceptible population, the exposed population, the infected population, and the recovered population, respectively. Note that the exposed population and the infected population refer to the asymptomatic and symptomatic persons, respectively. Hence, it turns out that, as observed in COVID-19 epidemic, the exposed may also spread the disease. Here we neglect both the newborn and the deceased individuals after reaching the maximum life, by assuming that these populations have small sizes and do not contribute essentially to the epidemic transmission during the time period (0, T). Thus, in our model the natality rate and the natural mortality rate (which is not related to the disease) are zero. The nonnegative coefficients  $\beta_i$  and  $\beta_e$  depend on space, time and the total living population

$$n := s + e + i + r. \tag{1.8}$$

We also assume that this epidemic evolves also by a diffusive process in which healthy and infected individuals spread with different diffusion coefficients  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$  and  $\kappa_r$  varying with space, time and the total living population. The diffusion coefficients are assumed to be bounded from below and from above by positive constants, so that the above system enjoys a parabolic character. Finally,  $\phi_r$ ,  $\phi_e$  and  $\sigma$  are fixed positive constants and  $\gamma$  is a nonnegative function that depends just on time.

For the system (1.1)–(1.6) we can prove a general existence result by exploiting the analysis performed in [3] and successfully applying the Schauder fixed point theorem. Further, we discuss two uniqueness and continuous dependence results in a reduced setting.

Moreover, in this paper the interest is particularly focused on the control of the individual movements, characterized by the time and space dependent controls given by the diffusion coefficients  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$  and  $\kappa_r$ , in order to maintain the infected population (both e and i) at a minimum level in a certain subdomain  $\Omega_C \subset \Omega$ . More exactly, having the map of the initial values of the infected populations in the disjunct subdomains  $\Omega_j$  of  $\Omega$ , we want to control the diffusion of the infected individuals from these subdomains in the time interval (0,T), such that, by limiting the individual movements by optimal policies, to preserve a certain domain with a lower density of infected. To this end, we consider that  $\Omega$  is represented as a reunion of subdomains  $\Omega_j$ , j = 1, ..., m, which can be referred as geographic areas, such that

$$\Omega = \bigcup_{j=1}^{m} \Omega_{j}, \quad \Omega_{j} \cap \Omega_{k} = \emptyset \quad \text{if} \quad j, k \in \{1, ..., m\}, \ j \neq k,$$

$$\Omega_{j} \text{ is measurable and } |\Omega_{j}| > 0 \text{ for } j = 1, ..., m. \tag{1.9}$$

Hence, we aim to control the diffusion within the cylindrical domains  $Q_j = \Omega_j \times (0, T)$ , j = 1, ..., m, via the controls  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$ ,  $\kappa_r$  that are assumed to have constant values in the domains  $Q_j$ , represented by

$$\kappa_{s}(x,t) = \sum_{j=1}^{m} u_{j}^{s} \chi_{Q_{j}}(x,t), \quad \kappa_{e}(x,t) = \sum_{j=1}^{m} u_{j}^{e} \chi_{Q_{j}}(x,t),$$

$$\kappa_{i}(x,t) = \sum_{j=1}^{m} u_{j}^{i} \chi_{Q_{j}}(x,t), \quad \kappa_{r}(x,t) = \sum_{j=1}^{m} u_{j}^{r} \chi_{Q_{j}}(x,t), \quad (x,t) \in Q$$
(1.10)

where  $\chi_{Q_j}$  is the characteristic function of  $Q_j$ , j=1,...,m. Obviously, the controls are the scalars  $u_j^s$ ,  $u_j^e$ ,  $u_j^i$ ,  $u_j^r$ , but, for the sake of brevity, in most of our calculations we shall refer to the functions  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$ ,  $\kappa_r$ .

Then, assuming that  $\Omega_C$  is a measurable set and denoting  $Q_C = \Omega_C \times (0, T)$ , for a given positive coefficient  $\alpha$  we introduce the cost functional

$$\mathcal{J}(\kappa_s, \kappa_e, \kappa_i, \kappa_r, e, i) = \frac{1}{2} \int_{Q_C} (|e|^2 + |i|^2) + \frac{\alpha}{2} \int_{Q} (|\kappa_s|^2 + |\kappa_e|^2 + |\kappa_i|^2 + |\kappa_r|^2)$$
 (1.11)

and state the control problem by

minimize  $\mathcal{J}(\kappa_s, \kappa_e, \kappa_i, \kappa_r, e, i)$  under the constraint  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r) \in \mathcal{U}_{ad}$ 

subject to (1.1)–(1.6), where

$$\mathcal{U}_{ad} = \mathcal{U}_{ad}^s \times \mathcal{U}_{ad}^e \times \mathcal{U}_{ad}^i \times \mathcal{U}_{ad}^r \tag{1.12}$$

and each factor (which is the set of the admissible components of the controls) has the structure

$$\mathcal{U}_{ad}^{s} := \left\{ \kappa_{s} := \sum_{j=1}^{m} u_{j}^{s} \chi_{Q_{j}} : u_{j}^{s} \in [u_{\min}^{s,j}, u_{\max}^{s,j}] \text{ for } j = 1, \dots, m \right\}$$
 (1.13)

for some fixed closed intervals  $[u_{\min}^{s,j}, u_{\max}^{s,j}]$ ,  $j=1,\ldots,m$ . Here, only the first set  $\mathcal{U}_{ad}^s$  is precisely defined, but  $\mathcal{U}_{ad}^e$ ,  $\mathcal{U}_{ad}^i$ ,  $\mathcal{U}_{ad}^r$  are completely analogous. Next, for the results obtained for this optimal control problem, we refer the reader to the sequel and, in particular, to Section 4.

The paper is organized as follows. In the next section, we list our assumptions and notations and state our results. The proofs of our results regarding the existence and uniqueness for the state system (1.1)–(1.6) and the continuous dependence of the solution on the controls are given in Section 3, while Section 4 is devoted to the study of the optimal control problem.

### 2 Statement of the problem and results

In this section, we state precise assumptions and present our results. First of all, the set  $\Omega \subset \mathbb{R}^d$ , d=1,2,3, is assumed to be bounded, connected and smooth. Next, if X is a Banach space, then  $\|\cdot\|_X$  denotes its norm, with the only exception of the space H defined below and the  $L^{\infty}$  spaces constructed on  $\Omega$ , (0,T) and Q, whose norms will be indicated by  $\|\cdot\|$  (i.e., without any subscript) and by  $\|\cdot\|_{\infty}$ , respectively. Moreover, for simplicity, we use the same symbol for the norm in X and that in any power of X. We also introduce

$$H := L^2(\Omega)$$
 and  $V := H^1(\Omega)$  (2.1)

and adopt the framework of the Hilbert triplet  $(V, H, V^*)$  obtained by identifying H with a subspace of the dual space  $V^*$  of V in the usual way, namely, in order that  $\langle z, v \rangle = \int_{\Omega} zv$  for every  $z \in H$  and  $v \in V$ , where  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $V^*$  and V.

Now, we list the assumptions on the structure of the system. We recall (1.7) for the definition of Q and we postulate that

$$\kappa_s, \, \kappa_e, \, \kappa_i, \, \kappa_r : Q \times \mathbb{R} \to \mathbb{R}$$
 are Carathéodory functions satisfying
$$\kappa_* \leq \kappa_s(x, t, \zeta), \, \kappa_e(x, t, \zeta), \, \kappa_i(x, t, \zeta), \, \kappa_r(x, t, \zeta) \leq \kappa^*$$
for a.a.  $(x, t) \in Q$ , every  $\zeta \in \mathbb{R}$  and some positive constants  $\kappa_*$  and  $\kappa^*$ 

$$\beta_i, \, \beta_e : Q \times \mathbb{R} \to \mathbb{R}$$
 are Carathéodory functions satisfying
$$\alpha_i : Q \in \mathbb{R} \to \mathbb{R}$$
are Carathéodory functions satisfying

$$0 \le \beta_i(x, t, \zeta), \beta_e(x, t, \zeta) \le \beta^*$$

for a.a. 
$$(x,t) \in Q$$
, every  $\zeta \in \mathbb{R}$  and some positive constant  $\beta^*$  (2.3)

$$\phi_e, \, \phi_r \text{ and } \sigma \text{ are positive constants}$$
 (2.4)

 $\gamma \in L^{\infty}(0,T)$  satisfies  $0 \le \gamma(t) \le \gamma^*$ 

for a.a. 
$$t \in (0, T)$$
 and some positive constant  $\gamma^*$ . (2.5)

**Remark 2.1.** We recall that  $f: Q \times \mathbb{R} \to \mathbb{R}$  is a Carathéodory function if

$$(x,t) \mapsto f(x,t,\zeta)$$
 is measurable on  $Q$  for every  $\zeta \in \mathbb{R}$ ,  $\zeta \mapsto f(x,t,\zeta)$  is continuous on  $\mathbb{R}$  for a.a.  $(x,t) \in Q$ .

