Lectures on Partial Differential Equations: A Supplement for a First Course

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Preface

If I were to start writing out all my grievances with university mathematics education right now, I'm sure I'd be finishing up around the time booze cruises to Neptune become an affordable way for the eternally bored urban middle class to waste a Sunday afternoon. Brevity being the soul of wit, then, I shall endeavour only to address one particular grievance: I find the provision of hard, concrete examples (in other words, case studies) in undergraduate mathematics to be severely lacking. Wrestling with long, tough examples and really internalizing their connections to real life is perhaps the only way I have been able to develop quantitative problem-solving skills and thereby obtain some degree of self-confidence as a mathematician. Let the scoffing Bourbakists scoff and mutter "Babylonian" under their breath as they pass by me in the hallowed halls of the Bahen Centre, I know what has worked for me and what I believe will work for many more mathematics students in the future. It is with this problem-solving-centric point of view that I have developed the following modest scrapbook of lectures and problems on the basic theory of partial differential equations.

This document is a collection of tutorials for a year-long first course in partial differential equations (PDEs) arising from my time serving as a teaching assistant for MATH 351 at the University of Toronto from September 2021 to April 2022. The course was attended by third-year students in mathematics and mathematical physics, most of whom had more than a passing acquaintance with δ's and ϵ's. The professor in charge of MATH 351 was the amazing Catherine Sulem, whose legendary intellect is matched by her outstanding warmth, kindness, and patience. Catherine indeed teaches students by building their problem-solving skills, as I have espoused above, but in my weekly tutorials I really wanted to challenge the students and take them even further with intense case studies. The required textbook for the course was Walter Strauss' introductory book [27], so in the lectures below the reader will find many references to this text. So, any student taking a course from Strauss' book and looking for more in-depth examples may find this document useful. In summary, these notes do not constitute a complete set of lectures for a first course in PDEs, but rather a supplement to such a course. In culinary terms, the lectures and textbook readings from a first course in PDEs would be the entrée, and my own notes would be the digestif.

Now come the excuses and caveats! First, these notes are definitely a work in progress, and there are definitely going to be many typos and arithmetic errors. Though it pains me to admit it, there are likely also some errors of a more substantial nature hidden throughout the document. Additionally, there are some statements missing context since such context was provided in the 2021-2022 lectures. Over time, I will go through the document and try to

rectify this issue, making the whole set of notes self-contained. The reader who finds mistakes of any kind in these notes is encouraged to email me at adam.morgan[at]mail.utoronto.ca, and I will do my best to fix the issue promptly.

I would like to thank my students for making this course an absolute joy to work on. Thanks are also due to Catherine Sulem for reasons I have already indicated, and to Fabio Pusateri and Gordon Swaters, from whom I have (lovingly) stolen some of the problems given below. Finally, I am also eternally grateful to my wife Kristen Côté for her undying emotional and LaTeXsupport.

Contents

Chapter 1

Classification of Linear Second-Order PDEs in Two Variables

1.1 Introduction

So far in lectures, you have at least briefly seen the following second-order linear PDEs:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{(Laplace's equation)}, \\ u_{tt} - u_{xx} &= 0 \quad \text{(wave equation)}, \\ u_t - u_{xx} &= 0 \quad \text{(heat equation/diffusion equation)}. \end{aligned}$$

Today, we prove a theorem that shows the above three equations are, in a certain sense, prototypical of all linear second-order PDEs with constant, real coefficients. This in turn gives us a recipe for simplifying such PDEs by (nearly) reducing them to a Laplace, wave, or heat equation. This material is also covered in [27, §1.6].

We start by setting up some notation. Consider a general homogeneous, linear, second-order, constant-coefficient PDE (with real coefficients) for a real-valued unknown $u(x, y)$:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0. \tag{1.1}$$

We define the **principal part** of the differential operator appearing above as

$$P = a_{11}\partial_x^2 + 2a_{12}\partial_x\partial_y + a_{22}\partial_y^2. \tag{1.2}$$

So, P consists only of the highest-order parts of the operator in question. It turns out that P contains all the information we need to (nearly) reduce (1.1) to one of the three prototypes presented above.

1.2 Classification Theorem

Before presenting the precise statement of our recipe for simplifying (1.1), we need a helpful lemma.

Lemma 1.2.1. *Let*

$$L\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

be a linear transformation. Then, we have

$$\begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} = L^{\mathrm{T}} \begin{bmatrix} \partial_\xi \\ \partial_\eta \end{bmatrix}. \tag{1.1}$$

Proof. Write L as a matrix with real entries:

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}.$$

Then, from the multivariable chain rule,

$$\begin{aligned} \partial_x &= \frac{\partial \xi}{\partial x}\partial_\xi + \frac{\partial \eta}{\partial x}\partial_\eta \\ &= L_{11}\partial_\xi + L_{21}\partial_\eta \end{aligned}$$

and

$$\partial_y = L_{12}\partial_\xi + L_{22}\partial_\eta.$$

Rewriting this in matrix notation gives

$$\begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} = \begin{bmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{bmatrix} \begin{bmatrix} \partial_\xi \\ \partial_\eta \end{bmatrix} = L^T \begin{bmatrix} \partial_\xi \\ \partial_\eta \end{bmatrix}.$$

□

Now, we present the **classification theorem**, which sorts PDEs of the form (1.1) into one of three families (elliptic, hyperbolic, or parabolic) depending on the coefficients of the principal operator P. Once a PDE is sorted into its family, the classification theorem also tells us if the PDE resembles the Laplace, wave, or heat equation, at least at highest order.

Theorem 1.2.2 (Classification Theorem, Two-Variable Version)**.** *Given the PDE* (1.1)*, define the* ***discriminant*** *by*

$$\mathcal{D} = a_{12}^2 - a_{11}a_{22}.$$

There exists a linear transformation

$$L\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

such that

1. *if* $\mathcal{D} < 0$*, then* (1.1) *becomes*

$$\left(\partial_\xi^2 + \partial_\eta^2\right) u + \{\textit{lower-order terms}\} = 0$$

and we call the PDE ***elliptic****;*

2. *if* $\mathcal{D} > 0$, *then* (1.1) *becomes*

$$\left(\partial_\xi^2 - \partial_\eta^2\right) u + \{\textit{lower-order terms}\} = 0$$

and we call the PDE ***hyperbolic****;*

3. *if* $\mathcal{D} = 0$, *then* (1.1) *becomes*

$$\partial_\xi^2 u + \{\textit{lower-order terms}\} = 0$$

and we call the PDE ***parabolic****.*

Notice that the classification theorem only uses information about the *principal part* of our PDE to determine if the equation is elliptic, hyperbolic, or parabolic. Additionally, this theorem tells us that (1.1) only agrees with the Laplace/wave/heat equation *at the principal level* after changing variables.

Proof. I only provide details for the hyperbolic case here: the reader can easily modify the proof to handle elliptic and parabolic equations (exercise: do this). We start by writing the principal operator in matrix notation:

$$P = \begin{bmatrix}\partial_x & \partial_y\end{bmatrix} \begin{bmatrix}a_{11} & a_{12} \\ a_{12} & a_{22}\end{bmatrix} \begin{bmatrix}\partial_x \\ \partial_y\end{bmatrix} \doteq \begin{bmatrix}\partial_x & \partial_y\end{bmatrix} A \begin{bmatrix}\partial_x \\ \partial_y\end{bmatrix}. \tag{1.2}$$

A is called the **coefficient matrix** of P. Note that A is a symmetric matrix with real entries, hence by the spectral theorem there exists an orthogonal 2×2 matrix Q and $\lambda_1, \lambda_2 \in \mathbb{R}$ so that

$$Q^{\mathrm{T}} A Q = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}. \tag{1.3}$$

Further, notice that by hypothesis

$$\mathcal{D} = -\det A = -\lambda_1 \lambda_2 > 0,$$

so the eigenvalues λ_i of A have opposite signs. By permuting x and y if necessary, assume $\lambda_2 < 0$. Then, define a matrix Λ by

$$\Lambda = \begin{bmatrix}\frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & \frac{1}{\sqrt{|\lambda_2|}}\end{bmatrix}$$

so that

$$\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix} = \Lambda^{\mathrm{T}} \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} \Lambda = (Q\Lambda)^{\mathrm{T}} A (Q\Lambda).$$

If we define

$$L = (Q\Lambda)^{\mathrm{T}}$$

then we discover

$$\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix} = L A L^{\mathrm{T}}. \tag{1.4}$$

Now, define new coordinates by

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix} = L\begin{bmatrix}x\\ y\end{bmatrix}.$$

By Lemma 1.2.1, we know

$$\begin{bmatrix}\partial_x\\ \partial_y\end{bmatrix} = L^{\mathrm{T}}\begin{bmatrix}\partial_\xi\\ \partial_\eta\end{bmatrix}$$

so (1.2) becomes

$$P = \begin{bmatrix}\partial_\xi & \partial_\eta\end{bmatrix} LAL^{\mathrm{T}}\begin{bmatrix}\partial_\xi\\ \partial_\eta\end{bmatrix}.$$

Using (1.4) then gives

$$P = \begin{bmatrix}\partial_\xi & \partial_\eta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\partial_\xi\\ \partial_\eta\end{bmatrix} = \partial_\xi^2 - \partial_\eta^2,$$

and the proof of the hyperbolic case is complete. □

From the proof of the classification theorem, we find that we can re-phrase the definitions of ellipticity et cetera in terms of the eigenvalues of the coefficient matrix of the principal operator P. Indeed, a definition in terms of eigenvalues is much more useful, since it allows us to extend the classification theorem to second-order linear PDEs in any number of variables. For full details on how this is done, see section 1.6 of Strauss' book.

1.3 Examples of Classification

"Classifying" a PDE means determining whether it is elliptic, hyperbolic, or parabolic. This is typically quite easy: we just identify the principal part of the equation, read off the coefficients, and do one line of arithmetic!

Example 1.3.1 (Telegrapher's Equation)**.** Consider the **telegrapher's equation**

$$u_{tt} + \delta u_t - u_{xx} = 0 \tag{1.1}$$

where $\delta > 0$ is a real constant. We want to classify this PDE.

To classify a PDE, all we need to do is look at the principal part. In this case, the principal operator is

$$P = \partial_t^2 - \partial_x^2,$$

which is precisely that of the wave equation. So, there is no more work to be done, and we know that (1.1) is *hyperbolic*.

Example 1.3.2 (Pinchover & Rubinstein Problem 3.2a)**.** We want to classify the PDE

$$u_{xx} + 6u_{xy} - 16u_{yy} = 0.$$

To start, we directly read off the coefficients:

$$\begin{aligned} a_{11} &= 1 \\ a_{12} &= 3 \quad \text{(CAREFUL!)} \\ a_{22} &= -16. \end{aligned}$$

Since $\mathcal{D} = a_{12}^2 - a_{12}a_{22} = 9 + 16 = 25 > 0$, we know by definition that the equation is *hyperbolic*.

Example 1.3.3 (Pinchover & Rubinstein Problem 3.5a)**.** Consider the variable-coefficient PDE

$$xu_{xx} - yu_{yy} + \frac{1}{2}(u_x - u_y) = 0.$$

Note that variability of the coefficients means we cannot necessarily reduce to one of the three prototypical equations by a linear change of variables. However, we can still determine the regions of xy space where the PDE is elliptic, hyperbolic, or parabolic by looking at the discriminant as a function of (x, y).

To start, we forget about the sub-principal terms and write down the principal coefficients:

$$\begin{aligned} a_{11}(x, y) &= x \\ a_{12}(x, y) &= 0 \\ a_{22}(x, y) &= -y. \end{aligned}$$

So, the discriminant function is

$$\mathcal{D}(x, y) = xy.$$

We conclude that this PDE is

- *hyperbolic* in the interior of the first and third quadrants and
- *elliptic* in the interior of the second and fourth quadrants.

On the coordinate axes, the discriminant vanishes, but strictly speaking the PDE reduces to an ODE on the axes so we shouldn't really be classifying it here.

1.4 Other Comments

Notice first that the classification theorem says nothing about PDEs with *complex* coefficients. This is a major limitation, leaving out such interesting PDEs as the free-particle Schrödinger equation

$$iu_t + \frac{1}{2}u_{xx} = 0.$$

As we shall see later on the course, solutions to the Schrödinger equation act altogether different from solutions to the Laplace, wave, and heat equations. So, not every second-order linear PDE worth looking at can be classified by the scheme outlined above, or modelled by any of the main three prototypes.

Also, the classification theorem appears to imply that the principal part of a PDE plays the most important role in determining that equation's behaviour, namely whether it "acts like" a Laplace, wave, or heat equation. However, there are many cases where sub-principal terms can dramatically affect the qualitative features of a PDE. For instance, solutions of the telegrapher's equation (1.1) do not conserve the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2 + u_x^2 \, \mathrm{d}x$$

while the wave equation ($\delta = 0$ case) does conserve this energy (exercise: prove this assuming the solution $u(t, x)$ decays rapidly in x). Additionally, for Schrödinger equations, adding a (sub-principal) first-order term can even prevent well-posedness! You will be able to see this ill-posedness mechanism, at least formally, after we learn about the Fourier transform.

Chapter 2

Basics of the Method of Characteristics

2.1 Method of Characteristics, Parametric Version

In Ch.1 of Strauss, you have learned how to solve the first-order linear PDE

$$a(t,x)u_t + b(t,x)u_x = 0. \tag{2.1}$$

by realizing that (2.1) is equivalent to the vanishing of the directional derivative of $u(t,x)$ in the direction

$$(a(t,x), b(t,x)) \in \mathbb{R}_t \times \mathbb{R}_x.$$

In other words, $u(t,x)$ is constant along implicitly defined curves $x(t)$ whose tangent vector field is given by $(a(t,x), b(t,x))$. Such curves obey the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{b(t,x(t))}{a(t,x(t))}. \tag{2.2}$$

Why? We know the tangent line to the curve at $(t,x(t))$ lies along $(a(t,x(t)), b(t,x(t)))$, hence the slope of this line (which is precisely $\mathrm{d}x/\mathrm{d}t$) is b/a. Solutions to (2.2) are called the **characteristic curves** (or simply **characteristics**) of (2.1). As you have seen already, we can reduce finding the general solution of (2.1) to the simpler problem of integrating the ODE (2.2) and finding the characteristics. This procedure of solution is called the **method of characteristics (MOC)**, and it is one of the few systematic analytical techniques you can deploy to exactly solve PDEs.

But, there is another approach to the MOC that is sometimes more user-friendly than integrating (2.2): instead of looking at characteristics as implicit curves in the xt plane, we can consider them as *parametric* curves $(t(s), x(s))$ (where here s is the parameter in question). You have already seen the basics of this point of view in lectures. Since we still

want the vector field (a, b) to be tangent to the characteristics, we must have

$$\frac{\mathrm{d}t}{\mathrm{d}s} = a(t(s), x(s)), \tag{2.3a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}s} = b(t(s), x(s)). \tag{2.3b}$$

The system of ODEs above is sometimes called the **characteristic system** of (2.1). Note that the characteristic system implies (at least formally) the implicit definition of a characteristic curve given by (2.2). Additionally, by the chain rule, (2.1), and (2.3) we find

$$\frac{\mathrm{d}}{\mathrm{d}s} u\left(t(s), x(s)\right) = 0,$$

so indeed the solution is still constant along characteristics.

We also need to label each distinct characteristic with a parameter τ, so our notation should really look like

$$(t(s;\tau), x(s;\tau)).$$

In the implicit version of MOC, the integration constant that arises from solving (2.2) plays the same role as τ. Recognize that τ picks out what characteristic we're on, and s gives (roughly) our displacement along that characteristic: we then have a way of re-parameterizing (t, x) space so that characteristics become coordinate lines

$$\tau = \text{constant}.$$

In practice, there is some freedom in determining the characteristic label τ. To keep life as simple as possible, τ should always be chosen based on where the solution to the PDE is already known. For example, if we are given an initial condition

$$u(0, x) = u_0(x)$$

then the τ-axis should be precisely $\{t = 0\}$. Incidentally, I recommend you use the parametric approach to MOC *only when you are given initial data* (or boundary data, etc): when you are interested in finding the general solution to a PDE, the implicit approach works a lot better. That being said, there are initial-value problems where the implicit approach is a lot cleaner, so generally speaking you should be comfortable with both methods.

All of this discussion is much easier to understand after a few examples:

Example 2.1.1 (Transport Equation Again)**.** Given $c \in \mathbb{R}$, consider the initial value problem for the transport equation,

$$\begin{cases} u_t + cu_x = 0 \\ u(0, x) = u_0(x) \end{cases}. \tag{2.4}$$

You already know the solution to this problem is

$$u(t, x) = u_0(x - ct),$$

which just means the graph of the initial state is transported without modification towards $\text{sgn}(c)\infty$ at constant speed c. We now re-derive this solution using the parametric form of MOC. The characteristic equations are

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1,$$
$$\frac{\mathrm{d}x}{\mathrm{d}s} = c.$$

These equations are easily integrated in terms of unknown functions of τ:

$$t(s;\tau) = s + t_0(\tau),$$
$$x(s;\tau) = cs + x_0(\tau).$$

To fix the t_0, x_0, we need to make a clever choice of τ. We want to treat $\{s = 0\}$ as the set where we already know the solution, but this is precisely $\{t = 0\}$. Therefore, we set the τ-axis equal to the x-axis: we label characteristics by their x-intercepts (see Figure 2.1 for a sketch). This means

$$t_0(\tau) = 0,$$
$$x_0(\tau) = \tau.$$

We conclude that $t = s$ and

$$\tau = x - ct. \tag{2.5}$$

Next, since

$$\frac{\mathrm{d}}{\mathrm{d}s} u\left(t(s;\tau), x(s;\tau)\right) = 0,$$

we necessarily have

$$u(t(s;\tau), x(s;\tau)) = f(\tau)$$

for an unknown function $f(\tau)$. Setting $s = 0$ gives

$$f(\tau) = u(0,\tau) = u_0(\tau).$$

We conclude that, in (s,τ) variables, the solution to the transport equation is

$$u(t(s;\tau), x(s;\tau)) = u_0(\tau).$$

We then use (2.5) to rewrite this solution in terms of the natural variables (t,x):

$$u(t,x) = u_0(x - ct).$$

Let's review our work so far. First, we wrote down and solved ODEs for the characteristic curves, lines in the xt plane where our solution $u(t,x)$ is constant. Then, since we are given data at $t = 0$, we chose to label characteristics by their intersections with the $\{t = 0\}$ axis, which we called τ. This choice amounts to parameterizing characteristics by $s = t$. Said differently, characteristics give us a new coordinate system in the xt plane: characteristics

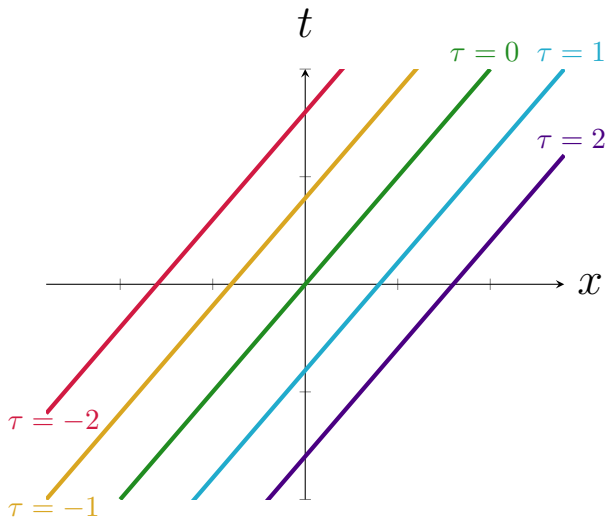

Figure 2.1: Various characteristics of the transport equation in xt plane. The characteristic lines are all labelled by their x-intercept τ.

are precisely the grid lines $\tau =$ constant, and the lines $t =$ constant become the grid lines $s =$ constant. See Figure 2.2 for an illustration. In particular, this picture tells us that the given initial data is "flowed out" from the $t = 0$ axis along the characteristics: to compute $u(t, x)$, all we need to do is find out what (unique!) characteristic passes through (t, x), then we can use the initial condition and the constancy of u along characteristics to finish. We see then that MOC works *because it tells us how to construct a coordinate system where solving our PDE becomes trivial.*

Remark. *In the language of differential geometry, we might say that the method of characteristics gives us a clean way of building a useful* ***foliation*** *of the space of independent variables (in the above example, the xt plane). For more discussion on first-order PDEs and characteristics from a geometric viewpoint, see Chapters 9 and 19 of* Introduction to Smooth Manifolds *(ed. 2) by John M. Lee (Springer, 2012). We won't need to use this geometric machinery in our course, however.*

Example 2.1.2 (Variable Coefficients)**.** To drive home the discussion of our previous example, let's solve the more complicated initial value problem

$$\begin{cases} u_t + txu_x = 0 \\ u(0, x) = u_0(x) \end{cases}. \tag{2.6}$$

We start by writing out the characteristic ODEs:

$$\begin{aligned} \frac{\mathrm{d}t}{\mathrm{d}s} &= 1, \\ \frac{\mathrm{d}x}{\mathrm{d}s} &= t(s)x(s), \\ \frac{\mathrm{d}u}{\mathrm{d}s} &= 0. \end{aligned}$$

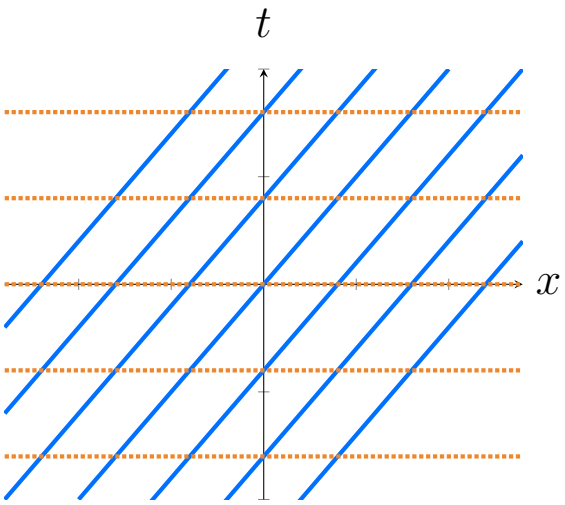

Figure 2.2: Sketch of the re-parameterization of the xt plane in terms of (s, τ) as suggested by the method of characteristics for the transport equation. The solid diagonal grid lines are characteristics $\tau =$ constant and the dotted horizontal grid lines are $s = t =$ constant.

To integrate the second equation, we first need to solve for $t(s)$. As in the previous example, since we are given the solution's values at $t = 0$, we should label the characteristics by their x-intercepts. That is, the τ-axis is precisely the x-axis. This means

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1 \Rightarrow t(s;\tau) = s.$$

Then, the ODE for $x(s;\tau)$ becomes

$$\frac{\mathrm{d}x}{\mathrm{d}s} = sx.$$

This separable ODE (together with the condition $x(0;\tau) = \tau$) is immediately integrated to yield

$$\tau = xe^{-\frac{1}{2}s^2}. \tag{2.7}$$

The characteristic curves for this PDE are then given by

$$(t(s;\tau), x(s;\tau)) = \left(s, \tau e^{\frac{1}{2}s^2}\right).$$

See Figure 2.3 for a sketch of a few of these curves. Since $\frac{\mathrm{d}u}{\mathrm{d}s} = 0$, we find that

$$u(t(s;\tau), x(s;\tau)) = f(\tau)$$

for an unknown function $f(\tau)$. Plugging in $s = 0$ and using (2.7), we find that the solution to our initial-value problem is

$$u(t, x) = u_0\left(xe^{-\frac{1}{2}t^2}\right).$$

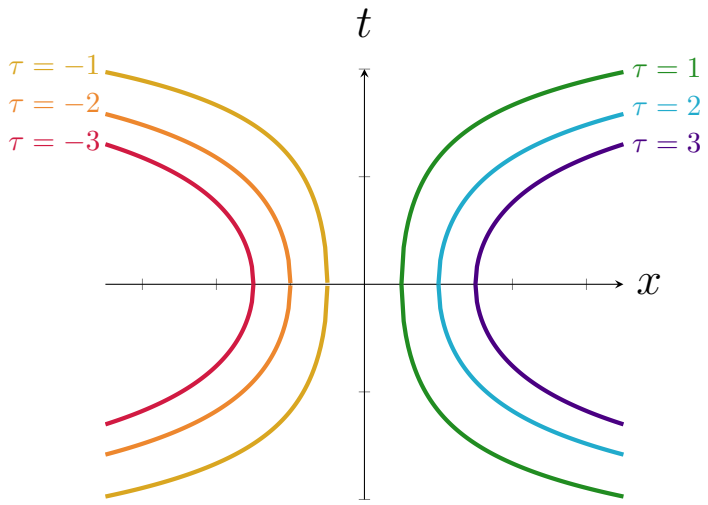

Figure 2.3: Various characteristics of $u_t + txu_x = 0$. The t axis is the $\tau = 0$ characteristic.

2.2 MOC Does Not Always Work!

In this section, we work through a brief example showing that MOC needs to be applied carefully, as it can fail even for very simple problems. The lesson here is to double-check that your given solution values are "consistent with" the characteristics of the PDE.

Example 2.2.1. Consider the Cauchy problem

$$\begin{cases} yu_x - xu_y = 0 \\ u|_{y=0} = x^3 \end{cases}. \tag{2.1}$$

We show that this problem does not admit a unique global solution $u(x, y)$. By "global", we mean defined for all $(x, y) \in \mathbb{R}^2$.

To start with, let's write the characteristics implicitly:

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{y}{x}. \tag{2.2}$$

This is a separable ODE, so solving it is trivial. In terms of an arbitrary constant R, the solution to this ODE is given implicitly by

$$x^2 + y^2 = R^2. \tag{2.3}$$

Therefore, the characteristics of the PDE in (2.1) are circles. In this case, the move to a new coordinate system where characteristics become grid lines is precisely the move to *polar coordinates*. In fact, the polar coordinate form of the PDE is

$$\partial_\theta u = 0,$$

which simply says that u is a radial function.

Now, let us focus on the $R = 1$ characteristic. Call this curve γ_1. The solution must be constant along γ_1, so to find $u|_{\gamma_1}$ all we need to do is evaluate u at one point along γ_1. Obviously, we know the solution at $(x, y) = (1, 0) \in \gamma_1$ since our data is given on the x-axis. Therefore,

$$u|_{\gamma_1} = x^3|_{(x,y)=(1,0)} = 1.$$

However, we also have $(-1, 0) \in \gamma_1$, which would imply

$$u|_{\gamma_1} = x^3|_{(x,y)=(-1,0)} = -1,$$

a contradiction. So, MOC gives a multiply-valued solution, which of course is not a solution at all!

We recognize that *the given data is not adapted to the characteristics of the governing PDE*. That is, the solution given on $\{y = 0\}$ is ambiguously defined on characteristics. One way to fix this is to cut out the "bad part" of the initial data curve. In other words, we only look for a solution valid on

$$\Omega_+ \doteq \mathbb{R}^2 - \left\{(x, y) \in \mathbb{R}^2 \mid y = 0, x < 0\right\}$$

or

$$\Omega_- = \mathbb{R}^2 - \left\{(x, y) \in \mathbb{R}^2 \mid y = 0, x > 0\right\}.$$

On $\Omega+$, we'd use the values of the solution on the positive x-axis to solve our PDE, and vice versa on Ω_-. Thus, we can still find a unique solution to (2.1) *provided* we don't demand the solution is also global.

Alternatively, we could simply ask for new initial conditions: as long as the solution prescribed on $\{y = 0\}$ is *even*, the solution given by MOC is unique and exists globally.

Remark. *(really more of a challenge exercise...) Remember from ODE theory that solutions to "generic" ODEs are only guaranteed to exist uniquely locally in time. Can you develop an analogous existence-uniqueness result for linear first order PDEs (with given initial data) using the method of characteristics? What does the above example tell you about the possibility of extending this result to find global-in-time solutions?*

Chapter 3

1D Wave Equation: An Example of D'Alembert's Formula in Action

Now, we switch gears and discuss the initial-value problem for the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad \forall\ (x,t) \in \mathbb{R}^2 \\ u|_{t=0} = f(x) \quad \forall\ x \in \mathbb{R} \\ u_t|_{t=0} = g(x) \quad \forall\ x \in \mathbb{R} \end{cases} . \tag{3.1}$$

In suitable units, (8.13) appears throughout mathematical physics when modelling

- the vibrations of an elastic string,
- the propagation of electromagnetic waves in a vacuum,
- gravity waves in a shallow layer of ideal, incompressible fluid,
- sound waves in a compressible fluid,
- and many other processes.

That being said, not every "wave" obeys the wave equation, as we shall come to see when we investigate dispersive PDEs.

In lectures, you have discovered that the solution to (8.13) can be expressed via **D'Alembert's formula**:

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ \mathrm{d}y. \tag{3.2}$$

From this expression, one easily discovers that the wave equation universally exhibits finite speed of propagation. Another nice consequence of D'Alembert's formula is that it allows us to define "solutions" of the wave equation even for non-smooth $f(x), g(x)$, as the next example demonstrates.

Example 3.0.1 (Plucked String, Strauss pp. 36 - 37)**.** Consider (8.13) with initial conditions given by

$$f(x) = \begin{cases} b\left(1 - \frac{|x|}{a}\right) & |x| < a, \\ 0 & \text{else} \end{cases}, \tag{3.3a}$$

$$g(x) = 0, \tag{3.3b}$$

where a, b are constants. See Figure 3.1 for a plot of $f(x)$, which takes the shape of a triangular bump. Physically, this initial setup could correspond to the plucking of a guitar string.

Our goal is to investigate the corresponding solution to the wave equation. Using D'Alembert's formula, we have

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]. \tag{3.4}$$

This means that, over time, the initial triangle splits into two smaller triangles moving towards $\pm\infty$ at speed c. You can see a movie of this happening on Quercus under the "Files" tab corresponding to our tutorial. Alternatively, Figure 3.2 shows a space-time filled contour plot illustrating this solution with a single image.

Before we go further, we need to address the elephant in the room: the solution to the wave equation given above *is not actually everywhere-differentiable*! So, strictly speaking (3.4) does not actually give a "solution" in the sense that we usually understand this term. However, D'Alembert's formula still gives a perfectly well-defined function, and from a physical point of view the solution appears to make sense. All in all, it doesn't seem fair to exclude $u(x,t)$ given in (3.4) as a solution just because it doesn't have enough derivatives: perhaps we should *define* a solution to (8.13) simply by D'Alembert's principle. This is our first step towards recognizing the importance of **weak solutions** to PDEs, perhaps the most important idea in modern (functional analysis-based) PDE theory.

Now that we definitely believe (3.4) actually gives a solution to the wave equation, let's describe the solution at

$$t = \frac{a}{2c}$$

as in Strauss. This time is too early for the initial state to have cleanly shed off two triangles, as we'll now demonstrate. By definition of $f(x)$, we have to split into several cases based on

$$|x \pm a/2| < a \quad \text{or} \quad |x \pm a/2| > a.$$

Strauss handles several of these cases explicitly. We'll look mostly at the ones he doesn't consider.

- If $x > 3a/2$, then $|x \pm a/2| > a$, hence $u(x,t) \equiv 0$ in this region.
- If $a/2 < x < 3a/2$, we find

$$0 < x - a/2 < a/2$$

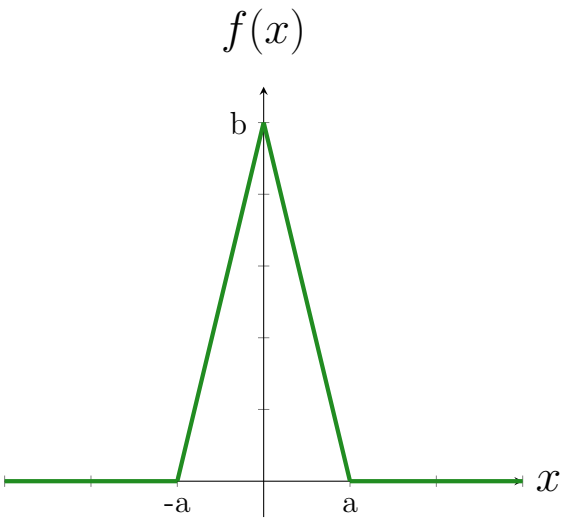

Figure 3.1: Plot of the initial state $f(x)$ in the plucked string example.

and

$$a < x + a/2 < 2a.$$

Therefore, only the right-moving term contributes here and we find

$$u(x,t) = \frac{1}{2}b\left(1 - \frac{(x-a/2)}{a}\right) = \frac{1}{2}\left(b - \frac{bx}{a} + \frac{b}{2}\right) = \frac{3b}{4} - \frac{bx}{2a}.$$

So, the solution is affine in this region. This corresponds to the outer edge of the right-moving small triangle.

- If $|x| < a/2$, then

$$-a < x - a/2 < 0$$

and

$$0 < x + a/2 < a.$$

In this case, both the left-moving and right-moving pieces contribute, and we have after some algebra (see Strauss) that

$$u(x,t) \equiv \frac{b}{2}.$$

By symmetry, then, we conclude that the graph of $u(x,t)$ is *trapezoidal* at $t = \frac{a}{2c}$. So, it takes a bit of time for the splitting into two triangular waves to "fully develop".

