MAT 186 Module E3: Limits of Riemann Sums
Section LEC 0108

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Plan for Today

1. Warm-Up Problem
2. Some theoretical discussion
3. Interpretive problems, discussion in groups
4. Numerical investigation: how good are Riemann sum approximations?

Warm-Up Problem

Problem

Consider the integral $I = \int_{-1}^{0} xe^{-x}\, \mathrm{d}x$. An imaginary student has submitted the following computation of I:

$$
\begin{aligned}
I &\approx \frac{1}{n}\sum_{k=1}^{n}\left(-1+\frac{k-1}{n}\right)e^{-\left(-1+\frac{k-1}{n}\right)} \\
&= \frac{e^{1+\frac{1-k}{n}}}{n}\sum_{j=0}^{n-1}\left(-1+\frac{j}{n}\right) \\
&= e^{1+\frac{1-k}{n}}\left(-1+\frac{1-\frac{1}{n}}{2}\right) \\
&\to -\frac{e}{2} \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

What do you think of this solution?

Theory Discussion: Integrability

- Recall: a function $f(x)$ is called **integrable** on $[a, b]$ if the Riemann sums associated to $f(x)$ converge as the subinterval width $\to 0$
- What does this mean *geometrically*? That the area under the graph of $f(x)$ is "well-defined"

Topic For Group Discussion

Can you think of a function that is not differentiable? Can you think of a function that is not integrable?

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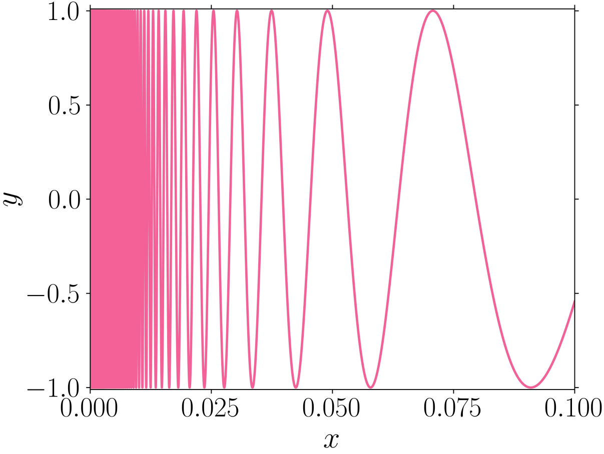
Topics For Group Discussion

1. Can you think of a function that is not differentiable?
2. Can you think of a function that is not integrable?

- **Differentiability is special, integrability is really, really not!** Even so, we mostly integrate only *continuous functions* in MAT 186.

Awful Functions that Are Still Integrable, Part 1

Consider $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$.



Still integrable on $[0, 1]$!

Even the **Thomae function** plotted below is integrable!

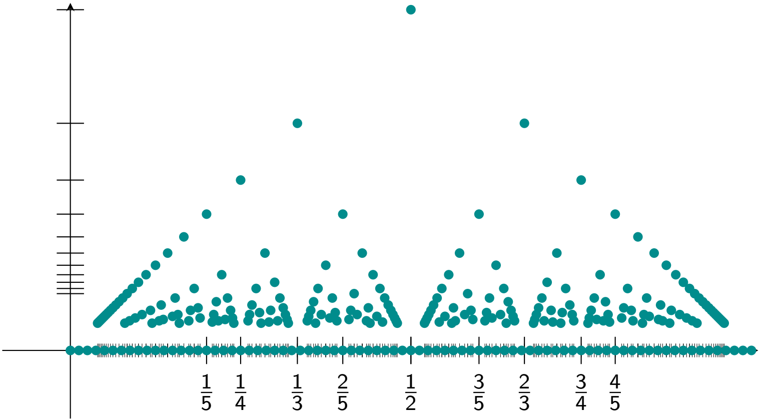


Image made using discussion here (click for source)

Interpretive Problem 1, Part 1

Problem

Suppose the two shorter edges of a right triangle are aligned with the x and y axes, so one corner of the triangle is at the origin. The length of the side aligned with the x-axis is a, and the length of the side aligned with the y-axis is b.

1. Express the hypotenuse of the triangle as the graph of a function $h(x)$ defined on $[0, a]$.
2. Suppose we chop the interval $[0, a]$ into n pieces of equal width. Write out the associated *left*-endpoint Riemann sum approximating the area of the triangle.

Solution to Interpretive Problem 1, Part 1

Solution to Interpretive Problem 1, Part 1

Interpretive Problem 1, Part 2

Problem

1. Prove that the Riemann sum you obtained evaluates to

$$\text{Riemann sum } = ab\left(1 - \frac{n-1}{2n}\right).$$

2. Conclude that the area of the triangle is given by $\frac{ab}{2}$.

Solution to Interpretive Problem 1, Part 2

Interpretive Problem 2

Problem (taken from Sean)

For any positive integer n, consider the sum

$$S_n = \frac{2}{n}\sum_{k=1}^{n}\sqrt{1-\left(-1+\frac{2k-2}{n}\right)^2}.$$

1. Find a continuous function $f(x)$ and $a, b \in \mathbb{R}$ such that

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} S_n.$$

2. Compute $\lim_{n\to\infty} S_n$.

Solution to Interpretive Problem 2

Solution to Interpretive Problem 2

How Good Are Riemann Sum Approximations? Part 1

- **Quadrature** = numerical approximation of integrals
- Let's study various approximations of the definite integral