If  $f: Q \times \mathbb{R}$  is a Carathéodory function we still term f the corresponding Nemytskii operator in the space of measurable functions on Q, i.e., if  $v: Q \to \mathbb{R}$  is measurable, we

use the following abbreviation

$$f(v)$$
 denotes the function  $Q \mapsto \mathbb{R}$  given by  $(x,t) \mapsto f(x,t,v(x,t))$ . (2.6)

We notice that

$$v_k \to v$$
 a.e. in  $Q$  implies that  $f(v_k) \to f(v)$  a.e. in  $Q$ . (2.7)

For the initial data, we postulate that

$$s_0, e_0, i_0, r_0 \in L^{\infty}(\Omega)$$
 are nonnegative. (2.8)

Then, the state system related to the control problem we want to discuss is the following: we look for a quadruplet (s, e, i, r) enjoying the regularity properties

$$s, e, i, r \in H^1(0, T; V^*) \cap L^2(0, T; V) \hookrightarrow C^0([0, T]; H)$$
 (2.9)

$$s, e, i, r \ge 0$$
 a.e. in  $Q$  (2.10)

$$s, e, i, r \in L^{\infty}(Q) \tag{2.11}$$

and satisfying the variational equations

$$\langle \partial_t s, v \rangle + \int_{\Omega} (\beta_i(n) \, si + \beta_e(n) \, se) \, v + \int_{\Omega} \kappa_s(n) \nabla s \cdot \nabla v - \int_{\Omega} \gamma r \, v = 0$$
 (2.12)

$$\langle \partial_t e, v \rangle - \int_{\Omega} (\beta_i(n) \, si + \beta_e(n) \, se) \, v + \int_{\Omega} (\sigma + \phi_e) e \, v + \int_{\Omega} \kappa_e(n) \nabla e \cdot \nabla v = 0 \qquad (2.13)$$

$$\langle \partial_t i, v \rangle + \int_{\Omega} \phi_r i \, v + \int_{\Omega} \kappa_i(n) \nabla i \cdot \nabla v - \int_{\Omega} \sigma e \, v = 0$$
 (2.14)

$$\langle \partial_t r, v \rangle - \int_{\Omega} (\phi_r i + \phi_e e) v + \int_{\Omega} \kappa_r(n) \nabla r \cdot \nabla v + \int_{\Omega} \gamma r v = 0$$
 (2.15)

a.e. in (0,T) and for every  $v \in V$ , where (cf. (1.8))

$$n := s + e + i + r \tag{2.16}$$

as well as the initial condition

$$(s, e, i, r)(0) = (s_0, e_0, i_0, r_0). (2.17)$$

Our first result regards the existence of a solution to this problem.

**Theorem 2.2.** Assume (2.2)–(2.5) on the structure of the system and (2.8) on the initial data. Then, there exists at least a quadruplet (s, e, i, r) satisfying (2.9)–(2.11) that solves problem (2.12)–(2.17) and satisfies the stability estimate

$$||(s, e, i, r)||_{H^1(0,T;V^*) \cap C^0([0,T];H) \cap L^2(0,T;V) \cap L^\infty(Q)} \le K_1$$
(2.18)

with some positive constant  $K_1 > 0$  that depends only on  $\Omega$ , T, the constants  $\kappa_*$ ,  $\kappa^*$ ,  $\beta^*$ ,  $\gamma^*$ ,  $\phi_e$ ,  $\phi_r$  and  $\sigma$ , and the initial data.

We cannot prove uniqueness in the above general setting, unfortunately, and we are able to show uniqueness only under suitable assumptions. Namely, we first suppose that

$$\kappa_s = \kappa_e = \kappa_i = \kappa_r$$
 do not explicitly depend on  $(x, t)$ . (2.19)

**Theorem 2.3.** Besides the assumptions of Theorem 2.2, assume (2.19). Then, problem (2.12)–(2.17) has a unique solution.

Under different assumptions we can prove both uniqueness and continuous dependence of the solution on the diffusion coefficients. We stress that we do not need any longer that these coefficients are the same. We recall that we have termed  $\zeta$  the last variable in the assumptions regarding coefficients and suppose that

the diffusion coefficients 
$$\kappa_s$$
,  $\kappa_e$ ,  $\kappa_i$  and  $\kappa_r$  do not depend on  $\zeta$  (2.20)  
 $|\beta_i(x,t,\zeta_1) - \beta_i(x,t,\zeta_2)| + |\beta_e(x,t,\zeta_1) - \beta_e(x,t,\zeta_2)| \le L|\zeta_1 - \zeta_2|$ 

for a.a. 
$$(x,t) \in Q$$
, every  $\zeta_1, \zeta_2 \in \mathbb{R}$  and some positive constant  $L$ . (2.21)

We will need a reinforcement of the above assumption only later on (when dealing with the contol problem). Nevertheless, we prefer to state it here. We use the notation

 $\beta_i'$  and  $\beta_e'$  denote the derivatives with respect to  $\zeta$  of  $\beta_i$  and  $\beta_e$ , respectively

(which exist almost everywhere due to Lipschitz continuity) and assume that

$$\beta_i'$$
 and  $\beta_e'$  are Carathéodory functions. (2.22)

For clarity, we write the form that problem (2.12)–(2.17) takes under assumption (2.20): the diffusion coefficients are just functions on Q and n enters just the coefficients  $\beta_e$  and  $\beta_i$ .

$$\langle \partial_t s, v \rangle + \int_{\Omega} (\beta_i(n) \, si + \beta_e(n) \, se) \, v + \int_{\Omega} \kappa_s \nabla s \cdot \nabla v - \int_{\Omega} \gamma r \, v = 0$$
 (2.23)

$$\langle \partial_t e, v \rangle - \int_{\Omega} (\beta_i(n) \, si + \beta_e(n) \, se) \, v + \int_{\Omega} (\sigma + \phi_e) e \, v + \int_{\Omega} \kappa_e \nabla e \cdot \nabla v = 0$$
 (2.24)

$$\langle \partial_t i, v \rangle + \int_{\Omega} \phi_r i \, v + \int_{\Omega} \kappa_i \nabla i \cdot \nabla v - \int_{\Omega} \sigma e \, v = 0$$
 (2.25)

$$\langle \partial_t r, v \rangle - \int_{\Omega} (\phi_r i + \phi_e e) v + \int_{\Omega} \kappa_r \nabla r \cdot \nabla v + \int_{\Omega} \gamma r v = 0$$
 (2.26)

$$(s, e, i, r)(0) = (s_0, e_0, i_0, r_0)$$
(2.27)

where n := s + e + i + r as before.

**Theorem 2.4.** Besides the assumptions of Theorem 2.2, assume (2.20)–(2.21). Then, problem (2.23)–(2.27) has a unique solution. Moreover, if  $\kappa_{sj}$ ,  $\kappa_{ej}$ ,  $\kappa_{ej}$  and  $\kappa_{rj}$ , j=1,2, are two choices of  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$  and  $\kappa_r$ , respectively, and  $(s_j, e_j, i_j, r_j)$  are the corresponding solutions, then the estimate

$$\begin{aligned} &\|(s_1, e_1, i_1, r_1) - (s_2, e_2, i_2, r_2)\|_{H^1(0,T;V^*) \cap C^0([0,T];H) \cap L^2(0,T;V)} \\ &\leq K_2 \|(\kappa_{s_1}, \kappa_{e_1}, \kappa_{i_1}, \kappa_{r_1}) - (\kappa_{s_2}, \kappa_{e_2}, \kappa_{i_2}, \kappa_{r_2})\|_{\infty} \end{aligned}$$
(2.28)

holds true with some positive constant  $K_2 > 0$  that depends only on  $\Omega$ , T, the constants  $\kappa_*$ ,  $\kappa^*$ ,  $\beta^*$ ,  $\gamma^*$ ,  $\phi_e$ ,  $\phi_r$  and  $\sigma$ , the Lipschitz constant L and the initial data.

The above result is crucial for the the control problem we introduce at once. The controls are the diffusion coefficients and we aim at controlling the exposed population and the infected population in a region  $\Omega_C \subset \Omega$ . Our theory needs the framework of Theorem 2.4. Besides this, we assume that

$$\Omega_C$$
 is a measurable subset of  $\Omega$  and  $Q_C := \Omega_C \times (0, T)$  (2.29)

$$\alpha$$
 is a positive constant. (2.30)

Then, the cost functional is defined by (1.11) for  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r, e, i) \in (L^2(Q))^6$ , in principle. However, in view of (1.9), (1.12) and (1.13), the set  $\mathcal{U}_{ad}$  of admissible controls satisfies the general properties

$$\mathcal{U}_{ad} \subset (L^{\infty}(Q))^4$$
 is convex and closed (2.31)

$$\kappa_* \leq \kappa_s, \, \kappa_e, \, \kappa_i, \, \kappa_r \leq \kappa^*$$
 a.e. in  $Q$ , for every  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r) \in \mathcal{U}_{ad}$ 

and some positive constants 
$$\kappa_*$$
 and  $\kappa^*$  (2.32)

the subspace span(
$$\mathcal{U}_{ad}$$
) is finite dimensional (2.33)

provided that

the variability intervals 
$$[u_{\min}^{s,j}, u_{\max}^{s,j}], [u_{\min}^{e,j}, u_{\max}^{e,j}],$$
  
 $[u_{\min}^{i,j}, u_{\max}^{i,j}], [u_{\min}^{r,j}, u_{\max}^{r,j}], j = 1, \dots, m, \text{ are all contained in } [\kappa_*, \kappa^*].$  (2.34)

Then, the control problem is the following

minimize 
$$\mathcal{J}(\kappa_s, \kappa_e, \kappa_i, \kappa_r, e, i)$$
, for  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r) \in \mathcal{U}_{ad}$ , where  $e$  and  $i$  are the components of the solution to problem (2.23)–(2.27) corresponding to the diffusion coefficients  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$  and  $\kappa_r$ . (2.35)

The existence of optimal controls and the necessary optimality conditions will be discussed in Section 4. We conclude this section by recalling the Young inequality

$$a b \le \delta a^2 + \frac{1}{4\delta} b^2$$
 for all  $a, b \in \mathbb{R}$  and  $\delta > 0$  (2.36)

which will be repeatedly used throughout the paper along with the Hölder and Schwarz inequalities, and by stating a general rule concerning the constants. The small-case symbol c stands for possibly different constants (whose actual values may change from line to line and even within the same line) that depend only on  $\Omega$ , T, the structure of the system, and the constants and the norms of the functions involved in the assumptions of the statements. In particular, the values of c are independent of the control variables  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$  and  $\kappa_r$ . A small-case symbol with a subscript like  $c_\delta$  indicates that the constant may depend on the parameter  $\delta$ , in addition. On the contrary, we mark precise constants that we can refer to by using different symbols, e.g., capital letters.

