

# Why is the Legendre Transformation Involution?

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## ABSTRACT

The question posed in the title is answered in terms of a simple pictorial argument that is manifestly symmetric between the two functions that are Legendre transform of each other.

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The Legendre transformation is a mathematical concept of great significance to physics. In mechanics and field theory it provides the transition between Hamiltonian and Lagrangian descriptions, and in thermodynamics it relates the different potentials. Nevertheless, with very few exceptions (notably [1]), it is usually introduced just along the way, leaving the impression of a sleight of hand. The feeling that some essential point might be missing from the standard description provided the motivation for the present considerations. In the following the definition of the Legendre transform  $G(y)$  of a function  $F(x)$  and a simple argument for its involutivity will be given.

Let us assume that the function  $F(x)$  is continuously differentiable, with a derivative

$$f(x) := F'(x)$$

that is strictly monotonically increasing. Then the function  $f(x)$  can be inverted to  $g(y)$ :

$$y = f(x) \quad \Longleftrightarrow \quad x = g(y),$$

and the Legendre transform of  $F(x)$  is defined as

$$G(y) := [xy - F(x)]|_{x=g(y)}. \tag{1}$$

If one performs the same operation on  $G(y)$ :

$$z := G'(y), \quad H(z) := [yz - G(y)]|_{y=h(z)},$$

where  $h$  is the function inverse to  $G'$ , a very short calculation reveals that  $z = x$ ,  $h = f$  and  $H = F$ , i.e. one has returned to the original function. This is, of course, perfectly sufficient as a proof of involutivity, but a physicist would prefer a more intuitive explanation, ideally in terms of geometry. The standard geometric interpretation of the Legendre transform proceeds by considering the graph of the convex function  $F(x)$  and its tangents. This is a correct pictorial account of formula (1) which can be used to give a geometric proof (see, e.g., Arnold [2]), but it does not make the symmetry between  $F$ ,  $f$  and  $x$  on one side and  $G$ ,  $g$  and  $y$  on the other side manifest. Let us therefore look at the graph of the monotonic function  $f(x)$  instead.

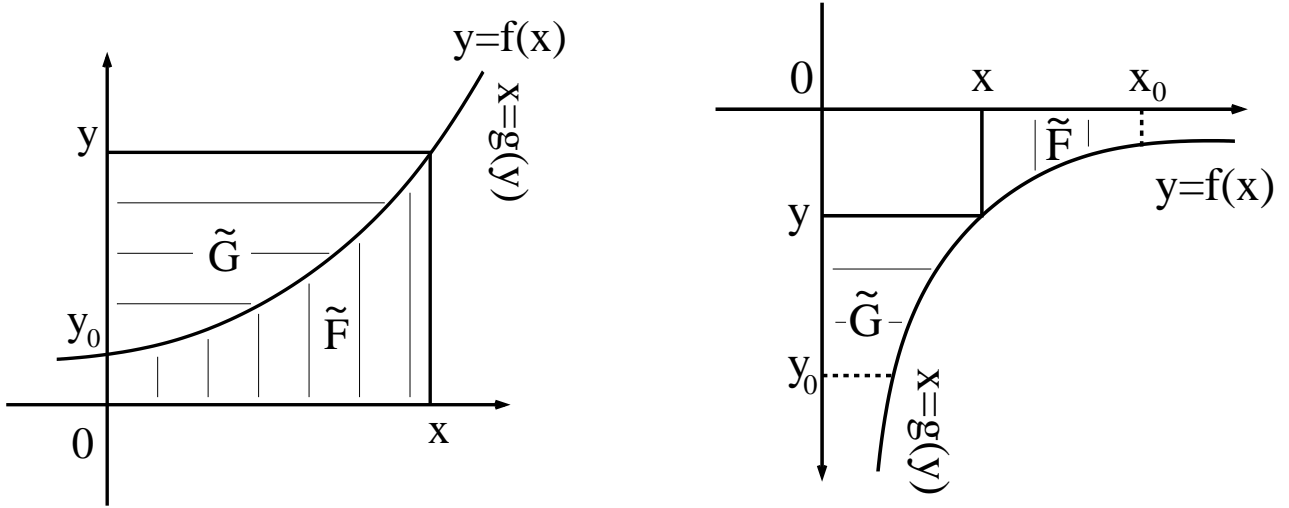


Figure 1: The graph of  $y = f(x)$  for the cases (a)  $x > 0, y > 0$  and (b)  $x > 0, y < 0$