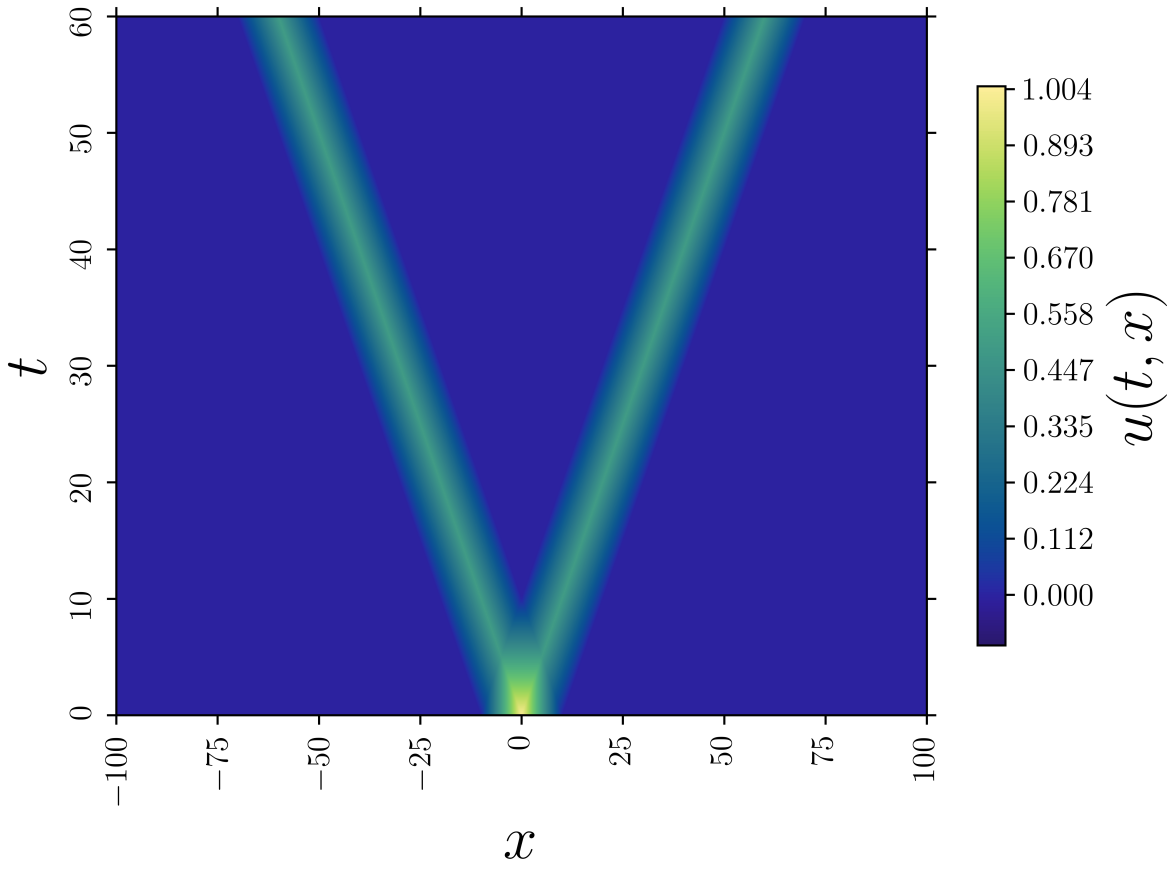

Figure 3.2: Space-time filled contour plot of the solution to the wave equation in the plucked string example. We see that the initial data eventually splits into two small triangular waves with identical, constant speed but opposite directions of motion.

Chapter 4

Linear and Nonlinear Diffusion via Similarity Solutions

4.1 Brief Summary of Essential Properties of the Heat/Diffusion Equation

In lectures and Strauss' book [27], you have seen the following expression for the solution of the diffusion equation:

Theorem 4.1.1 (Representation Formula in 1D)**.** *The unique solution to the initial-value problem*

$$\begin{cases} u_t - \kappa u_{xx} = 0 & \forall\ (x,t) \in \mathbb{R} \times (0,\infty) \\ u|_{t=0} = u_0(x) \end{cases} \tag{4.1}$$

is given by the formula

$$u(x,t) = \frac{1}{2\sqrt{\pi\kappa t}} \int_{-\infty}^{\infty} u_0(x')\ e^{-\frac{|x-x'|^2}{4\kappa t}}\ \mathrm{d}x'. \tag{4.2}$$

The function

$$S(x,t) \doteq \frac{1}{2\sqrt{\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}}$$

appearing in the representation formula is called the **heat kernel**. Other names for $S(x,t)$ are the **diffusion kernel**, the **fundamental solution to the heat equation**, or the **Green's function of the heat equation**.

We can write the solution of (4.1) as a convolution of the initial state against the heat kernel:

$$u(x,t) = \int_{-\infty}^{\infty} u_0(x')\ S(x-x',t)\ \mathrm{d}x'.$$

See Figure 4.1 for a plot of $S(x,t)$ for various t: as t gets larger, the peak of the graph becomes smaller, and the width of the graph becomes larger. In other words, evolving the heat kernel

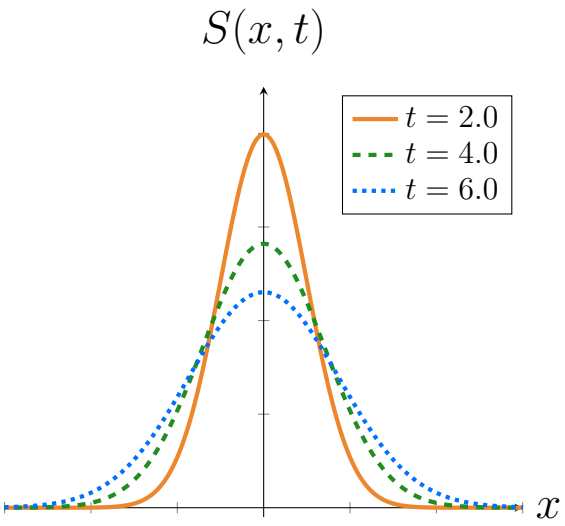

Figure 4.1: Plot of the heat kernel $S(x,t)$ at various times t for $\kappa = 1/4$. As time elapses, the central peak of the Gaussian gets smaller, and the width of the curves increases more slowly.

in time amounts to squishing down a Gaussian hump in a mass-conserving fashion. We also find that, as more time elapses, the aforementioned shallowing and widening of the graph becomes slower. This is a manifestation of a universal feature of the heat equation, namely the tendency to damp out regions of rapid change faster than regions of slow-to-moderate change: we'll be able to quantify this behaviour after learning about Fourier series.

The representation formula has several immediate consequences of profound physical and mathematical importance:

- **Pointwise decay**: if $\int_{-\infty}^{\infty} u_0(x)\,\mathrm{d}x < \infty$, then
$$\lim_{t\to\infty} u(x,t) = 0 \quad \text{uniformly in } x \in \mathbb{R}.$$
- **Smoothing effect**: $u(x,t)$ is smooth in x for all $t > 0$, even if $u_0(x)$ is discontinuous.
- **Infinite speed of propagation**: if $u_0(x) \geq 0$ is not identically 0, then $u(x,t) > 0\ \forall\ x \in \mathbb{R}, t > 0$ (so information about $u_0(x)$ being positive is transmitted throughout *all space, instantaneously*).

If you have not read about these results in Strauss, you should take a few minutes and prove each point as a little analysis exercise. To get a better intuitive idea of how the above three features appear in solutions to (4.1), have a look at the movies posted on quercus: `diffusion_movie_spike.mp4` shows how an initial "plucked string-like" initial state behaves while evolving under the diffusion equation, and `diffusion_movie_noisy_sine.mp4` does the same, but for an initial state given by a sine curve perturbed by (very noticeable!) high-amplitude Gaussian noise. While looking at each movie, try and see if you can identify how the images exemplify each of the three properties above.

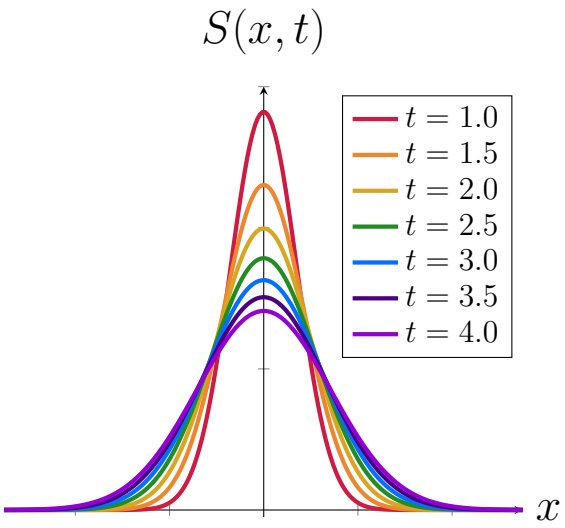

Figure 4.2: An expanded version of Figure 4.1 where more time steps are plotted.

Remark. *Actually, pointwise decay is hard to identify in* `diffusion_movie_noisy_sine.mp4` *because this demo only runs heat flow for a very short time. Additionally, both movies are based on a numerical code requiring u to vanish at two specified boundary points, so while you can still see infinite speed of propagation, it doesn't quite take the precise form we used earlier; rather, $u(x,t) > 0$ only holds for x in the* interior *of the computational domain.*

4.2 Derivation of the Heat Kernel via Similarity Method

In Strauss' book, the heat kernel was derived in a bit of a strange way: you looked for a special function Q solving the heat equation with step-function initial conditions, then differentiated Q with respect to x to find the heat kernel. Here, I want to re-derive the heat kernel using a more concrete physical example and a more systematic solution procedure.

Consider a long, thin wire made of a homogeneous and thermally conductive material. At time $t = 0$, the wire is at a reference temperature we may as well call 0 degrees. Now, suppose we quickly poke some fixed point $x = 0$ along the wire with a very hot needle at time $t = 0$. We assume the needle is so thin that its tip has zero width. Our goal in this section is to show that the resulting temperature distribution in the wire $u(x,t)$ is given precisely by the heat kernel $S(x,t)$. We use a **similarity method**, which leverages the scaling symmetry of a PDE (in this, the heat equation) to reduce to an ODE that can be solved with a few clever manipulations. Similarity methods are extremely powerful for solving both simple, linear PDEs as well as extremely nasty nonlinear PDEs (see the example in the next section). For a more systematic treatment of scaling symmetries and similarity solutions (which is far outside the scope of this course), see for instance the book of Barenblatt [2].

4.2.1 Point Source Initial Data

We know that the temperature $u(x,t)$ in the wire described above obeys the heat equation, but we still need to determine an appropriate initial condition. Since we assumed the hot needle (read: the *only* source of heat in the problem!) was very thin, we can reasonably say that all the initial temperature needs to be piled into the single point $x = 0$ and nowhere else. This means

$$u(x,0) = 0 \quad \forall\ x \neq 0.$$

Further, from thermal physics, we know there is a constant α (with suitable units) so that the total thermal energy in the rod at time t is given by

$$E(t) = \alpha \int_{-\infty}^{\infty} u(x,t)\ \mathrm{d}x.$$

In particular, to have a physically interesting finite-energy solution, we should always ask for u to be integrable in x. For simplicity, we may as well scale our variables so $E(0) = 1$. In particular, since $E(t)$ is necessarily constant if $u(x,t)$ obeys the heat equation and decays at infinity (can you prove this?), we know that $E(t) \equiv 1$ in this case. In other words, $u(x,t)$ is always a probability density function

Remark. *In PDE theory, lots of integrals are referred to as "energies". Always make sure you understand what this means for the specific problem at hand, otherwise you may become lost and confused.*

So, in summary, our initial state must satisfy

$$u(x,0) = 0 \quad \forall\ x \neq 0, \text{ and}$$
$$\int_{-\infty}^{\infty} u(x,t)\ \mathrm{d}x = 1.$$

Such a $u(x,0)$ cannot be a function: any sensible integration theory would tell us that a function vanishing everywhere except one point must integrate to zero. However, the above two conditions can still be imposed on our solution, at least if we're careful and impose them in the limit $t \to 0^+$ rather than at $t = 0$. This leads to the following definition:

Definition 4.2.1. *Consider a continuous function*

$$u(x,t) \colon \mathbb{R} \times (0,\infty) \to \mathbb{R}.$$

Suppose that

$$\lim_{t \to 0^+} u(x,t) = 0 \quad \forall\ x \neq 0$$

and

$$\lim_{t \to 0^+} \int_{-\infty}^{\infty} u(x,t)\ \mathrm{d}x = 1.$$

Then, we say that $u|_{t=0} = \delta(x)$. *In words, we say* $u(x,t)$ ***equals the delta function*** $\delta(x)$ ***at*** $t = 0$, *or* $u(x,0)$ ***is a point source***.

By the above definition, the so-called delta function $\delta(x)$ is not really a function, but instead a shorthand for dealing with initial data that is too densely concentrated to be dealt with via elementary calculus. Heuristically, $\delta(x)$ can be treated as if it obeys

$$\delta(x) = \begin{cases} +\infty & x = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

This expression is great for building intuition but useless for rigorous analysis, so deploy it wisely.

Also, we'll need another fact about the delta function in the sequel: for any smooth function $f(x)$, we have

$$\int_{-\infty}^{+\infty} f(x)\ \delta(x)\ \mathrm{d}x = f(0). \tag{4.2}$$

This can be seen by adding and subtracting $f(0)$ in the integrand:

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)\ \delta(x)\ \mathrm{d}x &= \int_{-\infty}^{+\infty} (f(x) - f(0) + f(0))\ \ \delta(x)\ \mathrm{d}x \\ &= \int_{-\infty}^{+\infty} (f(x) - f(0))\ \ \delta(x)\ \mathrm{d}x + f(0) \int_{-\infty}^{+\infty} \delta(x)\ \mathrm{d}x \\ &= \int_{-\infty}^{+\infty} (f(x) - f(0))\ \ \delta(x)\ \mathrm{d}x + f(0), \end{aligned}$$

where to get to the last line we have used $\int_{-\infty}^{+\infty} \delta(x)\ \mathrm{d}x = 1$. Now, since $\delta(x) = 0$ for all $x \neq 0$, the integrand in the first term above is identically zero, and (4.2) follows.

Remark. *When dealing with $\delta(x)$ rigorously, one actually* defines *$\delta(x)$ by demanding that* (4.2) *holds for all smooth $f(x)$ satisfying some decay condition as $x \to \pm\infty$. The business of handling "non-functions" like $\delta(x)$ head-on with proper mathematical rigor is called* ***distribution theory****. To properly deal with the hot needle problem using distribution theory, then, we would have to show*

$$\lim_{t \to 0^+} \int_{-\infty}^{\infty} S(x - x', t)\ f(x')\ \mathrm{d}x' = f(x)$$

for all nice enough f, where we recall $S(x,t)$ is the heat kernel. This is done carefully in [11, Ch. 2].

With this new definition in mind, we can phrase the initial-value problem modelling the temperature distribution in the wire as

$$\begin{cases} u_t - \kappa u_{xx} = 0 \quad \forall\ (x,t) \in \mathbb{R} \times (0, \infty) \\ \qquad u|_{t=0} = \delta(x) \end{cases}, \tag{4.3}$$

where κ is an absolute constant with suitable units depending on the material properties of the wire.

4.2.2 Uniqueness Theorem and Similarity Solution

Using the scaling symmetry of the diffusion equation, we can actually solve (4.3) exactly! The idea is to use scaling to reduce (4.3) to an ODE for an unknown function $V(z)$ of the so-called **similarity variable**

$$z = \frac{x}{\sqrt{\kappa t}}.$$

The ODE for $V(z)$ can then be solved with nothing more complicated than the method of integrating factors.

In order to justify this procedure, however, we'll need to show that our mathematical model (4.3) actually has unique solutions.

Theorem 4.2.2 (Heat Kernel Uniqueness Theorem)**.** *The constrained initial-value problem* (4.3) *has at most one solution.*

Proof. The proof runs along the same lines of the usual uniqueness proof for solutions of the heat equation (energy method): the only difference is a subtlety related to Lebesgue integration theory that we won't need to bother with in this course. □

So, if we find *one* solution to (4.3), we know it must be the *only* solution.

Now, we are ready to solve (4.3) through scaling. Recall that the diffusion equation admits a scaling transformation given by

$$u(x,t) \mapsto \tilde{u}(x,t) \doteq u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right), \quad \lambda > 0.$$

That is, if u satisfies the diffusion equation then so does $\tilde{u}$. Additionally, by linearity we also know for any $\mu \in \mathbb{R}$ that $v \doteq \mu\tilde{u}$ satisfies the diffusion equation as well. Now, u and v both satisfy the same PDE, but a formal change-of-variables computation using (4.2) shows that

$$v|_{t=0} = \mu\lambda\delta(x).$$

However, u and v will both satisfy (4.3) in the special case

$$\mu = \frac{1}{\lambda}.$$

Applying theorem 4.2.2, we conclude that if $u(x,t)$ solves (4.3) then we must have

$$u(x,t) = \frac{1}{\lambda} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right) \quad \forall\ \lambda > 0, \quad \forall\ (x,t) \in \mathbb{R} \times (0,\infty).$$

This expression tells us that our solution is always **self-similar**: the graph of the solution at time t is a suitably stretched graph of the solution at time t/λ^2 (do you believe this in light of Figure 4.1?). Notice that self-similarity arises because of both the scaling symmetry (and linearity) of the governing PDE as well as the very particular initial condition under

consideration. Additionally, we need to prove solutions are unique before we have any hope of proving self-similarity.

Here comes the really clever trick: if we choose $\lambda = \sqrt{\kappa t}$, then self-similarity gives

$$u(x,t) = \frac{1}{\sqrt{\kappa t}} u\left(\frac{x}{\sqrt{\kappa t}}, \frac{1}{\kappa}\right)$$

(picking $\lambda = \sqrt{t}$ is the more obvious way to simplify the RHS, but it turns out the κ is required a bit later to make some computations work). Now we're really in business! The unknown function u on the right-hand side depends only on the single variable

$$z \doteq \frac{x}{\sqrt{\kappa t}}. \tag{4.4}$$

z is called the **similarity variable** for this problem, since it arose from noticing that $u(x,t)$ was self-similar. So, the PDE can be translated to an ODE for

$$V(z) \doteq u\left(z, \frac{1}{\kappa}\right).$$

We now have to compute lots of derivatives to get a usable ODE for $V(z)$. From here, a lot of the computations look like those you encountered when deriving the heat kernel according to Strauss. To start with, we find

$$\begin{aligned} u_t &= -\frac{\kappa}{2}(\kappa t)^{-3/2} V(z) + (\kappa t)^{-1/2}\left[-\frac{\kappa}{2} x (\kappa t)^{-3/2}\right] V'(z) \\ &= -\frac{1}{2}(\kappa t)^{-1/2} t^{-1} [V(z) + z V'(z)] \\ &= -\frac{1}{2}(\kappa t)^{-1/2} t^{-1} \frac{\mathrm{d}}{\mathrm{d}z}[z V(z)]. \end{aligned}$$

Then, we turn to the spatial derivatives:

$$\begin{aligned} u_x &= (\kappa t)^{-1} V'(z) \\ \Rightarrow u_{xx} &= \kappa^{-1} (\kappa t)^{-1/2} t^{-1} V''(z). \end{aligned}$$

Putting all this together and using that u obeys the diffusion equation, we find

$$\begin{aligned} 0 &= u_t - \kappa u_{xx} \\ &= -(\kappa t)^{-1/2} t^{-1} \left\{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}[z V(z)] + V''(z)\right\}. \end{aligned}$$

We conclude that $V(z)$ obeys the ODE

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}[z V(z)] + V''(z) = 0. \tag{4.5}$$

Integrating (4.5), we find that

$$\frac{1}{2} z V(z) + V'(z) = \text{constant}. \tag{4.6}$$

Now, we must determine the constant on the right-hand side. Notice that

$$zV(z) = xu(x,t).$$

Since $u|_{t=0} = \delta(x)$, the above implies

$$\lim_{t\to 0^+} zV(z) = 0 \quad \forall\ x \neq 0.$$

This in turn means

$$\lim_{z\to\pm\infty} zV(z) = 0.$$

In light of the above constraint, taking the limit $z \to \infty$ in (4.6) gives

$$\lim_{z\to\pm\infty} V'(z) = \text{constant}.$$

However, we want $u(x,t)$ to be integrable in space, so $V(z)$ needs to be integrable in z. Thus we should demand

$$\lim_{z\to\pm\infty} V'(z) = 0,$$

whence the constant in (4.6) vanishes. We are left with solving

$$0 = \frac{1}{2}zV(z) + V'(z),$$

which is easily accomplished by introducing an integrating factor $e^{\frac{z^2}{4}}$. We conclude that

$$V(z) = V(0)e^{-\frac{1}{4}z^2}. \tag{4.7}$$

To determine $V(0)$ and complete the problem, we use that $u(x,t)$ is a probability density function to discover

$$\begin{aligned} 1 &= \frac{1}{\sqrt{\kappa t}}\int_{-\infty}^{\infty} V\left(\frac{x}{\sqrt{\kappa t}}\right)\ \mathrm{d}x \\ &= \int_{-\infty}^{\infty} V(z)\mathrm{d}z \\ &= V(0)\int_{-\infty}^{\infty} e^{-\frac{1}{4}z^2}\mathrm{d}z \\ &= 2V(0)\int_{-\infty}^{\infty} e^{-w^2}\mathrm{d}w. \end{aligned}$$

Now, recall that

$$\int_{-\infty}^{\infty} e^{-w^2}\mathrm{d}w = \sqrt{\pi},$$

and so $V(0) = \frac{1}{2\sqrt{\pi}}$. In summary,

$$V(z) = \frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}z^2}. \tag{4.8}$$

This in turn implies that the unique solution to (4.3) is given by

$$u(x,t) = \frac{1}{2\sqrt{\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}} = S(x,t). \tag{4.9}$$

So, we have obtained the heat kernel via

1. carefully defining what we mean by a "point source" initial condition
2. using the scaling symmetry of the heat equation to discover a similarity variable and reducing the heat equation to an ODE.

Both of these ideas will be incredibly useful throughout your studies of PDE theory.

Before concluding this section, let's discuss what happens if our hot needle pokes some general $x' \in \mathbb{R}$ instead of the origin. The relevant initial-value problem for the temperature $u(x,t)$ is

$$\begin{cases} u_t - \kappa u_{xx} = 0 \quad \forall\ (x,t) \in \mathbb{R} \times (0,\infty) \\ u|_{t=0} = \delta(x - x') \\ \displaystyle\int_{-\infty}^{\infty} u(x,t)\ \mathrm{d}x = 1 \quad \forall\ t \in (0,\infty) \end{cases} . \tag{4.10}$$

That is, the only thing that changes from (4.3) is the initial condition. Note that the definition of $u|_{t=0} = \delta(x - x')$ is just to replace $x \neq 0$ with $x \neq x'$ in definition 4.2.1. Then, we simply translate space variables according to

$$x \mapsto x - x'$$

to reduce (4.10) to (4.3). Therefore, the solution to (4.10) is

$$u(x,t) = S(x - x', t) = \frac{1}{2\sqrt{\pi\kappa t}} e^{-\frac{|x-x'|^2}{4\kappa t}}. \tag{4.11}$$

4.3 Similarity Method for a Nonlinear Heat Equation

To illustrate the wide applicability of similarity methods, I now describe how to obtain a (weak) similarity solution to a family of *nonlinear* heat equations. At this point in the course, you don't have to know what a "weak solution" precisely means, but just understand that it refers to a "solution" to a PDE that fails to be everywhere-differentiable (recall the plucked string example from tutorial 2 for a canonical example).

For a fixed number $\gamma > 1$, the PDE we investigate reads

$$u_t - \partial_{xx}\left(u^\gamma\right) = 0 \qquad \forall\ (x,t) \in \mathbb{R} \times (0,\infty). \tag{4.1}$$

First, notice that we can rewrite (4.1) as

$$u_t - \partial_x\left(\gamma u^{\gamma-1} u_x\right) = 0.$$

Revisiting the derivation of the heat equation as a model of heat conduction or particle diffusion, we see that we can think of the above PDE as describing diffusion with a state-dependent coefficient: the "diffusivity" here is

$$\kappa \doteq \gamma u^{\gamma-1}.$$

So, by analogy with the linear heat equation, we expect diffusion to occur faster in regions where u is larger (since a larger κ makes the heat kernel spread out more quickly than a small κ).

The type of nonlinear diffusion described by (4.1) appears naturally in ecological modelling. Suppose $u(x,t)$ is the number density of insects (all of the same species) inhabiting an essentially one-dimensional environment (say, a tree). As the density of insects at a particular point x increases, it becomes harder for each of them to access resources due to competition with other individuals. Intuitively, then, we expect that $u(x,t)$ will gradually widen over time in regions where $u(x,t)$ is large. This phenomenon is called **population pressure**. In 1980, Shigesada used a higher-dimensional version of (4.1) (with $\gamma = 2$) as a simple model for the diffusion of insects through a homogeneous environment, and found that special solutions of this IVP provided good fits for relevant experimental data on the dispersal of antlion populations [24]. In fact, some of the special, ecologically important solutions of (4.1) used by Shigesada are precisely the similarity solutions we'll obtain today! See Murray's book [22, §11.3] for a literature review on the importance of similarity solutions for nonlinear diffusion equations appearing in ecology. Additionally, (4.1) also describes the flow of a gas through a one-dimensional porous medium (for this reason, (4.1) is often called the **porous medium equation**), so our work today will have applications to many different scientific problems.

Now that the PDE under consideration has been motivated, let's try to find a similarity solution to the problem

$$\begin{cases} u_t - \partial_{xx}\left(u^\gamma\right) = 0 & \forall\ (x,t) \in \mathbb{R} \times (0,\infty) \\ u|_{t=0} = \delta(x). \end{cases} \tag{4.2}$$

We don't have the tools yet to show that this solution to (4.2) is unique, but we'll at least be able to get *one* solution via similarity methods. We will still have to assume that u and its derivatives decay rapidly as $|x| \to \infty$, though (exercise: under this assumption, can you shows that solutions to (4.2) have "conserved mass"?).

The first order of business is to find a scaling symmetry of the PDE. For to-be-determined numbers a, b, c, we consider the scaling transformation

$$u(x,t) \mapsto \tilde{u}(x,t) = \lambda^c u\left(\lambda^a x, \lambda^b t\right).$$

We are going to relate a, b, c so that $\tilde{u}$ still solves (4.1). We have

$$\begin{aligned} \tilde{u}_t &= \lambda^{b+c} u_t\left(\lambda^a x, \lambda^b t\right) \\ \partial_{xx}\left(\tilde{u}^\gamma\right) &= \lambda^{\gamma c + 2a}\left(u^\gamma\right)_{xx}\left(\lambda^a x, \lambda^b t\right). \end{aligned}$$

Then, if u solves (4.1), we have

$$\tilde{u}_t - \partial_{xx}\left(\tilde{u}^\gamma\right) = \lambda^{b+c} u_t\left(\lambda^a x, \lambda^b t\right) - \lambda^{\gamma c + 2a}\left(u^\gamma\right)_{xx}\left(\lambda^a x, \lambda^b t\right) = 0$$

provided $b + c = \gamma c + 2a$ which implies

$$b = 2a - (1-\gamma)c. \tag{4.3}$$

So, in order for $\tilde{u}$ to be a solution to (4.2), we must have that a, b, c obey the compatibility condition (4.3). Henceforth, we simplify by picking $b = 1$, whence

$$c = \frac{2a-1}{1-\gamma}.$$

Actually, in order to ensure

$$u|_{t=0} = \tilde{u}|_{t=0} = \delta(x),$$

we should choose

$$a = c = \frac{1}{1+\gamma}$$

following the discussion in the previous section. Therefore, our rescaled solution is

$$\tilde{u}(x,t) = \lambda^a u\left(\lambda^a x, \lambda t\right), \quad a = \frac{1}{1+\gamma}, \quad \forall\ \lambda > 0. \tag{4.4}$$

If we pick $\lambda = t^{-1}$, then, we find that the right-hand side of (4.4) becomes

$$t^{-a} u\left(x t^{-a}, 1\right).$$

Then, defining

$$\begin{aligned} z &\doteq x t^{-a}, \\ V(z) &\doteq u(z,1), \end{aligned}$$

we find that our similarity solution to (4.2) must have the form

$$u(x,t) = t^{-a} V(z). \tag{4.5}$$

Now, we have to crank out lots of derivatives:

$$\begin{aligned} \partial_t u &= -a t^{-a-1} V(z) + t^{-a} V'(z)\left(-a x t^{-a-1}\right) \\ \partial_x u^\gamma &= \gamma V^{\gamma-1}(z) V'(z) t^{-a(1+\gamma)} \\ \partial_x^2 u^\gamma &= \gamma(\gamma-1) V^{\gamma-2}(z)\left(V'(z)\right)^2 t^{-a(2-\gamma)} + \gamma V^{\gamma-1}(z) V''(z) t^{-a(2-\gamma)}. \end{aligned}$$

If u satisfies (4.1), then, we must have

$$-a t^{-a-1} V(z) - a z t^{-a-1} V'(z) = \gamma(\gamma-1) V^{\gamma-2}(z)\left(V'(z)\right)^2 t^{-a(2+\gamma)} + \gamma V^{\gamma-1}(z) V''(z) t^{-a(2+\gamma)}.$$

Notice that

$$a + 1 = 1 + \frac{1}{\gamma+1} = \frac{\gamma+2}{\gamma+1} = a\left(\gamma+2\right)$$

so the above ODE reduces to

$$-aV(z) - azV'(z) = \gamma(\gamma-1)V^{\gamma-2}(z)\left(V'(z)\right)^2 + \gamma V^{\gamma-1}(z)V''(z).$$

Re-arrangine a bit, we arrive that the equation

$$0 = \left(zV(z) + \frac{1}{a}\left(V^{\gamma}(z)\right)'\right)'. \tag{4.6}$$

Integrating this equation, we find

$$\text{constant} = zV(z) + \frac{1}{a}\left(V^{\gamma}(z)\right)'.$$

By taking the limit $z \to \infty$ as with our derivation of the heat kernel, we find that the constant on the left-hand side is 0, giving us

$$0 = zV(z) + \frac{\gamma}{a}V^{\gamma-1}(z)V'(z). \tag{4.7}$$

Now is where we start to deviate from the derivation of the heat kernel a bit (and, in doing so, going from the world of classical, smooth solutions to the world of weak solutions). First, notice that $V(z) \equiv 0$ solves the above ODE: obviously, we don't want our solution to vanish everywhere, so we ignore this case. However, in the region where $V \neq 0$, (4.7) becomes

$$-z = \frac{\gamma}{a}V^{\gamma-2}(z)V'(z).$$

This separable equation is quickly integrated to yield

$$V(z) = \left(A - \frac{a(\gamma-1)}{2\gamma}z^2\right)^{\frac{1}{\gamma-1}}, \tag{4.8}$$

where A is some constant. However, the $V(z)$ given above vanishes at

$$|z| = z_0 \doteq \sqrt{\frac{2\gamma A}{a(\gamma-1)}},$$

so to ensure consistency with our argument we only assume $V(z)$ is given by (4.8) in the region $|z| < z_0$. Outside of this interval, (4.7) is nonlinear and non-separable, so it doesn't look too amenable to a closed-form solution. Thus, for $|z| \geq z_0$ we simply set $V(z) = 0$, which gives a globally continuous (but non-differentiable at $z = \pm z_0$) solution to (4.7): if we define $(\xi)_+ \doteq \max(\xi, 0)$ then our solution $V(z)$ looks like

$$V(z) = \left(A - \frac{a(\gamma-1)}{2\gamma}z^2\right)_+^{\frac{1}{\gamma-1}}.$$

In terms of our original variables x, t, then, the similarity solution to (4.2) is

$$u(x,t) = \frac{1}{t^a}\left(A - \frac{a(\gamma-1)}{2\gamma}\frac{x^2}{t^{2a}}\right)_+^{\frac{1}{\gamma-1}}, \tag{4.9}$$

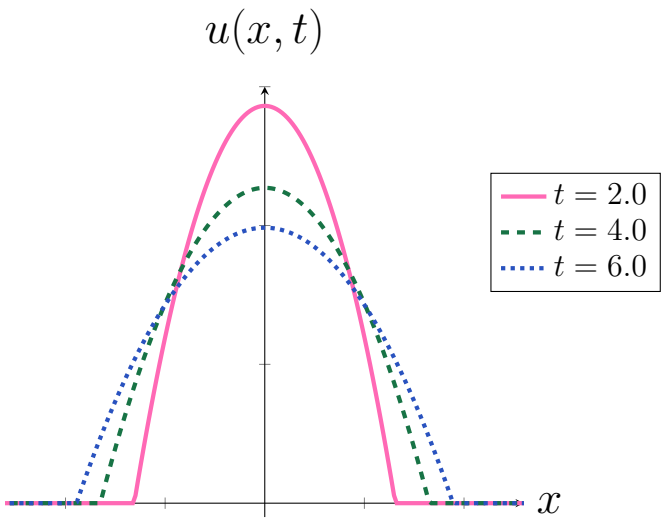

Figure 4.3: Plot of the ZKB solution at various times t. As time elapses, the peak of the parabolic hump decreases, and the interface between the hump and the region $\{u = 0\}$ moves further and further away from the origin.

where $a = 1/(\gamma + 1)$ and A is determined from the point-source constraint

$$\lim_{t \to 0^+} \int_{-\infty}^{+\infty} u(x,t)\, \mathrm{d}x = 1.$$

(4.9) is called the **Zel'dovich-Kompaneetz-Barenblatt (ZKB) solution** to (4.2). In the Shigesada insect dispersal model (with $\gamma = 2$), the ZKB solution with point-source ICs explicitly becomes

$$u(x,t) = \frac{1}{t^{1/3}} \left(\frac{3^{1/3}}{4} - \frac{x^2}{12t^{2/3}} \right)_+ .$$

See Figure 4.3 for a plot of the above function at various time steps. We see that the $t = 1$ parabolic bump is squished out in a similar fashion to the heat kernel: the amplitude of the solution decreases while the graph widens. However, there is a marked difference from the ZKB solution and the heat kernel, namely that the heat kernel is nonzero *everywhere*, while the ZKB solution features two propagating interfaces carrying all the "information" about nonzero initial data. These interfaces move at finite speed given explicitly by

$$c = \sqrt{\frac{2a\gamma A}{\gamma - 1}}\, t^{a-1}$$

(exercise: prove this! Also answer whether the interface motion becomes faster or slower as more time elapses... does Figure 4.3 substantiate your analysis?). In particular, the ZKB solution shows that *nonlinear* diffusions do not necessarily feature infinite speed of propagation!. This is, of course, in marked contrast to the linear case.

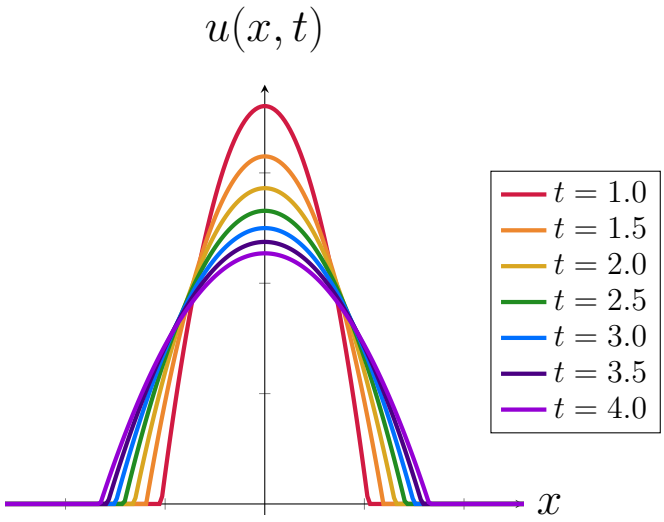

Figure 4.4: Expanded version of Figure 4.3 with more time steps plotted.

