MAT 186 Module D2: Differential Equations and their Solutions
Section LEC 0108

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Plan for Today

1. Lecture on module D2, w/ interpretive problems throughout.

Differential Equations Defined

Definition

A **(first-order) ordinary differential equation (ODE)** for an uknown function $y(t)$ is an expression of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y(t), t)$$

where $f(t, y)$ is a given function of *two* real variables.

A function $y(t)$ obeying the ODE above is called a **solution** to the ODE.

- Note: the "=" sign is equality of functions!
- It's OK for the right-hand side to be a function of *only* y (we then say the ODE is **autonomous**) or *only* t.

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Examples of ODEs

Example

$$y'(t) = 0.$$

For any fixed $y_0 \in \mathbb{R}$, any constant function $y(t) = y_0$ is a solution to the above ODE.

Examples of ODEs

Example (Exponential Growth/Decay)

For a fixed $r \in \mathbb{R}$, consider the ODE

$$p'(t) = rp(t).$$

For any $p_0 \in \mathbb{R}$, the function

$$p(t) = p_0 e^{rt}$$

is a solution to the ODE.

Exponential Decay

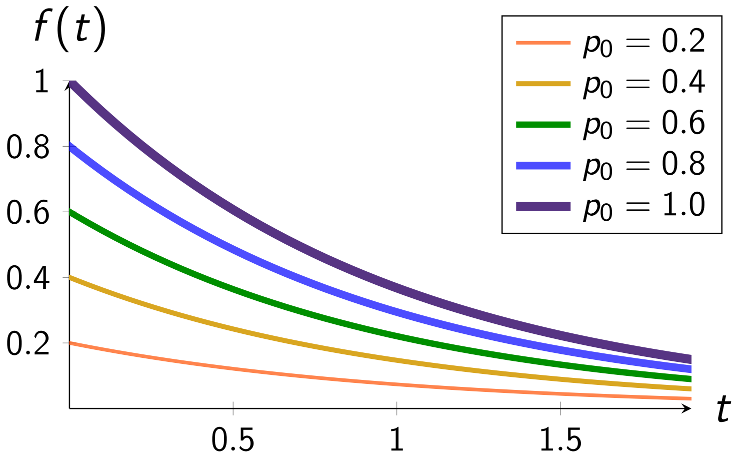

Figure: Various solutions of $p' = -p$ with different choices of p_0. The thicker the line, the larger p_0.

Interpretive Problem

Problem

For fixed $r, K > 0$, consider the **logistic equation** from population biology:

$$p'(t) = rp(t)\left(1 - \frac{p(t)}{K}\right).$$

The solution $p(t)$ models the density of organisms in a particular environment w/ a fixed, finite amount of resources. Show that, for any fixed $p_0 \geq 0$, the function

$$p(t) = \frac{p_0 K e^{rt}}{K + p_0\left(e^{rt} - 1\right)}$$

solves the logistic equation.

Solution to Interpretive Problem

Logistic Growth Fit to Gause's *P. Aurelia* Data

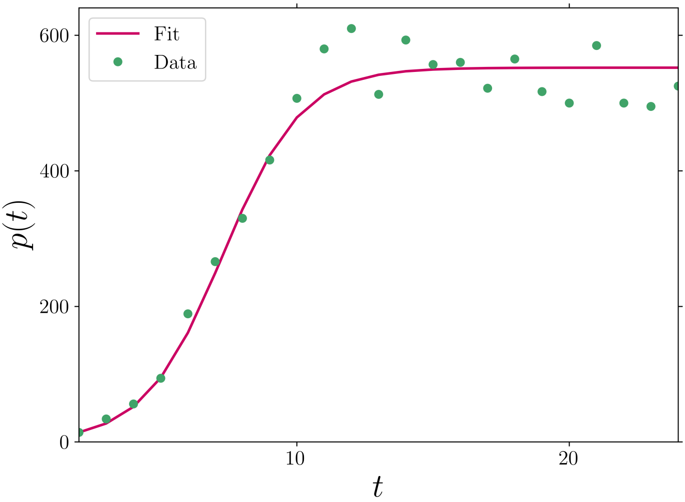

Figure: Fitting soln. of the logistic equation to Gause's data on growth of a population of *P. Aurelia* supplied w/ a constant amount of resources: the data allows us to estimate the correct r, K.