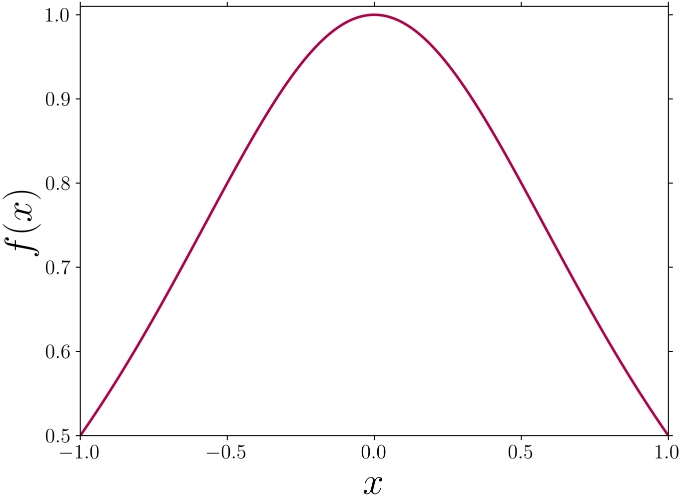
$$A = \int_{-1}^{+1} \frac{\mathrm{d}x}{1+x^2}.$$

- Fundamental Theorem of Calculus implies

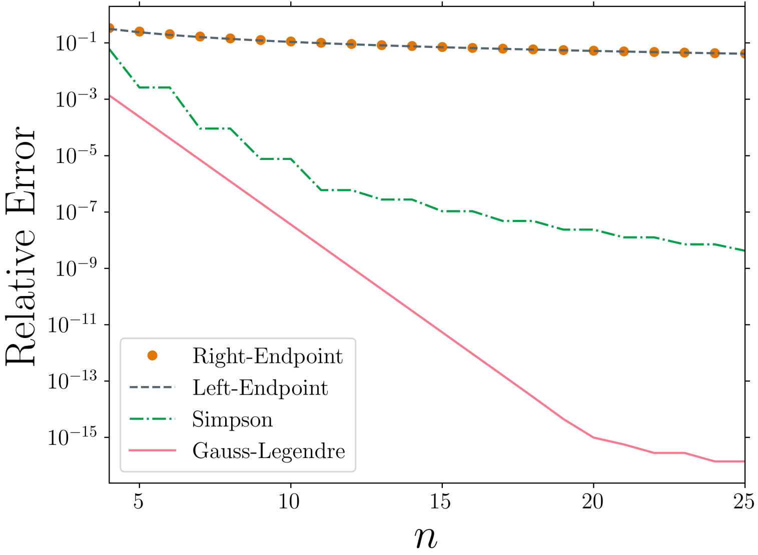
$$A = \frac{\pi}{2},$$

  so we have an exact answer to compare w/ our quadrature approximations!
- REMEMBER: comparing approximations against a known standard is a great way to assess the quality of a numerical method before you test the method "in the field".

The integrand $f(x) = \left(1 + x^2\right)^{-1}$ is a nice function:

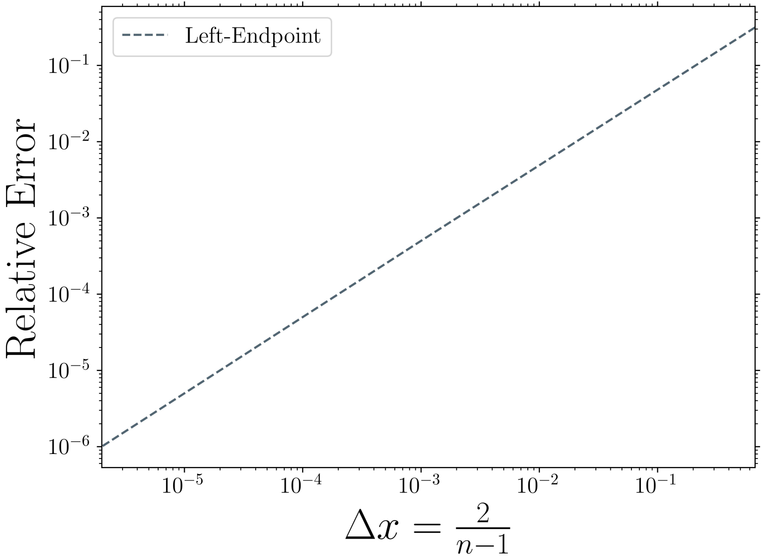


How Good Are Riemann Sum Approximations? Part 3



Here, n = number of subintervals + 1.

How Good Are Riemann Sum Approximations? Part 4



Error in left/right endpt. approx $\propto \Delta x$

Takeaways

- Lots of different ways to numerically approximate integrals!
- Riemann sums are conceptually very clear and quite useful when it comes to model derivation (see the next round of PCEs!),
  BUT such approximations tend to converge quite slowly!
- To actually get a usable numerical approximation of an integral, a more sophisticated method such as Simpson's rule, Gauss-Legendre quadrature, or adaptive quadrature should be applied.

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