Chapter 5

Estimating the Age of the Earth

5.1 Introduction

In this tutorial, we'll see boundary effects and source terms in action as we use the heat equation to estimate the age of the Earth. The presentation here draws largely from [18] and [30]. For a modern physical discussion of heat conduction in the Earth, see [10].

5.2 Kelvin's Estimate

In the nineteenth century, Kelvin attempted to approximately compute the age of the Earth using the tools of thermal physics. As described in [18], his approach was roughly based on the assumptions below:

- at time $t = 0$, the entire Earth is at a uniform reference temperature $T_0 = 1200°\text{C}$ (roughly, the melting temperature of rock);
- the temperature at the Earth's surface is at a reference value 0;
- the Earth's curvature has a negligible effect on the distribution of temperature throughout the crust;
- the Earth's crust is materially homogeneous, so the thermal conductivity is constant throughout the surface;
- there are no chemical reactions driving the production of heat within the Earth.

Under these simplifying assumptions, the Earth can roughly be modelled by the vertical half-space

$$\{z \mid 0 \leq z \leq \infty\}.$$

We treat $z = 0$ as the Earth's surface, and as $z \to +\infty$ we get closer and closer to the planet core (technically, it may be more natural to use the negative half-space $(-\infty, 0]$ instead, but this introduces an extra negative sign that's awkward to deal with). Then, we can determine the temperature of the Earth $T(z,t)$ by solving the initial-boundary value problem

$$\begin{cases} T_t - \kappa T_{zz} = 0 \\ \quad T|_{z=0} = 0 \\ \quad T|_{t=0} = T_0 \end{cases} \tag{5.1}$$

where κ is a material constant determined from geological observations. According to [30], an accepted value for κ is

$$\kappa = 6 \times 10^{-3} \text{ cm}^2/\text{s}.$$

Now, how can we use a solution of (5.1) to estimate the age of the Earth? Again we appeal to geological field observations: an empirical value for the temperature gradient at Earth's surface at the present day is given by

$$\gamma = T_z|_{z=0}(t) = \left(3 \times 10^{-4}\right) {}^\circ\text{C/cm}.$$

In geophysics, γ is called the **geothermal gradient**. Since γ is known, we can solve the above equation for t to determine the age of the Earth.

Our first task is to solve (5.1). Recall from lectures that we can solve PDEs on half-spaces using the **method of reflection**, also known as the **method of images** (you may have encountered the latter nomenclature in a course on electromagnetic theory). This involves introducing an "odd extension" of $T(z,t)$ to $\{z \in \mathbb{R}\}$ according to

$$\widetilde{T}(z,t) \doteq \begin{cases} T(z,t) & z \geq 0 \\ T(-z,t) & z < 0. \end{cases}$$

Now, $\widetilde{T}(z,t)$ is odd in z for all $t > 0$, so it automatically vanishes at $z = 0$. In turn,

$$\left(\widetilde{T}|_{z\geq 0}\right)\Big|_{z=0} = T(z,t)|_{z=0} = 0.$$

In other words, we can solve (5.1) by solving

$$\begin{cases} \widetilde{T}_t - \kappa \widetilde{T}_{zz} = 0 \\ \quad \widetilde{T}|_{t=0}(z) = \begin{cases} T_0 & z \geq 0 \\ -T_0 & z < 0 \end{cases} \end{cases} \tag{5.2}$$

and then defining $T(z,t) \doteq \widetilde{T}(z,t)|_{z\geq 0}$. However, this can easily be done using the representation formula for the heat equation:

$$\begin{aligned} \widetilde{T}(z,t) &= T_0 \int_0^\infty S(z - z', t) \, \mathrm{d}z' - T_0 \int_{-\infty}^0 S(z - z', t) \, \mathrm{d}z' \\ &= T_0 \int_0^\infty S(z - z', t) - S(z + z', t) \, \mathrm{d}z', \end{aligned}$$

where we recall that

$$S(z,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{z^2}{4\kappa t}}$$

gives the heat kernel over the entire real line. In other words, we know that the heat kernel for $[0,\infty)$ is

$$G(z,z',t) \doteq S(z-z',t) - S(z+z',t). \tag{5.3}$$

Therefore, we have found that the solution of (5.1) is

$$T(z,t) = T_0 \int_0^\infty G(z,z',t)\ \mathrm{d}z'. \tag{5.4}$$

By doing some simple Gaussian integrals (exercise: review these computations!) we find that the above expression can be simplified to

$$T(z,t) = T_0 \operatorname{erf}\left(\frac{z}{\sqrt{4\kappa t}}\right). \tag{5.5}$$

Recall that the error function is defined by the definite integral

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-w^2}\ \mathrm{d}w.$$

See Figure 5.1 for a plot of this solution at various time steps. This tells us that the planetary core $z = \infty$ stays at T_0, but near the surface $z = 0$ there is a buffer region where the temperature quickly transitions from T_0 to 0. As time elapses, the size of this buffer region increases, corresponding to gradual cooling of the Earth's interior. Notice also that the solution is always smooth for $t > 0$, which we saw was a universal feature of heat flow on the real line.

Since we know the geothermal gradient γ at the present day, we can roughly estimate the age of the Earth by computing $T_z|_{z=0}(t)$, then solving

$$T_z|_{z=0}(t) = \gamma$$

for t. A direct computation yields

$$\begin{aligned}
\gamma &= T_z|_{z=0} \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} \left[e^{-z^2/4\kappa t}\right]_{z=0} \\
&= \frac{T_0}{\sqrt{\pi\kappa t}}.
\end{aligned}$$

Let us define

$$a \doteq \frac{\gamma\sqrt{\pi\kappa}}{T_0} \approx 1.085 \times 10^{-15/2}\ \frac{1}{\sqrt{\text{seconds}}}.$$

Then,

$$t = a^{-2} \approx 30 \text{ million years}.$$

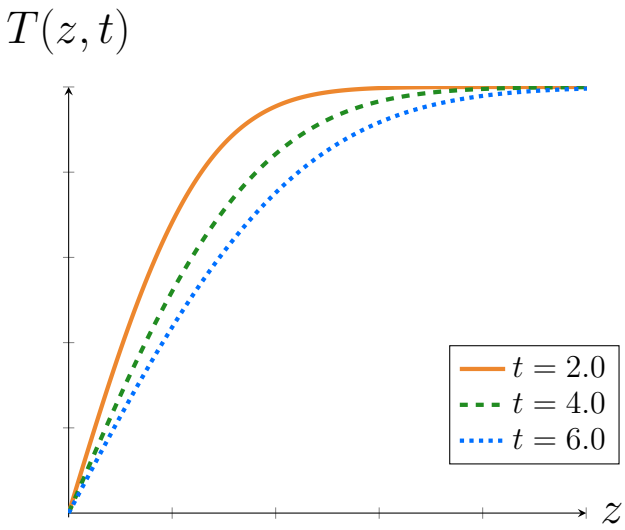

Figure 5.1: Plot of Kelvin's solution for the Earth's temperature at various time steps. The horizontal asymptote is T_0.

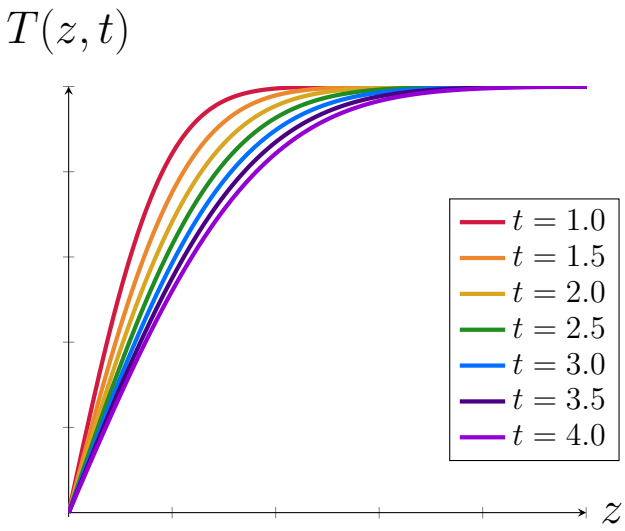

Figure 5.2: An expanded version of Figure 5.1 where more time steps are plotted.

Thus, under Kelvin's assumptions, we have roughly estimated the age of the earth as around 30 million years. Even in the nineteenth century, this result was known to be problematic: this estimate was at odds with geological observations, and appeared to prohibit sufficient time for Darwinian evolution to develop complex life. So, some physical process is clearly not captured by the assumptions listed at the beginning of this section!

5.3 Modelling Heat Sources within the Earth

In formulating the mathematical model (5.1), we ignored the possibility that there is some chemical reaction within the Earth generating heat. In the later part of the nineteenth century, however, an alternative mechanism for heat generation was discovered: *radioactivity*. That is, in light of radioactive decay of rock within the Earth, we should have a source of heat in our problem.

Accordingly, we want to pick a suitable function $f(z,t)$ describing heat creation by radioactivity within the Earth. Then, we solve the *inhomogeneous* problem

$$\begin{cases} T_t - \kappa T_{zz} = f(z,t) \\ \quad T|_{z=0} = 0 \\ \quad T|_{t=0} = T_0 \end{cases}, \tag{5.1}$$

compute the geothermal gradient, and solve for t to determine the age of the Earth.

Before experimenting with some simple choices of source function $f(z,t)$, we describe how to solve (5.1) for a general $f(z,t)$ using our knowledge of the half-line heat kernel $G(z,z',t)$. Our strategy is to use **Duhamel's principle**. This consists of the following recipe:

1. Write $w = T - T^h$, where T^h solves the homogeneous problem (5.1). Then, we know that w solves
$$\begin{cases} w_t - \kappa w_{zz} = f(z,t) \\ \quad w|_{z=0} = 0 \\ \quad w|_{t=0} = 0 \end{cases}.$$

2. Re-define our spatial differential operator by $A = \kappa \partial_z^2$, whence the above becomes
$$\begin{cases} w_t - Aw = f(z,t) \\ \quad w|_{z=0} = 0 \\ \quad w|_{t=0} = 0 \end{cases}. \tag{5.2}$$

3. Using the **method of integrating factors** from ODE theory, we (formally) know that
$$\partial_t \left(e^{-tA} w \right) = -e^{-tA} A w + e^{-tA} w_t = e^{-tA} f.$$

(exercise: if A is an $N \times N$ matrix, prove that $\partial_t e^{tA} = e^{tA}A$, and so "justify" the above formula). Integrating with respect to t tells us that the solution to (5.2) is given by

$$\begin{aligned} w(z,t) &= e^{tA}w|_{t=0} + \int_0^t e^{(t-s)A} f(z,s)\ \mathrm{d}s \\ &= \int_0^t e^{(t-s)A} f(z,s)\ \mathrm{d}s. \end{aligned} \tag{5.3}$$

4. Remembering the definition of w and that T^h is given by (5.5), we know that

$$T(z,t) = T_0 \operatorname{erf}\left(\frac{z}{\sqrt{4\kappa t}}\right) + \int_0^t e^{(t-s)A} f(z,s)\ \mathrm{d}s. \tag{5.4}$$

Therefore, to completely solve (5.1), it remains to sensibly define e^{tA} for $A = \kappa\partial_z^2$. We proceed by analogy with the matrix exponential: if A is an $N \times N$ real matrix and $x(t)$ is an $\mathbb{R}^n$-valued function of time, then the ODE

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} - Ax(t) = 0 \\ x(0) = x_0 \end{cases} \tag{5.5}$$

is solved by

$$x(t) = e^{tA}x_0, \quad e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k.$$

Now, when A is a linear *differential operator* instead of simply a matrix, we can imagine the PDE

$$v_t(z,t) - Av(z,t) = 0$$

as an ODE in "function space": in other words, $t \mapsto v(z,t)$ is a curve in a real vector space of suitably nice functions of z. By analogy, then, it makes sense to define $e^{tA}v_0(z)$ for a suitably nice function $v_0(z)$ as the solution to the initial-value problem

$$\begin{cases} v_t(z,t) - Av(z,t) = 0 \\ v|_{t=0} = v_0 \end{cases}.$$

When we want to impose a homogeneous boundary condition on $e^{tA}v_0(z)$ (as in our geochronology problem), we should instead define the exponential as the solution to the initial-boundary-value problem

$$\begin{cases} v_t(z,t) - Av(z,t) = 0 \\ v|_{z=0} = 0 \\ v|_{t=0} = v_0 \end{cases}.$$

In the case $A = \kappa\partial_z^2$, then, we know by earlier discussion that

$$e^{tA}v_0(z) = v(z,t) = \int_0^{\infty} G(z,z',t)\ v_0(z')\ \mathrm{d}z', \tag{5.6}$$

where $G(z, z', t)$ is defined by (5.3). We conclude that (5.4) can be written more precisely as

$$T(z,t) = T_0 \operatorname{erf}\left(\frac{z}{\sqrt{4\kappa t}}\right) + \int_0^t \int_0^\infty G(z, z', t-s)\ f(z', s)\ \mathrm{d}z'\ \mathrm{d}s. \tag{5.7}$$

Remark. *Our definition of the exponential of a differential operator as the solution to an ODE in a space other than $\mathbb{R}^n$ generalizes nicely to other contexts. For instance, the exponential on a general Lie group is also defined by solving an ODE, with an element of the corresponding Lie algebra serving as initial data.*

Remark. *Viewing a PDE as an ODE on a function space is the starting point for the* ***semigroup approach*** *to solving initial-value problems. In this context, we say that the family of integral operators $\left\{e^{\kappa t \partial_z^2} \mid t \geq 0\right\}$ form the* ***semigroup generated by*** *$\kappa\partial_z^2$. This set formally has the structure of an abelian semigroup (abelian group without inverses) upon noticing that*

$$e^{\kappa t_1 \partial_z^2} e^{\kappa t_2 \partial_z^2} = e^{\kappa t_2 \partial_z^2} e^{\kappa t_1 \partial_z^2} = e^{\kappa(t_1+t_2)\partial_z^2}$$

(exercise: convince yourself this is true, at least in the case without a boundary condition). We cannot upgrade this to a group structure, since the heat equation is not well-posed for $t \in (-\infty, 0]$. You will learn more about semigroups in graduate courses on functional analysis and PDE theory.

Now that we have a representation formula for a solution of (5.1) given by (5.7), we can start playing around with particular choices of source function $f(z,t)$. For simplicity, today we only consider time-independent sources $f = f(z)$. Let's begin with the simplest choice

$$f(z) \equiv B\kappa$$

with B a constant: according to [30], a suitable empirical value for B is $B = (3.25 \times 10^{-10})^\circ\,\mathrm{C/cm^2}$. Such a choice of $f(z)$ corresponds to a uniform distribution of radioactive rock through the whole interior of the Earth. Using our earlier computations of $\int_0^\infty G(z, z', t)\ \mathrm{d}z'$, we find

$$T(z,t) = T_0 \operatorname{erf}\left(\frac{z}{\sqrt{4\kappa t}}\right) + B\kappa \int_0^t \operatorname{erf}\left(\frac{z}{\sqrt{4\kappa(t-s)}}\right)\ \mathrm{d}s.$$

Then, we form the equation for the geothermal gradient:

$$\begin{aligned}
\gamma &= T_Z|_{z=0} \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} + \frac{B\kappa}{\sqrt{\pi\kappa}} \int_0^t \frac{\mathrm{d}s}{\sqrt{t-s}} \\
&= \frac{1}{\sqrt{\pi\kappa}} \left(T_0 t^{-1/2} + B\kappa t^{1/2} \int_0^1 \tau^{-1/2}\ \mathrm{d}\tau \right) \\
&= \frac{1}{\sqrt{\pi\kappa}} \left(T_0 t^{-1/2} + 2B\kappa t^{1/2} \right).
\end{aligned}$$

Then, t must satisfy the quadratic equation

$$0 = 4T_0\kappa^2 B^2 t^2 + \left(4B\kappa T_0 - a^2\right) t + 1.$$

However, this equation clearly has no real roots. Accordingly, $f \equiv$ constant cannot provide a realistic model of radioactive heat sources inside the Earth. We have arrived at the scientific conclusion that *the concentration of radioactive rock within our planet varies with distance from the core*. For any alternative discussion of why $f \equiv$ constant is not a good model, see [30].

Since we know the source function f cannot be constant, let us take the simplest non-constant $f(z)$ possible:

$$f(z) = \begin{cases} B\kappa & 0 \le z \le H \\ 0 & z > H. \end{cases}$$

This corresponds to all radioactive material being concentrated in a layer of thickness H below the surface of the Earth. Again, we can take $B = (3.25 \times 10^{-10})^\circ\,\mathrm{C/cm^2}$. For the layer thickness H, we assume H is on the order of the mean thickness of the Earth's crust, which is roughly 10 km (about 1/640 of the Earth's radius). Using (5.7), we can write the temperature distribution corresponding to this f as

$$T(z,t) = T_0 \operatorname{erf}\left(\frac{z}{\sqrt{4\kappa t}}\right) + B\kappa \int_0^t \int_0^H G(z, z', t-s)\, \mathrm{d}z'\, \mathrm{d}s. \tag{5.8}$$

The equation for the geothermal gradient then reads

$$\begin{aligned}
\gamma &= \frac{T_0}{\sqrt{\pi\kappa t}} + B\kappa \int_0^t \int_0^H \partial_z G(z, z', t)|_{z=0}\, \mathrm{d}z'\, \mathrm{d}s \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} + \frac{B\kappa}{2\sqrt{\pi}} \int_0^t \int_0^H \left(\kappa(t-s)\right)^{-3/2} z' e^{-(z')^2/4\kappa(t-s)}\, \mathrm{d}z'\, \mathrm{d}s \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} - \frac{B\kappa}{2\sqrt{\pi}} \int_0^t \int_0^H \left(\kappa(t-s)\right)^{-3/2} \left[2\kappa(t-s)\right] \frac{\partial}{\partial z'} \left[e^{-(z')^2/4\kappa(t-s)}\right] \mathrm{d}z'\, \mathrm{d}s \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} - \frac{B\kappa}{\sqrt{\pi}} \int_0^t \left(\kappa(t-s)\right)^{-1/2} \left[e^{-(z')^2/4\kappa t}\right]_{z=0}^{H} \mathrm{d}s \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} + \frac{B\kappa}{\sqrt{\pi}} \int_0^t \left(\kappa(t-s)\right)^{-1/2} \left[1 - e^{-H^2/4\kappa(t-s)}\right] \mathrm{d}s \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} - \frac{B\kappa}{\sqrt{\pi}} \int_t^0 (\kappa\tau)^{-1/2} \left[1 - e^{-H^2/4\kappa\tau}\right] \mathrm{d}\tau \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} + \frac{B\kappa}{\sqrt{\pi}} \int_0^t (\kappa\tau)^{-1/2} \left[1 - e^{-H^2/4\kappa\tau}\right] \mathrm{d}\tau \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} + \frac{2B\sqrt{\kappa}}{\sqrt{\pi}} t^{1/2} - \frac{B\sqrt{\kappa}}{\sqrt{\pi}} \int_0^t \tau^{-1/2}\, e^{-H^2/4\kappa\tau}\, \mathrm{d}\tau.
\end{aligned}$$

If we change variables according to

$$\sigma^2 = \frac{H^2}{4\kappa\tau}$$

then

$$2\sigma \, \mathrm{d}\sigma = -\frac{H^2}{4\kappa\tau^2}\mathrm{d}s = \frac{\sigma^2}{\tau} \, \mathrm{d}\tau$$

hence

$$\mathrm{d}\tau = \frac{2\tau}{\sigma} \, \mathrm{d}\sigma.$$

We then find

$$\gamma = \frac{T_0}{\sqrt{\pi\kappa t}} + \frac{2B\sqrt{\kappa}}{\sqrt{\pi}} t^{1/2} - \frac{BH}{\sqrt{\pi}} \int_{\frac{H}{2\sqrt{\kappa t}}}^{\infty} \frac{e^{-\sigma^2}}{\sigma^2} \, \mathrm{d}\sigma.$$

An integration by parts then shows that

$$\int_{\zeta}^{\infty} \frac{e^{-\sigma^2}}{\sigma^2} \, \mathrm{d}\sigma = \frac{e^{-\zeta^2}}{\zeta^2} - 2\int_{\zeta}^{\infty} e^{-\sigma^2} \, \mathrm{d}\sigma = \frac{e^{-\zeta^2}}{\zeta} + \sqrt{\pi}\left(\mathrm{erf}\left(\zeta\right) - 1\right)$$

(exercise: prove this! See [30] if you get stuck). We conclude that

$$\begin{aligned}
\gamma &= \frac{T_0}{\sqrt{\pi\kappa t}} + \frac{2B\sqrt{\kappa}}{\sqrt{\pi}} t^{1/2} - \frac{BH}{\sqrt{\pi}} \left[\frac{e^{-\zeta^2}}{\zeta} + \sqrt{\pi}\left(\mathrm{erf}\left(\zeta\right) - 1\right) \right]_{\zeta = \frac{H}{2\sqrt{\kappa t}}} \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} + \frac{2B\sqrt{\kappa}}{\sqrt{\pi}} t^{1/2} - \frac{BH}{\sqrt{\pi}} \left[\frac{e^{-H^2/4\kappa t}}{\frac{H}{2\sqrt{\kappa t}}} + \sqrt{\pi}\left(\mathrm{erf}\left(\frac{H}{2\sqrt{\kappa t}}\right) - 1\right) \right] \\
&= \frac{T_0}{\sqrt{\pi\kappa t}} + \frac{2B\sqrt{\kappa}}{\sqrt{\pi}} t^{1/2} \left(1 - e^{-H^2/4\kappa t}\right) - BH\left(\mathrm{erf}\left(\frac{H}{2\sqrt{\kappa t}}\right) - 1\right).
\end{aligned}$$

For large t (ie. $t = 10^9$ years, the true age of the Earth) and $H \approx 10$km, the above is approximately equal to

$$BH \approx \left(3 \times 10^{-4}\right)^{\circ} \mathrm{C/cm},$$

which is the actual geothermal gradient γ (see [30] for full details of the argument). Accordingly, assuming all radioactive rock is concentrated in a small layer near the Earth's surface gives a very good model for the actual age of our planet.

Chapter 6

Chebyshev Series

6.1 Introduction

In lectures and the textbook [27, §5.5], you have learned about the **Gibbs phenomenon**, which precludes the Fourier series of a discontinuous periodic function defined on $\mathbb{R}$ from converging nicely at points of discontinuity. This leads us to ask if there is an *alternative series expansion* for functions on a compact interval that do not admit a continuous periodic extension. In this tutorial, we shall see that there is such an alternative expansion, called the **Chebyshev series**. While the Fourier series of a smooth function on $[-1, 1]$ extended to $\mathbb{R}$ may fail to converge to $f(x)$ at the endpoints, the Chebyshev series of such a function converges uniformly to $f(x)$ on $[-1, 1]$!

The Chebyshev series of a nice function $f\colon [-1, 1] \to \mathbb{R}$ takes the form

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

where the $T_n(x)$ are special degree n polynomials called **Chebyshev polynomials**. That is, instead of expanding in sines or cosines (eigenfunctions of the linear operator $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with appropriate BCs), we expand in well-chosen polynomials. This is arguably a lot simpler than a Fourier expansion.

Chebyshev polynomials have myriad applications throughout applied mathematics, and in a single hour I can only talk about a few interesting properties of these beautiful functions. To help you discover the wide and wonderful world of Chebyshev polynomials for yourself, I'll give a brief survey of some good introductory literature on this subject:

- [18, 32] give excellent general accounts of Chebyshev polynomials and their use in approximation theory;
- the books [5, 13, 31] all describe how to use Chebyshev polynomials to design startlingly accurate numerical methods for PDEs;

- [33] (a masterpiece every applied math student should read!) describes how Chebyshev polynomials appear in numerical linear algebra.

6.2 Chebyshev Polynomials

We begin by introducing the Chebyshev polynomials and investigating some of their basic properties.

Definition 6.2.1. *The* ***Chebyshev polynomial of degree*** n *is the unique degree* n *polynomial function*

$$T_n \colon [-1, 1] \to \mathbb{R}$$

for which the nonlinear equation below holds:

$$T_n(\cos\theta) = \cos(n\theta) \quad \forall\ \theta \in [0, \pi]. \tag{6.1}$$

By inspection, we quickly verify that

$$T_0(x) = 1 \quad \text{and} \tag{6.2a}$$
$$T_1(x) = x, \tag{6.2b}$$

Before going further, we should make sure that all the other $T_n(x)$'s are actually polynomials. We start by proving this easy lemma:

Lemma 6.2.2. *$T_n(x)$ satisfies the recurrence relation*

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Proof. Using the trig identities

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

we find that

$$\begin{aligned} T_{n+1}(x) &= \cos((n+1)\theta) \\ &= \cos n\theta\cos\theta - \sin n\theta\sin\theta \\ &= \cos n\theta\cos\theta - [\cos((n-1)\theta) - \cos n\theta\cos\theta] \\ &= 2\cos n\theta\cos\theta - \cos((n-1)\theta) \\ &= 2\cos\theta T_n(x) - T_{n-1}(x). \end{aligned}$$

Using (6.2b), we're done. □

By the above lemma, the Chebyshev polynomials are manifestly unique. Also, with the lemma in hand, a trivial proof by induction shows

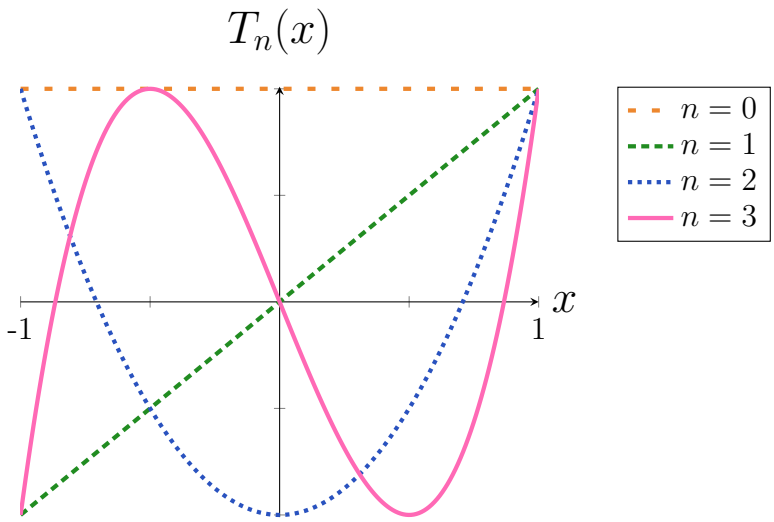

Figure 6.1: Plot of first few Chebyshev polynomials.

Corollary 6.2.3. *$T_n(x)$ is actually a polynomial of degree n.*

□

We have found that the constraint (6.1) indeed defines a degree n polynomial. See figure 6.1 for a sketch of the first four Chebyshev polynomials. For an alternative but equivalent definition of the Chebyshev polynomials, see exercise 6.2.1. Note also that the recurrence relation from lemma 6.2.2 implies that $T_n(x)$ can be numerically computed very easily, even for large n, using a simple `for` loop.

Remark. *One may wonder why we denote the Chebyshev polynomials by T_n instead of C_n. There are two reasons. First, c and C are customarily reserved to denote constants, so confusion may arise from having a family of functions called C_n hanging around. Second, in older French and German mathematics literature, Chebyshev's name was transliterated from its proper Cyrillic spelling in such a way that it started with a "T".*

Exercise 6.2.1. *Let $p_n : \mathbb{C} \mapsto \mathbb{C}$ be defined by*

$$p_n(z) = z^n.$$

Show that, for all z on the unit circle $U(1)$,

$$\mathrm{Re}\left(p_n(z)\right) = T_n\left(\mathrm{Re}(z)\right).$$

Thus the Chebyshev polynomial T_n is essentially just the real part of the complex monomial z^n restricted to $U(1)$.

Exercise 6.2.2. *Show that $|T_n(x)| \leq 1$ for all $x \in [-1, 1]$.*

6.3 Approximating Non-Periodic Functions via Chebyshev Polynomials

Next, I describe one way that Chebyshev polynomials can be used to approximate functions on $[-1, 1]$. Our discussion is loosely based on [5, §2.11], with many of the proofs and background results coming from [27, §5.4-5.5].

We begin by restricting the class of functions we seek to approximate: throughout the remainder of this document, $f\colon [-1, 1] \to \mathbb{R}$ denotes a C^1 function. That is, $f(x)$ is continuous and differentiable everywhere, and its derivative is also continuous everywhere. I reiterate that we do not need to assume that $f(x)$ is the restriction of a continuous periodic function defined on $\mathbb{R}$.

Let's formally write $f(x)$ as an infinite linear combination of Chebyshev polynomials:

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x), \quad x \in [-1, 1].$$

We call this expression the **Chebyshev series for** $f(x)$. How would we prove such a sum actually converges to $f(x)$, and how would we find the correct expansion coefficients a_n? The trick is to recall the definition (6.1) relating T_n to the cosine function on $[0, \pi]$. In particular, $\cos\theta$ is invertible on this interval. So, by changing variables according to

$$x = \cos\theta$$

in the formal Chebyshev series of $f(x)$ written above, we find

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n T_n(\cos\theta), \quad \theta \in [0, \pi].$$

Using (6.1), the above becomes

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta), \quad \theta \in [0, \pi].$$

In other words, the Chebyshev series of $f(x)$ is precisely the Fourier cosine series of $f(\cos\theta)$. This is awesome news: $f(\cos\theta)$ is periodic *regardless of whether or not* $f(x)$ *is periodic*, so its cosine series should have excellent convergence properties. Additionally, we can compute a_n using the familiar trig orthogonality rules:

$$a_n = \begin{cases} \frac{1}{\pi} \int_0^{\pi} f(\cos\theta) \ \mathrm{d}\theta & n = 0, \\ \frac{2}{\pi} \int_0^{\pi} f(\cos\theta) \ \cos(n\theta) \ \mathrm{d}\theta & n > 0. \end{cases} \tag{6.1}$$

Thus we have sketched a proof that the Chebyshev series of $f(x)$ converges uniformly everywhere on $[-1, 1]$. To make this proof rigorous, we only need to show

- the Fourier cosine series of $f(\cos\theta)$ converges pointwise to $f(\cos\theta)$ on $[0,\pi]$ (so the Chebyshev series of f actually makes sense), and that
- the Fourier cosine series of $f(\cos\theta)$ converges uniformly on $[0,\pi]$.

To simplify notation, let's define

$$F(\theta) \doteq f(\cos\theta). \tag{6.2}$$

Notice that

- $F(\theta)$ is continuously differentiable on $[0,\pi]$;
- $F(\theta)$ extends to an even, continuously differentiable, 2π-periodic function on $\mathbb{R}$ (since $\cos\theta$ does);
- $F'(\theta)$ is also 2π-periodic.

$F(\theta)$ is therefore the nicest kind of function we could possibly ask for, from the perspective of Fourier convergence theory. In particular, we can take advantage of many of the results you've already seen in lectures and the textbook. To start, let's explain why an $F(\theta)$ satisfying the above three properties is pointwise-equal to its cosine series.

Theorem 6.3.1. *Suppose $F(\theta)$ is a 2π-periodic even function that is continuously differentiable everywhere on $\mathbb{R}$, with $F'(\theta)$ also having period 2π. Then, $F(\theta)$ is equal to its Fourier cosine series pointwise, everywhere on $\mathbb{R}$.*

Proof. This is am immediate consequence of [27, Theorem 5.4.4∞], since for an even function the full Fourier series reduces to the Fourier cosine series. □

To establish uniform convergence of the cosine series of $f(\cos\theta)$, it suffices to to prove the following theorem (which is part of [27, Exercise 5.5.10, p. 146]) and apply it to $F(\theta) = f(\cos\theta)$:

Theorem 6.3.2. *Suppose $F(\theta)$ is a 2π-periodic even function that is continuously differentiable everywhere, with $F'(\theta)$ also having period 2π. Then, the Fourier cosine series of $F(\theta)$ converges uniformly on $[0,\pi]$.*

We need a helpful inequality before we prove the above theorem in earnest:

Proposition 6.3.3 (Cauchy-Schwarz Inequality for Series)**.** *For any real sequences a_n, b_n, we have*

$$\sum_{n=0}^{\infty} a_n b_n \leq \left(\sum_{n=0}^{\infty} a_n^2\right)^{1/2} \left(\sum_{n=0}^{\infty} b_n^2\right)^{1/2} \tag{6.3}$$

provided the sums on the right-hand side converge.

Proof. We follow the hint in [27, Exercise 5.5.5]. We first consider the case where a_n, b_n are both zero for $n \geq N$, so all sums in the claim are finite. We then want to prove

$$\left(\sum_{n=0}^{N} a_n b_n\right)^2 \leq \left(\sum_{n=0}^{N} a_n^2\right)\left(\sum_{n=0}^{N} b_n^2\right). \tag{6.4}$$

Let's simplify things by introducing the compact notation

$$\begin{aligned} a &= \sum_{n=0}^{N} a_n^2, \\ b &= \sum_{n=0}^{N} a_n b_n, \\ c &= \sum_{n=0}^{N} b_n^2. \end{aligned}$$

Then, define a smooth function $\varphi(t)$ by

$$\varphi(t) = \sum_{n=0}^{N} (a_n + t b_n)^2 = at^2 + 2bt + c \geq 0.$$

A quick calculus exercise shows that the minimum of φ occurs at

$$t_0 = -\frac{b}{a}.$$

Therefore,

$$0 \leq \varphi(t_0) = c - \frac{b^2}{a}.$$

Rearranging the above and using the definitions of a, b, c, we recover (6.4).

