MAT 186 Module D1: Linear Approximation
Section LEC 0108

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Plan for Today

1. Two Warm-Up Problems
2. Interpretive Problems, discussion in groups
3. Code demo

Warm-Up Problem 1

Problem

Explain why the small angle approximation for $\sin\theta$ makes sense.

Warm-Up Problem 2

Problem

If z denotes the (positive) depth below sea level, $\rho(z)$ denotes the density of seawater at depth z, $P(z)$ denotes the pressure of seawater at depth z, and $g \approx 10m/s^2$ denotes the acceleration due to gravity on Earth, then the **equation of hydrostatic balanace** tells us that

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \rho(z)g.$$

At a depth of $2km$, the density of seawater is about $1.03285kg/m^3$, and the pressure of seawater is about 197.4 atmospheres (1 atmosphere $= 101325\frac{kg}{ms^2}$ of pressure).

Estimate the pressure of seawater at a depth of $2.1km$.

Solution to Warm-Up Problem 2

Interpretive Problem 1

Problem

Consider the periodic function

$$f(x) = \begin{cases} \sqrt{x} & x \in [0,1) \\ \sqrt{x-1} & x \in [1,2) \\ \sqrt{x-2} & x \in [2,3) \\ \ldots \end{cases}$$

defined on $[0,\infty)$. Can linear approximation be used to estimate $f(x)$ for $x \approx 32$?

Solution to Interpretive Problem 1

Interpretive Problem 2, Part 1

Problem

Estimate $\sqrt{16.05}$ without using a calculator. Then, compute the error between your approximate value and the value produced by a calculator.

Solution to Interpretive Problem 2, Part 1

Interpretive Problem 2, Part 2

Problem

In 60 CE, the mathematician Heron described an algorithm for approximating $\sqrt{A}$.

1. Pick an initial guess $x_0 \approx \sqrt{A}$.
2. Iteratively define a sequence of approximations $x_n \approx \sqrt{A}$ via

$$x_{n+1} = \frac{1}{2}\left(\frac{A}{x_n} + x_n\right).$$

The more iterations we perform, the better the approximation $x_n \approx \sqrt{A}$ becomes.

Take $A = 16.05$ and use an initial guess of $x_0 = 4$.

- Show that the Heron approximation x_1 is same approximation we got using linearization.
- Compute x_2. Is the answer a better or worse approximation to $\sqrt{16.05}$ than x_1?

Solution to Interpretive Problem 2, Part 2

- In 1740, Thomas Simpson devised a numerical method for approximately finding the roots of a given differentiable function $f(x)$, based on earlier ideas of Newton. Owing to the weirdness of history, this algorithm is called **Newton's method**.
- Like Heron's method for approximating $\sqrt{A}$, Newton's method is iterative:
  1. pick an initial guess x_0 for the root;
  2. iteratively update the guess via

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

- Under certain conditions, one can show the approximation gets better and better with each successive iteration.

Interpretive Problem 3

Problem

Derive Newton's method for solving

$$f(x) = 0$$

from scratch by following the recipe below.

1. Assume you have constructed x_n already, and that it **is very close to** the true root x_*.
2. Explain why we want to have have $x_{n+1} = x_*$.
3. Find an expression for x_{n+1} in terms of x_n if we indeed have $x_{n+1} = x_*$.

Solution to Interpretive Problem 3

Extra Problem

Problem

- Explain why computing $\sqrt{A}$ is equivalent to finding the roots of a particular quadratic function $f(x)$.
- Show that Heron's method is a special case of Newton's method.

Solution to Extra Problem

Code Demo: Using Newton's Method

To the Jupyter notebook!