# 3 Well-posedness

Well-posedenss in a particular case is proved in [3, Thm 2.1]. We present this result in the form of a lemma.

**Lemma 3.1.** Assume (2.2)–(2.5) on the structure of the system and that all the coefficients  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$ ,  $\kappa_r$ ,  $\beta_i$  and  $\beta_e$  depend just on space and time and are independent of the last variable. Moreover, assume that the initial data satisfy (2.8). Then, there exists a unique quadruplet (s, e, i, r) satisfying (2.9)–(2.11) and solving problem (2.12)–(2.17). Moreover, the stability estimate

$$\|(s, e, i, r)\|_{H^1(0,T;V^*) \cap C^0([0,T];H) \cap L^2(0,T;V) \cap L^\infty(Q)} \le K_0 \tag{3.1}$$

holds true with some constant  $K_0 > 0$  that depends only on  $\Omega$ , T, the constants  $\kappa_*$ ,  $\kappa^*$ ,  $\beta^*$ ,  $\gamma^*$ ,  $\phi_e$ ,  $\phi_r$  and  $\sigma$ , and the initial data.

We use this result in our proofs of Theorems 2.2 and 2.3.

#### 3.1 Existence

We prove Theorem 2.2 by applying the Schauder fixed point theorem. Hence, we have to choose a proper subset  $\mathcal{K}$  of some Banach space  $\mathcal{V}$  that is convex and closed, and the continuous map  $\Phi: \mathcal{K} \to \mathcal{K}$  in order that  $\Phi(\mathcal{K})$  is relatively compact and a fixed point of  $\Phi$  is a solution to the problem we want to solve. To this end, we set

$$\mathcal{V} := L^2(0, T; H)$$
 and  $\mathcal{K} := \{ v \in \mathcal{V} : 0 \le v \le 4 K_0 \text{ a.e. in } Q \}$  (3.2)

where  $K_0$  is the constant that appears in (3.1). Then, we define  $\Phi$  as follows. For every  $z \in \mathcal{K}$ , we consider the system given by the equations (2.12)–(2.15) where n is replaced by z in the argument of  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$ ,  $\kappa_r$ ,  $\beta_i$  and  $\beta_e$ , complemented by the initial conditions (2.17). Then, we can apply the lemma: this system has a unique solution that we term  $(s^z, e^z, i^z, r^z)$  and also satisfies (3.1). So, we set

$$\Phi(z) := s^z + e^z + i^z + r^z. \tag{3.3}$$

Then,  $\Phi(z)$  is nonnegative. Moreover,  $\|\Phi(z)\|_{\infty} \leq \|s^z\|_{\infty} + \|e^z\|_{\infty} + \|i^z\|_{\infty} + \|r^z\|_{\infty} \leq 4 K_0$ . Therefore  $\Phi(z) \in \mathcal{K}$ . Furthermore, it is clear that any fixed point of  $\Phi$  is a solution to the original problem (2.12)–(2.17). To conclude the proof, it remains to show that  $\Phi$  is continuous and that  $\Phi(\mathcal{K})$  is relatively compact. The latter immediately follows since (3.1) applied to  $(s^z, e^z, i^z, r^z)$  implies that  $\Phi(z)$  is bounded in  $H^1(0, T; V^*) \cap L^2(0, T; V)$ , so that we can apply the Aubin–Lions lemma (see, e.g., [12, Thm. 5.1, p. 58]) and conclude that  $\Phi(\mathcal{K})$  is relatively compact. Now, we prove that  $\Phi$  is continuous. To this end, we fix  $z \in \mathcal{K}$  and a sequence  $\{z_k\}$  in  $\mathcal{K}$  that converges to z and we show that  $\Phi(z_k)$  converges to  $\Phi(z)$ . To simplify the notation, we write  $(s_k, e_k, i_k, r_k)$  in place of  $(s^{z_k}, e^{z_k}, i^{z_k}, r^{z_k})$ . We also set  $n_k := \Phi(z_k) = s_k + e_k + i_k + r_k$ . By applying (3.1) to  $(s_k, e_k, i_k, r_k)$  and owing to standard compactness arguments, we deduce that

$$(s_k, e_k, i_k, r_k) \to (s, e, i, r) \tag{3.4}$$

weakly in  $(H^1(0,T;V^*) \cap L^2(0,T;V))^4$  and weakly star in  $L^{\infty}(Q)^4$ , for some limiting quadruplet (s,e,i,r) and for some subsequence. We prove that s+e+i+r coincides with  $\Phi(z)$ . Once this is shown, since the limit has been characterized, we also conclude that the whole sequence  $\Phi(z_k)$  converges to  $\Phi(z)$  and the proof is complete. To prove

that s + e + i + r coincides with  $\Phi(z)$ , is suffices to show that (s, e, i, r) is the solution to the system given by the equations (2.12)–(2.15) where n is replaced by z in the argument of  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$ ,  $\kappa_r$ ,  $\beta_i$  and  $\beta_e$ , complemented by the initial conditions (2.17). The latter are clearly satisfied. Let us discuss just the first equation since the others are analogous. An equivalent formulation of our thesis is the following time-integrated version:

$$\int_0^T \langle \partial_t s(t), v(t) \rangle dt + \int_Q (\beta_i(z) si + \beta_e(z) se) v + \int_Q \kappa_s(z) \nabla s \cdot \nabla v - \int_Q \gamma r v = 0$$
for every  $v \in L^2(0, T; V)$ . (3.5)

By the Aubin-Lions lemma, the convergence (3.4) is also strong in  $L^2(0,T;H)$  and, without loss of generality, almost everywhere in Q. Then, by Remark 2.1, we have that  $\beta_i(z_k)$  converges to  $\beta_i(z)$  a.e. in Q, and analogous conclusions hold regarding  $\beta_e$  and  $\kappa_s$ . Since, a.e. in Q, it holds that

$$|\beta_i(z_k)s_ki_k| \le \beta^*K_0^2$$
,  $|\beta_e(z_k)s_ke_k| \le \beta^*K_0^2$  and  $|\kappa_s(z_k)\nabla v| \le \kappa^*|\nabla v|$ 

we can apply the Lebesgue dominated convergence theorem and deduce that

$$\beta_i(z_k)s_ki_k \to \beta_i(z)si$$
 and  $\beta_e(z_k)s_ke_k \to \beta_e(z)se$  strongly in  $L^2(0,T;H)$   
 $\kappa_s(z_k)\nabla v \to \kappa_s(z)\nabla v$  strongly in  $(L^2(0,T;H))^d$ .

This, the weak convergence of  $\partial_t s_k$ ,  $\nabla s_k$  and  $r_k$  given by (3.4), and the boundedness of  $\gamma$  ensure that we can pass to the limit in the analogue of (3.5) related to  $z_k$  and  $(s_k, e_k, i_k, r_k)$ , i.e.,

$$\int_0^T \langle \partial_t s_k(t), v(t) \rangle dt + \int_Q (\beta_i(z_k) s_k i_k + \beta_e(z_k) s_k e_k) v + \int_Q \kappa_s(z_k) \nabla s_k \cdot \nabla v - \int_Q \gamma r_k v = 0$$

and obtain (3.5) itself.

#### 3.2 Uniqueness and continuous dependence

This section is devoted to the proof of Theorems 2.3 and 2.4.

**Proof of Theorem 2.3.** We start with a preliminary observation. The assumptions on the diffusion coefficients given in the statement say that  $\kappa_s(n)$ ,  $\kappa_e(n)$ ,  $\kappa_i(n)$  and  $\kappa_r(n)$  are replaced by  $\kappa(n)$  where  $\kappa: \mathbb{R} \to \mathbb{R}$  is continuous and satisfies  $\kappa_* \leq \kappa(\zeta) \leq \kappa^*$  for every  $\zeta \in \mathbb{R}$ . Then, by adding all equations (2.12)–(2.15) to each other, we obtain that

$$\langle \partial_t n, v \rangle + \int_{\Omega} \nabla \widehat{\kappa}(n) \cdot \nabla v = 0$$
 a.e. in  $(0, T)$ , for every  $v \in V$  (3.6)

where we have set

$$\widehat{\kappa}(\zeta) := \int_0^{\zeta} \kappa(\xi) \, d\xi \quad \text{for } \zeta \in \mathbb{R}.$$

By taking v=1, we deduce that

$$\int_{\Omega} n(t) = \int_{\Omega} n(0) \quad \text{for every } t \in [0, T]. \tag{3.7}$$

From this, we derive a uniqueness result. Assume that  $n_1, n_2 \in H^1(0, T; V^*) \cap L^2(0, T; V)$  are two solutions to (3.6) with the same initial value. Then  $\int_{\Omega} (n_1 - n_2)(t) = 0$  for every  $t \in [0, T]$ . By also recalling that  $\Omega$  is connected, it is not difficult to show that there exists a unique  $N \in C^0([0, T]; V)$  that satisfies

$$\int_{\Omega} N(t) = 0 \quad \text{and} \quad \int_{\Omega} \nabla N(t) \cdot \nabla v = \int_{\Omega} (n_1 - n_2)(t) \, v \quad \text{for all } t \in [0, T] \text{ and } v \in V.$$

Moreover, N belongs to  $C^0([0,T]; H^2(\Omega))$  since  $n_1 - n_2 \in C^0([0,T]; H)$  and  $\Omega$  is smooth. At this point, we write (3.6) for  $n_1$  and  $n_2$ , take the difference and test it by N. Since  $n_1 - n_2 = -\Delta N$ , we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\nabla N|^2+\int_{\Omega}(\widehat{\kappa}(n_1)-\widehat{\kappa}(n_2))(n_1-n_2)=0 \quad \text{a.e. in } (0,T).$$

As  $\kappa$  is positive,  $\widehat{\kappa}$  is an increasing function, so that the above time derivative is non-positive. Since  $n_i(0) = n_2(0)$  implies that N(0) = 0, we conclude that  $|\nabla N|^2$  vanishes identically. Thus, N is space independent. Moreover, N has also zero mean value, and consequently we conclude that N = 0, whence  $n_1 = n_2$ .