To cover the case of infinite sums, we carefully take the limit $N \to \infty$. If

$$\sum_{n=0}^{\infty} a_n^2, \quad \sum_{n=0}^{\infty} b_n^2 < \infty$$

then for any $N \geq 0$ we have by (6.4) that

$$\left(\sum_{n=0}^{N} a_n b_n\right)^2 \leq \left(\sum_{n=0}^{N} a_n^2\right)\left(\sum_{n=0}^{N} b_n^2\right) \leq \left(\sum_{n=0}^{\infty} a_n^2\right)\left(\sum_{n=0}^{\infty} b_n^2\right).$$

That is, we have

$$\left(\sum_{n=0}^{N} a_n b_n\right)^2 \leq \left(\sum_{n=0}^{\infty} a_n^2\right)\left(\sum_{n=0}^{\infty} b_n^2\right) < \infty.$$

Taking square roots across this inequality and then passing to the limit $N \to \infty$, the proof is complete. □

Proof of Theorem 6.3.2. The proof follows along the exact same lines as [27, pp. 140-142]. First, since $F(\theta)$ is even, we know $F'(\theta)$ is *odd*. Since $F'(\theta)$ is also continuous, it is pointwise-equal to its Fourier sine series on $(0, \pi)$ (using [27, Theorem 5.4.4∞]):

$$F'(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta \quad \forall\ \theta \in (0, \pi),$$

where b_n are determined by the usual orthogonality relations as

$$b_n = \frac{2}{\pi} \int_0^{\pi} F'(\theta)\ \sin n\theta\ \mathrm{d}\theta.$$

Now, let a_n be the Fourier cosine coefficients of $F(\theta)$ for $n \geq 1$:

$$a_n = \frac{2}{\pi} \int_0^{\pi} F(\theta)\ \cos n\theta\ \mathrm{d}\theta.$$

Integrating by parts in the above gives

$$a_n = \frac{2}{\pi} \left[\frac{1}{n} F(\theta)\ \sin n\theta \right]_{\theta=0}^{\pi} - \frac{1}{n} b_n = -\frac{1}{n} b_n \quad \forall\ n \geq 1. \tag{6.5}$$

Using (6.5), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n \cos n\theta| &\leq \sum_{n=0}^{\infty} |a_n| \\ &\leq |a_0| + \sum_{n=1}^{\infty} \frac{1}{n}\ |b_n|. \end{aligned}$$

Using the Cauchy-Schwarz inequality for series (proposition 6.3.3) in the above, we find

$$\sum_{n=0}^{\infty} |a_n \cos n\theta| \leq |a_0| + \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{1/2}.$$

Both of the infinite series on the right-hand side converge (for the series involving b_n^2, this follows from Bessel's inequality [27, p. 132] since $\{\sin n\theta\}_{n \geq 1}$ is an orthogonal set of functions on $[0, \pi]$), so we conclude that the cosine series of $F(\theta)$ converges absolutely. Thus for every $\epsilon > 0$ there exists $N \geq 1$ so that

$$\left| \sum_{n=0}^{\infty} |a_n| - \sum_{n=0}^{N} |a_n| \right| = \sum_{n=N+1}^{\infty} |a_n| < \epsilon. \tag{6.6}$$

Now, choose an $\epsilon > 0$ and pick N sufficiently large so that (6.6) holds. Define

$$S_N(\theta) \doteq \sum_{n=0}^{N} a_n \cos n\theta.$$

Using (6.6) we find that, for any $\theta \in [0, \pi]$,

$$|F(\theta) - S_N(\theta)| = \left| \sum_{n=N+1}^{\infty} a_n \cos n\theta \right| \leq \sum_{n=N+1}^{\infty} |a_n| < \epsilon.$$

Note how we have tacitly used theorem 6.3.1 in order to say that $F(\theta)$ is pointwise-equal to its cosine series. Since $\epsilon > 0$ and θ were chosen arbitrarily, we conclude that

$$\lim_{N \to \infty} \sup_{\theta \in [0,\pi]} |F(\theta) - S_N(\theta)| = 0,$$

so $S_N \to F$ uniformly and we're all done. □

Remark. *In* [27, pp. 141-142]*, the proof of the uniform convergence result for a full Fourier series has a small error: in the comment below equation (15), the author says that the periodicity of f has been used to get rid of the boundary terms appearing in the integration by parts. As we saw here in our discussion of the cosine series, however, periodicity does not factor into this part of the argument at all! Rather, when dealing with the* sine terms *in the full Fourier series, one needs to use periodicity to kill the boundary contributions. However, we still needed to assume a periodic function here in order to guarantee pointwise convergence.*

We can now conclude the following beautiful theorem on convergence of Chebyshev series:

Theorem 6.3.4. *Suppose $f : [-1, 1] \to \mathbb{R}$ is continuously differentiable. Then,*

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x) \quad \forall\ x \in [-1, 1]$$

with a_n given by (6.1)*. Further, the series converges uniformly in* $[-1, 1]$*.*

Proof. Apply theorems 6.3.1, 6.3.2 to

$$F(\theta) = f(\cos\theta).$$

□

So, by expanding a sufficiently nice function in terms of Chebyshev polynomials instead of trig functions, we circumvent the Gibbs phenomenon! Thus truncating a Chebyshev series guarantees a reliable way to approximate a function by a linear combination of simple functions. That being said, in practice one typically uses Chebyshev polynomials to approximate functions through **interpolation** rather than truncating the Chebyshev series. This is because interpolation is computationally cheaper than numerically approximating the integrals a_n. In fact, Chebyshev-based interpolation is a critical ingredient in many numerical methods for PDEs (see the references mentioned in the introduction).

Exercise 6.3.1. *Under what conditions on a_n, b_n does* (6.3) *hold with equality?*

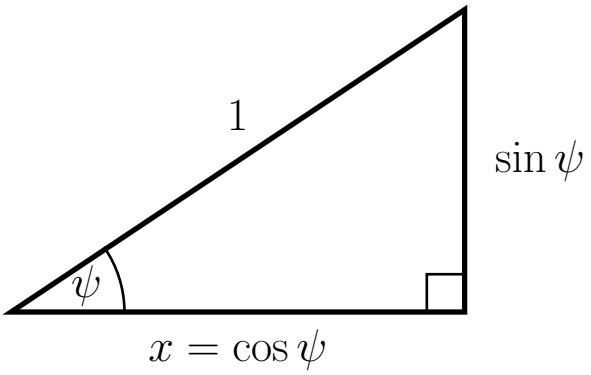

Figure 6.2: A right triangle that illustrates $\sin(\cos^{-1} x) = \sqrt{1-x^2}$ when $x \in (0,1]$ and cosine is restricted to $[0,\pi]$.

6.4 Spatial Representation of the Expansion Coefficients & Chebyshev Orthogonality

We have seen that the coefficients of a Chebyshev series can be computed by (6.1), but in a sense this formula is a bit unsatisfying since it involves integration with respect to θ rather than x. Let's therefore change variables (6.1) according to

$$x = \cos\theta.$$

In the $n = 0$ case, this gives

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi} f(\cos\theta)\ \mathrm{d}\theta \\ &= -\int_1^{-1} f(x)\ \frac{\mathrm{d}x}{\pi\sin(\cos^{-1} x)} \\ &= \int_{-1}^{1} f(x)\ \frac{\mathrm{d}x}{\pi\sin(\cos^{-1} x)}. \end{aligned} \tag{6.1}$$

The denominator

$$W(x) \doteq \pi\sin\left(\cos^{-1} x\right)$$

doesn't look very friendly, so let's try to simplify it. First, recall that $\cos^{-1} x \in [0,\pi]$ by convention, so $W(x) \geq 0$. In other words, $\psi = \cos^{-1} x$ means $W(x) = \sin\psi \in [0,1]$. Figure 6.2 relates all this information in the case $x > 0$. From this figure and the Pythagorean theorem, we discover

$$W(x) = \pi\sqrt{1-x^2}. \tag{6.2}$$

A similar argument shows that (6.2) holds for $x < 0$ (the case $x = 0$ is obvious), see exercise 6.4.1. If we define the **weight function** by

$$w(x) = \frac{1}{W(x)} = \frac{1}{\pi\sqrt{1-x^2}} \tag{6.3}$$

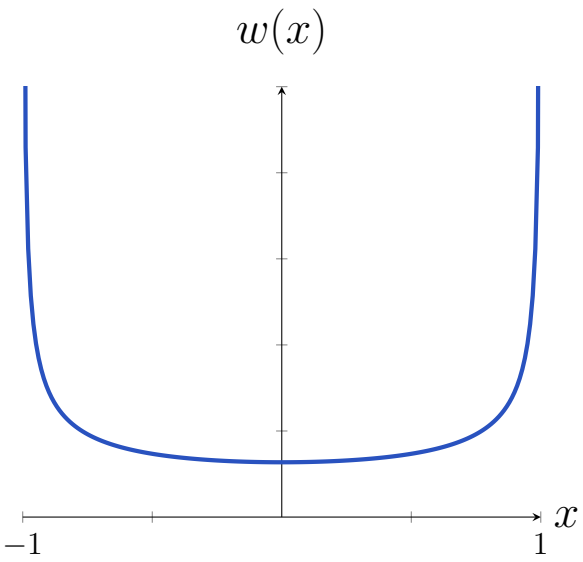

Figure 6.3: Plot of the weight function $w(x)$. $w(x)$ grows without bound near $x = \pm 1$.

(see figure 6.3 for a plot of $w(x)$), we can write (6.1) as

$$a_0 = \int_{-1}^{1} f(x)\ w(x)\ \mathrm{d}x. \tag{6.4}$$

We have therefore reduced computation of a_0 to performing an integral over the original domain $[-1, 1]$. Before performing the analogous manipulations for a_n, $n > 0$, we introduce some new notation.

Definition 6.4.1. *Let $\phi\colon [-1, 1] \to \mathbb{R}$ be continuous. We define* ***the integral of $\phi(x)$ with respect to the Chebyshev measure*** $\mathrm{d}\mu(x)$ *by*

$$\int_{-1}^{1} \phi(x)\ \mathrm{d}\mu(x) \doteq \int_{-1}^{1} \phi(x)\ w(x)\ \mathrm{d}x.$$

To use sloppy notation, the Chebyshev measure can be written as

$$\mathrm{d}\mu(x) = w(x)\ \mathrm{d}x.$$

The integral of $\phi(x)$ wrt $\mathrm{d}\mu(x)$ can be thought of as the usual integral of $\phi(x)$, but with certain x-values contributing more to the integral than others. In particular, points near the boundary are "weighed" much more heavily than points near $x = 0$ owing to the blowup of $w(x)$ at $x = \pm 1$. Note also that

$$\int_{-1}^{1} \mathrm{d}\mu(x) \doteq \int_{-1}^{1} w(x)\ \mathrm{d}x = 1,$$

so $w(x)$ can be thought of as a **probability density function**, or $\mathrm{d}\mu(x)$ can be thought of as a **probability measure**.

Remark. *$w(x)$ is a special case of the β-distribution appearing in statistics.*

Using the Chebyshev measure, we can write (6.4) as

$$a_0 = \int_{-1}^{1} f(x)\ \mathrm{d}\mu(x). \tag{6.5}$$

Now, we turn to the higher coefficients. Using (6.1), we get for $n > 1$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\ \cos(n\theta)\ \mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\ T_n(\theta)\ \mathrm{d}\theta \\ &= -2\int_1^{-1} f(x)\ T_n(x)\ w(x)\ \mathrm{d}x \\ &= 2\int_{-1}^{1} f(x)\ T_n(x)\ \mathrm{d}\mu(x). \end{aligned} \tag{6.6}$$

Putting (6.6) and (6.6) together, we have completely reduced the computation of the Chebyshev coefficients to integrals over $[-1, 1]$:

$$a_n = \begin{cases} \int_{-1}^{1} f(x)\ \mathrm{d}\mu(x) & n = 0 \\ \\ 2\int_{-1}^{1} f(x)\ T_n(x)\ \mathrm{d}\mu(x) & n > 0. \end{cases} \tag{6.7}$$

Now, (6.7) has several cosmetic similarities to (6.1): both forms gives expressions for a_n as L^2 inner products of $f(x)$ with some family of functions, either $\cos n\theta$ or $T_n(x)$, although the measure used to define the product is different ($\mathrm{d}x$ for Fourier cosine and $\mathrm{d}\mu(x)$ for Chebyshev). In particular, (6.7) merely says that the a_n can be computed by performing an L^2 projection of $f(x)$ onto $T_n(x)$ with respect to $\mathrm{d}\mu(x)$.

The above discussion naturally leads us to ask if the Chebyshev polynomials satisfy an orthogonality condition, like the Fourier basis functions, that would could have also used to derive (6.7) by taking L^2 inner products across the expression

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x).$$

Indeed, we have

Proposition 6.4.2. *The Chebyshev polynomials sastisfy the orthogonality relation*

$$\int_{-1}^{1} T_n(x)\ T_m(x)\ \mathrm{d}\mu(x) = \begin{cases} 0 & m \neq n \\ 1 & m = n = 0 \\ \frac{1}{2} & m = n > 0. \end{cases} \tag{6.8}$$

Proof. This follows immediately from using (6.1) in conjunction with

$$\frac{1}{\pi}\int_0^\pi \cos n\theta \ \cos m\theta \ \mathrm{d}x = \begin{cases} 0 & m \neq n \\ 1 & m = n = 0 \\ \frac{1}{2} & m = n > 0. \end{cases}$$

□

We have discovered that the Chebyshev polynomials actually constitute an orthogonal system of functions on $[-1, 1]$, provided we use the Cheybshev measure to compute our inner products. In particular, this allows us to establish a nice L^2 convergence theory for Chebyshev series by simply adapting the abstract arguments outlined in [27, §5.4] (see exercise 6.4.2 to get a start on building this theory).

There are many other families of orthogonal polynomials (bearing the names of Legendre, Hermite, and other continental luminaries) having profound importance in mathematical physics. For an excellent introduction to the general theory of orthogonal polynomials, see [15, Ch. 8].

Exercise 6.4.1. *Draw a picture to show that* (6.2) *holds for* $x < 0$.

Exercise 6.4.2. *Let* $\mathcal{V}$ *denote the real vector space space of continuous real-valued functions on* $[-1, 1]$. *Define a subspace of* $\mathcal{V}$ *by*

$$\mathcal{V}_{Cheb} \doteq \left\{ \phi(x) \in \mathcal{V} \ \middle| \ \int_{-1}^{1} |\phi(x)|^2 \ \mathrm{d}\mu(x) < \infty \right\},$$

where $\mathrm{d}\mu(x)$ *denotes the Chebyshev measure.*

a) Show that $\mathcal{V}_{Cheb}$ *is actually a subspace.*

b) Does $\mathcal{V} = \mathcal{V}_{Cheb}$*?*

c) Put a norm on $\mathcal{V}_{Cheb}$ *by defining*

$$\|\phi\|_{L^2_\mu} \doteq \left(\int_{-1}^{1} |\phi(x)|^2 \ \mathrm{d}\mu(x) \right)^{1/2}.$$

You may assume without proof that this is actually a norm. Prove that, for any $N > 0$, *the function*

$$(b_1, b_2, ..., b_N) \mapsto \left\| \phi(x) - \sum_{n=0}^{N} b_n T_n(x) \right\|_{L^2_\mu}$$

is minimized when $b_n =$ *the* n^{th} *Chebyshev coefficient of* $\phi(x)$. *Hint: try to do as little work as possible!*

Chapter 7

The Neumann Problem for Poisson's Equation

7.1 Introduction

Throughout this document, let $\Omega \subseteq \mathbb{R}^n$ be a connected, bounded, open set with a smooth, connected boundary $\partial\Omega$. The closure of Ω is denoted by $\overline{\Omega}$. Denote the unit outward normal on $\partial\Omega$ by $\hat{\mathbf{n}}$. Suppose $f\colon \Omega \to \mathbb{R}$ and $g\colon \partial\Omega \to \mathbb{R}$ are two given continuous functions. In this tutorial we study the BVP

$$\Delta u(x) = -f(x) \quad \forall\ x \in \Omega, \tag{7.1a}$$

$$\left.\frac{\partial u}{\partial \hat{\mathbf{n}}}\right|_{\partial\Omega} = \quad g(x) \quad \forall\ x \in \partial\Omega. \tag{7.1b}$$

The question of when we can solve this problem uniquely is surprisingly complex, and a complete understanding of it requires some knowledge of the physics underlying (11.3). If you want more of a concrete connection to the textbook, my objective today is to make sure you understand the trivial exercise [27, problem 6.1.11] very, very deeply.

The methods we'll develop to understand when (11.3) can be solved (and when it can be solved uniquely) are critical tools for the modern applied mathematician. To emphasize this point, I have included a lengthy exercise at the end of the tutorial explaining how such methods can be used to understand the basic structure of a medical imaging problem.

7.2 Background: the Diffusion Equation as a Conservation Law

The smell of cooking reaches your nostrils from far across the house. Your hands slowly become warm as you hold them out in front of a lit fire. A population of animals gradually

spreads out to fill its habitat. These are all examples of **diffusion**, the tendency of substances to flow from regions of high concentration to regions of low concentration. In this section, we'll talk a bit about the relationship between this qualitative picture of diffusion and the actual diffusion equation you've seen so far in lectures and the textbook. This will lead nicely into a well-rounded discussion of (11.3).

Consider the inhomogeneous diffusion equation posed on $\Omega \subseteq \mathbb{R}^n$: for some constant κ called the **diffusivity** with appropriate units to be discussed below,

$$U_t - \kappa \Delta U = \mathcal{F}. \tag{7.1}$$

We'll introduce particular boundary conditions on the unknown $U(x,t)$ a bit later. Physically, (7.1) models the diffusion of a pollutant through some medium. For concreteness, you can take the pollutant to be coloured dye and the medium to be water. $U(x,t)$ is then interpreted as the mass density of pollutant, so its units should be

$$[U] = \frac{\text{mass}}{\text{volume}} = \text{kg m}^{-n}.$$

In particular, we find

$$\text{total mass of pollutant in } \Omega \text{ at time } t \doteq M(t) = \int_\Omega U(x,t)\ \mathrm{d}x.$$

For consistency across (7.1), then, we must have that the units of f and κ are given by

$$[\mathcal{F}] = \text{kg m}^{-n}\ \text{s}^{-1}, \tag{7.2a}$$
$$[\kappa] = \text{m}^2\text{s}^{-1}, \tag{7.2b}$$

see exercise 7.2.1.

Now, what qualitative physical information can we extract from (7.1)? To find out, start by integrating the PDE over Ω and using the definition of $M(t)$:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_\Omega \mathcal{F}(x)\ \mathrm{d}x + \kappa \int_\Omega \Delta U(x,t)\ \mathrm{d}x. \tag{7.3}$$

Using the divergence theorem, we can express the second term on the right-hand side as

$$\begin{aligned}\int_\Omega \Delta U(x,t)\ \mathrm{d}x &= \int_\Omega \nabla \cdot (\nabla U(x,t))\ \mathrm{d}x \\ &= \oint_{\partial\Omega} \nabla U(x,t) \cdot \hat{\mathbf{n}}\ \mathrm{d}S(x) \\ &= \oint_{\partial\Omega} \frac{\partial U}{\partial \hat{\mathbf{n}}}\ \mathrm{d}S(x),\end{aligned} \tag{7.4}$$

where $\mathrm{d}S(x)$ denotes the surface measure on $\partial\Omega$. Next, let's define the **mass flux** by

$$\mathbf{F}(x,t) \doteq -\kappa \nabla U(x,t). \tag{7.5}$$

With this definition at home, we can combine (7.4) and (7.3) to discover

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_\Omega \mathcal{F}(x)\ \mathrm{d}x - \int_\Omega \mathbf{F}(x,t)\cdot\hat{\mathbf{n}}\ \mathrm{d}S(x). \tag{7.6}$$

The above formula is called the **equation of mass conservation**, and it summarizes the most important physical information contained in (7.1). It tells us that the mass of pollutant contained in Ω can change only due to

- **the creation of mass within** Ω: this is reflected by the "source term" $\int_\Omega \mathcal{F}(x)\ \mathrm{d}x$;
- **the passage of mass through** $\partial\Omega$**:** this is reflected by the "boundary flux term" $\oint \mathbf{F}(x,t)\cdot\hat{\mathbf{n}}\ \mathrm{d}S(x)$.

From a physical point of view, this is obvious! There is no way mass can magically show up inside Ω unless we either put it there through $\mathcal{F}$, or it was carried in through the boundary (you should stop and do exercise 7.2.2 now). In plain language, (7.6) reads

change in amount of mass = mass created/destroyed + mass brought in/taken out through boundary.

Before going further, let's discuss the flux a bit more. We take $\mathcal{F} = 0$ in this paragraph for simplicity. Note that if $\nabla U(x,t)\cdot\hat{\mathbf{n}} \geq 0$ over all of $\partial\Omega$ then by (7.6) we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\int_\Omega \mathbf{F}(x,t)\cdot\hat{\mathbf{n}}\ \mathrm{d}S(x) \geq 0.$$

So, if U is increasing across $\partial\Omega$ at time t then there will be a gain of mass in Ω near time t. This agrees with our macroscopic understanding of diffusion as the passage of a pollutant from regions of high concentration to regions of low concentration. Similarly, if U is decreasing across $\partial\Omega$, then $\frac{\mathrm{d}M}{\mathrm{d}t} \leq 0$ since this time Ω is the region of relatively high concentration. This discussion gives post-hoc justification for the choice of $\mathcal{F} = -\kappa\nabla U$ as our flux function. Indeed, if this course was more about mathematical modelling, we would have obtained (7.1) by defining our flux this way (in physics language this means we are using **Fick's law**), demanding that mass be conserved over an arbitrary domain Ω to derive (7.6), and finally using arbitrariness of Ω and a bit of vector calculus to obtain (7.1).

Finally, we move on to discuss boundary conditions. Of course, there are many possible boundary conditions of physical interest depending on the particular application one wants to address, but for today we only focus on two canonical examples.

1. **Dirichlet BC**: $U|_{\partial\Omega}(x) = g(x)$. This (obviously!) means that we are keeping the boundary of our domain at a prescribed density given at any particular $x \in \partial\Omega$ by $g(x)$.
2. **Neumann BC**: $\frac{\partial U}{\partial \hat{\mathbf{n}}}|_{\partial\Omega}(x) = g(x)$. The interpretation of this BC is a little bit more subtle, but not by much. We know from Fick's law that that

$$\mathbf{F}|_{\partial\Omega}(x) = -\kappa g(x).$$

Therefore, $g(x)$ is directly proportional to the flux at $x \in \partial\Omega$. Therefore, imposing a Neumann condition amounts to prescribing how much stuff is leaving through the boundary. In particular, if $g(x) = 0$ then $\mathbf{F}|_{\partial\Omega} = 0$ and *no pollutant can leave* Ω.

Exercise 7.2.1. *Convince yourself that* (7.2) *holds.*

Exercise 7.2.2. *Give a common-sense physical argument for why should we only expect terms involving the normal component of* $\mathbf{F}$ *to appear in the equation of mass conservation.*

7.3 A Necessary Condition for Existence of a Solution to the Neumann Problem

Now, we use our understanding of the diffusion equation as a physical model in order to deduce a necessary condition for the BVP (11.3) to admit a solution. First, we have to explain how this BVP is actually related to a diffusion problem!

Notice that we can think of any solution to Poisson's equation

$$\Delta u = -f$$

as a **steady-state solution** or **equilibrium solution** of the diffusion equation. That is, suppose $U(x,t)$ satisfies

$$U_t - \kappa \Delta U = \kappa f(x). \tag{7.1}$$

(plus some initial and boundary conditions) and the limiting mass density

$$u(x) \doteq \lim_{t\to\infty} U(x,t)$$

always exists: we'll soon see that this supposition is quite restrictive. Then, by taking the limit $t \to \infty$ in (7.1), we see that u satisfies $\Delta u = -f$ (don't bother yourself with issues of exchanging derivatives and limits, just focus on getting an intuitive understanding). Further, provided the BCs on $U(x,t)$ are *time-independent*, we can also take the limit $t \to \infty$ to obtain corresponding BCs on the steady-state $u(x)$. In particular, if we impose a Neumann condition on $U(x,t)$ (read: prescribe the mass flux on the boundary) in the form

$$\left.\frac{\partial U}{\partial \hat{\mathbf{n}}}\right|_{\partial\Omega}(x) = g(x)$$

then by taking formal limits again we have

$$\left.\frac{\partial u}{\partial \hat{\mathbf{n}}}\right|_{\partial\Omega}(x) = g(x).$$

In summary, we have found that (11.3) describes the long-time behaviour of pollutant diffusion through Ω subject to a prescribed boundary flux and taking into account internal

sources and sinks of mass via $f(x)$. Note that this loose argument depended very strongly on assuming that the source function $f(x)$ and the BC $g(x)$ were both time-independent.

How can we guarantee that such a limiting distribution exists, at least for the Neumann problem? Let's think about the diffusion problem

$$U_t - \kappa \Delta U(x) = \kappa f(x) \quad \forall\ x \in \Omega \tag{7.2a}$$

$$\left.\frac{\partial U}{\partial \hat{\mathbf{n}}}\right|_{\partial\Omega}(x) = g(x) \tag{7.2b}$$

$$U|_{t=0} = U_0(x). \tag{7.2c}$$

Our strategy is to look at how much mass is eventually going to end up Ω due to competition or cooperation between the source/sink term $f(x)$ and the prescribed flux BC $g(x)$. Using the BCs, we know the net amount of mass flowing through through $\partial\Omega$ at any time t is given by

$$M_{\text{bdry}} = -\kappa \oint_{\partial\Omega} g(x)\ \mathrm{d}S(x).$$

Recall that if $M_{\text{bdry}} > 0$, we interpret this statement as meaning that $|M_{\text{bdry}}|$ units of mass leave the disc. Similarly, the amount of mass gained or lost at any given time time due only to the source term is

$$M_{\text{int}} = \kappa \int_{\Omega} f(x)\ \mathrm{d}x.$$

Here comes the punchline: if

$$M_{\text{bdry}} \neq M_{\text{int}}$$

we *do not* have a steady-state $u = \lim_{t\to\infty} U$. To see this, assume first that $M_{\text{bdry}} < 0$, then mass keeps on piling up inside Ω indefinitely unless $M_{\text{int}} < 0$ to suck some excess mass out: if $M_{\text{int}} \geq 0$ then any steady-state would have to contain an infinite mass of pollutant, which is ludicrous. However, if $M_{\text{int}} < 0$ and $|M_{\text{int}}| > |M_{\text{bdry}}|$, then there is a net loss of mass at each time and so eventually all mass will be sucked out of the disc. In turn, this implies that the boundary flux must tend to 0 and we arrive at a contradiction. In either case, the notion of a solution to the steady-state problem does not make physical sense unless we have perfect balance between the mass destroyed due to internal sinks and the mass transported inside through the boundary: $M_{\text{bdry}} = M_{\text{int}}$. A symmetric argument allows us to remove the restriction $M_{\text{bdry}} < 0$.

The discussion of the previous paragraph implies that we can't find a steady-state solution to (7.2) unless

$$-\int_{\Omega} f(x)\ \mathrm{d}x = \oint_{\partial\Omega} g(x)\ \mathrm{d}S(x). \tag{7.3}$$

This amounts to saying that there can be no solution to the Neumann BVP (11.3) unless (7.3) holds. So, to get mathematical insight into a particular BVP, we have used physical reasoning about *another* BVP! Now that we have a really solid intuitive understanding of *why* we need to impose a compatibility condition between the forcing term and the Neumann BC in order to solve (11.3), we want to transform this intuition into a rigorous result. This is not very hard:

Proposition 7.3.1 (Necessary Condition for Existence)**.** *If there exists some* $u: \overline{\Omega} \to \mathbb{R}$ *satisfying* (11.3)*, then we must have*

$$-\int_\Omega f(x)\ \mathrm{d}x = \oint_{\partial\Omega} g(x)\ \mathrm{d}S(x). \tag{7.4}$$

Proof. By integrating the PDE in (11.3) and using the divergence theorem, we find

$$\begin{aligned} -\int_\Omega f(x)\ \mathrm{d}x &= \int_\Omega \Delta u(x)\ \mathrm{d}x \\ &= \int_\Omega \nabla\cdot(\nabla u(x))\ \mathrm{d}x \\ &= \oint_{\partial\Omega} \nabla u(x)\cdot\hat{\mathbf{n}}\ \mathrm{d}S(x) \\ &= \oint_{\partial\Omega} \frac{\partial u}{\partial\hat{\mathbf{n}}}\ \mathrm{d}S(x) \\ &= \oint_{\partial\Omega} g(x)\ \mathrm{d}S(x), \end{aligned}$$

which is what we wanted to show. □

7.4 A Sufficient Condition for Uniqueness

Our only hope of obtaining a solution to (11.3) is to demand that the conditions of proposition 7.3.1 hold. However, this condition is not enough to guarantee a solution to (11.3) exists, or that it is unique. Existence can be handled in at least two ways:

1. if the geometry of Ω is simple enough (ie. it is a rectangle/cube/hypercube, or a circle/sphere, or a cylinder,...) we can explicitly construct a solution via separation of variables;
2. for a general domain Ω, one may not be able to write down a solution (even in Fourier series form!), but one can still prove a solution exists using tools from functional-analytic PDE theory (specifically, the **Lax-Milgram theorem** gives existence of a "weak" solution, and the **principle of elliptic regularity** guarantees this solution is smooth... see [11, ch. 5,6] for a complete discussion).

Today, we'll skip the existence question and instead focus on uniqueness, but in the near future you'll learn about how to establish existence using the first method outlined above.

First, we must face the unfortunate fact that solutions of (11.3) are *not unique*. To see this, suppose u solves (11.3) and c is any constant. Then, $w \doteq u + c$ satisfies

$$\begin{aligned} \Delta w &= -f \quad \forall\ x \in \Omega, \\ \left.\frac{\partial w}{\partial\hat{\mathbf{n}}}\right|_{\partial\Omega} &= \quad g \quad \forall\ x \in \partial\Omega. \end{aligned}$$

So if we have one solution, we have infinitely many solutions! Thus (11.3) is not even well-posed.

To fix this non-uniqueness issue, we need to find another constraint to impose on solutions. Suppose u_1, u_2 solve (11.3). Then,

$$v \doteq u_1 - u_2$$

solves

$$\Delta v = 0 \quad \forall\ x \in \Omega, \tag{7.1a}$$

$$\left.\frac{\partial v}{\partial \hat{\mathbf{n}}}\right|_{\partial\Omega} = 0 \quad \forall\ x \in \partial\Omega. \tag{7.1b}$$

We can easily characterize all solutions to the above BVP:

Lemma 7.4.1. *Suppose $v\colon \overline{\Omega} \to \mathbb{R}$ satisfies (7.1). Then, v is constant throughout Ω.*

Proof. Since energy methods gave us so much mileage when dealing with uniqueness theory for evolution equations, let's try an energy argument here. Multiplying the PDE by v and integrating gives

$$\begin{aligned}
0 &= \int_\Omega v\Delta v \ \mathrm{d}x \\
&= \int_\Omega \nabla\cdot(v\nabla v) - |\nabla v|^2 \ \mathrm{d}x \\
&= -\int_\Omega |\nabla v|^2 \ \mathrm{d}x + \int_{\partial\Omega} v\frac{\partial v}{\partial \hat{\mathbf{n}}} \ \mathrm{d}S(x) \\
&= -\int_\Omega |\nabla v|^2 \ \mathrm{d}x.
\end{aligned}$$

This can only hold if $\nabla v = 0$ throughout Ω. But, since we assumed at the start of the document that Ω was connected, this means v is constant. □

This lemma in turn tells us that any two solutions to (11.3) differ by a constant. So, once we impose a constraint that fixes this constant, we've secured uniqueness. A canonical choice of constraint is simply to fix the value of the surface integral of u to 0. Of course, in applications to mass diffusion, we want the value of this integral to be nonzero since dealing with negative mass density is not really physical. However, by appropriately translating the solution and BCs, we can easily recover a nonnegative density u. We summarize this discussion with another proposition.

Proposition 7.4.2 (Sufficient Condition for Uniqueness)**.** *Suppose u satisfies (11.3). Then, u is unique among the class of solutions satisfying*

$$\oint_{\partial\Omega} u \ \mathrm{d}S(x) = 0. \tag{7.2}$$

□

Exercise 7.4.1. *In this exercise we investigate a variant of* (11.3) *appearing in medical tomography. Consider a domain* $\Omega \subseteq \mathbb{R}^3$ *made of an electrically conductive material: in real-life medical applications* Ω *is a section of the human body, say the heart. The tendency of material inside* Ω *to support an electrical current is quantified by an empirically determined function called the* ***conductivity*** $\sigma\colon \overline{\Omega} \to (0, \infty)$*. If we assume we are given the conductivity* $\sigma(x)$*, then we can relate the current* $\mathbf{J}(x)$ *in* Ω *to the electric field* $\mathbf{E}(x)$ *in* Ω *via* ***Ohm's law****:*

$$\mathbf{J} = \sigma \mathbf{E}. \tag{7.3}$$

Also, electromagnetic theory tells us that the electric field in Ω *is also coupled to the magnetic field* $\mathbf{H}(x)$ *in* Ω *(created by the current* $\mathbf{J}$*) through the* ***equilibrium Maxwell equations****:*

$$\nabla \times \mathbf{E} = 0 \tag{7.4a}$$
$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{7.4b}$$

a) Suppose we apply a fixed current $J_0 \hat{\mathbf{n}}$ *on* $\partial\Omega$*, amounting to sending current* into *the body. Show then that the electric field satisfies the first-order boundary-value problem*

$$\nabla \cdot (\sigma \mathbf{E}) = 0 \quad \forall\ x \in \Omega \tag{7.5a}$$
$$\sigma \mathbf{E} \cdot \hat{\mathbf{n}}|_{\partial\Omega} = J_0 \quad \forall\ x \in \partial\Omega \tag{7.5b}$$

and explain why this problem cannot be used (on its own at least) to determine $\mathbf{E}$*.*

b) If (7.5) *holds, compute* $\oint_{\partial\Omega} J_0(x)\ \mathrm{d}S(x)$*.*

c) Explain why there exists $u\colon \overline{\Omega} \to \mathbb{R}$ *so that*

$$\mathbf{E} = \nabla u.$$

u *is called the* ***electrostatic potential*** *or* ***voltage****.*

d) Show that u *obeys*

$$\nabla \cdot (\sigma \nabla u)\,(x) = 0 \quad \forall\ x \in \Omega, \tag{7.6a}$$
$$\sigma \frac{\partial u}{\partial \hat{\mathbf{n}}}\bigg|_{\partial\Omega} = J_0 \quad \forall\ x \in \partial\Omega. \tag{7.6b}$$

Can this problem on its own determine $\mathbf{E}$*?*

e) Modify proposition 7.3.1 to obtain a necessary condition for existence of a solution to (7.6)*.*

f) Show that the solution of (7.6) *is unique provided the surface voltage satisfies*

$$\oint_{\partial\Omega} u\ \mathrm{d}S(x) = 0.$$

Do we lose some physical information in imposing this constraint? Explain why or why not in 1-2 brief sentences.