Examples of ODEs

- ODEs can also involve higher derivatives of the unknowns!
- **Order** of an ODE = highest number of time differentiations of the unknown $y(t)$.

Example (Simple Harmonic Motion)

For fixed $\omega \in \mathbb{R}$,

$$x''(t) + \omega^2 x(t) = 0.$$

This is a *second-order* ODE since it involves $x''(t)$ (two time derivatives).

This ODE is solved by functions of the form

$$x(t) = A\cos\omega t + B\sin\omega t$$

for any constant parameters $A, B \in \mathbb{R}$.

Examples of ODEs

Example (van der Pol Oscillator)

For fixed $\mu \in \mathbb{R}$, consider the van der Pol (vdP) equation:

$$x''(t) - \mu \left(1 - x^2\right) x'(t) + x(t) = 0.$$

This ODE models non-sinusoidal oscillations in circuits with vacuum tubes. The difficult nonlinear term means vdP must be solved numerically!

van der Pol Oscillator

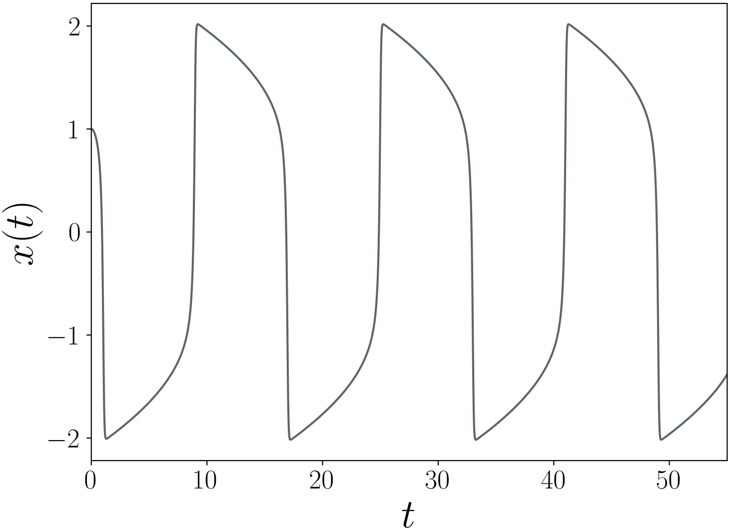

Figure: Numerical solution of vdP with $\mu = 8$.

“The” Solution or “A” Solution?

- Many example so far involve a family of solns. depending on constant parameters. Punchline: solutions to ODEs are NOT unique.
- However, in science & engineering, one usually places one or more *conditions* on solns. The condition fixes a unique solution! (well, see module D3...)

Example

The **initial-value problem**

$$\begin{cases} y'(t) = 2y(t) \\ y(0) = 5 \end{cases}$$

has the unique solution

$$y(t) = 5e^{2t}.$$

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Equilibrium Solutions

Definition

Consider the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y(t), t).$$

A constant function

$$y(t) = y_0 \quad \forall\ t$$

solving this ODE is called a **equilibrium solution**.

- "Equilibrium" = time no longer matters.
- Any equilibrium solution $y(t) = y_0$ satisfies the *algebraic* equation

$$0 = f(y_0, t) \quad \forall\ t.$$

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Equilibrium Example

Example

For a fixed $r \in \mathbb{R}$, consider the ODE

$$p'(t) = rp(t).$$

Find all equilibrium solutions to this ODE.

Interpretive Problem

Problem

For fixed $r, K > 0$, consider the logistic equation

$$p' = rp\left(1 - \frac{p}{K}\right).$$

Find all equilibrium solutions to this ODE.

Finding Equilibria Numerically

- Consider an autonomous ODE

$$y'(t) = f(y).$$

Recall: any equilibria $y(t) = y_0$ satisfy the *nonlinear algebraic* equation

$$0 = f(y_0)$$

- If $f(y_0)$ is a very bad function, algebraic eqn. can only be solved approximately.
- Use Newton's method!
- There are other root-finding routines (inexact Newton, quasi-Newton, homotopy methods,...) built into both graphing websites like Wolfram Alpha and more sophisticated software like MATLAB or NumPy/SciPy.

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Interpretive Problem

Problem

Consider the ODE

$$x'(t) = \cos(x) - x.$$

- Prove that there exists an equilibrium solution x_{eq} to the above ODE. *Hint: draw a picture of the right-hand side $f(x)$.*
- Using any computer software/website of your choice, find x_{eq} to three decimal places.