At this point, we can start our uniqueness proof. We pick two solutions  $(s_j, e_j, i_j, r_j)$ , j = 1, 2. By applying the above observation to  $n_j = s_j + e_j + i_j + r_j$ , we conclude that  $n_1 = n_2$ . It follows that  $\kappa_s(n_1) = \kappa_s(n_2)$  and that similar equalities hold for  $\kappa_e$ ,  $\kappa_i$ ,  $\kappa_r$ ,  $\beta_i$  and  $\beta_e$ , i.e., these coefficients can be considered as given functions on Q. Therefore, we can apply the uniqueness part of Lemma 3.1 and conclude that  $(s_1, e_1, i_1, r_1)$  and  $(s_2, e_2, i_2, r_2)$  coincide.

**Proof of Theorem 2.4.** We pick two choices of the diffusion coefficients as in the statement and a pair of arbitrary corresponding solutions. We accordingly define  $n_1$  and  $n_2$  and set for brevity

$$\kappa_s := \kappa_{s1} - \kappa_{s2}, \quad \kappa_e := \kappa_{e1} - \kappa_{e2}, \quad \kappa_i := \kappa_{i1} - \kappa_{i2} \quad \text{and} \quad \kappa_r := \kappa_{r1} - \kappa_{e2}$$
 $s := s_1 - s_2, \quad e := e_1 - e_2, \quad i := i_1 - i_2, \quad r := r_1 - r_2 \quad \text{and} \quad n = n_1 - n_2.$ 

Then, we write equations (2.23)–(2.26) for both coefficients and solutions, take the differences, test them by s, e, i and r, respectively, and rearrange a little. We obtain that

$$\begin{split} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} |s|^2 + \int_{\Omega} \kappa_{s1} |\nabla s|^2 = -\int_{\Omega} \kappa_s \nabla s_2 \cdot \nabla s \\ &- \int_{\Omega} \left( \beta_i(n_1) - \beta_i(n_2) \right) s_1 \, i_1 \, s - \int_{\Omega} \beta_i(n_2) |s|^2 \, i_1 - \int_{\Omega} \beta_i(n_2) s_2 \, i \, s \\ &- \int_{\Omega} \left( \beta_e(n_1) - \beta_e(n_2) \right) s_1 \, e_1 \, s - \int_{\Omega} \beta_e(n_2) |s|^2 \, e_1 - \int_{\Omega} \beta_e(n_2) s_2 \, e \, s + \int_{\Omega} \gamma r \, s \, , \\ &\frac{1}{2} \frac{d}{dt} \int_{\Omega} |e|^2 + \int_{\Omega} \kappa_{e1} |\nabla e|^2 = -\int_{\Omega} \kappa_e \nabla e_2 \cdot \nabla e \\ &+ \int_{\Omega} \left( \beta_i(n_1) - \beta_i(n_2) \right) s_1 \, i_1 \, e + \int_{\Omega} \beta_i(n_2) s \, i_1 \, e + \int_{\Omega} \beta_i(n_2) s_2 \, i \, e \\ &+ \int_{\Omega} \left( \beta_e(n_1) - \beta_e(n_2) \right) s_1 \, e_1 \, e + \int_{\Omega} \beta_e(n_2) s \, e_1 \, e + \int_{\Omega} \beta_e(n_2) s_2 \, |e|^2 - (\sigma + \phi_e) \int_{\Omega} |e|^2 \, , \end{split}$$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |i|^2 + \int_{\Omega} \kappa_{i1} |\nabla i|^2 = -\int_{\Omega} \kappa_i \nabla i_2 \cdot \nabla i - \int_{\Omega} \phi_r |i|^2 + \sigma \int_{\Omega} e i,$$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |r|^2 + \int_{\Omega} \kappa_{r1} |\nabla r|^2 = -\int_{\Omega} \kappa_r \nabla r_2 \cdot \nabla r + \int_{\Omega} (\phi_r i + \phi_e e) r - \int_{\Omega} \gamma |r|^2.$$

At this point, we add these equalities to each other. The left-hand side we obtain is bounded from below by

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (|s|^2 + |e|^2 + |i|^2 + |r|^2) + \kappa_* \int_{\Omega} (|\nabla s|^2 + |\nabla e|^2 + |\nabla i|^2 + |\nabla r|^2)$$

and we have to estimate the right-hand side from above. For a while, the values of the constants denoted by c can also depend on the solutions we have considered. Some terms are nonpositive and can be neglected. Let us treat just one of the terms involving gradients since the others are analogous. By the Young inequality, we have a.e. in (0,T) that

$$-\int_{\Omega} \kappa_s \nabla s_2 \cdot \nabla s \leq \|\kappa_s\|_{\infty} \int_{\Omega} |\nabla s_2| |\nabla s| \leq \frac{\kappa_*}{2} \int_{\Omega} |\nabla s|^2 + c \|\kappa_s\|_{\infty}^2 \|s_2\|_V^2.$$

Since  $s_2 \in L^2(0,T;V)$ , we deduce that

$$-\int_{Q_t} \kappa_s \nabla s_2 \cdot \nabla s \le \frac{\kappa_*}{2} \int_{Q_t} |\nabla s|^2 + c \|\kappa_s\|_{\infty}^2$$

where  $Q_t : \Omega \times (0, t)$ . Let us now treat some of the terms involving products. By (2.21) and the Hölder and Young inequalities, we have a.e. in (0, T) that

$$-\int_{\Omega} (\beta_i(n_1) - \beta_i(n_2)) s_1 i_1 s \le L \|s_1\|_{\infty} \|i_1\|_{\infty} \int_{\Omega} |n| |s| \le c \int_{\Omega} (|s|^2 + |e|^2 + |i|^2 + |r|^2).$$

The next are even easier:

$$-\int_{\Omega} \beta_{i}(n_{2})|s|^{2} i_{1} - \int_{\Omega} \beta_{i}(n_{2})s_{2} i s \leq \beta^{*} ||i_{1}||_{\infty} \int_{\Omega} |s|^{2} + \beta^{*} ||s_{2}||_{\infty} \int_{\Omega} |i| |s|$$

$$\leq c \int_{\Omega} (|s|^{2} + |i|^{2}).$$

Since no new difficulty arises in estimating the other terms, we quickly conclude. After integrating with respect to time and owing to the initial conditions, we can apply the Gronwall lemma and obtain that

$$\|(s, e, i, r)\|_{C^0([0,T];H)\cap L^2(0,T;V)} \le c \|(\kappa_s, \kappa_e, \kappa_i, \kappa_r)\|_{\infty}.$$
(3.8)

Here, as said before, the value of c can depend on the solutions we have considered. Nevertheless, we can apply (3.8) in the case  $\kappa_s = \kappa_e = \kappa_i = \kappa_r = 0$  and obtain that (s, e, i, r) = (0, 0, 0, 0). Since the solutions entering (3.8) were arbitrary, this proves uniqueness. Therefore, the solutions we have considered in the above proof are those given by our existence theorem and thus satisfy the stability estimate (2.18). Hence, the norms we have bounded by c in the derivation can be bounded by the constant  $K_1$  that appears in (2.18). Hence, (3.8) actually holds with a constant c that has the same dependence that the constant  $K_2$  of the statement has. It remains to prove that

$$\|\partial_t(s, e, i, r)\|_{L^2(0, T; V^*)} \le c \|(\kappa_s, \kappa_e, \kappa_i, \kappa_r)\|_{\infty}$$
(3.9)

with a similar constant c. We just consider the first component. We come back to (2.23) written for both coefficients and solutions and take the difference. Then, with an arbitrary  $v \in L^2(0,T;V)$ , we test the latter by v and have that

$$\langle \partial_t s, v \rangle + \int_{\Omega} (\beta_i(n_1) s_1 i_1 - \beta_i(n_2) s_2 i_2 + \beta_e(n_1) s_1 e_1 - \beta_e(n_2) s_2 e_2) v$$
  
+ 
$$\int_{\Omega} (\kappa_{s_1} \nabla s_1 - \kappa_{s_2} \nabla s_2) \cdot \nabla v - \int_{\Omega} \gamma r v = 0 \quad \text{a.e. in } (0, T).$$