*In **electrical impedance tomography (EIT)**, a small voltage $u_0(x)$ is applied to $\partial\Omega$, and the resulting surface current $J_0(x)$ is measured. Using these current measurements, one reconstructs the conductivity function σ. This in turn lends a great deal of medical insight into, for instance, whether or not the electrical activity in someone's heart is healthy. We now formulate the mathematical problem underlying EIT and investigate some basic principles/formulas related to this problem.*

g) Explain why, in EIT, the voltage u inside Ω obeys the Dirichlet problem

$$\nabla \cdot (\sigma \nabla u)(x) = 0 \quad \forall\, x \in \Omega, \tag{7.7a}$$

$$u|_{\partial\Omega} = u_0 \quad \forall\, x \in \partial\Omega. \tag{7.7b}$$

Further, show that u is unique (see [27, exercise 6.1.10]*).*

*h) Define the **Dirichlet-to-Neumann operator** Λ_σ associated to Ω with conductivity σ by*

$$\Lambda_\sigma \colon u_0 \mapsto J_0.$$

In other words, Λ_σ takes in a given applied surface voltage and uses the unique solution of (7.7) to output the resulting surface current. In one sentence, explain why the name "Dirichlet-to-Neumann" operator is appropriate.

i) Briefly explain why EIT can mathematically be described as constructing an "inverse" map

$$\Lambda_\sigma \mapsto \sigma.$$

*This means EIT is an **inverse problem**, with (7.7) being the corresponding **forward problem** (determining $u_0 \mapsto J_0$, knowing σ).*

*j) If $f, g \colon \partial\Omega \to \mathbb{R}$ then we define their **inner product** by*

$$\langle f, g \rangle = \oint_{\partial\Omega} f(x)\ g(x)\ \mathrm{d}S(x).$$

*Suppose $u, v \colon \overline{\Omega} \to \mathbb{R}$ and that u satisfies (7.7). Show that Λ_σ can be written in the **weak form***

$$\langle \Lambda_\sigma u|_{\partial\Omega}, v|_{\partial\Omega} \rangle = -\int_\Omega \sigma(x)\ \nabla u \cdot \nabla v\ \mathrm{d}x. \tag{7.8}$$

k) Use the weak form to explain why Λ_σ has nonlinear dependence on σ. Thus EIT is a nonlinear inverse problem!

*l) Define a sensible notion of a **Neumann-to-Dirichlet operator** $\mathcal{R}_\sigma$ for this problem. Be careful to explain any constraints you need to impose on the domain of $\mathcal{R}_\sigma$ to ensure this operator is well-defined. Hint: use parts a) - f).*

m) Show that $\Lambda_\sigma \mathcal{R}_\sigma = 1$, the identity operator acting on functions defined on $\partial\Omega$.

n) Define an operator P acting on functions defined on $\partial\Omega$ by

$$Pf \doteq \frac{1}{Area\,(\partial\Omega)} \fint_{\partial\Omega} f(x)\ \mathrm{d}S(x).$$

Show that $\mathcal{R}_\sigma \Lambda_\sigma = 1 - P$, so the Neumann-to-Dirichlet operator is not the two-sided inverse of the Dirichlet-to-Neumann operator.

In practice, the above relationships between Λ_σ and $\mathcal{R}_\sigma$ can be leveraged to numerically solve the forward problems associated to EIT, a baby step towards solving the full nonlinear inverse problem. Actually attacking the inverse problem properly requires a lot of functional analysis. See [21] *for an introductory text on the subject.*

Chapter 8

Mathematics of the Clarinet

8.1 Problem Description

In this tutorial, we study a mathematical model for the acoustics of a B$^\flat$ clarinet: from now on, we'll suppress the B$^\flat$ in referring to the instrument. The clarinet is nearly a perfect cylinder. In particular, there are no bends in its body like in an alto saxophone. The simplicity of the clarinet's geometry allows us to model clarinet acoustics via an initial-boundary-value problem for the wave equation. The presentation here is inspired by an example in [9, §2.7]. For further discussion on clarinet acoustics, see the excellent educational materials at `phys.unsw.edu.au/jw/clarinetacoustics.html` (click here in the PDF version to visit this site).

8.2 Derivation of the Model Equations

First, we use the equations of fluid mechanics to explain why the wave equation is a suitable model for the motion of air inside a clarinet. I have given plenty of references to classic books on compressible flow, so any students interested in this material will have no trouble getting started in learning more about this fascinating subject.

8.2.1 Variables and Primitive Equations

For simplicity, we treat a clarinet as a thin, symmetrical pipe of constant diameter. This requires assuming all the keys are being held down simultaneously (see exercise 8.2.1). To a reasonable approximation, then, we can model a clarinet as a one-dimensional spatial domain. Then, the only space variable in the problem is position along the axis of the clarinet, which we denote by $x \in [0, \ell]$ for a suitable length ℓ (in the ball park of 0.5 metres). Since sound involves motion, we naturally must also include time t as one of our manipulated variables. We worry about specifying boundary and initial conditions a bit later.

To adequately describe sound waves within a clarinet, we need to introduce three responding field variables:

- the **mass density field** $\rho(x,t)$, with units of mass/length;
- the **pressure** $p(x,t)$, with units of force/length;
- the **(Eulerian) velocity** $u(x,t)$, with units of speed (length/time).

Additionally, we'll need two positive parameters $A > 0$, $\gamma \in (1, 5/3)$. An acceptable value of γ for air at room temperature is $\gamma = 1.4$. The motion of air in the clarinet is then governed by the 1D **compressible Euler equations** for an ideal gas [8, 19]:

$$\begin{cases} \rho_t + \partial_x(\rho u) = 0, \\ \rho u_t + \rho u u_x + p_x = 0, \\ p = A\rho^\gamma. \end{cases} \tag{8.1}$$

The first equation is called the **continuity equation**, representing the infinitesimal version of mass conservation for the fluid (see exercise 8.2.2). The second equation is called the **momentum equation**, and it really just says that Newton's second law is satisfied for each small parcel of fluid. Finally, the third equation is the **isentropic equation of state** for an ideal gas, determined from empirical evidence. You may be more familiar with the ideal gas equation of state in the form

$$pV = nR_0T \tag{8.2}$$

where V is the volume occupied by the air, T is its temperature, n is the number of moles of gas present, and R_0 is the gas constant (in appropriate units). In exercise 8.2.3, you will prove that both forms of the ideal gas equation of state are equivalent under some natural assumptions.

Exercise 8.2.1. *How does our model of the clarinet as an interval $[0, \ell]$ change if we do not suppose all the keys are held down? This is meant to be a bit of an open-ended question.*

Exercise 8.2.2. *Prove that a solution (ρ, p, u) to the Euler equations* (8.1) *satisfies the* ***principle of mass conservation****: for any finite interval $[a, b] \subseteq \mathbb{R}$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho(x,t)\ \mathrm{d}x = [\rho(x,t)u(x,t)]_{x=b}^{a}.$$

Interpret this result physically (you may wish to consider $u > 0, u < 0$ separately).

Exercise 8.2.3. *In this exercise you show, following* [8, §3,4]*, that the third equation in* (8.1) *is equivalent to* (8.2)*.*

a) Let m be the mass of one mole of air molecules. Define the ***specific volume*** *of air in the instrument to be*

$$\tau \doteq \text{ volume occupied by air, per unit mass.}$$

If we let $R = R_0/m$, show that (8.2) *takes the more convenient form*

$$T = \frac{p\tau}{R}. \tag{8.3}$$

b) Let S denote the entropy of our sample of air (you don't need to know the physical meaning of entropy to do this exercise) and let $E = E(S, \tau)$ denote its internal energy per unit mass. The ***first law of thermodynamics*** *states that, for any small increments $\Delta S, \Delta\tau$, the corresponding energy increment ΔE satisfies*

$$\Delta E = T\Delta S - p\Delta\tau.$$

Use this and (8.3) *to argue that E satisfies the linear, first-order PDE*

$$RE_S + \tau E_\tau = 0. \tag{8.4}$$

c) Use the method of characteristics to show that the general solution of (8.4) *is*

$$E(S, \tau) = h\left(\tau H\right), \tag{8.5}$$

with $h(z)$ an arbitrary function and $H = H(S) = e^{-S/R}$.

d) Let $z = \tau H$ denote the characteristic variable from part c. Show that the pressure and temperature obey

$$p = -Hh'(z) \tag{8.6a}$$

$$T = -\frac{z}{R}h'(z). \tag{8.6b}$$

e) Under some natural physical assumptions, $h(z)$ is strictly decreasing *and* convex. *Argue why this implies we can write $E = E(T)$.*

f) Define the ***speed of sound*** *by*

$$c \doteq \sqrt{\frac{\partial p}{\partial \rho}}. \tag{8.7}$$

Define a function $\gamma(T)$ by

$$\gamma(T) = 1 + R\frac{\mathrm{d}T}{\mathrm{d}E}. \tag{8.8}$$

Use (8.6) *to show that*

$$c^2 = \gamma(T)RT. \tag{8.9}$$

Hint: first use the inverse function theorem to compute $\frac{\mathrm{d}T}{\mathrm{d}E}$ in terms of τ, H, z, and $h(z)$.

Now, suppose there is a positive constant K so that

$$E = KT.$$

This means we are assuming air is ***polytropic****. Then, $\gamma = 1 + RK$.*

g) Use (8.5) *and* (8.6) *in conjunction with the polytropicity hypothesis to show that*

$$h(\tau H) = \left(\tau \frac{H}{H_0}\right)^{1-\gamma},$$

where $H_0 > 0$ *is an integration constant.*

h) Use (8.6) *and part g to find a function* $A(S)$ *so that*

$$p = A(S)\tau^{-\gamma}.$$

i) Conclude that, when entropy is constant, the third equation in (8.1) *indeed holds.*

8.2.2 Linearization

The nonlinear system (8.1) is impossible to solve analytically for a generic set of initial and boundary conditions. To make some headway with our problem, then, we should first try to simplify these equations. We'll accomplish this via **linearizing** (8.1).

Let's start by noticing that, for a constant ρ_0, the pair

$$(\rho, u) = (\rho_0, 0) \tag{8.10}$$

solves (8.1). We have omitted the pressure since it can be found from ρ via the equation of state. Since this special solution is time-independent, it's called an **equilibrium solution** of (8.1). Next, we look for solutions to (8.1) that have the form

$$(\rho, u) = (\rho_0 + \epsilon\widetilde{\rho}, \epsilon\widetilde{u})$$

where $\epsilon > 0$ is a small parameter. This amounts to weakly perturbing the equilibrium solution described above. Plugging this ansatz into (8.1) gives

$$\begin{cases} \epsilon\widetilde{\rho}_t + \epsilon^2\widetilde{u}\widetilde{\rho}_x + \epsilon\left(\rho_0 + \epsilon\widetilde{\rho}\right)\widetilde{u}_x = 0, \\ \epsilon\left(\rho_0 + \epsilon\widetilde{\rho}\right)\widetilde{u}_t + \epsilon^2\widetilde{u}\widetilde{u}_x + \epsilon A\gamma\left(\rho_0 + \epsilon\widetilde{\rho}\right)^{\gamma-1}\widetilde{\rho}_x = 0. \end{cases}$$

Definition 8.2.1. *Given* $n \in \mathbb{R}$*, we say a function* $a(\epsilon)$ *is* $o(\epsilon^n)$ *(in words,* $a(\epsilon)$ *is* ***little-oh*** *of* ϵ^n*) if*

$$\lim_{\epsilon \to 0^+} \frac{a(\epsilon)}{\epsilon^n} = 0.$$

Upon collecting terms, using Taylor expansion to simplify $(\rho_0 + \epsilon\widetilde{\rho})^{\gamma-1}$, and applying the definition of sound speed (8.7), the above system takes the form

$$\begin{cases} \widetilde{\rho}_t + \rho_0\widetilde{u}_x + o(1) = 0 \\ \rho_0\widetilde{u}_t + c^2(\rho_0)\widetilde{\rho}_x + o(1) = 0. \end{cases} \tag{8.11}$$

We have tacitly assumed that $\tilde{\rho}, \tilde{u}$, and their derivatives) are uniformly bounded in ϵ as $\epsilon \to 0^+$: this does not come for free from the mathematics, and must be imposed on grounds of physical reasonableness.

Now, how can we get a more friendly system of PDEs from (8.11)? Notice that every term in this system that is not inside an $o(1)$ is *linear*. Therefore, if we could suitable justify ignoring the $o(1)$ terms, we'd have a linear system, which is certainly a lot easier to deal with. By definition of the little-oh symbol, the $o(1)$ terms will be very small if we take $\epsilon \ll 1$. This corresponds to our sound wave/perturbation of the equilibrium state (8.10) having a very small amplitude. Of course, since we're modelling the sound made by a clarinet rather than a jet engine, we expect that our waves will indeed have small amplitude (see exercise 8.2.5 for some deeper discussion on the small-amplitude hypothesis). Accordingly, on physical grounds it is perfectly sensible to assume that the $o(1)$ terms in (8.11) can be ignored. Doing this and dropping $\sim$'s then yields the *linear* system

$$\begin{cases} \rho_t + \rho_0 u_x = 0, \\ u_t + \dfrac{c^2(\rho_0)}{\rho_0}\rho_x = 0. \end{cases} \tag{8.12}$$

Note that dropping the $o(1)$ terms is *not the same* as taking the limit $\epsilon \to 0^+$, since this completely kills the perturbation and laves us only with the boring equilibrium.

In exercise 8.2.4, you are asked to show that (8.12) can be reduced to the single PDE

$$u_{tt} - c^2 u_{xx} = 0, \tag{8.13}$$

which is precisely the 1D wave equation! By now, we have seen various methods for dealing with the wave equation, meaning that we're in good shape for making predictions about the physical problem at hand!

Exercise 8.2.4. *Show that* (8.12) *implies* (8.13).

Exercise 8.2.5. *In our derivation of the wave equation as a model for sound propagation, we assumed that sound waves necessarily had a very small amplitude. As it turns out, in* $3D$ *you can actually obtain a (highly!) nonlinear variant of the wave equation valid for sound waves of arbitrary amplitude, provided you assume there are no vortices in the air through which sound is travelling. Read* [4, pp. 84-86] *and fill in any missing details to derive this nonlinear wave equation.*

8.2.3 Boundary Conditions

Since we're dealing with a PDE that's second-order in space, we'll need to impose two boundary conditions. Recall that we denote position along the clarinet's axis by $x \in [0, \ell]$: $x = 0$ corresponds to the mouthpiece of the instrument, while $x = \ell$ corresponds to the bell. Let's tackle each of these boundary points one at a time:

- At the mouthpiece $x = 0$, the opening is so small that it can reasonably be ignored. In other words, we suppose that the column of air inside the clarinet is closed at $x = 0$. Now, $u(x, t)$ is the velocity of whatever air particle happens to be at position x at time t (pointing along the axis of the clarinet). We clearly should not allow any air to pass through $x = 0$, since we are treating the mouthpiece as perfectly closed. This is the same as saying that the velocity of any fluid particle at $x = 0$ must vanish. We conclude that the correct BC at $x = 0$ is the Dirichlet condition

$$u(0, t) = 0.$$

- As for the bell $x = \ell$, begin by letting ρ_0 denote the density of static air, as in our derivation of the wave equation from the compressible Euler system. Suppose the density of air outside the clarinet is constant at ρ_0. Since the bell is open, $x = \ell$ is the interface between the air inside the clarinet and the air outside the clarinet. Accordingly, we should demand that the density is continuous across this interface, giving

$$\rho(\ell, t) = \rho_0.$$

  Using the first equation in (8.12), this gives a homogeneous Neumann BC on the velocity:

$$u_x(\ell, t) = 0.$$

We conclude that the correct boundary conditions for modelling sound in a clarinet are *mixed* Dirichlet-Neumann BCs.

Remark 8.2.2. *The above explanation of the open-pipe boundary condition (the BC at $x = \ell$) may have left you a bit unsatisfied: if we know from high school that a sound wave is an oscillation in an air pressure field (and therefore, by the ideal gas equation of state, the density field), then shouldn't our choice of BCs confine all sound inside the clarinet? The answer to this apparent conundrum is that sound involves* not *just an oscillation in the pressure field, but the velocity field as well. In particular, information about fluid particles inside the clarinet moving is transferred outside the clarinet through the bell; such behaviour is certainly not prohibited by the Neumann BC on velocity. Thus, air outside the clarinet starts to move, and this is why we can actually hear the sound the clarinet makes. For more on this point, see the excellent discussion on* `physics.stackexchange.com/questions/256008` *(click the text in the PDF version to access this page).*

Exercise 8.2.6. *What would the correct BCs be if we were interested in studying a flute instead of a clarinet?*

8.3 Solution à la Fourier

If we choose units so that sounds speed is $c \equiv 1$, then we have seen that the motion of air inside a clarinet is governed by the IBVP

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \forall\ (x,t) \in (0,\ell) \times (0,\infty) \\ u(x,0) = u_0(x) \quad \forall\ x \in (0,\ell) \\ u_t(x,0) = v_0(x) \quad \forall\ x \in (0,\ell) \\ u(0,t) = 0 \quad \forall\ t \in [0,\infty) \\ u_x(\ell,t) = 0 \quad \forall\ t \in [0,\infty) \end{cases} \tag{8.1}$$

for a suitable choice of $u_0(x), v_0(x)$. Since we're dealing with a *bounded* spatial domain here, we should look for a solution by the Fourier method:

- separate variables according to $u(x,t) = X(x)T(t)$,
- solve the spatial eigenvalue problem, and
- form the solution to (8.1) as a Fourier series.

Once we've done this, we'll discuss the separated solutions a bit more deeply to discover some musically interesting information about the clarinet.

8.3.1 Computation of the Series Solution

We already know that separating variables gives

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda^2$$

for $\lambda > 0$. In exercise 8.3.1, you are asked to show that a nonnegative separation constant is impossible. Let's now attack the spatial eigenvalue problem

$$\begin{cases} X''(x) = -\lambda^2 X(x) \quad \forall\ x \in (0,\ell) \\ X(0) = 0 \\ X'(\ell) = 0 \end{cases} . \tag{8.2}$$

The general solution to the ODE above is

$$X(x) = A\cos\lambda x + B\sin\lambda x.$$

Applying the BCs gives the linear system

$$\begin{aligned} 0 &= A \\ 0 &= -A\lambda\sin\lambda\ell + B\lambda\cos\lambda\ell. \end{aligned}$$

These equations immediately imply that $A = 0$. If $\lambda = 0$, then $X \equiv 0$ which is not allowed (0 is never an eigenvector!). Accordingly, we need $B \neq 0$ and

$$\lambda \ell = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, 3, \ldots .$$

That is, the spatial eigenvalues are

$$-\lambda_n^2 = -\left(\frac{(2n+1)\pi}{2\ell}\right)^2, \quad n = 0, 1, 2, 3, \ldots .$$

Generally, it's actually more convenient to write the above as

$$-\lambda_m^2 = -\left(\frac{m\pi}{2\ell}\right)^2, \quad m = 1, 3, 5, \ldots .$$

with associated eigenfunctions.

$$X_m(x) = B_m \sin\left(\frac{m\pi}{2\ell}x\right), \quad m = 1, 3, 5, \ldots \tag{8.3}$$

where B_m is a constant. The first few eigenfunctions X_1, X_3, X_5 are plotted in figure 8.1.

Now that we've solved the spatial problem, let's turn to the temporal problem. The solution to the temporal problem is

$$T_m(t) = C_m \cos\left(\frac{m\pi}{2\ell}t\right) + D_m \sin\left(\frac{m\pi}{2\ell}t\right), \quad m = 1, 3, 5, \ldots . \tag{8.4}$$

If we ignore the initial conditions, we conclude that the separated solutions to (8.1) are, upon redefining our constants,

$$u_m = A_m \cos\left(\frac{m\pi}{2\ell}t\right) \sin\left(\frac{m\pi}{2\ell}x\right) + B_m \sin\left(\frac{m\pi}{2\ell}t\right) \sin\left(\frac{m\pi}{2\ell}x\right).$$

The cases $A_m = 0$ or $B_m = 0$ correspond to **standing waves** inside the clarinet. We'll discuss the qualitative properties of standing waves more carefully a bit later.

To take the given initial conditions into account, we must use a *sum* of separated solutions. That is, we look for $u(x,t)$ in the form of a Fourier series

$$u(x,t) = \sum_{\substack{m \geq 1 \\ m \text{ odd}}} A_m \cos\left(\frac{m\pi}{2\ell}t\right) \sin\left(\frac{m\pi}{2\ell}x\right) + B_m \sin\left(\frac{m\pi}{2\ell}t\right) \sin\left(\frac{m\pi}{2\ell}x\right). \tag{8.5}$$

The Fourier coefficients A_m, B_m can then be determined in terms of u_0 and v_0 using the following trigonometric orthogonality condition: for all $m, n = 1, 3, 5, \ldots$,

$$\int_0^\ell \sin\frac{n\pi x}{2\ell} \sin\frac{m\pi x}{2\ell} \, \mathrm{d}x = \frac{\ell}{2}\delta_{mn}. \tag{8.6}$$

See exercise 8.3.2. Using this formula, we easily arrive at (for $m = 1, 3, 5, ...$)

$$\begin{cases} A_m = \dfrac{2}{\ell} \displaystyle\int_0^\ell u_0(x) \ \sin\left(\frac{m\pi}{2\ell}x\right) \ \mathrm{d}x \\ B_m = \dfrac{2}{m\pi} \displaystyle\int_0^\ell v_0(x) \ \sin\left(\frac{m\pi}{2\ell}x\right) \ \mathrm{d}x \end{cases}, \tag{8.7}$$

and if we can compute these integrals then we have a clean solution to (8.1) (see the exercises below for some examples where these computations can indeed be done).

Exercise 8.3.1. *Show that the eigenvalues μ of*

$$\begin{cases} X''(x) = \mu X(x) \quad \forall \ x \in (0, \ell) \\ X(0) = 0 \\ X'(\ell) = 0 \end{cases} \tag{8.8}$$

satisfy $\mu < 0$, thereby substantiating our earlier computations with $\mu = -\lambda^2$.

Exercise 8.3.2. *Prove* (8.6) *using the identity*

$$2 \sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta).$$

Exercise 8.3.3. *Verify that the solution to* (8.1) *with initial data*

$$\begin{aligned} u_0(x) &= \sin\left(\frac{m\pi}{2\ell}x\right), \quad n = 1, 3, 5, ... \\ v_0(x) &= 0 \end{aligned}$$

is a single standing wave with amplitude 1*.*

Exercise 8.3.4. *Explicitly solve* (8.1) *with initial data*

$$\begin{aligned} u_0(x) &= \frac{4}{5} \sin\left(\frac{\pi}{2\ell}x\right) + \frac{1}{10} \sin\left(\frac{5\pi}{2\ell}x\right), \\ v_0(x) &= \sin\left(\frac{3\pi}{2\ell}x\right). \end{aligned}$$

8.3.2 Deeper Discussion of Eigenvalues and Standing Waves

In this subsection, we discuss the spatial eigenvalues and standing wave solutions more thoroughly: this is where most of the physically and musically relevant conclusions about clarinet acoustics are identified.

We recall that the eigenfunctions are

$$X_m = B_m \sin\left(\frac{m\pi}{2\ell}x\right) \quad m = 1, 3, 5, ...$$

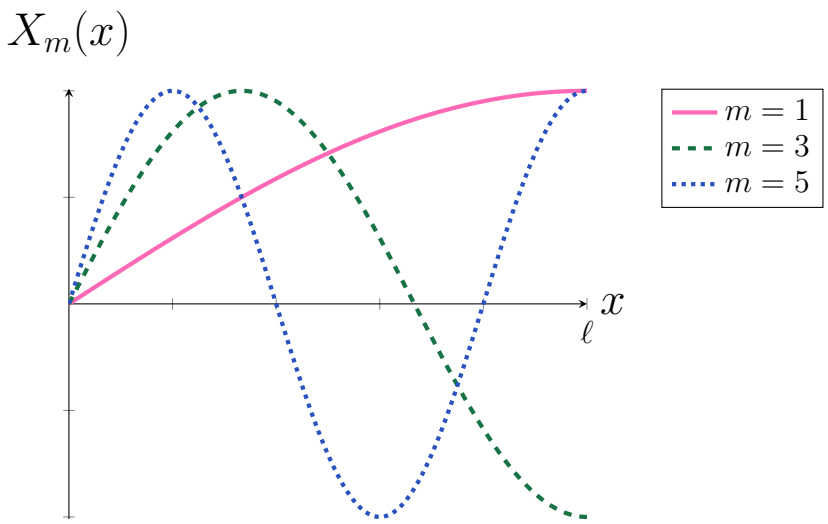

Figure 8.1: Plot of the first few spatial eigenfunctions of the clarinet. The eigenfunctions vanish at the mouthpiece $x = 0$, and have a local extremum at the bell $x = \ell$.

for any nonzero $B_m \in \mathbb{R}$. Often, we wish to choose

$$B_m = \sqrt{\frac{2}{\ell}}$$

so that the eigenfunctions have **unit L^2 norm**:

$$\|X_m\|_{L^2}^2 \doteq \int_0^\ell X_m(x)^2 \, \mathrm{d}x = 1 \iff B_m = \sqrt{\frac{2}{\ell}}.$$

For this discussion, however, we'll simply choose $B_m = 1$. See figure 8.1 for a plot of X_1, X_3, and X_5. Note how the closed-pipe condition at $x = 0$ and the open-pipe condition at $x = \ell$ are clearly visible in each eigenfunction.

Let us now imagine we blow into the clarinet, still with all the keys held down, to excite a single standing wave. Following exercise 8.3.3, this corresponds to solving (8.1) with initial data given by

$$u_0(x) = \sin\left(\frac{m\pi}{2\ell}x\right), \quad n = 1, 3, 5, \ldots$$
$$v_0(x) = 0.$$

Using the computations in the previous subsection, the velocity of air within the clarinet is then given by

$$u_m(x,t) = \sin\left(\frac{m\pi}{2\ell}t\right)\sin\left(\frac{m\pi}{2\ell}x\right).$$

Then, the temporal frequency of the motion (number of oscillations per unit time) is clearly given by

$$\omega_m = \frac{m\pi}{2\ell} = \sqrt{|-\lambda_m^2|}.$$

In particular, frequency is the square root of a spatial eigenvalue: this is a quasi-universal truth in mathematical physics. So, as m increases, the frequency of the sound we create increases. We hear this as a higher-pitched noise: indeed, as m increases indefinitely, the sound emitted from the clarinet would become unbearable (see exercise 8.3.5 for more on this).

Now, the musical note we hear when the m^{th} mode is excited is precisely the frequency ω_m associated to the standing wave u_m written above. The standing wave corresponding to the smallest frequency/lowest note ω_1 is called the **fundamental mode**. The higher m solutions correspond to what are called **overtones** in music: $m = 3$ is called the **second overtone**, $m = 5$ is called the **third overtone**, et cetera. A musician can play an overtone in the clarinet by a technique known as **overblowing**, essentially a controlled way to force high-energy air into the instrument (see again exercise 8.3.5).

Suppose now that a clarinetist starts a song by playing the fundamental mode. Then, they gradually blow harder to transition into the second overtone. The ratio of final to initial frequencies is then

$$\frac{\omega_3}{\omega_1} = 3.$$

In music theory, a note ω_b is said to be **one octave above** another note ω_a if

$$\frac{\omega_b}{\omega_a} = 2.$$

We know from experience that notes separated by an octave sound pretty much the same. For a clarinet, however, the second overtone and the fundamental are *not* separated by an octave. So, as a clarinetist overblows to go from $m = 1$ to $m = 3$, we don't just hear a (very boring!) octave jump. In precise musical terms, we jump an **octave and a fifth**. This behaviour, which cannot be observed in other wind instruments like the flute cf. exercise 8.3.8, allows a clarinetist to make interesting noises without doing any keywork. For a demonstration of this, see the video linked at `youtu.be/JgtxURnPnDk?t=83` (click) : the segment of this video most relevant to our discussion on overtones lasts from about 1:25-5:31.

Before ending, I want you to notice that looking at the behaviour of individual standing waves/spatial eigenfunctions gave us enough physical information to make concrete and interesting predictions about clarinet acoustics. In other words, we didn't really need the *entire* Fourier series (for a general initial state) to say something cool about our problem. This is an important truth you should take forward into your PDEs career: it's generally easier to get answers to some physical problem by looking at the qualitative behaviour of the first few eigenfunctions, rather than staring at the Fourier series expression for generic initial data. Further, in my opinion, a numerical simulation is immeasurably more useful than an infinite Fourier series when it comes to making real physical predictions (though some of the best numerical simulations are essentially based on truncating Fourier series, so I certainly don't mean to imply that learning Fourier series is useless!). Before the end of the course, I hope to explain how you can write your own code to (approximately) solve problems like (8.1) on the computer.

Exercise 8.3.5. *Define an energy functional associated to air in the clarinet according to*

$$E[u] \doteq \frac{1}{2}\int_0^\ell u_x(x,t)^2 + u_t^2(x,t)\ \mathrm{d}x.$$

a) Show that, if $u(x,t)$ solves (8.1), then $E[u]$ is constant.

b) Compute $E[u]$ when

$$u(x,t) = \sin\left(\frac{m\pi}{2\ell}t\right)\sin\left(\frac{m\pi}{2\ell}x\right)$$

and show that energy increases with m. So, the higher the note we play, the more energy the associated wave carries.

Exercise 8.3.6. *Suppose we hold down only the first three keys on the clarinet (about a third of the way down the length of the instrument) and blow lightly into the mouthpiece to excite the fundamental mode. Does the resulting sound have a pitch that is higher or lower than that heard when all keys are held down? You probably want to do exercise 8.2.1 before attempting this one.*

Exercise 8.3.7. *What is the shortest clarinet a typical adult human can hear* any *note on?*

Exercise 8.3.8. *Re-do everything in section 3, replacing "clarinet" with "flute" (you should do exercise 8.2.6 first). What qualitative differences do you notice in the eigenfunctions? How much higher is the fundamental mode of the flute than that of the clarinet?*

Exercise 8.3.9.

a) Let's make a new instrument called a "clute" by taking the mouthpiece off of a clarinet and sticking it into one end of a flute. Will the clute sound more like a clarinet or a flute? Explain.

b) Alternatively, we can make a "flarinet" by attaching a clarinet bell to one end of a flute (no clarinet mouthpiece is involved). Will the flarinet sound more like a clarinet or a flute? Explain.

c) Watch Bryan Suits' video demonstration on clutes and flarinets at youtu.be/RGzjlOv88Ik (click). Do your answers to parts a and b agree with the video demonstration?

Chapter 9

The Imperfect Laplacian: Loss of Regularity and Ill-Posedness

9.1 Introduction

Throughout your mathematical life, you'll often hear Laplace's equation described as one of the nicest PDEs out there. This is indeed the case: you've already discussed the elegant maximum principle and indispensible mean-value property for Laplace's equation, and you've seen some beautiful connections between harmonic functions and complex analysis. However, just like us humans, no PDE is perfect. Even Laplace's equation can exhibit some pathological behaviour, as we'll discuss today through two examples. The first example shows that, if we do not pose Laplace's equation on a smooth domain, then the derivatives of our solution may behave quite badly near the boundary. In our second example, we discover that the Cauchy problem for Laplace's equation is severely ill-posed; in particular, the ill-posedness cannot be fixed by the easy methods we used to deal with ill-posedness of the Neumann problem for Poisson's equation. Interestingly, both of these nasty examples can be completely understood using simple separation-of-variables computations!

9.2 Loss of Regularity in the Wedge Problem

Fix a particular angle $\beta \in (0, 2\pi)$ and $a > 0$. Let us define a (nonsmooth) domain $\Omega \subseteq \mathbb{R}^2$ using polar coordinates:

$$\Omega \doteq \{0 < r < a, 0 < \theta < \beta\}. \tag{9.1}$$

Ω therefore represents a wedge when $\beta \leq \pi$ (see for example figure 9.1) or a "Pac-Man" shape when $\beta > \pi$ (see for example figure 9.2).

We then consider the boundary value problem of finding $u: \overline{\Omega} \to \mathbb{R}$ such that, for a given

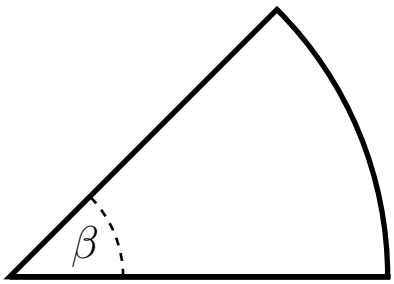

Figure 9.1: Example of a case where our domain is defined by $\beta < \pi$ (a wedge).

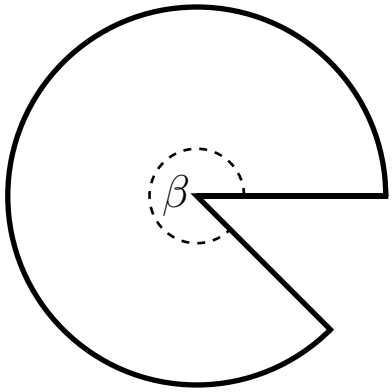

Figure 9.2: Example of a case where our domain is defined by $\beta > \pi$ (a "Pac-Man").

continuous function $g(\theta)$,

$$\begin{cases} \Delta u = 0 \quad \forall\ (r,\theta) \in \Omega \\ u|_{\theta=0} = 0 \\ u|_{\theta=\beta} = 0 \\ u|_{r=a} = g(\theta) \\ |u(r,\theta)| < \infty \quad \forall\ (r,\theta) \in \overline{\Omega}. \end{cases} \tag{9.2}$$

Solutions to this problem can be ill-behaved depending on the angle β subtended by the wedge, as the question below demonstrates.

Question 9.2.1.

a) Solve (9.2) *using polar coordinates.*

b) Show that, if $\beta > \pi$, then u_r is not necessarily bounded on $\overline{\Omega}$ for arbitrary $g(\theta)$.

c) Put a restriction on $g(\theta)$ guaranteeing that u_r is bounded on $\overline{\Omega}$.