We first assume that  $x$  and  $f(x)$  are positive; see the first diagram of Figure 1 which depicts the graph of  $f(x)$  over  $x$ . The same plot can be interpreted as the graph of  $g(y)$  with respect to the  $y$ -axis. Expressed in a symmetric manner, the figure shows the locus of all pairs  $(x, y)$  with  $y = f(x)$  or, equivalently,  $x = g(y)$ . Now consider the rectangle bounded by the coordinate axes and their parallels through such a point  $(x, y)$ . The area of that rectangle is  $A = xy$ , and the graph cuts it into two parts with areas  $\tilde{F}$  and  $\tilde{G}$ , respectively. From the figure it is immediately obvious that

$$\tilde{F} = \int_{x_0}^x f(\hat{x}) d\hat{x}, \quad \tilde{G} = \int_{y_0}^y g(\hat{y}) d\hat{y}, \quad \tilde{F} + \tilde{G} = xy,$$

with  $x_0 = 0$  if the graph intersects the  $y$ -axis in  $y_0 \geq 0$ , and  $y_0 = 0$  if the graph intersects the  $x$ -axis in  $x_0 \geq 0$ . Clearly  $\tilde{F}$  is a function of  $x$  with  $\tilde{F}'(x) = f(x) = F'(x)$ , hence

$$F(x) = \tilde{F}(x) + c, \quad G(y) = \tilde{G}(y) - c$$

for some real constant  $c$ . So  $F$  is, up to a constant, the area under the graph of  $f$ , and  $G$  is, up to minus that constant, the area under the graph of  $g$ , and the symmetry is manifest.

What if our assumptions  $x \geq 0, y \geq 0$  are not satisfied? For  $x \leq 0, y \leq 0$  the argument is essentially unmodified since  $(-x)(-y) = xy$ . If  $xy < 0$  consider the second part of Figure 1. Here we have fixed two arbitrary constant values  $x_0, y_0$  in such a way that  $x_0 > x > 0, y_0 < y < 0$  for the range of pairs  $(x, y)$  that we want to consider. Denote by  $A_0$  the area determined by the coordinate axes, the vertical line through  $x_0$ , the horizontal line through  $y_0$  and the graph. Then we have

$$A_0 = -xy + \tilde{F} + \tilde{G} \quad \text{with} \quad \tilde{F} = - \int_x^{x_0} f(\hat{x}) d\hat{x}, \quad \tilde{G} = \int_{y_0}^y g(\hat{y}) d\hat{y}.$$

Up to the constant  $A_0$  that can be absorbed in the redefinitions from  $\tilde{F}$  to  $F$  and from  $\tilde{G}$  to  $G$ ,  $\tilde{F}$  and  $\tilde{G}$  again add up to  $xy$ .

The fact that the present picture requires redefinitions of functions by constants is directly related to the interpretation of  $F(x)$  and  $G(y)$  as integrals of  $f(x)$  and  $g(y)$ , respectively. As always, integrals are well-defined only up to equivalences of the type  $F \sim \tilde{F}$ , with ‘ $\sim$ ’ meaning ‘equal up to a constant function’. This fits precisely with the physical interpretation, where the predictions do not change if quantities like the Hamiltonian or thermodynamic potentials are redefined by constants.

**Note added after completion:** I first presented this material in informal talks on March 15, 2012 in Vienna and on June 4, 2012 in Heidelberg. After writing it up in the form of the present manuscript, I became aware of [3] which is dated June 29, 2012 (submission) / August 22, 2012 (publication) and has some overlap in content. I am grateful to Johanna Knapp for pointing out this reference to me.

## References

- [1] R. K. P. Zia et al, *Making Sense of the Legendre Transform*, Am. J. Phys. 77 (2009) 614, arXiv:0806.1147.
- [2] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (second edition), Springer (1989).
- [3] H.-J. Hoffmann, *A new interpretation of Legendre’s transformation and consequences*, Mat.-wiss u. Werkstofftech. (2012) 43, No. 8.