By treating the products similarly as before, it is easy to prove that

$$\langle \partial_t s, v \rangle \le c \int_{\Omega} (|s| + |e| + |i| + |r|)|v| + c \int_{\Omega} |\nabla s| |\nabla v| + c \|\kappa_s\|_{\infty} \int_{\Omega} |\nabla s_1| |\nabla v|.$$

We immediately deduce that

$$\langle \partial_t s, v \rangle \le c \left( \|(s, e, i, r)\| \|v\| + \|\nabla s\| \|\nabla v\| + \|\kappa_s\|_{\infty} \|\nabla s_1\| \|\nabla v\| \right).$$

and by integrating over (0,T) we infer that

$$\int_{0}^{T} \langle \partial_{t} s(t), v(t) \rangle dt 
\leq c \left( \|(s, e, i, r)\|_{L^{2}(0,T;H)} + \|s\|_{L^{2}(0,T;V)} + \|\kappa_{s}\|_{\infty} \|s_{1}\|_{L^{2}(0,T;V)} \right) \|v\|_{L^{2}(0,T;V)}.$$

By applying the stability estimate to  $s_1$  and combining with (3.8), we conclude that

$$\int_0^T \langle \partial_t s(t), v(t) \rangle dt \le c \| (\kappa_s, \kappa_e, \kappa_i, \kappa_r) \|_{\infty} \| v \|_{L^2(0,T;V)}.$$

But this and the analogues for the other controls yield (3.9) since v is arbitrary in  $L^2(0,T;V)$ , and the proof is complete.

#### 4 The control problem

This section is devoted to the study of the control problem (2.35) presented in the Introduction. We first prove the existence of an optimal control. Then, we derive a first order necessary condition for optimality. Concerning the proof of this condition, we do not follow the standard technique based on the Fréchet differentiability of the control-to-state mapping, since the problem is particularly complicated. So, we prefer to deduce the necessary condition by directly acting on the first variation of the cost functional and proceeding in the direction of Gâteax derivatives. The condition we prove sounds as follows. If  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*)$  is an optimal control and  $(s^*, e^*, i^*, r^*)$  is the corresponding optimal state, then there holds the variational inequality

$$\int_{Q} \left\{ (\alpha \kappa_{s}^{*} - \nabla s^{*} \cdot \nabla p)(\kappa_{s} - \kappa_{s}^{*}) + (\alpha \kappa_{e}^{*} - \nabla e^{*} \cdot \nabla q)(\kappa_{e} - \kappa_{e}^{*}) \right. \\
+ (\alpha \kappa_{i}^{*} - \nabla i^{*} \cdot \nabla w)(\kappa_{i} - \kappa_{i}^{*}) + (\alpha \kappa_{r}^{*} - \nabla r^{*} \cdot \nabla z)(\kappa_{r} - \kappa_{r}^{*}) \right\} \\
\ge 0 \quad \text{for every } (\kappa_{s}, \kappa_{e}, \kappa_{i}, \kappa_{r}) \in \mathcal{U}_{ad}$$

where (p, q, w, z) is the solution to a proper adjoint problem (cf. (4.35)–(4.39)). We will see that the optimal coefficients are just one-dimensional projections of weighted mean values of suitable ingredients of the adjoint system on the closed intervals that enter the particular definition of  $\mathcal{U}_{ad}$ .

Besides the general assumptions (2.2)–(2.5) on the structure of the system and (2.8) on the initial data, we need to use (2.20)–(2.21) as well as (2.22). We recall the form (2.23)–(2.27) that the original state system takes under the above assumptions. It is understood that these assumptions are in force in the whole section and we do not recall them in our statements.

#### 4.1 Existence of an optimal control

**Theorem 4.1.** Assume (2.29)–(2.33). Then there exists an optimal control, i.e., there exist  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*) \in \mathcal{U}_{ad}$  such that, if  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r) \in \mathcal{U}_{ad}$  and  $(s^*, e^*, i^*, r^*)$  and (s, e, i, r) are the states corresponding to  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*)$  and  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r)$ , respectively, then there holds the inequality

$$\mathcal{J}(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*, e^*, i^*) \le \mathcal{J}(\kappa_s, \kappa_e, \kappa_i, \kappa_r, e, i). \tag{4.1}$$

Proof. We use the direct method. Let  $\Lambda \geq 0$  be the infimum of the cost functional subject to the constraints specified in (2.35), let  $\{(\kappa_{sk}, \kappa_{ek}, \kappa_{ik}, \kappa_{rk})\}$  be a minimizing sequence and let  $\{(s_k, e_k, i_k, r_k)\}$  be the sequence of the corresponding states. Since  $\mathcal{U}_{ad}$  is bounded in  $(L^{\infty}(Q))^4$  and span $(\mathcal{U}_{ad})$  is finite dimensional, the sequence of controls has a strongly converging subsequence, which we still term  $\{(\kappa_{sk}, \kappa_{ek}, \kappa_{ik}, \kappa_{rk})\}$  to simplify the notation. By applying the stability estimate (2.18) to the above sequence of states and owing to standard compactness results (in particular to the Aubin–Lions lemma), we find a convergence subsequence, still termed  $\{(s_k, e_k, i_k, r_k)\}$  to avoid a boring notation, in the related topology. We thus have that

$$\mathcal{J}(\kappa_{sk}, \kappa_{ek}, \kappa_{ik}, \kappa_{rk}, e_k, i_k) \to \Lambda \tag{4.2}$$

$$(\kappa_{sk}, \kappa_{ek}, \kappa_{ik}, \kappa_{rk}) \to (\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*) \quad \text{strongly in } (L^{\infty}(Q))^4$$

$$(s_k, e_k, i_k, r_k) \to (s^*, e^*, i^*, r^*)$$

$$(4.3)$$

weakly star in 
$$(H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q))^4$$
,  
strongly in  $(L^2(0,T;H))^4$  and a.e. in  $Q$  (4.4)

for some quadruplets  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*)$  and  $(s^*, e^*, i^*, r^*)$ . The former belongs to  $\mathcal{U}_{ad}$  since  $\mathcal{U}_{ad}$  is closed. We now prove that  $(s^*, e^*, i^*, r^*)$  is the state corresponding to  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*)$ , i.e., it solves (2.23)–(2.27) where one reads  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*)$  in place of  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r)$ . The initial condition (2.27) is satisfied since the sequence  $\{(s_k, e_k, i_k, r_k)\}$  also converges weakly in  $(C^0([0, T]; H))^4$ . We just prove that the first equation (2.23) is satisfied and we do not repeat this argument for the others. Namely, we prove that the limiting quadruplet satisfies the equivalent time-integrated version of (2.23), i.e.,

$$\int_{0}^{T} \langle \partial_{t} s^{*}(t), v(t) \rangle dt + \int_{Q} (\beta_{i}(n^{*}) s^{*}i^{*} + \beta_{e}(n^{*}) s^{*}e^{*}) v + \int_{Q} \kappa_{s}^{*}(n^{*}) \nabla s^{*} \cdot \nabla v$$

$$- \int_{Q} \gamma r^{*} v = 0 \quad \text{for every } v \in L^{2}(0, T; V) \tag{4.5}$$

where  $n^* := s^* + e^* + i^* + r^*$ . Thus, we fix  $v \in L^2(0,T;V)$ . By obviously defining  $n_k := s_k + e_k + i_k + r_k$ , we have that

$$|\beta_i(n_k) s_k i_k v| \le c |v|$$
 and  $|\beta_e(n_k) s_k e_k v| \le c |v|$  a.e. in  $Q$ 

so that

$$\beta_i(n_k) s_k i_k v \to \beta_i(n^*) s^* i^* v$$
 and  $\beta_e(n_k) s_k e_k v \to \beta_e(n^*) s^* e^* v$ 

strongly in  $L^2(0,T;H)$  by the Lebesgue dominated convergence theorem. On the other hand, (4.3) implies

$$\kappa_{sk} \nabla v \to \kappa_s^* \nabla v$$
 strongly in  $(L^2(0,T;H))^d$ 

and the analogues for the other controls. Therefore, one can pass to the limit as k tends to infinity in the analogue of (4.5) written for  $(\kappa_{sk}, \kappa_{ek}, \kappa_{ik}, \kappa_{rk})$  and the corresponding states and obtain (4.5) itself. By strong convergence, we also have that

$$\mathcal{J}(\kappa_{sk}, \kappa_{ek}, \kappa_{ik}, \kappa_{rk}, e_k, i_k) \to \mathcal{J}(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*, e^*, i^*)$$

whence  $\mathcal{J}(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*, e^*, i^*) = \Lambda$ . We conclude that  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*)$  is an optimal control and  $(s^*, e^*, i^*, r^*)$  is the corresponding state.