Solution:

a) This first part of the solution mostly follows [27, §6.3]. We employ separation of variables, looking first for solutions of the form

$$u(r,\theta) = R(r)\Theta(\theta).$$

Since the polar coordinate Laplacian takes the form

$$\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$$

we find that

$$-\frac{\Theta''}{\Theta} = \frac{r^2R'' + rR'}{R} = \lambda^2 \tag{9.3}$$

for a constant λ (see exercise 9.2.1). We focus on the angular equation first. Θ obeys the BVP

$$\begin{cases} \Theta'' = -\lambda^2\Theta, \\ \Theta|_{\theta=0} = 0 \\ \Theta|_{\theta=\beta} = 0 \end{cases} \tag{9.4}$$

Immediately, we know the solution to this problem is

$$\Theta_n(\theta) = \text{constant} \times \sin\left(\frac{n\pi}{\beta}\theta\right), \quad \lambda = \lambda_n = \frac{n\pi}{\beta},\ n = 1, 2, 3, ...$$

Now that we've found the eigenvalues, we can turn to the radial equation:

$$\begin{cases} r^2R'' + rR' + \lambda_n^2 R = 0, \\ |R| < \infty. \end{cases} \tag{9.5}$$

We look for solutions in the form $R = r^a$. Plugging into the above equation, we find that $|a| = |\lambda_n|$. To ensure $|R| < \infty$, we only take the positive values of a. This yields

$$R_n(r) = \text{constant} \times r^{\frac{n\pi}{\beta}}.$$

Then, using the principle of superposition, we can express our solution $u(r, \theta)$ as a Fourier series with r-dependent coefficients:

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n\ r^{\frac{n\pi}{\beta}}\ \sin\left(\frac{n\pi}{\beta}\theta\right). \tag{9.6}$$

Next, we impose the boundary condition at $r = a$ to determine the expansion coefficients A_n:

$$g(\theta) = u(a, \theta) = \sum_{n=1}^{\infty} A_n\ a^{\frac{n\pi}{\beta}}\ \sin\left(\frac{n\pi}{\beta}\theta\right).$$

Using formulas familiar from our investigation of 1D boundary-value problems, we know then that the n^{th} Fourier sine series coefficient of g on $[0, \beta]$ is

$$A_n\ a^{\frac{n\pi}{\beta}} = \frac{2}{\pi}\int_0^{\beta} g(\theta)\ \sin\left(\frac{n\pi}{\beta}\theta\right)\ \mathrm{d}\theta.$$

We conclude that

$$A_n = \frac{2a^{-n\pi/\beta}}{\pi}\int_0^{\beta} g(\theta)\ \sin\left(\frac{n\pi}{\beta}\theta\right)\ \mathrm{d}\theta. \tag{9.7}$$

Combining (9.6) and (9.7) gives our complete solution to (9.2).

b) Using (9.6) we find that

$$u_r = \sum_{n=1}^{\infty} A_n \ \frac{n\pi}{\beta} \ r^{\frac{n\pi}{\beta}-1} \ \sin\left(\frac{n\pi}{\beta}\theta\right).$$

The $n = 1$ term includes $r^{\frac{\pi}{\beta}-1}$, which blows up as $r \to 0^+$ when $\beta > \pi$. Therefore, for such β, u_r may be unbounded on $\overline{\Omega}$.

c) We know that the $n = 1$ term is what causes u_r to become unbounded near $r = 0$. So, by picking $g(\theta)$ so that $A_1 = 0$, this troublesome term doesn't appear. Concretely, we may use (9.7) to express this constraint as

$$0 = \int_0^{\beta} g(\theta) \ \sin\left(\frac{\pi}{\beta}\theta\right) \ \mathrm{d}\theta. \tag{9.8}$$

If (9.8) holds, then u_r extends to a continuous, bounded function on $\overline{\Omega}$.

So, we have found that solutions to Laplace's equation on a nonsmooth domain Ω do not necessarily extend to smooth functions on $\overline{\Omega}$. In particular, the sharpness in u (severe lack of boundedness in u_r) is inherited from the sharpness of the wedge (the sharp point at $r = 0$). In the language of analysis, we say that our solution **loses regularity** when β exceeds the threshold value $\beta = \pi$: think of "regular" as meaning "very very good derivatives".

Remark. *When we demand that the underlying domain Ω and the boundary data $g(\theta)$ are smooth, it turns out that we can say quite a bit about smoothness of the corresponding harmonic function using the principle of* ***elliptic regularity*** [11, ch. 6].

Exercise 9.2.1. *Justify* (9.3) *by showing that we must have a positive separation constant.*

Exercise 9.2.2. *What geometric feature does Ω lose when we go from $\beta \leq \pi$ to $\beta > \pi$?*

9.3 Hadamard's Example

In this section, we discuss an example of Hadamard that shows simple problems involving Laplace's equation may still be ill-posed. We have already seen an example of an ill-posed Laplace equation when we discussed the Neumann problem for Poisson's equation on a smooth, bounded domain (week of November 29), but this ill-posedness could be fixed by simply adding another integral constraint to the problem to guarantee uniqueness. Hadamard's example exhibits a much more nefarious type of ill-posedness: there is a "butterfly effect" where arbitrarily small changes to the initial state lead to enormous changes in the solution!

Consider the Cauchy problem

$$\begin{cases} \Delta u = 0 \quad \forall \ (x, y) \in \mathbb{R}^2 \\ u|_{y=0} = 0 \\ u_y|_{y=0} = \dfrac{1}{n}\sin(nx). \end{cases} \tag{9.1}$$

This means we're imposing wave-equation-like initial conditions on Laplace's equation. We can solve (9.1) via separation of variables. Assume

$$u(x, y) = X(x)Y(y).$$

Plugging into Laplace's equation gives

$$0 = X''Y + XY'' \Rightarrow \frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)}.$$

The above expression can only hold if there exists a separation constant $\mu \in \mathbb{R}$ so that

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = \mu. \tag{9.2}$$

Thus we have converted the PDE into two ODEs:

$$X'' = \quad \mu X, \tag{9.3a}$$
$$Y'' = -\mu Y. \tag{9.3b}$$

We must now consider two cases depending on the sign of the separation constant μ.

- **Case 1 ($\mu > 0$)**: write $\mu = \lambda^2$ for $\lambda > 0$. Then, the general solutions of the ODEs in (9.3) are

  $$X(x) = A_1 \cosh(\lambda x) + A_2 \sinh(\lambda x),$$
  $$Y(y) = B_1 \cos(\lambda y) + B_2 \sin(\lambda y).$$

  Now, $u|_{y=0} = 0$ immediately implies $B_1 = 0$. For the other condition, we compute

  $$\begin{aligned} u_y|_{y=0} &= X(x)Y'(0) \\ &= [A_1 \cosh(\lambda x) + A_2 \sinh(\lambda x)] \lambda B_2 \\ &= \frac{1}{n} \sin(nx). \end{aligned}$$

  However, $\cosh(\lambda x), \sin(\lambda x)$, and $\sin(nx)$ are linearly independent, so there is no way the above expression can hold. We conclude that this case ($\mu > 0$) is *impossible*.

- **Case 2 ($\mu \leq 0$)**: write $\mu = -\lambda^2$ for $\lambda \geq 0$. The ODEs in (9.3) have general solutions

  $$X(x) = A_1 \cos(\lambda x) + A_2 \sin(\lambda x),$$
  $$Y(y) = B_1 \cosh(\lambda y) + B_2 \sinh(\lambda y).$$

  $u|_{y=0} = 0$ implies $B_1 = 0$, and the other condition yields

  $$\begin{aligned} u_y|_{y=0} &= X(x)Y'(0) \\ &= [A_1 \cos(\lambda x) + A_2 \sin(\lambda x)] \lambda B_2 \\ &= \frac{1}{n} \sin(nx). \end{aligned}$$

Using linear independence again, the above holds provided

$$\begin{aligned} A_1 &= 0, \\ \lambda A_2 B_2 &= \frac{1}{n}, \text{ and} \\ \lambda &= n. \end{aligned}$$

Consequently, the solution here is

$$\begin{aligned} u(x,y) = X(x)Y(y) &= A_2 B_2 \sin(\lambda x)\sinh(\lambda y) \\ &= \frac{1}{n^2}\sin(nx)\sinh(ny). \end{aligned} \tag{9.4}$$

So, the solution to (9.1) is given by (9.4).

To see that our work implies (9.1) is ill-posed, we investigate how our solution behaves as $n \to \infty$. For notational clarity, let us define

$$u_n(x,y) \doteq \frac{1}{n^2}\sin(nx)\sinh(ny). \tag{9.5}$$

Fix any $(x,y) \in \mathbb{R}^2$ such that $y \neq 0$ and

$$\frac{x}{\pi} \notin \mathbb{Q}.$$

If you want a concrete example, pick $(x,y) = (\pi^2, 1)$. In particular, this choice of x means

$$|\sin(nx)| > 0 \quad \forall\ n \geq 1.$$

Then, we find

$$\begin{aligned} \lim_{n\to\infty} |u_n(x,y)| &= \lim_{n\to\infty} n^{-2}\,|\sinh(ny)|\,|\sin(nx)| \\ &= \frac{1}{2}\lim_{n\to\infty} n^{-2}\ e^{n|y|}\ |\sin(nx)| \\ &= +\infty, \end{aligned}$$

since $e^{n|y|}$ blows up faster than n^2 (for instance, by L'Hopital's rule) and sine is bounded. We conclude that

$$\lim_{n\to\infty} |u_n(x,y)| = +\infty$$

on the complement of

$$\mathcal{S} = \left\{ y = 0 \text{ or } \frac{x}{\pi} \in \mathbb{Q} \right\}.$$

In the jargon of measure theory, $\mathcal{S}$ is said to be a **null set**: it has Lebesgue measure zero and therefore covers zero area (think if $\mathcal{S}$ as a collection of isolated dust particles lying on top of a piece of paper). So, as $n \to \infty$, $u_n(x,y)$ blows up *almost everywhere*: we cannot even make sense of the "pointwise limit" as an element of $L^\infty(\mathbb{R}^2)$, the space of Lebesgue-measurable functions that are finite almost everywhere!

Why does this blowup imply ill-posedness of (9.1)? First notice that the Cauchy data obeys

$$\frac{1}{n}\sin(nx) \to 0 \text{ uniformly as } n \to \infty.$$

Accordingly, the limiting case of (9.1) is

$$\begin{cases} \Delta u = 0 & \forall\ (x,y) \in \mathbb{R}^2 \\ u|_{y=0} = 0 & \\ u_y|_{y=0} = 0 & \end{cases}, \tag{9.6}$$

which admits the trivial solution $u \equiv 0$. However, $u_n(x,y)$ obviously cannot converge to

$$u \equiv 0$$

in any reasonable sense. Accordingly, there is no way the data-to-solution correspondence can be continuous. By Hadamard's definition of well-posedness, therefore, (9.4) is not well-posed.

In practice, it's useful to think of the continuity requirement in the well-posedness definition as a very weak version of **stability**. In this problem, we have effectively shown that the trivial solution $u \equiv 0$ of (9.6) is unstable: if we pick $n \gg 1$, then the Cauchy data in (9.1) is very close to zero (pointwise), but the solution u_n is very far from 0 (pointwise).

Remark. (9.1) *is one of the two most famous examples of a seriously ill-posed Cauchy problem, the other example being the backward-in-time heat equation: running backwards for a finite time can be OK, but for an infinite time you will run into nasty problems as all your mass is packed into a tiny set.*

Chapter 10

The Acoustical Kirchhoff Formula

10.1 Introduction

In this tutorial we derive the **Kirchhoff representation formulas** for an elliptic PDE called the **Helmholtz equation**. These formulas will cover cases when our PDE is posed either inside our outside some bounded domain: in the outside case, we need to impose a special "boundary condition at infinity". The Kirchhoff formulas have several rich consequences we investigate as well. All of this analysis is motivated by the problem of justifying why sonar works (we'll see below that Helmholtz's equation describes the spatial behaviour of typical sounds waves). In a future tutorial, we'll use the tools established today to help prove that a sonar instrument can, in principle, describe the shape of object hidden under the water using only measurements of sound waves on the ocean's surface. The presentation of the technical material here is heavily based off that in Colton and Kress' excellent book on scattering theory [7, §2.1-2.2].

10.2 The Helmholtz Equation in Exterior Domains: Motivation and Basic Ideas

10.2.1 Motivation: Hunting *Red October*

The military submarine *Red October* has gone rogue, and we must find it quickly to ensure the safety of the world! Suppose we only have access to ships equipped with acoustic measurement devices, specifically sonar instruments. These measurement devices can detect the surface signature of sound waves produced by an object deep under the ocean: the ship carrying an acoustic instrument sends out an **incident sound wave** into the ocean, the incident wave bounces off whatever objects are under the waves, and the returning **scattered sound wave** is measured back at the ship. Before we rely on sonar to help us catch *Red October*, we should of course ask if it can possibly work.

Question 10.2.1. *Can the measurements from our instruments be used to detect the presence of* Red October*?*

Of course, if you know a fishing enthusiast who's rich enough to afford a fish-finder, you've probably seen firsthand that sonar actually (mostly) works, so the answer to this question is definitely "yes". I want to show you how we can establish this affirmative answer using PDE methods. This will take several tutorials, spread out over the semester. Today, we'll be focused on establishing that every scattered wave has a characteristic **far-field pattern** that approximately describes its behaviour at distances very far from the scattering obstacles (read: at the surface of the ocean where our measuring devices are). In later tutorials, we prove that the correspondence taking scattering obstacles to far-field patterns is unique, and this is mostly enough to answer Question 10.2.1.

10.2.2 PDE Description of Sound Waves

The first step in our theoretical investigation of sonar is to formulate a PDE model for the propagation of sound waves in water. From our earlier discussion on clarinet acoustics, we heuristically expect that such sound waves will obey the wave equation, with a characteristic speed c depending on the thermodynamic properties of water. This is indeed the case: as with the clarinet problem, you can see this for yourself by linearizing the compressible Euler equations. The two big differences between clarinet acoustics and underwater acoustics are

1. unlike air, water is not approximately an ideal gas, so the definition of sound speed we gave for the clarinet problem has to be modified slightly (specifically, we now define squared sound speed by
$$c^2(\rho_0) = \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}$$
where p is pressure, ρ is density, and ρ_0 is the mean density of water at rest) ;
2. the geometry of the clarinet allowed us to reduce the problem down to a single spatial dimension, but for sound waves in water we have to keep the problem fully 3D.

With these considerations in mind, the equation governing the evolution of density $\rho(x,t)$ in the fluid is then the wave equation with signal speed $c^2(\rho_0)$: it t denotes time and $x \in \mathbb{R}^3$ denotes spatial position, then
$$\rho_{tt} - c^2(\rho_0)\Delta_x \rho = 0. \tag{10.1}$$
The next stage in the modelling process is to simplify the wave equation even more by separating variables in a particular way: for a fixed **temporal frequency** ω, assume
$$\rho(x,t) = \cos(\omega t)\ u(x).$$
This means that ρ is a **time-harmonic** wave. By now, you should have enough comfort with separated solutions of the wave equation in 1D to be unsurprised by this ansatz. Now,

dealing with trig functions is a lot easier if we switch to complex variables:

$$\cos(\omega t) = \mathrm{Re}\left(e^{-i\omega t}\right).$$

Since the wave equation is linear and has real coefficients, there is no harm in treating ρ and u as *complex-valued* for the purposes of computations, as long as we make sure to take only their real parts at the end of the day when we need a real value of density. This means that time-harmonic waves can also be written as

$$\rho(x,t) = e^{-i\omega t}u(x).$$

The spatial component u satisfies the elliptic equation

$$0 = \left(\Delta_x + \frac{\omega^2}{c^2(\rho_0)}\right)u.$$

If we define the **wavenumber** or **spatial frequency** of our sound wave by

$$k \doteq \frac{\omega}{c}$$

then we find that $u(x)$ obeys the **Helmholtz equation**:

$$\left(\Delta_x + k^2\right)u = 0. \tag{10.2}$$

This is really just the problem of finding a given eigenfunction of Δ_x knowing a particular eigenvalue k^2 *a priori*.

10.2.3 Fundamental Solution of the Helmholtz Equation

Our first task is to understand the simplest nontrivial solutions to (10.2). Since the differential operator in this PDE is rotationally invariant, it makes sense to focus on solutions with only radial dependence:

Lemma 10.2.2. *The two independent radial solutions of the Helmholtz equation that are valid away from the origin are*

$$u_\pm = \frac{e^{\pm ikr}}{r}.$$

Proof. See exercise 10.2.1. □

A question arises: which one of the two radial solutions is "more natural"? Recall that we really want u to correspond to a scattered sound wave radiating off an obstacle and towards the sea surface (which we can approximate as spatial infinity). To see this, let's consider the density fields corresponding to $u_\pm$:

$$\begin{aligned}\rho_\pm &= e^{-i\omega t}u_\pm \\ \Rightarrow \mathrm{Re}\left(\rho_\pm\right) &= \frac{1}{r}\cos\left(-\omega t \pm kr\right).\end{aligned}$$

So, u_+ (and only u_+!) gives rise to a wave that travels radially outward towards infinity over time. Since we want to study scattered waves going off to infinity, we should therefore focus our attention only on u_+; we'll go further down this path and actually prescribe a type of pseudo-boundary condition that selects only outgoing waves in the next subsection. With this discussion in mind, our fundamental solution for Helmholtz should be built using only u_+:

Definition 10.2.3. *The **fundamental solution of the Helmholtz equation** is given by*

$$\Phi(x,y) \doteq \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} \quad \forall\, x \neq y \in \mathbb{R}^3. \tag{10.3}$$

The normalization 4π is chosen only to simplify the representation formulas we derive. By Lemma 10.2.2, we clearly have

Lemma 10.2.4. *For each fixed $y \in \mathbb{R}^3$,*

$$\left(\Delta_x + k^2\right)\Phi(x,y) = 0 \quad \forall\, x \neq y.$$

□

10.2.4 Waves in Exterior Domains: Main Ideas and Definitions

Now, we know that our problem domain isn't really all of $\mathbb{R}^3$, since we want to add obstacles like *Red October* that cannot be penetrated by sound waves. So, we must formulate the Helmholtz equation in exterior domains, that is, $\mathbb{R}^3$ minus some nice enough set. The purpose of this subsection is to carefully define an exterior domain and to introduce appropriate "boundary conditions at spatial infinity" that ought to be imposed on solutions to Helmholtz posed on such exterior domains.

Our first task is to define what sort of obstacles we want to allow.

Definition 10.2.5. *We call an open set $\Omega \subseteq \mathbb{R}^3$ a **domain** if*

1. *its closure $\overline{\Omega}$ is compact and $\partial\Omega$ is smooth, or*
2. *Ω is the interior of a polyhedron.*

Note that we say "obstacle" in physical language and "domain" in mathematical language. The reason for this double-jargon is that one must first study solutions of Helmholtz inside domains in order to study solutions of Helmholtz in exterior domains, and saying that we're solving Helmholtz "inside an obstacle" is a bit awkward.

With the definition of a domain taken care of, the corresponding definition of an exterior domain is easy:

Definition 10.2.6. *Given a domain Ω, we define its* ***exterior*** *by*

$$\operatorname{Ext}(\Omega) \doteq \mathbb{R}^3 - \overline{\Omega}.$$

Notice how we have defined the exterior so that it is always *open*. Additionally, the restrictions appearing in the definition of Ω allow us to use the main theorems of vector calculus (including Green's identities) when dealing with functions defined on Ω and $\operatorname{Ext}(\Omega)$.

Next, we want to prescribe appropriate "boundary conditions at infinity" on our solutions. Before doing so, we need to set up some notation that greatly simplifies dealing with the behaviour of functions near infinity:

Definition 10.2.7 (Order Symbols)**.** *Let $v\colon \operatorname{Ext}(\Omega) \to \mathbb{C}$ be any function. Pick $m \in \mathbb{N} \cup \{0\}$.*

1. *We say that*
$$v = \mathcal{O}\left(r^{-m}\right) \quad \text{as} \quad r \to \infty$$
if
$$-\infty < \lim_{r\to\infty} r^m |v| < \infty.$$
2. *We say that*
$$v = o\left(r^{-m}\right) \quad \text{as} \quad r \to \infty$$
if
$$\lim_{r\to\infty} r^m v = 0.$$

Here are some easy examples to help you get comfortable with the order symbols (see also exercise 10.2.2):

Example 10.2.8. Clearly, $r^{-m} = \mathcal{O}(r^{-m})$ and $r^{-m} = o(r^{-m-1})$.

Example 10.2.9. Since $|e^{ikr}| = 1$ for all $r \geq 0$, we find

$$\frac{e^{ikr}}{r} = \mathcal{O}\left(r^{-1}\right) \quad \text{as} \quad r \to \infty.$$

This implies the weaker result

$$\frac{e^{ikr}}{r} = o(1) \quad \text{as} \quad |x-y| \to \infty.$$

Finally, we turn to one of the most important and insightful definitions in scattering theory.

Definition 10.2.10. *Given a wavenumber k, we say that a function $v\colon \operatorname{Ext}(\Omega) \to \mathbb{C}$ satisfies the* ***Sommerfeld radiation condition*** *if*

$$\frac{\partial v}{\partial r} - ikv = o\left(r^{-1}\right) \quad \text{as} \quad r \to \infty. \tag{10.4}$$

At first glance, the Sommerfeld condition may look a bit unnatural, but think of it this way: for large r, Sommerfeld approximately says

$$\frac{\partial v}{\partial r} \approx ikv \Rightarrow v \approx e^{ikr}.$$

This means that, very roughly speaking, the Sommerfeld condition *only allows outgoing waves*! Following the discussion at the end of the previous subsection, then, it is reasonable to suppose that scattered waves are well-described by Sommerfeld. To further substantiate this claim, we have the following:

Lemma 10.2.11. *$\frac{e^{ikr}}{r}$ satisfies the Sommerfeld radiation condition.*

Proof. By direct computation with $\phi = \frac{e^{ikr}}{r}$,

$$\frac{\partial \phi}{\partial r} - ik\phi = -\frac{1}{r}\phi = \mathcal{O}(r^{-2}) = o\left(r^{-1}\right).$$

□

It's likewise easy to check that the inwardly-moving radial solution $\frac{e^{-ikr}}{r}$ does not satisfy Sommerfeld.

Finally, we come to the definition of a solution to the Helmholtz equation in an exterior domain, which we recall physically corresponds to a scattered wave returning to our sonar device.

Definition 10.2.12. *If $u \in C^2\left(\overline{\text{Ext}\,(\Omega)}\right)$ satisfies the Helmholtz equation in $\text{Ext}\,(\Omega)$ as well as the Sommerfeld radiation condition (10.4), then we call $u(x)$ an **exterior solution** of the Helmholtz equation in $\text{Ext}\,(\Omega)$.*

With a bit more care, one can relax the definition of an exterior solution to allow

$$u(x) \in C^2\left(\text{Ext}\,(\Omega)\right) \cap C\left(\overline{\text{Ext}\,(\Omega)}\right),$$

but we won't worry about this in our investigation.

Exercise 10.2.1. *Prove Lemma 10.2.2.*

Exercise 10.2.2. *Generalize the definition of the order symbols to cover cases like*

$$v = \mathcal{O}\left(w\right), \quad v = o(w)$$

where w is a given continuous and nonvanishing function on $\text{Ext}\,(\Omega)$. Use your generalized definition to describe the asymptotic behaviour of the rapidly decaying radial solutions of

$$\left(\Delta_x - k^2\right) u = 0, \quad x \in \mathbb{R}^3.$$

10.3 The Kirchhoff Representation Formula

Having sensibly defined our notion of solution, we now turn to obtaining representation formulas for solutions to the Helmholtz equation. We need to consider two different cases: solutions within a domain Ω, and exterior solutions in $\mathrm{Ext}\,(\Omega)$. Fortunately, the two representation formulas for the interior/exterior cases only differ by a sign, so memorizing both expressions is not difficult. The interior case follows largely by mimicking the proof of the representation formula for harmonic functions defined in a domain, while the exterior case essentially just amount to applying a clever limiting argument to the interior case. Once we have these representation formulas at hand, we'll prove some fascinating results about solutions to the Helmholtz equation in the next section. These nice results are crucial to mathematically establishing a nice answer to Question 10.2.1.

Throughout the sequel, $\mathbf{n}$ denotes the unit *outward* normal to $\partial\Omega$. In the case where Ω is polyhedral, we note that $\mathbf{n}$ is not defined on the edges of $\partial\Omega$. Additionally, the open ball of radius R centred at x is denoted by $B(x, R)$.

10.3.1 Interior Case

Theorem 10.3.1 (Kirchhoff Representation Formula, Interior Case)**.** *Let $u(x) \in C^2\left(\overline{\Omega}\right)$ be a solution to the Helmholtz equation in Ω. Then, we have*

$$u(x) = \oint_{\partial\Omega} -u(y)\frac{\partial\Phi}{\partial\mathbf{n}}(x,y) + \Phi(x,y)\frac{\partial u}{\partial\mathbf{n}}(y)\ \mathrm{d}S(y) \quad \forall\ x \in \Omega.$$

Proof. We mostly follow the proof of the analogous representation formula for harmonic functions given in Strauss' book [27, pp. 186-187]. First, notice that we can't just plug into the second Green's identity because $\Phi(x, y)$ behaves badly for $x \approx y$. So, we need to be more careful. Start by fixing $x \in \Omega$ and picking $\epsilon < \frac{1}{2}\mathrm{dist}\,(x, \partial\Omega)$. Cut out a ball of radius ϵ around x to define the modified domain

$$\Omega_\epsilon \doteq \Omega - \overline{B(x,\epsilon)}.$$

Since all $y \in \Omega_\epsilon$ are far enough away from x, $\Phi(x, y)$ is perfectly smooth (in y) on Ω_ϵ and we are free to apply Green's second identity. This gives

$$\int_{\Omega_\epsilon} u(y)\Delta_y\Phi(x,y) - \Phi(x,y)\Delta_y u(y)\ \mathrm{d}y = \oint_{\partial\Omega_\epsilon} u(y)\frac{\partial\Phi}{\partial\mathbf{n}}(x,y) - \Phi(x,y)\frac{\partial u}{\partial\mathbf{n}}(y)\ \mathrm{d}S(y).$$

Since both u and Φ satisfy the Helmholtz equation in Ω_ϵ (derivatives with respect to y), the above reduces to

$$0 = \oint_{\partial\Omega_\epsilon} u(y)\frac{\partial\Phi}{\partial\mathbf{n}}(x,y) - \Phi(x,y)\frac{\partial u}{\partial\mathbf{n}}(y)\ \mathrm{d}S(y).$$

Further, since

$$\partial\Omega_\epsilon = \partial\Omega \cup \partial B(x,\epsilon),$$

we can further refine the above to

$$\oint_{\partial B(x,\epsilon)} u(y)\frac{\partial \Phi}{\partial \mathbf{n}}(x,y) - \Phi(x,y)\frac{\partial u}{\partial \mathbf{n}}(y)\ \mathrm{d}S(y) = \oint_{\partial \Omega} u(y)\frac{\partial \Phi}{\partial \mathbf{n}}(x,y) - \Phi(x,y)\frac{\partial u}{\partial \mathbf{n}}(y)\ \mathrm{d}S(y).$$

Notice how we have used that the *outward normal* on $\partial\Omega_\epsilon$ restricts to the *inward normal* on $\partial B(x,\epsilon)$ (draw a picture!), so the signs in the above expression are correct. Then, finishing the proof amounts to showing

$$\lim_{\epsilon \to 0} \oint_{\partial B(x,\epsilon)} u(y)\frac{\partial \Phi}{\partial \mathbf{n}}(x,y) - \Phi(x,y)\frac{\partial u}{\partial \mathbf{n}}(y)\ \mathrm{d}S(y) = -u(x). \tag{10.1}$$

Towards this goal, let's define

$$I_\epsilon^1 = \oint_{\partial B(x,\epsilon)} u(y)\frac{\partial \Phi}{\partial \mathbf{n}}(x,y)\ \mathrm{d}S(y), \tag{10.2}$$

$$\tag{10.3}$$

$$I_\epsilon^2 = -\oint_{\partial B(x,\epsilon)} \Phi(x,y)\frac{\partial u}{\partial \mathbf{n}}(y)\ \mathrm{d}S(y). \tag{10.4}$$

It turns out that controlling I_ϵ^2 is easier, so we'll handle this term first. Notice that, since $u \in C^2\left(\overline{\Omega}\right)$, we know there exists $M > 0$ so that

$$\left|\frac{\partial u}{\partial \mathbf{n}}(y)\right| \leq M \quad \forall\ y \in \partial B(x,\epsilon).$$

Then, we straightforwardly estimate

$$\begin{aligned}
0 \leq \left|I_\epsilon^2\right| &\leq \oint_{\partial B(x,\epsilon)} |\Phi(x,y)|\ \left|\frac{\partial u}{\partial \mathbf{n}}(y)\right|\ \mathrm{d}S(y) \\
&\leq M \oint_{\partial B(x,\epsilon)} |\Phi(x,y)|\ \mathrm{d}S(y) \\
&\leq M\epsilon^{-1}\ \mathrm{Area}\left(B(x,\epsilon)\right) \\
&= 4\pi M\epsilon \\
&\to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

So, I_ϵ^2 contributes nothing to (10.1) in the limit.

We can now focus on the slightly more difficult integral I_ϵ^1. Since $\mathbf{n}$ for $\partial B(x,\epsilon)$ simply points radially outward from x, we find

$$\begin{aligned}
I_\epsilon^1 &= \oint_{\partial B(x,\epsilon)} u(y)\ \left[\frac{\partial \Phi}{\partial r}\right]_{r=\epsilon}\ \mathrm{d}S(y) \\
&= \frac{1}{4\pi}\oint_{\partial B(x,\epsilon)} u(y)\ \left[e^{ikr}\left(\frac{ik}{r} - \frac{1}{r^2}\right)\right]_{r=\epsilon}\ \mathrm{d}S(y) \\
&= -e^{ik\epsilon}\left(1 - ik\epsilon\right)\left\{\frac{1}{4\pi\epsilon^2}\oint_{\partial B(x,\epsilon)} u(y)\ \mathrm{d}S(y)\right\}.
\end{aligned}$$

We recognize that the term in curly braces is the mean of $u(y)$ over the surface $\partial B(x,\epsilon)$, which converges to $u(x)$ as $\epsilon \to 0$ since u is continuous near x. We conclude that $I_\epsilon^1 \to -u(x)$, so the proof is done. □

Remark. *The interior Kirchhoff formula is the starting point for scalar* ***diffraction theory****, the study of how sound or light is affected as is passes through a small slit or aperture. The phenomenology of diffraction is fascinating, but seriously discussing any examples would take more time than I am prepared to sacrifice. Those who wish to see a non-rigorous account of how Kirchhoff can be used to predict diffraction patterns may consult* [12, Ch. 5] *(see also* [3, Ch. 6] *for a better account of the actual computations involved).*

10.3.2 Exterior Case

Theorem 10.3.2 (Kirchhoff Representation Formula, Exterior Case)**.** *Let $u(x)$ be an exterior solution to the Helmholtz equation in* $\mathrm{Ext}\,(\Omega)$*. Then, we have*

$$u(x) = \oint_{\partial\Omega} u(y)\frac{\partial\Phi}{\partial\mathbf{n}}(x,y) - \Phi(x,y)\frac{\partial u}{\partial\mathbf{n}}(y)\ \mathrm{d}S(y) \quad \forall\ x \in \mathrm{Ext}\,(\Omega)\,,$$

where $\mathbf{n}$ *is outward relative to* Ω*, not* $\mathrm{Ext}\,(\Omega)$*.*

Note the sign discrepancy between the interior and exterior cases! Can you see why we expect the negative sign right off the bat?

Proof. The strategy of the proof is to be lazy: we already have a representation formula for the case of a bounded domain, so we want to try and approximate $\mathrm{Ext}\,(\Omega)$ by a bounded domain and then prove the approximation converges in a suitable limit. To this end, we pick $R > 0$ and define a new (bounded!) domain

$$\Omega_R \doteq \mathrm{Ext}\,(\Omega) \cap B(0,R).$$

We assume $R \gg 1$ so that $\Omega \subseteq B(0,R)$. Intuitively, we have "$\Omega_R \to \mathrm{Ext}\,(\Omega)$ as $R \to \infty$". Applying the interior Kirchhoff formula in Ω_R and using that the outward normal on $\partial\Omega_R$ restricts to the inward normal on $\partial\Omega$, we have

$$u(x) = \oint_{\partial\Omega} u(y)\frac{\partial\Phi}{\partial\mathbf{n}}(x,y) - \Phi(x,y)\frac{\partial u}{\partial\mathbf{n}}(y)\ \mathrm{d}S(y) - \oint_{\partial B(0,R)} u(y)\frac{\partial\Phi}{\partial\mathbf{n}}(x,y) - \Phi(x,y)\frac{\partial u}{\partial\mathbf{n}}(y)\ \mathrm{d}S(y),$$

where we understand that $\mathbf{n}$ is the unit outward normal in each integral. Clearly, then, our main goal is to show that the integral over $\partial B(0,R)$ vanishes as $R \to \infty$. Since we have not yet used that our solution must satisfy the Sommerfeld condition (10.4), we should re-write

this integral suggestively:

$$\oint_{\partial B(0,R)} u(y)\frac{\partial \Phi}{\partial \mathbf{n}}(x,y) - \Phi(x,y)\frac{\partial u}{\partial \mathbf{n}}(y)\ \mathrm{d}S(y) = \frac{1}{4\pi}\oint_{\partial B(0,R)} u(y)\ \left[\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right) - ik\left(\frac{e^{ikr}}{r}\right)\right]_{r=R}\ \mathrm{d}S(y)$$
$$- \frac{1}{4\pi}\oint_{\partial B(0,R)} \Phi(x,y)\ \left(\frac{\partial u}{\partial r} - iku\right)(y)\ \mathrm{d}S(y)$$

$$= I_R^1 - I_R^2.$$

I_R^2 is the easier term to control, so let's deal with this one first. Using the Cauchy-Schwarz inequality for functions, we have

$$\left|I_R^2\right| \leq \left(\oint_{\partial B(0,R)} |\Phi(x,y)|^2\ \mathrm{d}S(y)\right)\ \left(\oint_{\partial B(0,R)} \left|\frac{\partial u}{\partial r} - iku\right|^2 (y)\ \mathrm{d}S(y)\right).$$

Since Φ is bounded above by R^{-1} on $\partial B(0,R)$ and u satisfies the Sommerfeld condition, we have

$$\left|I_R^2\right| \leq \left(4\pi R^2\right) R^{-2} \oint_{\partial B(0,R)} o\left(R^{-2}\right)\ \mathrm{d}S(y) \leq \left(16\pi^2 R^2\right) \times o\left(R^{-2}\right).$$

We conclude that

$$\lim_{R\to\infty} I_R^2 = 0. \tag{10.5}$$

Next, we deal with I_R^1. Another application of Cauchy-Schwarz, in conjunction with the computations presented in the proof of Lemma 10.2.11, gives

$$\left|I_R^1\right| \leq \left(\oint_{\partial B(0,R)} |u(y)|^2\ \mathrm{d}S(y)\right)\ \left(\oint_{\partial B(0,R)}\ \left[\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right) - ik\left(\frac{e^{ikr}}{r}\right)\right]_{r=R}^2\ \mathrm{d}S(y)\right)$$
$$= \mathcal{O}\left(R^{-2}\right) \times \oint_{\partial B(0,R)} |u(y)|^2\ \mathrm{d}S(y).$$

So, we have reduced the problem to proving that

$$\lim_{R\to\infty} \oint_{\partial B(0,R)} |u(y)|^2\ \mathrm{d}S(y) < \infty. \tag{10.6}$$

We accomplish this in a clever roundabout way. First, note that we can apply Green's first identity to the functions $u, \overline{u}$ (the overhead bar denotes complex conjugation) on Ω_R to find

$$\int_{\Omega_R} |\nabla u|^2 - k^2|u|^2\ \mathrm{d}y = \oint_{\partial\Omega_R} u(y)\ \frac{\partial \overline{u}}{\partial \mathbf{n}}(y)\ \mathrm{d}S(y) = \oint_{\partial B(0,R)} u(y)\ \frac{\partial \overline{u}}{\partial \mathbf{n}}(y)\ \mathrm{d}S(y) - \oint_{\partial\Omega} u(y)\ \frac{\partial \overline{u}}{\partial \mathbf{n}}(y)\ \mathrm{d}S(y).$$

Taking imaginary parts across this identity yields

$$\oint_{\partial\Omega} \mathrm{Im}\left(u(y)\ \frac{\partial \overline{u}}{\partial \mathbf{n}}(y)\right)\ \mathrm{d}S(y) = \oint_{\partial B(0,R)} \mathrm{Im}\left(u(y)\ \frac{\partial \overline{u}}{\partial \mathbf{n}}(y)\right)\ \mathrm{d}S(y) \quad \forall\ R \gg 1. \tag{10.7}$$

Next, we use that u satisfies the Sommerfeld radiation condition to say

$$\begin{aligned} 0 &= \lim_{R\to\infty} \oint_{\partial B(0,R)} \left| \frac{\partial u}{\partial r} - iku \right|^2 (y)\ \mathrm{d}S(y) \\ &= \lim_{R\to\infty} \oint_{\partial B(0,R)} \left| \frac{\partial u}{\partial r} \right|^2 + k^2|u|^2 + 2k\mathrm{Im}\left(u\frac{\partial \overline{u}}{\partial r} \right)\ \mathrm{d}S(y). \end{aligned}$$

Re-arranging and using (10.7), we discover

$$\lim_{R\to\infty} \oint_{\partial B(0,R)} \left| \frac{\partial u}{\partial r} \right|^2 + k^2|u|^2\ \mathrm{d}S(y) = -2k \oint_{\partial\Omega} \mathrm{Im}\left(u(y)\ \frac{\partial \overline{u}}{\partial \mathbf{n}}(y) \right)\ \mathrm{d}S(y) < \infty. \tag{10.8}$$

Since the integrand on the leftmost side is non-negative, we must have

$$\lim_{R\to\infty} \oint_{\partial B(0,R)} \left| \frac{\partial u}{\partial r} \right|^2\ \mathrm{d}S(y), \quad \lim_{R\to\infty} \oint_{\partial B(0,R)} |u|^2\ \mathrm{d}S(y) < \infty.$$

In particular, (10.6) holds and we're all done. □

Exercise 10.3.1. *Suppose that Ω is a domain and $u(x)$ is a solution to the Helmholtz equation on Ω. Assume there is an open subset $\Gamma \subseteq \partial\Omega$ with*

$$u|_\Gamma = \left. \frac{\partial u}{\partial \mathbf{n}} \right|_\Gamma = 0.$$

Prove that $u \equiv 0$ in Ω.

