# MAT 186 Module D1: Linear Approximation Section LEC 0108

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# Plan for Today

- Two Warm-Up Problems
- 2 Interpretive Problems, discussion in groups
- Code demo

# Warm-Up Problem 1

#### **Problem**

Explain why the small angle approximation for  $\sin \theta$  makes sense.

# Warm-Up Problem 2

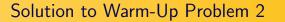
#### **Problem**

If z denotes the (positive) depth below sea level,  $\rho(z)$  denotes the density of seawater at depth z, P(z) denotes the pressure of seawater at depth z, and  $g\approx 10m/s^2$  denotes the acceleration due to gravity on Earth, then the **equation of hydrostatic balanace** tells us that

$$\frac{\mathrm{d}P}{\mathrm{d}z}=\rho(z)g.$$

At a depth of 2km, the density of seawater is about  $1.03285kg/m^3$ , and the pressure of seawater is about 197.4 atmospheres (1 atmosphere =  $101325\frac{kg}{ms^2}$  of pressure).

Estimate the pressure of seawater at a depth of 2.1km.



## Interpretive Problem 1

#### **Problem**

Consider the periodic function

$$f(x) = \begin{cases} \sqrt{x} & x \in [0,1) \\ \sqrt{x-1} & x \in [1,2) \\ \sqrt{x-2} & x \in [2,3) \\ \dots \end{cases}$$

defined on  $[0, \infty)$ . Can linear approximation be used to estimate f(x) for  $x \approx 32$ ?



# Interpretive Problem 2, Part 1

#### **Problem**

Estimate  $\sqrt{16.05}$  without using a calculator. Then, compute the error between your approximate value and the value produced by a calculator.



## Interpretive Problem 2, Part 2

#### **Problem**

In 60 CE, the mathematician Heron described an algorithm for approximating  $\sqrt{A}$ .

- **1** Pick an initial guess  $x_0 \approx \sqrt{A}$ .
- 2 Iteratively define a sequence of approximations  $x_n \approx \sqrt{A}$  via

$$x_{n+1} = \frac{1}{2} \left( \frac{A}{x_n} + x_n \right).$$

The more iterations we perform, the better the approximation  $x_n \approx \sqrt{A}$  becomes.

Take A = 16.05 and use an initial guess of  $x_0 = 4$ .

- Show that the Heron approximation  $x_1$  is same approximation we got using linearization.
- Compute  $x_2$ . Is the answer a better or worse approximation to  $\sqrt{16.05}$  than  $x_1$ ?



# Interpretive Problem 3: Background

- In 1740, Thomas Simpson devised a numerical method for approximately finding the roots of a given differentiable function f(x), based on earlier ideas of Newton. Owing to the weirdness of history, this algorithm is called **Newton's method**.
- Like Heron's method for approximating  $\sqrt{A}$ , Newton's method is iterative:
  - pick an initial guess  $x_0$  for the root;
  - 2 iteratively update the guess via

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

• Under certain conditions, one can show the approximation gets better and better with each successive iteration.

## Interpretive Problem 3

#### **Problem**

Derive Newton's method for solving

$$f(x) = 0$$

from scratch by following the recipe below.

- **1** Assume you have constructed  $x_n$  already, and that it is very close to the true root  $x_*$ .
- 2 Explain why we want to have have  $x_{n+1} = x_*$ .
- **3** Find an expression for  $x_{n+1}$  in terms of  $x_n$  if we indeed have  $x_{n+1} = x_*$ .



### Extra Problem

#### **Problem**

- Explain why computing  $\sqrt{A}$  is equivalent to finding the roots of a particular quadratic function f(x).
- Show that Heron's method is a special case of Newton's method.



# Code Demo: Using Newton's Method

To the Jupyter notebook!