Solution to Interpretive Problem

Relaxation or Repulsion?

- Note: if a soln starts at an equilibrium point, it stays there forever!
- What happens if we start close to an equilibrium point? What happens if we start far from an equilibrium point?
- Let's look at the logistic equation

$$p' = p(1-p)$$

numerically...

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Relaxation/Attraction

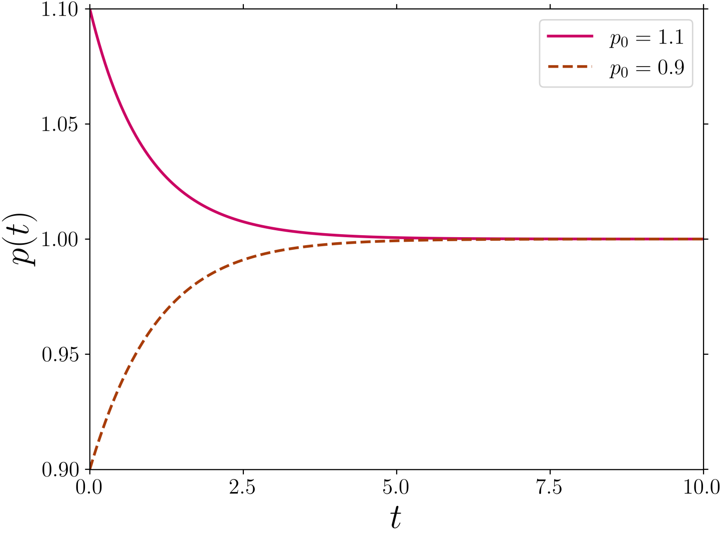

Figure: The nonzero equilibrium K is attractive (or *stable*).

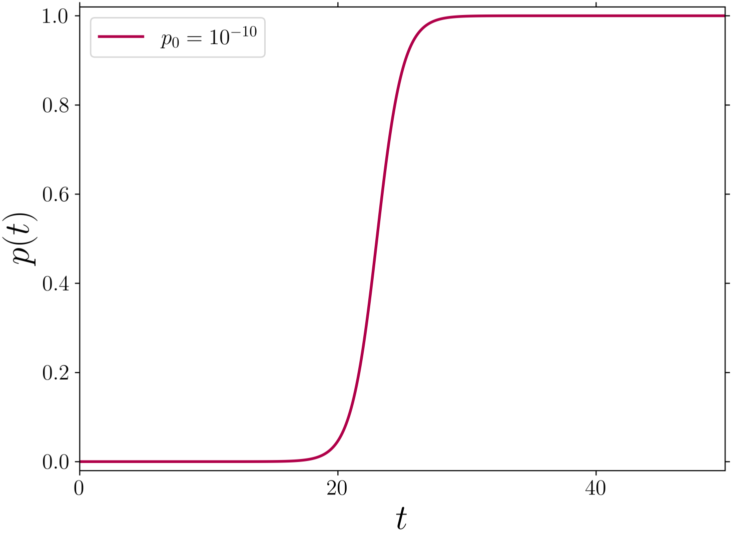

Figure: The zero equilibrium is repulsive (or *unstable*).

A Graphical Method

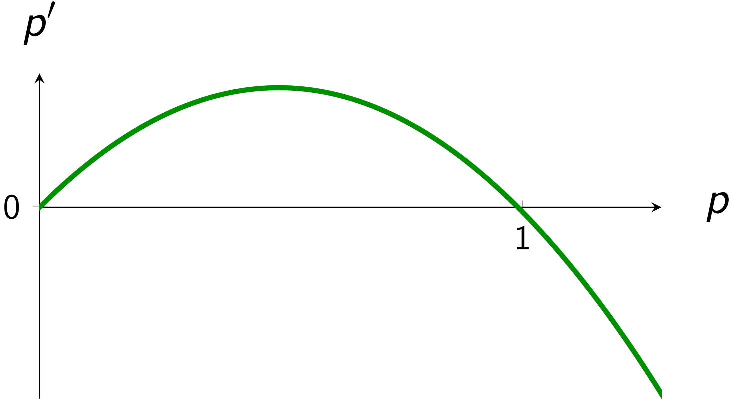

Figure: Plot of p' as a function of p for the logistic equation. How is p changing at an instant where $p < 1$? $p > 1$?