Exercise 10.3.2. *Can you "weaken" the Sommerfeld radiation condition on an exterior solution $u(x)$ and still recover the exterior Kirchhoff formula?*

10.4 Applications of the Kirchhoff Formula

Now, we mine the representation formulas for all they're worth! First, we use these formulas to show (with a little help from the theory of harmonic functions) that interior and exterior solutions of Helmholtz are automatically *analytic*. Then, we combine the representation formulas with a bit of good old-fashioned Taylor expansion to show the existence of a far-field pattern characterizing a given scattered wave (more precise results on uniqueness of the far-field pattern will have to wait for a future tutorial).

10.4.1 Analyticity Results

To prove that solutions to the Helmholtz equation are always analytic, we start by quoting a classical result on harmonic functions.

Theorem 10.4.1. *If $u(x)$ is harmonic in an open set $U \subseteq \mathbb{R}^n$ then $u(x)$ is analytic in U.*

Proof. This is a consequence of the mean value property of harmonic functions. The proof is a bit long and technical (relying on careful estimates of the derivatives of $u(x)$) so I won't go through it here. Interested readers may consult the proof in [11, Ch. 2]. □

Knowing that harmonic implies analytic makes it easy to show that the fundamental solution of Helmholtz is analytic:

Corollary 10.4.2. *Fix any $y \in \mathbb{R}^3$. For any $\epsilon > 0$, $\Phi(x, y)$ is analytic on*

$$A_\epsilon \doteq \left\{ x \in \mathbb{R}^3 \mid |x - y| > \epsilon \right\}.$$

Proof. Using the well-known Taylor series of the analytic function $z \mapsto e^z$, we can write

$$4\pi\Phi(x, y) = |x - y|^{-1} + \sum_{m=1}^{\infty} \frac{(ik)^m}{m!} |x - y|^{m-1}.$$

So, it suffices to show that $x \mapsto |x - y|^{-1}$ is analytic on A_ϵ. However, since this function is harmonic on A_ϵ, an application of Theorem 10.4.1 is all we need to finish the proof. □

Once we learn a bit more about functions on the sphere, we'll be able to produce a useful power series expression for $x \mapsto |x - y|^{-1}$ and, therefore, $\Phi(x, y)$. If you're particularly anxious to see these power series expansions soon, have a look at [7, §2.3] or [16, Ch. V].

Now, we arrive at the main punchline of this subsection.

Corollary 10.4.3. *Let Ω be a domain. If $u(x)$ is an interior (resp. exterior) solution of the Helmholtz equation with $u \in C^2\left(\overline{\Omega}\right)$ (resp. $u \in C^2\left(\overline{\mathrm{Ext}\left(\Omega\right)}\right)$) then $u(x)$ is analytic.*

Proof. We prove the theorem only for the interior case, as the exterior case is essentially the same. Fix any $x \in \Omega$ and $\epsilon < \frac{1}{2}\mathrm{dist}\left(x, \partial\Omega\right)$. Applying the Kirchhoff representation formula on $B(x, \epsilon)$ gives

$$u(x) = \oint_{\partial B(x,\epsilon)} -u(y) \frac{\partial \Phi}{\partial \mathbf{n}}(x, y) + \Phi(x, y) \frac{\partial u}{\partial \mathbf{n}}(y) \, \mathrm{d}S(y)$$

Analyticity of $u(x)$ in $B(x, \epsilon)$ then follows from Corollary 10.4.2. Since $x \in \Omega$ was arbitrary, the proof is complete. □

In later tutorials, we'll use the above in conjunction with the well-known identity theorem for analytic functions to help justify the efficacy of sonar.

10.4.2 Existence of Far-Field Patterns

Next, we prove that every sound wave has a "signature" at spatial infinity. This requires us to first establish a few asymptotic formulas for the Euclidean distance and the fundamental solution of Helmholtz.

Definition 10.4.4. *Given $x \neq 0 \in \mathbb{R}^3$,*

$$\hat{x} \doteq x/|x|.$$

Lemma 10.4.5. *As long as y stays in a bounded set, we have the expansion*

$$|x-y| = |x| - \hat{x}\cdot y + \mathcal{O}\left(|x|^{-1}\right) \quad \text{as} \quad |x| \to \infty.$$

Proof. By Taylor-expanding the function

$$z \mapsto \sqrt{a + x^{-2}} \quad (a \geq 0 \text{ fixed})$$

we find

$$\begin{aligned} |x-y| &= |x| + \left(\frac{-2x\cdot y + |y|^2}{2|x|}\right) + \mathcal{O}\left(|x|^{-2}\right) \\ &= |x| - \hat{x}\cdot y + \mathcal{O}\left(|x|^{-1}\right). \end{aligned}$$

□

Lemma 10.4.6. *As long as y stays in a bounded set, we have the expansions*

$$\Phi(x,y) = \frac{e^{ik|x|}}{4\pi|x|}\left[e^{-ik\hat{x}\cdot y} + \mathcal{O}\left(|x|^{-1}\right)\right] \quad \text{as} \quad |x| \to \infty,$$

$$\frac{\partial \Phi(x,y)}{\partial \mathbf{n}(y)} = \frac{e^{ik|x|}}{4\pi|x|}\left[\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \mathbf{n}(y)} + \mathcal{O}\left(|x|^{-1}\right)\right] \quad \text{as} \quad |x| \to \infty.$$

Proof. For the first expression, we start by combining the Taylor expansion of

$$z \mapsto \frac{1}{a+z} \quad (a \geq 0 \text{ fixed})$$

with Lemma 10.4.5 to find

$$4\pi\Phi(x,y) = \frac{e^{ik|x|}}{|x|}\left(e^{ik\left[-\hat{x}\cdot y + \mathcal{O}(|x|^{-1})\right]} + \mathcal{O}(|x|^{-2})\right).$$

Then, we use the Taylor series of $z \mapsto e^z$ to write

$$4\pi\Phi(x,y) = \frac{e^{ik|x|}}{|x|}\left(e^{-ik\hat{x}\cdot y}\right)\left(1 + \frac{ik}{|x|} + \mathcal{O}(|x|^{-2})\right) = \frac{e^{ik|x|}}{|x|}\left(e^{-ik\hat{x}\cdot y}\right)\left(1 + \mathcal{O}(|x|^{-1})\right),$$

hence we've established the first expression. The second expression then follows automatically since y only multiplies terms that are at worst $\mathcal{O}(1)$. □

Corollary 10.4.7. *Let $u(x)$ be an exterior solution to the Helmholtz equation. Then, there exists a C^2 function $u_\infty : \mathbb{S}^2 \to \mathbb{C}$ such that*

$$u(x) = \frac{e^{ik|x|}}{|x|}\left[u_\infty\left(\frac{x}{|x|}\right) + \mathcal{O}\left(|x|^{-1}\right)\right] \quad \text{as} \quad |x| \to \infty.$$

*u_∞ is called the **far-field pattern** of $u(x)$, and is given explicitly by*

$$u_\infty(\hat{x}) = \frac{1}{4\pi}\oint_{\partial\Omega} u(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \mathbf{n}(y)} - e^{-ik\hat{x}\cdot y}\frac{\partial u}{\partial \mathbf{n}(y)}\, \mathrm{d}S(y). \tag{10.1}$$

Proof. This follows immediately from combining the exterior Kirchhoff formula with Lemma 10.4.6. Remember that the boundary integral is taken with respect to y, so it doesn't affect the order of the $\mathcal{O}\left(|x|^{-1}\right)$ error terms from Lemma 10.4.6. □

By definition, the far-field pattern only depends on the angular variables in a spherical coordinate parameterization of $\mathbb{R}^3$. This result therefore tells us that, for large $|x|$, $u(x)$ actually looks like a truly separated solution! You may take this as a further example of why we often start solving PDEs by separating variables.

10.4.3 More Discussion of the Sonar Problem

We can now formulate two relatively precise questions relevant to hunting *Red October*. The first question is a sharpened version of Question 10.2.1:

Question 10.4.8. *Does each far-field pattern correspond to one and only one domain? In other words, can we tell if an undersea object is* Red October *or not based only on knowing its corresponding far-field pattern?*

The answer to this question is "yes"! Using separation of variables in spherical coordinates and analyticity of exterior solutions, we'll be able to at least show that polyhedral domains are uniquely determined by the corresponding far-field patterns. So, if we approximate *Red October* by a rectangular prism, we're in good shape. We'll return to this question before the course ends.

Of course, such an abstract solvability result is nice, but to really hone in on the rogue submarine, we need an answer to this final question:

Question 10.4.9. *Can a domain Ω be reconstructed via a stable and efficient numerical algorithm given sufficiently many samples of its corresponding far-field pattern? In other words, can we write a computer program that interfaces with a (possibly imperfect) acoustic measuring device to determine (in a timely fashion) if* Red October *is nearby?*

Unfortunately, a satisfying answer to this question is far outside the scope of our course. After you get comfortable with basic functional analysis and numerical methods, have a look at [7, Ch. 2-5] to get a modern perspective on this problem. But, for a first course, even just proving that the problem can be solved is something to get excited about! So, I hope you're anxious to see how the story concludes in a few months.

Chapter 11

Green's Functions in Spherical Geometry via the Method of Images

11.1 Introduction

In this tutorial I explain the **method of images**, a useful tool for finding the Green's function of a highly symmetric spatial domain. The method of images is essentially the reflection method in slightly different garb: generally speaking, the actual nuts-and-bolts implementation of both methods is quite similar. However, students who like thinking about PDEs physically may find the method of images a bit easier to work with, so I want to make sure everyone sees the details at least once in this course. The presentation is largely focused around two related concrete examples: the determination of the Green's function for the interior or exterior of a ball in $\mathbb{R}^3$. For other nice presentations of this material, see the books of Craig [9, §4.5] and Kellogg [16, Ch. IX].

11.2 Green's Functions via Electrostatics

To begin, I present some basic ideas from electrostatics: in my opinion, the how-and-why of the method of images is clearest in this particular physical context. First, I describe how Poisson equations appear in electromagnetic theory as constraints on the **potential** determined by a given distribution of electrical charge. Then, I demonstrate that the Green's function of a domain Ω represents (up to a constant factor) the potential of a **point charge** trapped inside a **grounded conductor** whose exterior wall coincides with $\partial\Omega$. From here, we'll be in a good spot to deeply understand the method of images later on. For a gentle physical introduction to further topics in electrostatics (and to fill in details I skim here), see [14, Ch. 2-4]. Note that our presentation goes against most modern physics texts and implicitly uses so-called Gaussian units in order to avoid the introduction of universal constants.

11.2.1 Basic Background in Electrostatics

Let $\Omega \subseteq \mathbb{R}^3$ be an open set (bounded or unbounded) with at least a piecewise smooth boundary. Electrostatics, as the name suggests, describes non-moving but electrically charged particles occupying Ω. Recall from living in the twenty-first century that electric charges come in two flavours, positive or negative. Suppose we are given the distribution of electric charges inside Ω via a **charge density function** $\rho(x)$, with units of charge per unit volume: $\rho(x) \gg 1$ where there are a lot of positively charged particles clustered near x, $\rho(x) \ll -1$ where there are a lot of negatively charged particles clustered near x, and $\rho(x) \approx 0$ either where there are very few particles near x, or all the particles near x are split in sign at a nearly $50-50$ ratio. If the total electric charge contained in Ω is q, then we must have

$$\int_\Omega \rho(x)\ \mathrm{d}x = q.$$

The main problem of electrostatics is to determine the **(static) electric field** $\mathbf{E}\colon \Omega \to \mathbb{R}^3$ created by the charge distribution $\rho(x)$. Think of the electric field this way: if we place a new charged particle with charge Q at $x \in \Omega$, then for a short time (before the new particle starts to influence the existing charge density) the force experienced by this particle is $Q\mathbf{E}(x)$. Quantitatively, $\mathbf{E}(x)$ is determined using the following principle:

Principle 11.2.1 (Maxwell's Equations for Electrostatics)**.** *Suppose Ω is occupied by static electric charges with known density $\rho\colon \Omega \to \mathbb{R}$. Then, the electric field $\mathbf{E}\colon \Omega \to \mathbb{R}^3$ inside Ω obeys the system of first-order PDEs*

$$\nabla \cdot \mathbf{E} = 4\pi\rho(x) \quad \textit{(Gauss' Law),} \tag{11.1a}$$

$$\nabla \times \mathbf{E} = 0 \quad \textit{(Ampère's Law).} \tag{11.1b}$$

We all know that scalars are much easier to work with than vectors. So, our first step in solving the Maxwell equations of electrostatics is to reduce these PDEs to a *single, scalar* PDE. Fortuantely, this is simple.

Corollary 11.2.2 (Potential Formulation of Electrostatics)**.** *Suppose that $\Omega \subseteq \mathbb{R}^3$ is simply connected. If the vector field $\mathbf{E}\colon \Omega \to \mathbb{R}^3$ obeys* (11.1)*, then there exists a C^2 function*

$$\phi\colon \Omega \to \mathbb{R}$$

*called the **potential** such that*

$$\mathbf{E} = -\nabla\phi. \tag{11.2}$$

In particular, ϕ satisfies the Poisson equation

$$\Delta\phi = -4\pi\rho(x) \quad \forall\ x \in \Omega. \tag{11.3}$$

Proof. Since Ω is simply connected, we may combine Ampère's law with de Rham's theorem from differential topology to obtain the existence of the potential ϕ. (11.3) then follows from plugging (11.2) into Gauss' law. □

Now, what boundary conditions should the potential ϕ satisfy? With a bit of physics, one can obtain some general boundary conditions on electric fields that, in turn, give rise to boundary conditions on the potential ϕ [14, §2.3.5]. For today, however, we only deal with the special case where $\partial\Omega$ is a **conductor** (a material that permits charged particles to flow through it with zero resistance) that is held at a known constant potential. In physics language, one uses the term **grounded** to describe such a conductor. Thus, ϕ obeys a simple Dirichlet boundary condition

$$\phi|_{\partial\Omega} = \phi_0 = \text{constant}.$$

But, we can make our lives even easier by noticing that potentials differing only by a constant give rise to the same electric field! In other words, two potentials are physically equivalent if their difference is constant. Therefore, we can without loss of generality demand that our potential satisfies a *homogeneous* Dirichlet BC. In summary, for a simply connected region $\Omega \subseteq \mathbb{R}^3$ enclosed by a grounded conductor and containing a charge distribution $\rho(x)$, the electric potential ϕ inside Ω obeys the Dirichlet boundary value problem

$$-\Delta\phi(x) = 4\pi\rho(x) \quad \forall\ x \in \Omega, \tag{11.4a}$$

$$\phi|_{\partial\Omega} = 0. \tag{11.4b}$$

Of course, you have already learned that the solution to (11.4) can be written in terms of the Green's function $G(x, x')$ of Ω as [27, p. 190]

$$\phi(x) = -4\pi \int_\Omega \rho(y)\ G(x, y)\ \mathrm{d}y. \tag{11.5}$$

11.2.2 Green's Function = Potential of a Point Charge

With rudimentary electrostatics under our belts, let's try and answer the following concrete question: suppose a domain Ω with a grounded conducting boundary is occupied by a single tiny particle with electric charge $q \in \mathbb{R}$ located at a fixed position $x' \in \Omega$. What is the potential $\Phi_q(x, x')$ produced by this single charged particle?

Now, (11.5) indicates that we can find $\Phi_q(x, x')$ provided we can express the charge density $\rho(x)$ for this particular problem. What properties should $\rho(x)$ have?

- First, since the particle is very small, we can reasonably suppose it occupies a single point in space. In other words, we demand that

$$\rho(x) = 0 \quad \forall\ x \neq x'. \tag{11.6}$$

- Additionally, since we know the electric charge of our particle is q, $\rho(x)$ must satisfy the integral constraint

$$\int_\Omega \rho(x)\ \mathrm{d}x = q. \tag{11.7}$$

If $\rho(x)$ obeys the two conditions above, we say that we are dealing with a **point charge**. Of course, no actual function can satisfy both conditions (11.6) and (11.7) simultaneously: we discussed this issue when we derived the heat kernel last semester. Thus we expect that the point charge density is not a function, but rather a Dirac δ-function.

Definition 11.2.3. *The **Dirac delta function** $\delta(x)$ is defined by the following requirement: given any differentiable (and, if Ω is not bounded, rapidly decaying) function $\varphi\colon \Omega \to \mathbb{R}$ and $x' \in \Omega$,*

$$\int_\Omega \varphi(x)\ \delta(x - x')\ \mathrm{d}x = \varphi(x'). \tag{11.8}$$

By choosing $\varphi \equiv 1$ in the case where Ω is bounded, we find that $\delta(x)$ formally obeys the normalization condition

$$\int_\Omega \delta(x - x')\ \mathrm{d}x = 1 \quad \forall\ x' \in \Omega. \tag{11.9}$$

In particular, it's often useful to pretend that $\delta(x)$ obeys

$$\delta(x - x') = \begin{cases} +\infty & x = x', \\ 0 & \text{otherwise.} \end{cases} \tag{11.10}$$

I said this in an earlier tutorial, but it bears repeating: the above expression is great for building intuition but useless for rigorous analysis, so deploy it wisely. I also emphasize that $\delta(x)$ makes perfect sense as long as it appears underneath an integral sign.

We then find that the charge density of a point particle located at $x' \in \Omega$ with charge q is given by

$$\rho(x) = q\delta(x - x').$$

Therefore, the potential of a point charge $\Phi_q(x, x')$ formally obeys the Poisson BVP

$$-\Delta_x \Phi_q(x, x') = 4\pi q \delta(x - x') \quad \forall\ x \in \Omega, \tag{11.11a}$$

$$\Phi_q(x, x')|_{x \in \partial\Omega} = 0. \tag{11.11b}$$

Using the representation formula (11.5) then yields

$$\Phi_q(x, x') = -4\pi q \int_\Omega \delta(y - x')\ G(x, y)\ \mathrm{d}y$$

If $x \neq x'$, then $G(x, x')$ is differentiable in x. Using the definition of the δ-function then gives

$$\Phi_q(x, x') = -4\pi q G(x, x') \quad \forall\ x \neq x'.$$

We conclude that the potential of a point charge inside Ω is precisely a constant times the Green's function of Ω! Thus we can use the terms "potential of a point charge" and "Green's function" essentially interchangeably. In particular, we can formally characterize the Green's function of Ω as the solution of the Dirichlet problem

$$\Delta G(x, x') = \delta(x - x') \quad \forall\ x \in \Omega, \tag{11.12a}$$

$$G(x, x')|_{x \in \partial\Omega} = 0. \tag{11.12b}$$

Example 11.2.4. Consider the case where our domain is all of $\mathbb{R}^3$. The potential of a point charge located at x' is then given by the fundamental solution of the Laplace equation:

$$\text{potential} = -4\pi q \times \frac{-1}{4\pi|x-x'|} = \frac{q}{|x-x'|}.$$

11.3 Green's Function Inside & Outside a Ball

With our new physical knowledge at hand, we can obtain a neat and intuitively pleasing alternative to the reflection method used in Strauss' book [27, §7.4] to compute the Green's function for various simple domains. This new scheme is called the **method of images** or the **method of image charges**. We'll learn the method of images by seeing it in action as we explicitly compute the Green's function for the interior and exterior of a solid ball in $\mathbb{R}^3$.

11.3.1 The Method of Images

Pick a fixed radius R and define a solid open ball by

$$B \doteq \left\{ x \in \mathbb{R}^3 \;\middle|\; |x| < R \right\}.$$

We also define the exterior of the ball by

$$\text{Ext}\,(B) = \mathbb{R}^3 - \overline{B}.$$

Since finding a Green's function is equivalent to finding a point charge potential, we can determine the Green's function of B and $\text{Ext}\,(B)$ by finding the potential of a point charge with $q = 1$ placed inside B or $\text{Ext}\,(B)$, assuming as always that the boundary is grounded. If $\Omega = B$ or $\text{Ext}\,(B)$, then by (11.11) the problem to be solved is

$$-\Delta_x H(x,x') = 4\pi\delta(x-x') \quad \forall\ x \in \Omega, \tag{11.1a}$$

$$H(x,x')|_{x\in\partial\Omega} = 0. \tag{11.1b}$$

See exercise 11.3.1 for information on a minor detail we have glossed over above.

How can we come up with an explicit expression for $H(x,x')$? All we really know is the geometry of our domain (B or $\text{Ext}\,(B)$) and the potential in the case of no grounded boundaries (namely $|x-x'|^{-1}$). As it turns out, this is all we need to apply the method of images! The recipe is written below.

1. Take away the grounded conducting surface along $\partial\Omega$ so our particle's potential defines a function on all of $\mathbb{R}^3$, call it

   $$\Phi_1(x,x') = |x-x'|^{-1}.$$

   $\partial\Omega$ is now a purely geometric object.

2. Place a point particle with charge $Q < 0$ at a point $\sigma(x') \in \text{Ext}\,(\Omega)$. This new particle is called an **image charge**. Call the potential of the image charge

$$\Phi_Q(x, x') = Q|x - \sigma(x')|^{-1}.$$

Thus the total potential of the two-particle system is, by linearity of the Poisson equation,

$$\Phi(x, x') = \Phi_1(x, x') + \Phi_Q(x, \sigma(x')), \quad x \in \mathbb{R}^3.$$

3. Adjust $\sigma(x')$ and $|Q|$ so that the potential of the original charge and the image charge exactly cancel each other out along $\partial\Omega$. Notice then that $\Phi|_{x\in\partial\Omega} = 0$, and that by construction all singularities of $\Phi - \Phi_1$ live outside of Ω.

4. By uniqueness of Green's functions [27, Exercise 7.3.1], we conclude that

$$H(x, x') = \Phi(x, x')|_{x\in\Omega}.$$

What a stunningly clever idea! Step 3 is the only one you should be worried about: how do we know there is a nice correspondence $x' \mapsto \sigma(x')$ that can give rise to $\Phi|_{x\in\partial\Omega} = 0$? For a general Ω, there is no such mapping. For very symmetric domains such as a half-space (quotient of $\mathbb{R}^3$ by the $\mathbb{Z}_2$ action $(x, y, z) \mapsto (x, y, -z)$) or the ball B and its exterior (both invariant under the obvious action of the orthogonal group $\mathcal{O}(3)$), however, $\sigma(x')$ can be constructed with just a little bit of patience. The next subsection is focused on computing $\sigma(x')$ for B; naturally, the same transformation should work for $\text{Ext}\,(B)$ as well. You may wish to do exercise 11.3.2 first to get a feel for how to find $\sigma(x')$ in an easier situations.

11.3.2 Topological Issues, Spherical Inversion, and the Kelvin Transform

Here, we derive the image charge Q and the original-to-image position mapping $x' \mapsto \sigma(x')$ for our ball B of radius R, as well as its exterior. Owing to the symmetry between the interior and exterior problems, we want $\sigma\colon B \to \text{Ext}\,(B)$ to be a bijection. This way we can solve the interior problem first, then simply invert σ and change charges appropriately to solve the exterior problem! As we'll see, we need to relax this requirement a little for practical purposes, but it provides a nice guiding light for the first few steps of our investigation.

Let's dive into the computations. In light of the bijection requirement above, we focus on the interior problem (determining the potential with an original charge inside B) first. Using the recipe laid out in the previous subsection, we know the total potential Φ created by the original and image charge together satisfies

$$0 = \Phi(x, x') = |x - x'|^{-1} + Q|x - \sigma(x')|^{-1} \quad \forall\ x \quad \text{s.t.} \quad |x| = R. \tag{11.2}$$

Choose the special point

$$x = \frac{R}{|x'|}x'.$$

Re-arranging (11.2) and using $Q < 0$ then tells us that

$$Q = -\frac{|\sigma(x')| - R}{R - |x'|}. \tag{11.3}$$

(11.3) gives a necessary condition for the method of images to work in this case. This formula gives us a certain amount of freedom in choosing the image position $\sigma(x')$, which is good news since we already imposed the harsh constraint of σ being a bijection. So, as long as we can build a bijection $B \to \mathrm{Ext}\,(B)$, we're all set to solve the problem.

Unfortunately, we have to make a teeny-tiny compromise and only demand that σ is a bijection between $B - \{0\}$ and $\mathrm{Ext}\,(B)$. Why does 0 cause trouble? The reason is topological. Obviously σ should also be continuous with continuous inverse, or in other words a **homeomorphism**. Now, any closed spherical surface inside B can be continuously shrunken down to a point without leaving B. In the language of algebraic topology, this means the **second homology group** of B is trivial. However, a closed spherical surface in $\mathrm{Ext}\,(B)$ can only be shrunk down to a point if it does not enclose the origin. This means that the second homology group of $\mathrm{Ext}\,(B)$ has a single generator (it is isomorphic to $\mathbb{Z}$). Of course, every kindergartener knows that homeomorphisms induce isomorphisms on homology groups, so since 0 and $\mathbb{Z}$ obviously aren't isomorphic we conclude B and $\mathrm{Ext}\,(B)$ aren't homeomorphic.

However, we can easily construct a homeomorphism $B - \{0\} \to \mathrm{Ext}\,(B)$ by **spherical inversion**:

$$\sigma(x') \doteq \frac{R^2}{|x'|^2} x', \quad x' \in B - \{0\} \tag{11.4}$$

To see why spherical inversion is so natural, look at the case $R = 1$ first. Notice that $\sigma(x')$ extends to a map on $\overline{B} - \{0\}$ fixing the sphere ∂B: this explains the name "spherical inversion". In fact, $\sigma(x')$ is its own inverse, so it's very easy to compute with! In the sequel, then, we refer to σ as a map from $\mathbb{R}^3 - \{0\}$ to itself. Therefore, as long as we don't put our image charge at the origin (the perfect center of the domain enclosed by our grounded conductor), we can define a useful original-to-image position map σ: see exercise 11.3.3 for some food for thought about this point.

Using the formula for spherical inversion in conjunction with (11.3), we find that the charge of our image particle should be

$$Q = -\frac{R}{|x'|}. \tag{11.5}$$

This motivates the following definition:

Definition 11.3.1. *Let $u\colon \mathbb{R}^3 - \{0\} \to \mathbb{R}$ be a continuous function and let $R > 0$. Define its* ***Kelvin transform with respect to the sphere of radius*** *R by*

$$\mathcal{K}[u](y) \doteq \frac{R}{|y|} u(\sigma(y)),$$

where σ denotes spherical inversion.

Therefore, the potential of the image particle can be written as

$$\Phi_Q(x,x') = -\mathcal{K}\left[\Phi_1(\cdot,x')\right](x),$$

where notation has been chosen to reflect that we Kelvin-transform only in the x-variable. The total potential created by the presence of both the original and the image particle is, then,

$$\Phi(x,x') = \Phi_1(x,x') - \mathcal{K}\left[\Phi_1(\cdot,x')\right](x) = |x-x'|^{-1} - \frac{R}{|x'|}\left|x - \frac{R^2}{|x'|^2}x'\right|^{-1}. \tag{11.6}$$

In exercise 11.3.4, you are asked to verify that indeed $\Phi(x,x')|_{x\in\partial B} = 0$: strictly speaking, we only checked this when x was parallel to x'. We conclude that the potential for a single particle of unit charge located at position x' *inside* a grounded sphere of radius R is

$$H(x,x') = \Phi(x,x')|_{|x|<R} = |x-x'|^{-1} - \frac{R}{|x'|}\left|x - \frac{R^2}{|x'|^2}x'\right|^{-1}.$$

Using invertibility of the original-to-image correspondence σ, we likewise find that the potential for a single particle of unit charge located at position x' *outside* a grounded sphere of radius R is given by the same expression! In purely mathematical terms, we can phrase this result as follows:

Theorem 11.3.2. *The Green's function for the interior or exterior of a ball of radius R is*

$$G(x,x') = -\frac{1}{4\pi|x-x'|} + \frac{1}{4\pi\left|\frac{|x'|}{R}x - \frac{R}{|x'|}x'\right|}. \tag{11.7}$$

11.3.3 Outline of the Reflection Method for this Example

I'll conclude by outlining how one would approach the above example via the reflection method, that is, without using the electrostatic interpretation of Green's functions and introducing an image charge. The procedure is quite similar to the method of images when it comes to technical details: you should interpret this to mean that the use of the method of images vs. the method of reflection is largely a matter of personal taste. In light of the similarity between the two methods, I'll be scant with details here and leave you to fill them in (exercise 11.3.6).

Of course, by exercise 11.3.3 we must still demand $x' \neq 0$. First, you need to extend $G(x,x')$ from $B - \{0\}$ to a function $\widetilde{G}(x,x')$ defined on $\mathbb{R}^3 - \{0\}$: the exterior case is so similar we ignore it here. A naïve way to perform this extension is via spherical inversion:

$$\widetilde{G}(x,x') = \begin{cases} G(x,x') & x \in B \\ G(\sigma(x),x') & x \notin B \end{cases}$$

One must then look at what PDE the extended function satisfies. Of course, $\widetilde{G}(x,x')$ is harmonic away from x' for $x \in B$. For $x \notin B$, however, $\widetilde{G}$ is not harmonic away from

$x \neq \sigma(x')$: the spherical inversion introduces some nasty coefficients we can't easily deal with. However, if we use the Kelvin transform to extend the Green's function according to

$$\widetilde{G}(x,x') = \begin{cases} G(x,x') & x \in B \\ \mathcal{K}\left[G(\cdot,x')\right](x) & x \notin B, \end{cases} \tag{11.8}$$

then $\widetilde{G}(x,x')$ is harmonic away from $x = x'$ and $x = \sigma(x')$! One can then represent $\tilde{G}(x,x')$ using the fundamental solution of Laplace's equation, and after unwinding notation (11.7) falls back out.

Exercise 11.3.1. *Verify (at least heuristically) that B and $\mathrm{Ext}\,(B)$ are simply connected, allowing us to actually use the potential formulation of Maxwell's equations for electrostatics in these cases.*

Exercise 11.3.2. *Use the method of images to find the Green's function for the half-space in $\mathbb{R}^3$. Hint: choose the map $x' \mapsto \sigma(x')$ and the image charge Q so the problem is as simple as possible.*

Exercise 11.3.3. *Why does the problem*

$$\begin{aligned} -\Delta_x H(x,0) &= 4\pi\delta(x) \quad \forall\ |x| < R, \\ H(x,0)|_{|x|=R} &= 0 \end{aligned}$$

have no solution? Conclude that a spherical conducting surface with a point charge placed directly at its centre cannot be grounded.

Exercise 11.3.4 ([14, Problem 3.8, p. 129])**.** *Pick $x' \in B - \{0\}$, $x \in \overline{B} - \{0\}$. Use the law of cosines to prove that the two-particle potential* (11.6) *vanishes on $|x| = R$. Hint: use that x' and its spherical inversion $\sigma(x')$ lie along the same line to help draw a useful triangle.*

Exercise 11.3.5. *What goes wrong with finding the Green's function for the interior/exterior of a ball if we assume that the image charge is $Q = -1$?*

Exercise 11.3.6. *Fill in the details for the derivation of the Green's function of a ball & its exterior via the method of reflection. Following* [11, Problem 2.11, p. 87]*, you should first prove that spherical inversion σ is conformal: in other words, for each fixed $x \in \mathbb{R}^3$, the Jacobian of σ at x is an orthogonal matrix. If you get stuck with the computations, see* [16, Ch. IX, §2].

