MAT 186 Module A3: Exponentials and Logarithms
Section LEC 0108

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Plan for Today

1. Warm-Up Problems: 1 min individually, 2 min groups
2. Class Discussion of Warm-Up Problems
3. Aside: Euler's number e as a limit
4. Background on the Interpretive Problem: modelling microorganism population growth w/ exponential fnc.
5. Interpretive Problem statement, discussion in groups
6. Afterword: exponential growth isn't the whole story

Warm-Up Problem 1

Problem (based on textbook ch.1 problem 283)

Your friend Ingrid wants to solve the equation below for x:

$$7^{3x-2} = 11.$$

She presents the following solution:

$$\begin{aligned} 7^{3x-2} &= 11 \\ \Rightarrow \ln\left(7^{3x-2}\right) &= \ln 11 \\ \Rightarrow (3x-2)\ln 7 &= \ln 11 \\ \Rightarrow 3x-2 &= \frac{\ln 11}{\ln 7} = \ln(11-7) = \ln 4 \\ \Rightarrow x &= \frac{1}{3}\left(2+\ln 4\right). \end{aligned}$$

Is Ingrid's solution correct? If not, where did she go wrong?

Warm-Up Problem 2

Problem

Solve for x:

$$1 = e^{-x/2}\sqrt{3e^{2x} - 4}.$$

Hint: if you get stuck, try setting $y = e^x$.

Aside: e as a Limit

- In the book and PCE, you've seen that

$$\left(1+\frac{1}{n}\right)^n \approx e \quad \text{for } n \gg 1.$$

- *Idea*: e appears "universally" when describing growth or decay (of a biological population, chemical substance, money in a savings account) taking place over **a large amount** of **very small periods**.
- Soon, we'll learn to *define* a limit operator in such a way that we can say rigorously

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

In fact, in MAT 157, e might be *defined* this way! Return to the above when discussing **Euler's numerical method** for solving ODEs.

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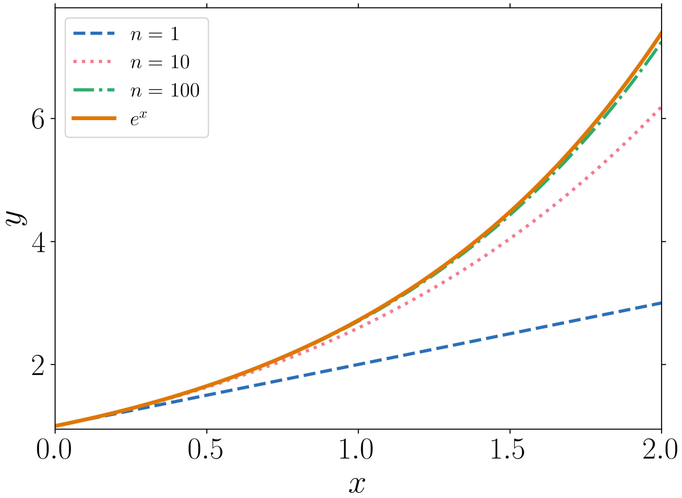
Aside: e as a Limit

Below, we plot $f_n(x) = \left(1 + \frac{x}{n}\right)^n$ to visualize how well it approximates e^x for large n.



Interpretive Problem: Background

- In the 1920s, the Russian biologist Georgy Gause became interested in the mathematical structure underlying population growth.
- Gause performed experiments on the growth of a laboratory population of microscopic organisms (*Paramecium aurelia*) over twenty-six days.
- Population began with only a few cells, and was provided with a **constant supply of nutrients**
- The data from this experiment, and other more complex experiments Gause performed, are available in his book *The Struggle for Existence* (which you can go get from Gerstein library!).

Paramecium Colony

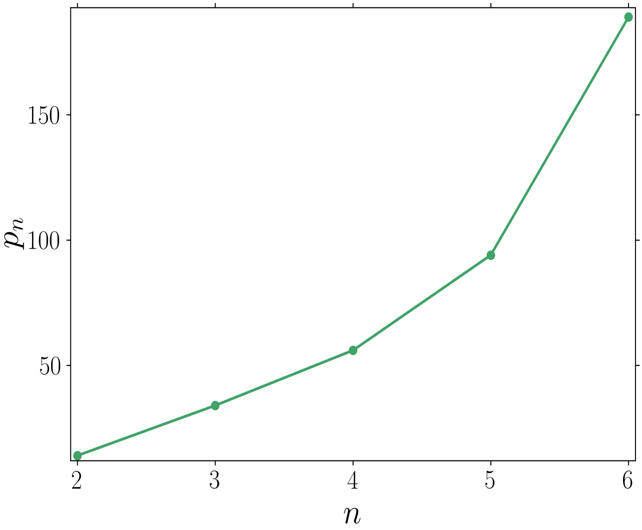
Figure: Image of a colony of paramecia from Diapteron's website (click).