#### 4.2 Necessary conditions for optimality

We fix an optimal control  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*)$  and the corresponding state  $(s^*, e^*, i^*, r^*)$ . Given any  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r) \in \mathcal{U}_{ad}$ , we associate the following linearized system

$$\langle \partial_{t}\xi, v \rangle + \int_{\Omega} \kappa_{s}^{*} \nabla \xi \cdot \nabla v + \int_{\Omega} (A_{1}\xi + B_{1}\eta + C_{1}\iota + D_{1}\rho)v$$

$$= -\int_{\Omega} (\kappa_{s} - \kappa_{s}^{*}) \nabla s^{*} \cdot \nabla v, \qquad (4.6)$$

$$\langle \partial_{t}\eta, v \rangle + \int_{\Omega} \kappa_{e}^{*} \nabla \eta \cdot \nabla v + \int_{\Omega} (A_{2}\xi + B_{2}\eta + C_{2}\iota + D_{2}\rho)v$$

$$= -\int_{\Omega} (\kappa_{e} - \kappa_{e}^{*}) \nabla e^{*} \cdot \nabla v, \qquad (4.7)$$

$$\langle \partial_{t}\iota, v \rangle + \int_{\Omega} \kappa_{i}^{*} \nabla \iota \cdot \nabla v + \int_{\Omega} (A_{3}\xi + B_{3}\eta + C_{3}\iota + D_{3}\rho)v$$

$$= -\int_{\Omega} (\kappa_{i} - \kappa_{i}^{*}) \nabla i^{*} \cdot \nabla v, \qquad (4.8)$$

$$\langle \partial_{t}\rho, v \rangle + \int_{\Omega} \kappa_{r}^{*} \nabla \rho \cdot \nabla v + \int_{\Omega} (A_{4}\xi + B_{4}\eta + C_{4}\iota + D_{4}\rho)v$$

$$= -\int_{\Omega} (\kappa_{r} - \kappa_{r}^{*}) \nabla r^{*} \cdot \nabla v \qquad (4.9)$$

$$(\xi, \eta, \iota, \rho)(0) = (0, 0, 0, 0) \tag{4.10}$$

where we have set

$$A_1 := \beta_i'(n^*)s^*i^* + \beta_i(n^*)i^* + \beta_e'(n^*)s^*e^* + \beta_e(n^*)e^*$$
(4.11)

$$B_1 := \beta_i'(n^*)s^*i^* + \beta_e(n^*)s^* + \beta_e'(n^*)s^*e^*$$
(4.12)

$$C_1 := \beta_i'(n^*)s^*i^* + \beta_i(n^*)s^* + \beta_e'(n^*)s^*e^*$$
(4.13)

$$D_1 := \beta_i'(n^*)s^*i^* + \beta_e'(n^*)s^*e^* - \gamma \tag{4.14}$$

$$A_2 := -A_1, \quad B_2 := -A_2 + \sigma + \phi_e, \quad C_2 := -C_1, \quad D_2 := -D_1 - \gamma$$
 (4.15)

$$A_3 := 0, \quad B_3 := -\sigma, \quad C_3 := \phi_r, \quad D_3 := 0$$
 (4.16)

$$A_4 := 0, \quad B_4 := -\phi_e, \quad C_4 := -\phi_r, \quad D_4 := \gamma.$$
 (4.17)

We notice at once that problem (4.6)–(4.10) has a unique solution

$$(\xi, \eta, \iota, \rho) \in (H^1(0, T; V^*) \cap L^2(0, T; V))^4$$

$$(4.18)$$

since it is a uniformly parabolic problem with bounded coefficients and known right-hand sides belonging to  $L^2(0,T;V^*)$  if we read (4.6)–(4.9) as an abstract equation in the framwork  $(V,H,V^*)^4$ . We have the following result:

**Proposition 4.2.** Assume (2.29)–(2.33) and let  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*)$  and  $(s^*, e^*, i^*, r^*)$  be an optimal control and the corresponding state, respectively. Then, for every  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r) \in U_{ad}$  there holds the inequality

$$\int_{Q_C} (\eta e^* + \iota i^*) 
+ \alpha \int_{Q} ((\kappa_s - \kappa_s^*) \kappa_s^* + (\kappa_e - \kappa_e^*) \kappa_s^* + (\kappa_i - \kappa_i^*) \kappa_s^* + (\kappa_r - \kappa_r^*) \kappa_s^*) \ge 0$$
(4.19)

where  $(\xi, \eta, \iota, \rho)$  is the solution to the linearized system associated with  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r)$ .

*Proof.* We fix the admissible control  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r)$  and the corresponding state (s, e, i, r) and, for every  $\lambda \in (0, 1)$ , we introduce the incremented control

$$(\kappa_s^{\lambda}, \kappa_e^{\lambda}, \kappa_i^{\lambda}, \kappa_r^{\lambda}) := (\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*) + \lambda \left( (\kappa_s, \kappa_e, \kappa_i, \kappa_r) - (\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*) \right)$$
(4.20)

which belongs to  $\mathcal{U}_{ad}$  since  $\mathcal{U}_{ad}$  is convex. We also set

$$(s^{\lambda}, e^{\lambda}, i^{\lambda}, r^{\lambda})$$
 is the state corresponding to  $(\kappa_s^{\lambda}, \kappa_e^{\lambda}, \kappa_i^{\lambda}, \kappa_r^{\lambda})$  (4.21)

and define the quotients

$$\xi_{\lambda} := \frac{s^{\lambda} - s^{*}}{\lambda}, \quad \eta_{\lambda} := \frac{e^{\lambda} - e^{*}}{\lambda}, \quad \iota_{\lambda} := \frac{i^{\lambda} - i^{*}}{\lambda} \quad \text{and} \quad \rho_{\lambda} := \frac{r^{\lambda} - r^{*}}{\lambda}.$$
 (4.22)

We prove that  $(\xi_{\lambda}, \eta_{\lambda}, \iota_{\lambda}, \rho_{\lambda})$  converges in a suitable topology to the solution  $(\xi, \eta, \iota, \rho)$  to the linearized system as  $\lambda$  tends to zero. First, we notice the regularity

$$(\xi_{\lambda}, \eta_{\lambda}, \iota_{\lambda}, \rho_{\lambda}) \in (H^1(0, T; V^*) \cap L^2(0, T; V))^4. \tag{4.23}$$

Then, we write the system these quotients solve. This is obtained as follows: we write (2.23)–(2.27) for the incremented control (4.20) and the corresponding state; then we do

the same for the optimal control and state; finally, we take the differences and divide by  $\lambda$ . With the notations

$$\beta_i^* := \beta_i(n^*)$$
 and  $\beta_i^{\lambda} := \beta_i(n^{\lambda})$  where  $n^{\lambda} := s^{\lambda} + e^{\lambda} + i^{\lambda} + r^{\lambda}$  (4.24)

and the analogous ones for  $\beta_e^*$  and  $\beta_e^{\lambda}$ , and observing that

$$\frac{\kappa_s^{\lambda} - \kappa_s^*}{\lambda} = \kappa_s - \kappa_s^*$$

and that analogous identities hold for the other similar fractions, we have that

$$\langle \partial_{t}\xi_{\lambda}, v \rangle + \int_{\Omega} \frac{1}{\lambda} (\beta_{i}^{\lambda} - \beta_{i}^{*}) s^{\lambda} i^{\lambda} v + \int_{\Omega} \beta_{i}^{*} (\xi_{\lambda} i^{\lambda} + s^{*} \iota_{\lambda}) v$$

$$+ \int_{\Omega} \frac{1}{\lambda} (\beta_{e}^{\lambda} - \beta_{e}^{*}) s^{\lambda} e^{\lambda} v + \int_{\Omega} \beta_{e}^{*} (\xi_{\lambda} e^{\lambda} + s^{*} \eta_{\lambda}) v$$

$$+ \int_{\Omega} \kappa_{s}^{*} \nabla \xi_{\lambda} \cdot \nabla v - \int_{\Omega} \gamma \rho_{\lambda} v = -\int_{\Omega} (\kappa_{s} - \kappa_{s}^{*}) \nabla s^{\lambda} \cdot \nabla v , \qquad (4.25)$$

$$\langle \partial_{t} \eta_{\lambda}, v \rangle - \int_{\Omega} \frac{1}{\lambda} (\beta_{i}^{\lambda} - \beta_{i}^{*}) s^{\lambda} i^{\lambda} v - \int_{\Omega} \beta_{i}^{*} (\xi_{\lambda} i^{\lambda} + s^{*} \iota_{\lambda}) v$$

$$- \int_{\Omega} \frac{1}{\lambda} (\beta_{e}^{\lambda} - \beta_{e}^{*}) s^{\lambda} e^{\lambda} v - \int_{\Omega} \beta_{e}^{*} (\xi_{\lambda} e^{\lambda} + s^{*} \eta_{\lambda}) v$$

$$+ (\sigma + \phi_{e}) \int_{\Omega} \eta_{\lambda} v + \int_{\Omega} \kappa_{e}^{*} \nabla \eta_{\lambda} \cdot \nabla v = -\int_{\Omega} (\kappa_{e} - \kappa_{e}^{*}) \nabla e^{\lambda} \cdot \nabla v , \qquad (4.26)$$

$$\langle \partial_{t} \iota_{\lambda}, v \rangle + \phi_{r} \int_{\Omega} \iota_{\lambda} v + \int_{\Omega} \kappa_{i}^{*} \nabla \iota_{\lambda} \cdot \nabla v$$

$$- \sigma \int_{\Omega} \eta_{\lambda} v = -\int_{\Omega} (\kappa_{i} - \kappa_{i}^{*}) \nabla i^{\lambda} \cdot \nabla v , \qquad (4.27)$$

$$\langle \partial_{t} \rho_{\lambda}, v \rangle - \phi_{r} \int_{\Omega} \iota_{\lambda} v - \phi_{e} \int_{\Omega} \eta_{\lambda} v + \int_{\Omega} \gamma \rho_{\lambda} v + \int_{\Omega} \kappa_{i}^{*} \nabla \iota_{\lambda} \cdot \nabla v$$

$$= -\int_{\Omega} (\kappa_{r} - \kappa_{r}^{*}) \nabla r^{\lambda} \cdot \nabla v , \qquad (4.28)$$

all the equations holding a.e. in (0,T) and for every  $v \in V$ . Moreover, the initial condition

$$(\xi_{\lambda}, \eta_{\lambda}, \iota_{\lambda}, \rho_{\lambda}) = (0, 0, 0, 0) \tag{4.29}$$

is also satisfied. In order to control the behavior as  $\lambda$  tends to zero, we perform an estimate. We notice that the stability estimate (2.18) holds for both  $(s^*, e^*, i^*, r^*)$  and  $(s^{\lambda}, e^{\lambda}, i^{\lambda}, r^{\lambda})$ . By applying the continuous dependence estimate (2.28), we have that

$$\|(s^{\lambda}, e^{\lambda}, i^{\lambda}, r^{\lambda}) - (s^{*}, e^{*}, i^{*}, r^{*})\|_{H^{1}(0,T;V^{*}) \cap L^{2}(0,T;V)}$$

$$\leq c \|(\kappa_{s}^{\lambda}, \kappa_{e}^{\lambda}, \kappa_{i}^{\lambda}, \kappa_{r}^{\lambda}) - (\kappa_{s}^{*}, \kappa_{e}^{*}, \kappa_{i}^{*}, \kappa_{r}^{*})\|_{\infty}$$

$$= c \lambda \|(\kappa_{s}, \kappa_{e}, \kappa_{i}, \kappa_{r}) - (\kappa_{s}^{*}, \kappa_{e}^{*}, \kappa_{i}^{*}, \kappa_{r}^{*})\|_{\infty} \leq c \lambda$$

$$(4.30)$$

whence

$$\|(\xi_{\lambda}, \eta_{\lambda}, \iota_{\lambda}, \rho_{\lambda})\|_{H^{1}(0,T;V^{*}) \cap L^{2}(0,T;V)} \le c.$$

$$(4.31)$$

Therefore, thanks to standard compactness results, we have that

$$(\xi_{\lambda}, \eta_{\lambda}, \iota_{\lambda}, \rho_{\lambda}) \to (\xi, \eta, \iota, \rho)$$
 weakly in  $(H^{1}(0, T; V^{*}) \cap L^{2}(0, T; V))^{4}$ 

for some quadruplet  $(\xi, \eta, \iota, \rho)$  satisfying (4.18) (here and later on, for a subsequence, in principle; however, after the proof we now perform, the whole family converges, by uniqueness). We prove that the limiting quadruplet solves problem (4.6)–(4.10). As usual, in letting  $\lambda$  tend to zero, we consider the time-integrated versions of both (4.25)–(4.29) and (4.6)–(4.10) with time dependent test functions  $v \in L^2(0,T;V)$ . For brevity, we do not treat all the terms of (4.25)–(4.29) since many of them can be dealt with very easily, in particular since

$$(s^{\lambda}, e^{\lambda}, i^{\lambda}, r^{\lambda}) \to (s^*, e^*, i^*, r^*)$$
 and  $(\xi_{\lambda}, \eta_{\lambda}, \iota_{\lambda}, \rho_{\lambda}) \to (\xi, \eta, \iota, \rho)$   
strongly in  $(L^2(0, T; H))^4$  and a.e. in  $Q$  (4.32)

the former by (4.30) and the latter thanks to the Aubin–Lions lemma (see, e.g., [12, Thm. 5.1, p. 58]). Namely, we confine ourselves to consider the integral involving  $\beta_i^{\lambda}$  (the same argument is used for  $\beta_e^{\lambda}$ ), which is the most delicate. For a fixed  $v \in L^2(0,T;V)$ , we prove that

$$\int_{Q} \frac{1}{\lambda} (\beta_i^{\lambda} - \beta_i^*) \, s^{\lambda} i^{\lambda} v \quad \to \quad \int_{Q} \beta_i'(n^*) (\xi + \eta + \iota + \rho) \, s^* i^* v \tag{4.33}$$

as  $\lambda$  tends to zero. We have a.e. in Q that

$$\frac{\beta_i^{\lambda} - \beta_i^*}{\lambda} = \frac{1}{\lambda} \int_{n^*}^{n^{\lambda}} \beta_i'(\zeta) d\zeta = \frac{1}{\lambda} \int_0^1 \beta_i'(n^* + \tau(n^{\lambda} - n^*))(n^{\lambda} - n^*) d\tau 
= \int_0^1 \beta_i'(n^* + \tau(n^{\lambda} - n^*)) d\tau (\xi_{\lambda} + \eta_{\lambda} + \iota_{\lambda} + \rho_{\lambda}).$$

On the other hand, (4.32) implies that  $n^{\lambda}$  converges to  $n^*$  a.e. in Q. Since  $\beta_i'$  is a Carathéodory function (cf. (2.22)) we have that  $\beta_i'(n^* + \tau(n^{\lambda} - n^*))$  converges to  $\beta_i'(n^*)$  a.e. in Q for every  $\tau \in (0,1)$ . Finally,  $\beta_i'$  is bounded by the Lipschitz constant L (see (2.21)). Hence, we have that

$$\int_0^1 \beta_i'(n^* + \tau(n^{\lambda} - n^*)) d\tau \quad \to \quad \int_0^1 \beta_i'(n^*) d\tau = \beta_i'(n^*)$$

by the Lebesgue dominated convergence theorem. Therefore

$$\frac{1}{\lambda}(\beta_i^{\lambda} - \beta_i^*)s^{\lambda}i^{\lambda} \quad \to \quad \beta_i'(n^*)(\xi + \eta + \iota + \rho)s^*i^* \quad \text{a.e. in } Q$$

as  $\lambda$  tends to zero. On the other hand, on account of (4.31), we also have that

$$\|(1/\lambda)(\beta_i^{\lambda} - \beta_i^*)\|_{L^2(0,T;H)} \le (L/\lambda)\|n^{\lambda} - n^*\|_{L^2(0,T;H)} = L\|\xi_{\lambda} + \eta_{\lambda} + \iota_{\lambda} + \rho_{\lambda}\|_{L^2(0,T;H)} \le c$$

whence also

$$\|(1/\lambda)(\beta_i^{\lambda} - \beta_i^*)s^{\lambda}i^{\lambda}\|_{L^2(0,T;H)} \le c.$$

Hence

$$\frac{1}{\lambda}(\beta_i^{\lambda} - \beta_i^*)s^{\lambda}i^{\lambda} \quad \to \quad \beta_i'(n^*)(\xi + \eta + \iota + \rho)s^*i^* \quad \text{weakly in } L^2(0, T; H)$$

and (4.33) follows. This concludes the proof that  $(\xi, \eta, \iota, \rho)$  is the solution to (4.6)–(4.10).

At this point, we are ready to prove (4.19). Due to optimality, we have that

$$\frac{\mathcal{J}(\kappa_s^{\lambda}, \kappa_e^{\lambda}, \kappa_i^{\lambda}, \kappa_r^{\lambda}, e^{\lambda}, i^{\lambda}) - \mathcal{J}(\kappa_s^{*}, \kappa_e^{*}, \kappa_i^{*}, \kappa_r^{*}, e^{*}, i^{*})}{\lambda} \ge 0 \quad \text{for every } \lambda \in (0, 1)$$
 (4.34)

and we aim at letting  $\lambda$  tend to zero in this inequality. We just consider two of the terms involved in (4.34), namely

$$\frac{1}{2} \int_{Q_C} \frac{|e^{\lambda}|^2 - |e^*|^2}{\lambda} \quad \text{and} \quad \frac{\alpha}{2} \int_{Q} \frac{|\kappa_s^{\lambda}|^2 - |\kappa_s^*|^2}{\lambda}$$

since the others are analogous. We have that

$$\frac{1}{2} \int_{Q_C} \frac{|e^{\lambda}|^2 - |e^*|^2}{\lambda} = \int_{Q_C} \frac{e^{\lambda} - e^*}{\lambda} \frac{e^{\lambda} + e^*}{2} = \int_{Q_C} \eta_{\lambda} \frac{e^{\lambda} + e^*}{2}$$

and this converges to

$$\int_{Q_C} \eta \, e^*$$

as  $\lambda$  tends to zero. Similarly

$$\frac{\alpha}{2} \int_{Q} \frac{|\kappa_{s}^{\lambda}|^{2} - |\kappa_{s}^{*}|^{2}}{\lambda} = \alpha \int_{Q} \frac{\kappa_{s}^{\lambda} - \kappa_{s}^{*}}{\lambda} \frac{\kappa_{s}^{\lambda} + \kappa_{s}^{*}}{2} = \alpha \int_{Q} (\kappa_{s} - \kappa_{s}^{*}) \frac{\kappa_{s}^{\lambda} + \kappa_{s}^{*}}{2}$$

converges to

$$\alpha \int_{Q} (\kappa_s - \kappa_s^*) \, \kappa_s^*.$$

Hence, (4.19) immediately follows.