Gause's Data for Days 2-6

Day	Density
0	2.0
1	NaN
2	14.0
3	34.0
4	56.0
5	94.0
6	189.0

"Density" means "population density" = number of cells per cm^3

Gause's Data for Days 2-6



$n =$ num. days elapsed, $p_n =$ num. of cells per cm^3 at day n

Interpretive Problem

Problem

The goal of this problem is to try to find a function $p(t)$ that approximately reproduces Gause's measured data points (n, p_n) for $n = 2, \ldots, 6$; that is, we should have $p(n) \approx p_n$.

1. Heuristically explain why it's reasonable to model $p(t)$ by a function of the form

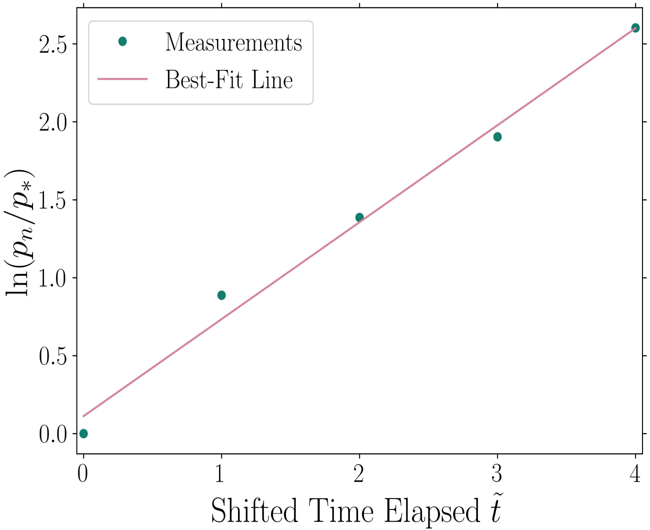
$$p(t) = p_* e^{r(t-2)} \quad \text{for } t \geq 2$$

   where $p_*, r > 0$ are constant parameters.

2. Determine p_* from the data.
3. Explain how you would use the measurements (n, p_n) and a line of best fit to estimate r.

Interpretive Problem: Solution

Implementing the Fit



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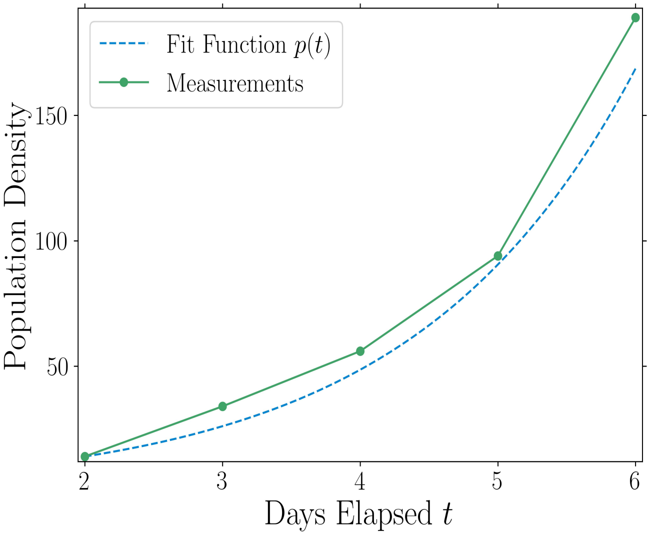
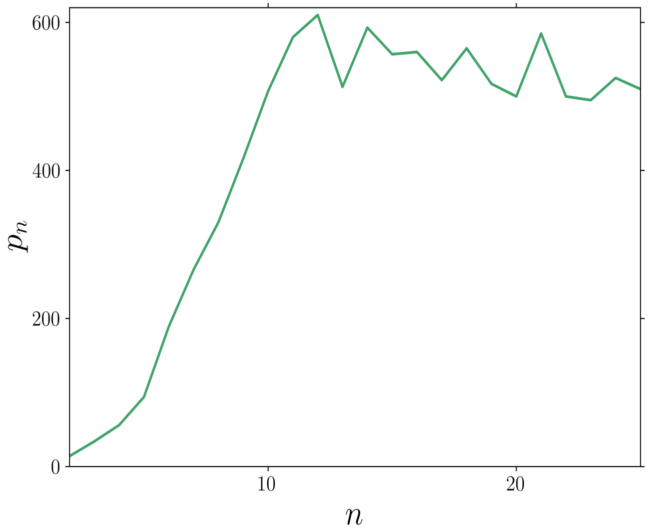


Here, $p(t) = 14e^{r_{\text{best}}(t-2)}$ where r_{best} is the slope of the line of best fit from the previous slide. The fit seems pretty good!

The Whole Story: Gause's Data for the Days 2-26



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Interpreting Gause's Data

- Population initially grows rapidly, then oscillates about a "carrying capacity" of about 500
- Fast growth quickly levelling off = **logistic growth**
- Interpretation: eventually there are so many organisms that they must compete for the finite amount of resources available $\Rightarrow$ exponential growth is not perpetually sustainable

So, while exponential growth is useful for partially understanding the data, it doesn't tell the whole story.

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