The result just proved is not satisfactory. Indeed, the linearized problem (4.6)–(4.10) is involved infinitely many times since  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r)$  is arbitrary in  $\mathcal{U}_{ad}$ . This trouble is bypassed by the introduction of a proper adjoint problem. This is formally obtained by testing the equations (4.6)–(4.9) by p, q, z and w, respectively, integrating by parts in time and collecting the terms involving  $\xi$ ,  $\eta$ ,  $\iota$  and  $\rho$ , respectively. It is the following

$$-\langle \partial_t p, v \rangle + \int_{\Omega} \kappa_s^* \nabla p \cdot \nabla v + \int_{\Omega} (A_1 p + A_2 q + A_3 w + A_4 z) v = 0$$
 (4.35)

$$-\langle \partial_t q, v \rangle + \int_{\Omega} \kappa_e^* \nabla q \cdot \nabla v + \int_{\Omega} (B_1 p + B_2 q + B_3 w + B_4 z) v = \int_{\Omega_C} e^* v \tag{4.36}$$

$$-\langle \partial_t w, v \rangle + \int_{\Omega} \kappa_i^* \nabla w \cdot \nabla v + \int_{\Omega} (C_1 p + C_2 q + C_3 w + C_4 z) v = \int_{\Omega_C} i^* v$$
 (4.37)

$$-\langle \partial_t z, v \rangle + \int_{\Omega} \kappa_r^* \nabla z \cdot \nabla v + \int_{\Omega} (D_1 p + D_2 q + D_3 w + D_4 z) v = 0$$
 (4.38)

$$(p,q,w,z)(T) = (0,0,0,0)$$
(4.39)

all the equations holding a.e. in (0,T) and for every  $v \in V$ , where  $A_j$ ,  $B_j$ ,  $C_j$  and  $D_j$ ,  $j = 1, \ldots, 4$ , are defined in (4.11)–(4.17). This is a backward linear parabolic problem with bounded coefficients. Thus, it has a unique solution

$$(p,q,rw,z) \in (H^1(0,T;V^*) \cap L^2(0,T;V))^4 \tag{4.40}$$

which actually is more regular. A satisfactory and explicit necessary condition for optimality is given in the following theorem, which is our last result. The precise form of the set  $\mathcal{U}_{ad}$  specified in the Introduction is assumed.

**Theorem 4.3.** Assume (2.29)–(2.33) and (1.9), (1.12), (1.13), (2.34). Let  $(\kappa_s^*, \kappa_e^*, \kappa_i^*, \kappa_r^*)$  be an optimal control of the form

$$\kappa_s(x,t) = \sum_{j=1}^m u_{s,j}^* \chi_{Q_j}(x,t), \quad \kappa_e(x,t) = \sum_{j=1}^m u_{e,j}^* \chi_{Q_j}(x,t),$$

$$\kappa_i(x,t) = \sum_{j=1}^m u_{i,j}^* \chi_{Q_j}(x,t), \quad \kappa_r(x,t) = \sum_{j=1}^m u_{r,j}^* \chi_{Q_j}(x,t), \quad (x,t) \in Q$$
(4.41)

and let  $(s^*, e^*, i^*, r^*)$  denote the corresponding state. Also, let (p, q, w, z) be the solution to the adjoint problem (4.35)–(4.39). Then, noting that  $\int_{\mathcal{O}} \chi_{Q_i} = |\Omega_i| T$ , setting

$$\mu_j^s := \frac{\int_Q (\nabla s^* \cdot \nabla p) \,\chi_{Q_j}}{|\Omega_j| \, T}, \quad \mu_j^e := \frac{\int_Q (\nabla e^* \cdot \nabla q) \,\chi_{Q_j}}{|\Omega_j| \, T},$$

$$\mu_j^i := \frac{\int_Q (\nabla i^* \cdot \nabla w) \,\chi_{Q_j}}{|\Omega_j| \, T}, \quad \mu_j^r := \frac{\int_Q (\nabla r^* \cdot \nabla z) \,\chi_{Q_j}}{|\Omega_j| \, T}$$

$$(4.42)$$

and recalling that  $[u_{\min}^{s,j}, u_{\max}^{s,j}]$ ,  $[u_{\min}^{e,j}, u_{\max}^{e,j}]$ ,  $[u_{\min}^{i,j}, u_{\max}^{i,j}]$ ,  $[u_{\min}^{r,j}, u_{\max}^{r,j}]$  are the variability intervals, all contained in  $[\kappa_*, \kappa^*]$ , for  $j = 1, \ldots, m$ , it turns out that

$$u_{s,j}^*$$
 is the one-dimensional projection of  $\mu_j^s/\alpha$  on  $[u_{\min}^{s,j}, u_{\max}^{s,j}]$ , i.e.,
$$u_{s,j}^* = \max\{\min\{u_{\max}^{s,j}, \mu_j^s/\alpha\}, u_{\min}^{s,j}\} \quad \text{for } j = 1, \dots, m$$
(4.43)

$$u_{e,j}^*$$
 is the one-dimensional projection of  $\mu_j^e/\alpha$  on  $[u_{\min}^{e,j}, u_{\max}^{e,j}]$ , i.e., 
$$u_{e,j}^* = \max\{\min\{u_{\max}^{e,j}, \mu_j^e/\alpha\}, u_{\min}^{e,j}\} \quad \text{for } j = 1, \dots, m$$
 (4.44)

$$u_{i,j}^*$$
 is the one-dimensional projection of  $\mu_j^i/\alpha$  on  $[u_{\min}^{i,j}, u_{\max}^{i,j}]$ , i.e., 
$$u_{i,j}^* = \max\{\min\{u_{\max}^{i,j}, \mu_j^i/\alpha\}, u_{\min}^{i,j}\} \quad \text{for } j = 1, \dots, m$$
 (4.45)

$$u_{r,j}^*$$
 is the one-dimensional projection of  $\mu_j^r/\alpha$  on  $[u_{\min}^{r,j}, u_{\max}^{r,j}]$ , i.e.,  
 $u_{,j}^* = \max\{\min\{u_{\max}^{r,j}, \mu_j^r/\alpha\}, u_{\min}^{r,j}\}$  for  $j = 1, \dots, m$ . (4.46)

*Proof.* We fix  $(\kappa_s, \kappa_e, \kappa_i, \kappa_r) \in \mathcal{U}_{ad}$  and consider the associated linearized system (4.6)–(4.10). We test the equations by p, q, w and z, respectively, and integrate with respect to time over (0, T). At the same time, we test (4.35)–(4.38) by  $-\xi$ ,  $-\eta$ ,  $-\iota$  and  $-\rho$ ,

respectively, and integrate over (0, T). Finally, we add all the resulting equality to each other and rearrange. Due to the obvious cancellations that occur, we have that

$$\int_{0}^{T} (\langle \partial_{t} \xi(t), p(t) \rangle + \langle \partial_{t} p(t), \xi(t) \rangle) dt + \int_{0}^{T} (\langle \partial_{t} \eta(t), q(t) \rangle + \langle \partial_{t} q(t), \eta(t) \rangle) dt 
+ \int_{0}^{T} (\langle \partial_{t} \iota(t), w(t) \rangle + \langle \partial_{t} w(t), \iota(t) \rangle) dt + \int_{0}^{T} (\langle \partial_{t} \rho(t), z(t) \rangle + \langle \partial_{t} z(t), \rho(t) \rangle) dt 
= - \int_{Q} (\kappa_{s} - \kappa_{s}^{*}) \nabla s^{*} \cdot \nabla p - \int_{Q} (\kappa_{e} - \kappa_{e}^{*}) \nabla e^{*} \cdot \nabla q 
- \int_{Q} (\kappa_{i} - \kappa_{i}^{*}) \nabla i^{*} \cdot \nabla w - \int_{Q} (\kappa_{r} - \kappa_{r}^{*}) \nabla r^{*} \cdot \nabla z 
- \int_{Q_{C}} e^{*} \eta - \int_{Q_{C}} i^{*} \iota.$$

Note that all the involved functions belong to  $H^1(0,T;V^*) \cap L^2(0,T;V)$ . Then, owing to the well-known integration-by-parts formula and accounting for the initial and final conditions (4.10) and (4.39), we deduce that the contribution due to the first two lines of the above equality vanishes. By combining what remains with (4.19), we obtain

$$\int_{Q} \left\{ (\alpha \kappa_{s}^{*} - \nabla s^{*} \cdot \nabla p)(\kappa_{s} - \kappa_{s}^{*}) + (\alpha \kappa_{e}^{*} - \nabla e^{*} \cdot \nabla q)(\kappa_{e} - \kappa_{e}^{*}) + (\alpha \kappa_{i}^{*} - \nabla i^{*} \cdot \nabla w)(\kappa_{i} - \kappa_{i}^{*}) + (\alpha \kappa_{r}^{*} - \nabla r^{*} \cdot \nabla z)(\kappa_{r} - \kappa_{r}^{*}) \right\} \ge 0.$$
(4.47)

Next, by (1.12) observe that  $\mathcal{U}_{ad}$  is the product  $\mathcal{U}_{ad} = \mathcal{U}_{ad}^s \times \mathcal{U}_{ad}^e \times \mathcal{U}_{ad}^i \times \mathcal{U}_{ad}^r$  and that a control is admissible if and only if its components belong to the corresponding factors. Then, the variables  $\kappa_s$ ,  $\kappa_e$ ,  $\kappa_i$  and  $\kappa_r$  that enter (4.47) are as in (1.10) and independent from each other, so that the inequality (4.47) splits into four independent inequalities. In particular, we have

$$\int_{Q} (\alpha \kappa_{s}^{*} - \nabla s^{*} \cdot \nabla p) (\kappa_{s} - \kappa_{s}^{*}) \geq 0 \quad \text{for every } \kappa_{s} \in \mathcal{U}_{ad}^{s}$$

and, by virtue of (4.41) and (4.42), the above inequality reduces to

$$(\alpha u_{s,j}^* - \mu_j^s)(u_j^s - u_{s,j}^*) \ge 0$$
 for every  $u_j^s \in [u_{\min}^{s,j}, u_{\max}^{s,j}]$ , for  $j = 1, \dots, m$ .

This easily leads to (4.43). The same argumentation can be repeated for the deduction of (4.44)–(4.46).

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