Exercise 11.3.7 (adapted from [11, Problem 2.11, p. 87])**.**

a) Generalize spherical inversion and the Kelvin transform to $\mathbb{R}^n - \{0\}$.

b) Prove that n-dimensional spherical inversion is always conformal.

c) Prove that the n-dimensional Kelvin transform of a harmonic function is harmonic.

d) Are there values of n for which harmonicity is not invariant under conformal transformations? Compare this with the developments of the Monday, January 24 lecture.

Chapter 12

Practice With Bessel Functions

12.1 Introduction

In this tutorial, we review some essential properties of Bessel functions. Then, we go through two easy examples of how to use Bessel functions to solve physically motivated PDE problems. The first example involves the 2D version of the Helmholtz equation, and the second example involves an evolution equation appearing in the modelling of large-scale oceanic Rossby waves. For further examples of PDE problems involving Bessel functions, see for example [30]. For more theoretical discussion (beyond the scope of MAT351) on the connection between Bessel functions and the representation theory of the group of Euclidean motions of $\mathbb{R}^2$, see [34].

12.2 Review of Bessel Functions

The **Bessel equation of order** $\nu \in \{0, 1, 2, 3, ...\}$ is given by

$$x^2 y''(x) + xy'(x) + \left(x^2 - \nu^2\right) y(x) = 0. \tag{12.1}$$

Proposition 12.2.1. *There are two linearly independent, real-valued solutions $J_\nu(x), Y_\nu(x)$ to the ODE* (12.1) *valid on $(0, \infty)$, called the* ***Bessel functions of the first and second kinds****, respectively. Each of these functions has infinitely many real roots. Additionally, we have*

$$\lim_{x \to 0^+} J_\nu(x) < \quad \infty \quad \forall\ \nu, \quad \textit{and} \tag{12.2a}$$

$$\lim_{x \to 0^+} Y_\nu(x) = -\infty \quad \forall\ \nu. \tag{12.2b}$$

See figures 12.1 and 12.2 for plots of the first couple Bessel functions of each kind. Notice that, after $x \approx 5$, both kinds of Bessel function look like damped oscillations. This qualitative observation can be quantified using the following asymptotic expansions:

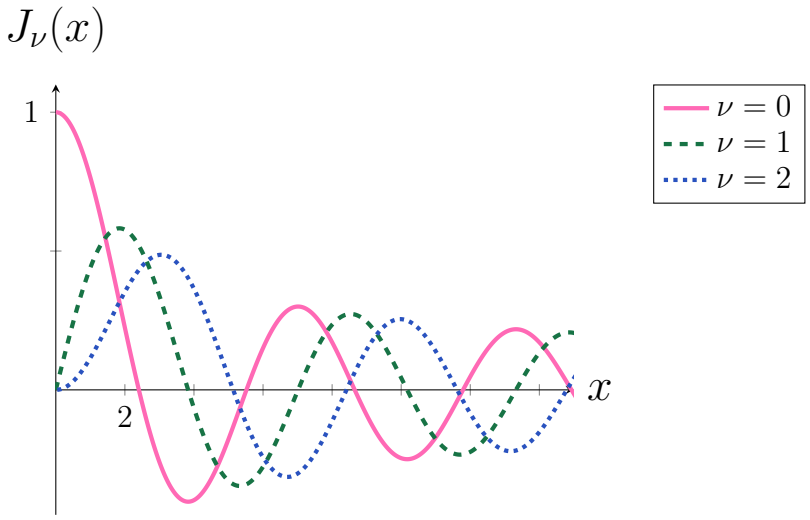

Figure 12.1: Plot of the first few Bessel functions of the first kind, $J_\nu(x)$. Note that all of these functions are finite at $x = 0$.

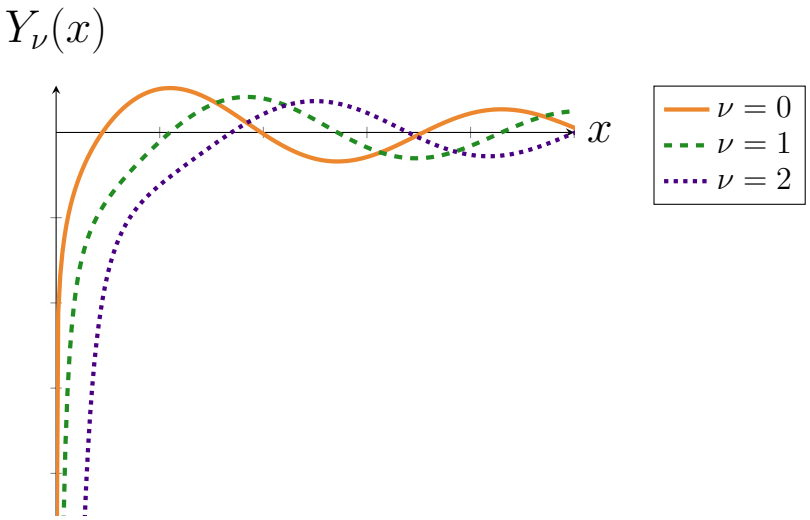

Figure 12.2: Plot of the first few Bessel functions of the second kind, $Y_\nu(x)$. Note that all of these functions diverge towards $-\infty$ as $x \to 0^+$.

Lemma 12.2.2 (Asymptotics for Bessel Functions)**.** *As $|x| \to \infty$, we have*

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(|x|^{-1}\right),$$
$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(|x|^{-1}\right).$$

Proof. See [20, §5.10-5.11] or [26, Appendix A]: the main thrust of the proof involves using methods of complex analysis to represent the Bessel functions as improper integrals. If you take a graduate course on asymptotic methods, you'll cover some systematic tools to obtain similar expansions of special functions. □

12.3 Example: Solving the Helmholtz Equation in 2D

Example 12.3.1. *For a fixed $k \in \mathbb{R}$, suppose a complex-valued function $\Phi: \mathbb{R}^2 - \{0\} \to \mathbb{C}$ satisfies the 2D Helmholtz equation subject to the Sommerfeld radiation condition:*

$$\begin{cases} \Delta\Phi(x,y) + k^2\Phi(x,y) = 0 & \forall\ x \in \mathbb{R}^2 - \{0\}, \\ \Phi_r - ik\Phi = o\left(r^{-1/2}\right) & as \quad r \to \infty. \end{cases} \tag{12.1}$$

Find $\Phi(x,y)$, and thereby determine the fundamental solution to the Helmholtz equation in the plane.

Solution:
First, we use the rotation-invariance of the Helmholtz operator

$$\Delta + k^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + k^2$$

to justify looking for a purely radial solution

$$\Phi(x,y) = \Phi(r).$$

Then, the Helmholtz equation reduces to the ODE

$$0 = r^2\Phi_{rr}(r) + r\Phi_r(r) + r^2k^2\Phi(r). \tag{12.2}$$

This is almost Bessel's equation of order 0! We want to get rid of the pesky extra k in the last term, so we change variables:

$$s = kr \Rightarrow \quad \Phi_r = k\Phi_s, \quad \Phi_{rr} = k^2\Phi_{ss}.$$

Implementing this change reduces our ODE to

$$0 = s^2\Phi_{ss} + s\Phi_s + s^2\Phi = 0. \tag{12.3}$$

This is actually (12.1) with $\nu = 0$, so we know that there exist constants $\alpha, \beta \in \mathbb{C}$ such that

$$\Phi = \alpha J_0(s) + \beta Y_0(s).$$

Changing back to the original variable r gives

$$\Phi(r) = \alpha J_0(kr) + \beta Y_0(kr). \tag{12.4}$$

Next, we need to determine α, β using the Sommerfeld condition. Since we're dealing with $r \gg 1$, we want to make use of the asymptotic expansions of J_0, Y_0 given in lemma 12.2.2. Using these expansions, we have

$$\Phi(r) = \sqrt{\frac{2}{\pi kr}} \left[\alpha \cos\left(kr - \frac{\pi}{4}\right) + \beta \sin\left(kr - \frac{\pi}{4}\right)\right] + \mathcal{O}(r^{-1}).$$

Then, we find

$$\begin{aligned} r^{1/2}\left[\Phi_r - ik\Phi\right] = &\sqrt{\frac{2}{2\pi}}\left(-\frac{1}{2}r^{-1}\right)\left[\alpha \cos\left(kr - \frac{\pi}{4}\right) + \beta \sin\left(kr - \frac{\pi}{4}\right)\right] \\ &+ \sqrt{\frac{2}{\pi k}}\left[-k\alpha \sin\left(kr - \frac{\pi}{4}\right) + \beta k \cos\left(kr - \frac{\pi}{4}\right)\right] \\ &- ik\sqrt{\frac{2}{\pi k}}\left[\alpha \cos\left(kr - \frac{\pi}{4}\right) + \beta \sin\left(kr - \frac{\pi}{4}\right)\right] \\ &+ \mathcal{O}(r^{-1/2}). \end{aligned}$$

To ensure that the right-hand side vanishes as $r \to \infty$, we need to have

$$\begin{aligned} -i\beta - \alpha &= 0, \\ \beta - i\alpha &= 0. \end{aligned}$$

This implies $\alpha = 1, \beta = i$, so we conclude

$$\Phi(r) = J_0(kr) + iY_0(kr). \tag{12.5}$$

If we define the **Hankel functions of the first and second kinds with order** ν by

$$H_\nu^{(1)}(x) \doteq J_\nu(x) + iY_\nu(x), \tag{12.6a}$$

$$H_\nu^{(2)}(x) \doteq J_\nu(x) - iY_\nu(x) \tag{12.6b}$$

(which clearly also give a fundamental set of solutions to (12.1)), then we can also write

$$\Phi(r) = H_0^{(1)}(kr). \tag{12.7}$$

In other words, the solution of (12.1) is given by a scaled version of the order 0 Hankel function of the first kind. Notice that we have the asymptotic expansion

$$\Phi(r) = \sqrt{\frac{2}{\pi kr}} e^{i\left(kr - \frac{\pi}{4}\right)} + \mathcal{O}\left(r^{-1}\right). \tag{12.8}$$

Using this expansion, we can easily double-check that the Sommerfeld radiation condition is satisfied. □

Exercise 12.3.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain. Write the solution of*

$$\Delta u(x,y) + k^2 u(x,y) = 0 \quad \forall\ (x,y) \in \Omega. \tag{12.9}$$

as an integral over $\partial\Omega$ involving $\Phi(x,y)$. Make sure to justify your formula rigorously.

Exercise 12.3.2. *Repeat the previous exercise, this time assuming $u(x)$ solves Helmholtz* outside Ω. *What conditions at infinity need to be imposed on u?*

12.4 Example: Rossby Waves in a Circular Ocean

Example 12.4.1. *Let's model the geometry of an ocean surrounded by land using a two-dimensional ball of radius $R > 0$:*

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\right\}.$$

One can use perturbation theory to show that, for two dimensionless physical parameters β, F, the free surface elevation $\phi(x,y,t)$ of large-scale Rossby waves obeys an IBVP of the form [23, Ch. 3]

$$\begin{cases} (\Delta - F)\,\phi_t + \beta\phi_x = 0 & \forall\ (x,y,t) \in \Omega \times (0,\infty), \\ \phi|_{t=0} = \phi_0(x,y) & \forall\ (x,y) \in \Omega, \\ \phi|_{\partial\Omega}(x,y) = 0 & \forall\ (x,y) \in \partial\Omega, \end{cases} \tag{12.1}$$

for a given initial state ϕ_0. The PDE appearing in the first line is called the ***linearized quasigeostrophic potential vorticity equation****.*

a) Introduce an a priori *unknown frequency parameter $\omega \in \mathbb{R}$ and make the ansatz*

$$\phi(x,y,t) = \psi(x,y)\cos\left(\frac{\beta x}{2\omega} + \omega t\right)$$

to transform (12.1) *into a time-independent elliptic eigenvalue problem.*

b) Explain why y does not appear in the cosine term of the above ansatz.

c) Assuming that $\psi(x,y)$ is rotationally symmetric, find the eigenvalues of the problem you wrote down in part a. Express your answer in terms of R and ζ_{mn} = the n^{th} zero of $J_m(x)$.

d) Show that there are countably many admissible frequencies ω_n, and write them out in terms of the spatial eigenvalues. This relationship between spatial eigenvalues and frequencies is called the ***dispersion relation*** *for* (12.1).

Solution:

a) We directly compute

$$\begin{aligned}
0 &= (\Delta - F)\,\phi_t + \beta\phi_x \\
&= -\omega\Delta\left(\psi\sin\left(\frac{\beta x}{2\omega}+\omega t\right)\right) + F\omega\psi\sin\left(\left(\frac{\beta x}{2\omega}+\omega t\right)\right) \\
&\quad + \beta\psi_x\cos\left(\frac{\beta x}{2\omega}+\omega t\right) - \frac{\beta^2}{2\omega}\psi\sin\left(\frac{\beta x}{2\omega}+\omega t\right) \\
&= -\omega\left[\Delta\psi\sin\left(\frac{\beta x}{2\omega}+\omega t\right) + \frac{\beta}{\omega}\psi_x\cos\left(\frac{\beta x}{2\omega}+\omega t\right) - \left(\frac{\beta}{2\omega}\right)^2\psi\sin\left(\frac{\beta x}{2\omega}+\omega t\right)\right] \\
&\quad + F\omega\psi\sin\left(\left(\frac{\beta x}{2\omega}+\omega t\right)\right) \\
&\quad + \beta\psi_x\cos\left(\frac{\beta x}{2\omega}+\omega t\right) - \frac{\beta^2}{2\omega}\psi\sin\left(\frac{\beta x}{2\omega}+\omega t\right) \\
&= \left\{\Delta\psi + \left(\left(\frac{\beta}{2\omega}\right)^2 - F\right)\psi\right\}\omega\sin\left(\frac{\beta x}{2\omega}+\omega t\right).
\end{aligned}$$

Then, if we define

$$\gamma^2 \doteq \left(\frac{\beta}{2\omega}\right)^2 - F, \tag{12.2}$$

(12.1) is transformed to the purely spatial eigenvalue problem

$$\begin{cases} \left(\Delta + \gamma^2\right)\psi = 0 & \forall\ (x,y)\in\Omega, \\ \psi|_{\partial\Omega}(x,y) = 0 & \forall\ (x,y)\in\partial\Omega. \end{cases} \tag{12.3}$$

b) No y appears in the ansatz because of the ϕ_x term in the governing PDE: there is a natural anisotropy in the model that should be reflected in any physically interesting solutions.

c) If $\psi(x,y)$ is rotationally symmetric, then $\psi(x,y) = \psi(r)$. Using the polar form of the Laplacian, this means the PDE in (12.3) reduces to

$$0 = r^2\psi_{rr} + r\psi_r + \gamma^2 r^2\psi.$$

From the previous example, we know then that $\psi(x,y)$ can be written as a (real) linear combination of Bessel functions:

$$\psi(x,y) = A\ J_0(\gamma r) + B\ Y_0(\gamma r). \tag{12.4}$$

Now, ψ mus be well-defined throughout the disc of radius R centred about the origin $r = 0$, so we know that $B = 0$ is necessary. Therefore, we find

$$\psi(x,y) = A\ J_0(\gamma r). \tag{12.5}$$

Using the BC at $r = R$, we find that

$$0 = J_0(\gamma R). \tag{12.6}$$

Therefore, γR must be equal to ζ_{0n} for some $n = 1, 2, ...$ We conclude that there is a countably infinite number of eigenvalues γ_n^2 for (12.3), and that these eigenvalues are given explicitly by

$$\gamma_n^2 = \left(\frac{\zeta_{0n}}{R}\right)^2, \quad n = 1, 2, 3, \ldots . \tag{12.7}$$

d) Using (12.2) and (12.7), we discover

$$\left(\frac{\beta}{2\omega}\right)^2 - F = \left(\frac{\zeta_{0n}}{R}\right)^2.$$

Re-arranging this expression yields

$$\omega_n = \frac{\beta}{2\sqrt{F + \left(\frac{\zeta_{0n}}{R}\right)^2}}, \quad n = 1, 2, 3, \ldots . \tag{12.8}$$

which is the desired dispersion relation.

Exercise 12.4.1. *Find the dispersion relation for Rossby waves in a rectangular ocean.*

Chapter 13

The Lorentz Group and the Wave Equation

In tutorial, some students asked me how a particular problem in HW 8 could be done without messy linear algebra computations involving a bunch of indices. Below is my response. I hope to convince you that you should start moving away from heavy index computations whenever possible.

Definition 13.0.1. *Consider the* 4×4 *matrix* η *defined below:*

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{13.1}$$

We say $L \in \mathrm{GL}(4, \mathbb{R})$ *is a* ***Lorentz transformation*** *if*

$$L^{-1} = \eta L^{\mathrm{T}} \eta. \tag{13.2}$$

The set of all Lorentz transformations is denoted by $\mathcal{O}(1,3)$*.*

Lemma 13.0.2. $\mathcal{O}(1,3)$ *forms a group under the operation of matrix multiplication.*

Proof. To prove the claim, we must show that

- the product of any two Lorentz transformations is a Lorentz transformation (closure under multiplication), and
- the inverse matrix of any Lorentz transformation is a Lorentz transformation (closure under inversion).

To prove closure under multiplication, suppose $L, M \in \mathcal{O}(1,3)$. A direct computation yields that

$$\eta^2 = \mathrm{Id}.$$

Then,

$$\eta\left(LM\right)^{\mathrm{T}}\eta = \eta M^{\mathrm{T}}L^{\mathrm{T}}\eta = \left(\eta M^{\mathrm{T}}\eta\right)\left(\eta L^{\mathrm{T}}\eta\right) = M^{-1}L^{-1} = \left(LM\right)^{-1}$$

which is what we wanted to show. For closure under inversion, note that for any $L \in \mathcal{O}(1,3)$

$$\left[\eta\left(L^{-1}\right)^{\mathrm{T}}\eta\right]^{-1} = \left[\eta\left(L^{\mathrm{T}}\right)^{-1}\eta\right]^{-1} = \eta^{-1}L^{\mathrm{T}}\eta^{-1} = \eta L^{\mathrm{T}}\eta = L^{-1}.$$

Taking the inverse of both sides completes the proof. □

Lemma 13.0.3. *Define a functional on* $\mathbb{R}^4$ *by*

$$m(\mathbf{v}) = \mathbf{v}^{\mathrm{T}}\eta\mathbf{v} = -v_0^2 + v_1^2 + v_2^2 + v_3^2, \quad \textit{where} \quad \mathbf{v} = \left(v_0, v_1, v_2, v_3\right)^{\mathrm{T}}.$$

$L \in \mathrm{GL}(4,\mathbb{R})$ *is contained in the subgroup* $\mathcal{O}(1,3)$ *if and only if* $m(L\mathbf{v}) = m(\mathbf{v})\ \forall\ \mathbf{v} \in \mathbb{R}^4$.

Proof. First, suppose that $L \in \mathcal{O}(1,3)$. We have

$$\begin{aligned} m\left(L\mathbf{v}\right) &= \left(L\mathbf{v}\right)^{\mathrm{T}}\eta L\mathbf{v} \\ &= \mathbf{v}^{\mathrm{T}}\left[L^{\mathrm{T}}\eta L\right]\mathbf{v}. \end{aligned}$$

By definition of a Lorentz transformation and $\eta^2 = \mathrm{Id}$, we find

$$L^{\mathrm{T}}\eta L = \eta$$

so the above becomes

$$m\left(L\mathbf{v}\right) = \mathbf{v}^{\mathrm{T}}\left[\eta\right]\mathbf{v} = m(\mathbf{v}),$$

so the $\Rightarrow$ implication is taken care of.

For the opposite implication, suppose $L \in \mathrm{GL}(4,\mathbb{R})$ preserves m:

$$m(\mathbf{v}) = m\left(L\mathbf{v}\right) \quad \forall\ \mathbf{v} \in \mathbb{R}^4.$$

In particular, the matrices of the quadratic forms $\mathbf{v} \mapsto m(\mathbf{v})$ and $\mathbf{v} \mapsto m\left(L\mathbf{v}\right)$ must agree. Using the work from the previous step,

$$\mathbf{v}^{\mathrm{T}}\eta\mathbf{v} = \mathbf{v}^{\mathrm{T}}\left[\eta\left(\eta L^{\mathrm{T}}\eta\right)L\right]\mathbf{v} \quad \forall\ \mathbf{v} \in \mathbb{R}^4.$$

Since the above holds for all $\mathbf{v} \in \mathbb{R}^4$, we conclude

$$\eta = \eta\left(\eta L^{\mathrm{T}}\eta\right)L,$$

which is readily re-arranged to obtain

$$L^{-1} = \eta L^{\mathrm{T}}\eta,$$

the defining characteristic of a Lorentz transformation. □

The above lemma tells us that the Lorentz transformations preserve the pseudo-norm $m(\mathbf{v})$, so such transformations generalize orthogonal transformations: this explains the notation $\mathcal{O}(1,3)$.

Lemma 13.0.4 (Gradient Transformation for Linear Changes of Coordinates)**.** *Suppose L is an invertible $n \times n$ matrix of real numbers. Consider a linear change of coordinates*

$$\begin{aligned} L\colon \mathbb{R}^n &\to \mathbb{R}^n \\ \mathbf{x} &\mapsto \mathbf{y}(\mathbf{x}) = L\mathbf{x}. \end{aligned}$$

Then, for any $u \in C^1(\mathbb{R}^n)$,

$$\nabla_{\mathbf{x}} (u \circ L)(\mathbf{x}) = L^{\mathrm{T}} (\nabla_{\mathbf{y}} u)(\mathbf{y}(x)),$$

where the subscripts on the gradient operators indicate what set of coordinates we differentiate with respect to. In operator form, the above can be written

$$\nabla_{\mathbf{x}} = L^{\mathrm{T}} \nabla_{\mathbf{y}}.$$

Notice how we are treating the operators $\nabla_{\mathbf{x}}$, $\nabla_{\mathbf{y}}$ like vectors in $\mathbb{R}^n$ with operator-valued entries, so multiplication of either of these operators by a matrix makes perfect sense.

Proof. By the chain rule from multivariable calculus, the pushforward of $u \circ L$ at any point $\mathbf{x} \in \mathbb{R}^n$ is given by

$$(u \circ L)_{*,\mathbf{x}} \mathbf{v} = u_{*,L\mathbf{x}} L_{*,\mathbf{x}} \mathbf{v} = u_{*,L\mathbf{x}} L\mathbf{v} \quad \forall\ \mathbf{v} \in \mathrm{T}_{\mathbf{x}}\mathbb{R}^n \simeq \mathbb{R}^n.$$

Using the relationship between pushforward and gradient, the above yields

$$[\nabla_{\mathbf{x}} (u \circ L)(\mathbf{x})]^{\mathrm{T}} \mathbf{v} = [(\nabla_{\mathbf{y}} u)(\mathbf{y}(x))]^{\mathrm{T}} L\mathbf{v} = \left[L^{\mathrm{T}} (\nabla_{\mathbf{y}} u)(\mathbf{y}(x))\right]^{\mathrm{T}} \mathbf{v} \quad \forall\ \mathbf{v} \in \mathrm{T}_{\mathbf{x}}\mathbb{R}^n \simeq \mathbb{R}^n.$$

We conclude that

$$[\nabla_{\mathbf{x}} (u \circ L)(\mathbf{x})]^{\mathrm{T}} = \left[L^{\mathrm{T}} (\nabla_{\mathbf{y}} u)(\mathbf{y}(x))\right]^{\mathrm{T}}$$

which yields the claim upon taking transposes. □

Proposition 13.0.5. *Suppose $u\colon \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}$ solves the wave equation*

$$u_{tt} - \Delta u = 0$$

and $L \in \mathcal{O}(1,3)$. Then, $U \doteq u \circ L$ also solves the wave equation.

Proof. We use the notation

$$\mathbf{x} = (t, x, y, z) \in \mathbb{R}_t \times \mathbb{R}^3_x = \mathbb{R}^4$$

and

$$\mathbf{y} \doteq L\mathbf{x}.$$

Notice that the wave equation for u can be rewritten in the form

$$0 = \left(\nabla_{\mathbf{x}}^{\mathrm{T}} \eta \nabla_{\mathbf{x}}\right) u.$$

We then have using lemma 13.0.4 and $\eta^2 = \mathrm{Id}$ that

$$\begin{aligned}
-\left(U_{tt} - \Delta U\right) &= \left(\nabla_{\mathbf{x}}^{\mathrm{T}} \eta \nabla_{\mathbf{x}}\right)(u \circ L) \\
&= \left(\nabla_{\mathbf{x}}^{\mathrm{T}} \eta\right) L^{\mathrm{T}} \nabla_{\mathbf{y}} u\left(\mathbf{y}(x)\right) \\
&= \nabla_{\mathbf{x}}^{\mathrm{T}} \left(\eta L^{\mathrm{T}} \eta\right) \eta \nabla_{\mathbf{y}} u\left(\mathbf{y}(x)\right) \\
&= \left(\nabla_{\mathbf{x}}^{\mathrm{T}} L^{-1}\right) \eta \nabla_{\mathbf{y}} u\left(\mathbf{y}(x)\right) \\
&= \left(\nabla_{\mathbf{y}}^{\mathrm{T}} L L^{-1}\right) \eta \nabla_{\mathbf{y}} u\left(\mathbf{y}(x)\right) \\
&= \nabla_{\mathbf{y}}^{\mathrm{T}} \eta \nabla_{\mathbf{y}} u\left(\mathbf{y}(x)\right) \\
&= -\left(u_{tt} - \Delta u\right)\left(\mathbf{y}(\mathbf{x})\right) \\
&= 0.
\end{aligned}$$

We could have skipped a few early steps using the operator expression of lemma 13.0.4, but I want to make sure you understand how both the original and transformed coordinates figure into this argument. □

In more formal language, the above result tells us that $\mathcal{O}(1,3)$ is a **symmetry group** for the wave equation.

Chapter 14

The Wacky World of Nonlinear Wave Equations

14.1 Introduction

In the past few lectures, you have studied the linear wave equation in $\mathbb{R}^3$, discussing finite speed of propagation and the derivation of Kirchhoff's representation formula. Today, we'll use these ideas and techniques to help understand two interesting examples in the theory of *nonlinear* wave equations. Nonlinear wave equations appear throughout theoretical physics, being especially important in (for example) high-amplitude acoustics, general relativity, and Yang-Mills-Higgs theories. For today, however, we won't talk about physics, instead focusing on some purely analytical aspects of two relatively simple wave equations (at least, simple compared to Yang-Mills): analyzing such toy PDEs is a crucial first step towards understanding the really cool physics stuff. The interested reader can find graduate-level introductions to the theory of nonlinear wave equations in [11, 29].

Throughout these notes, $B(x, R)$ denotes the ball of radius $R > 0$ centred at $x \in \mathbb{R}^3$.

14.2 Ill-Posedness for Some Algebraic Nonlinearities

Suppose I is a (possibly infinite) time interval containing 0. For p an odd integer, we study the IVP below:

$$\begin{cases} v_{tt} - \Delta v - |v|^{p-1}v = 0 \quad \forall\ (x,t) \in \mathbb{R}^3 \times I, \\ v|_{t=0} = f(x) \quad \forall\ x \in \mathbb{R}^3, \\ v_t|_{t=0} = g(x) \quad \forall\ x \in \mathbb{R}^3. \end{cases} \tag{14.1}$$

The PDE appearing in the above IVP is called the **focusing algebraic nonlinear wave equation (FNLW)**. As we'll see soon, FNLW has many interesting mathematical properties that dramatically illustrate how nonlinearity can give rise to strange behaviour. In particular, we'll prove that there are smooth, compactly supported initial data for which the

corresponding solution of (14.1) blows up to infinity in finite time! Our discussion in this section is based on [29, §3.2] (especially exercise 3.9).

Before we discuss finite-time blowup, we need a helpful fact concerning a nonlinear variant of the principle of causality for linear wave equations:

Proposition 14.2.1 (Finite Speed of Propagation for FNLW)**.** *Suppose v_1 and v_2 both satisfy FNLW in some time interval I. Further, suppose these two solutions and their time derivatives agree in $B(x_0, R)$ at time 0. Then, we must have*

$$v_1(x,t) = v_2(x,t) \quad \forall\ (x,t) \quad s.t. \quad |x - x_0| \leq R - |t| \quad and \quad t \in I.$$

Proof. The proof follows along similar lines to the analogous result for the linear wave equation, except nonlinearity makes the energy much more complicated. The interested reader may find a complete proof in [29, proposition 3.3]. □

To see what this result has to do with finite speed of propagation, suppose our initial data $v|_{t=0}$, $v_t|_{t=0}$ are compactly supported inside $B(0,1)$. Pick $x_0 \in \mathbb{R}^3$ with $|x_0| \gg 1$, then set $R = |x_0|/2$. In $B(x_0, R)$, the above tells us that v must agree with the zero solution at every x with

$$|x - x_0| \leq R - |t|.$$

That is, information about v not being identically zero cannot affect such positions x. As $|t|$ gets larger, however, there are fewer and fewer positions where we can guarantee the solution is zero: this reflects the "residue" of the initial data finally moving into $B(x_0, R)$. That is, we have to wait a finite amount of time to start to feel the effect of the initial state if we are very far from the origin. Additionally, notice that the proposition immediately yields uniqueness of solutions to (14.1).

Now, we're in a good position to start talking about blowup. Our first step towards constructing a blowup solution (that is, a solution to (14.1) which goes to infinity as $t \to t_0^-$ where t_0 is a fixed, finite time) is to focus on purely time-dependent solutions to FNLW. If $v(t)$ satisfies FNLW, then the ODE

$$v''(t) = |v|^{p-1} v \tag{14.2}$$

must hold. Plugging the ansatz

$$v(t) = c(t_0 - t)^q \quad \text{with} \quad t_0, c, q \quad \text{fixed}$$

(valid for $-\infty < t < t_0$) into the above, we obtain the system

$$q(q-1) = |c|^{p-1}, \tag{14.3a}$$
$$q - 2 = qp. \tag{14.3b}$$

In turn, this implies

$$q = \frac{-2}{p-1},$$

Therefore, we have a time-independent solution to FNLW,

$$v(t) = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}} (t_0 - t)^{\frac{-2}{p-1}} . \tag{14.4}$$

In particular,

$$\lim_{t \to t_0^-} v(t) = +\infty,$$

so $v(t)$ is a blowup solution. However, the initial data f, g are simply constant functions, and in the world of nonlinear wave equations we like our initial data to decay in space (you should know the reason for this by now!). So, $v_*(t)$ is not really a genuinely weird, mind-shattering blowup solution. That being said, we can use $v_*(t)$ to construct such a weird blowup solution, one with smooth and compactly supported initial data!

Theorem 14.2.2. *There exist smooth, compactly supported initial data f, g for which the corresponding solution to* (14.1) *blows up in finite time.*

Proof. Let $\varphi: \mathbb{R}^3 \to \mathbb{R}$ be a smooth bump function equal to 1 on $B(0,1)$ and compactly supported within $B(0,2)$ (see exercise 14.2.3). Choose any $t_0 > 0$ and some $R > t_0$. Let $v_*(t)$ be given by (14.4) with this particular choice of t_0. We use the initial data

$$f(x) = v_*(0)\ \varphi(x/R),$$
$$g(x) = v_*'(0)\ \varphi(x/R).$$

Suppose there exists a smooth global-in-time solution to (14.1) with these f, g and call it $u(x,t)$. Pick any $\epsilon > 0$. By finite speed of propagation (proposition 14.2.1),

$$u(x, t_0 - \epsilon) = v(t_0 - \epsilon) = \text{constant} \times \epsilon^{\frac{-2}{p-1}} \quad \forall\ x \quad \text{s.t.} \quad |x| \leq R - t_0 + \epsilon. \tag{14.5}$$

However, since $u(x,t)$ is smooth and exists for all time,

$$\lim_{t \to t_0^-} u(x,t) = \lim_{\epsilon \to 0^+} u(x, t_0 - \epsilon) < \infty \quad \forall\ x \in \mathbb{R}^d.$$

The above statement is contradicted by taking $\epsilon \to 0^+$ in (14.5), so we're done. □

Notice that the particular amplitudes of the initial data provided the mechanism for blowup! This suggests that the size of our initial data plays a huge role in determining the behaviour of solutions to a nonlinear PDE.

Exercise 14.2.1. *Find an energy functional conserved by FNLW. Is this energy strictly positive?*

Exercise 14.2.2. *Can you construct a variant of the blowup solution* (14.4) *for the* ***defocusing NLW***

$$v_{tt} - \Delta v + |v|^{p-1} v = 0?$$

Exercise 14.2.3. *Explicitly construct the bump function $\varphi(x)$ appearing in the proof of theorem 14.2.2. Consult a real analysis book or* [11] *if you get stuck.*

14.3 Small-data Global Existence for a Derivative-Nonlinear Wave Equation

The last section may have you very frightened: is blowup a generic feature of nonlinear wave equations? In this section, I'll attempt to help allay those fears by giving an example of a very nasty-looking nonlinear wave equation that admits global-in-time solutions for a suitable class of nice initial data. Specifically, we look at the IVP

$$\begin{cases} v_{tt} - \Delta v + (v_t)^2 - |\nabla_x v|^2 = 0 \quad \forall\ (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ v|_{t=0} = f(x) \quad \forall\ x \in \mathbb{R}^3, \\ v_t|_{t=0} = g(x) \quad \forall\ x \in \mathbb{R}^3. \end{cases} \tag{14.1}$$

We'll show that, if f, g are compactly supported and "small enough" in a suitable norm, then a $v(x,t) \in C^2(\mathbb{R}^3 \times \mathbb{R})$ solving (14.1) exists. This requires a clever change of variables to transform the problem to a *linear* wave equation which we can analyze using the Kirchhoff formula. The content here is essentially a complete solution of exercises 6.3 and 6.4 in [9].

Before beginning, we need to define some helpful norms we can use to help quantify what we mean by "small enough" initial data.

Definition 14.3.1.

- *We denote the set of continuous, bounded, real-valued functions on $\mathbb{R}^3$ by $C_b(\mathbb{R}^3)$. This becomes a normed real vector space when we define*

$$\|f\|_{C_b(\mathbb{R}^3)} \doteq \sup_{x \in \mathbb{R}^3} |f(x)|. \tag{14.2}$$

- *If $\mathbf{V}: \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field with entries in $C_b(\mathbb{R}^3)$, we define*

$$\|\mathbf{V}\|_{C_b(\mathbb{R}^3)} \doteq \sup_{x \in \mathbb{R}^3} |\mathbf{V}(x)|, \tag{14.3}$$

  where $|\mathbf{V}(x)|$ is the Euclidean norm of $\mathbf{V}(x)$.

- *We define $C_b^1(\mathbb{R}^3)$ to be the subspace of $C_b(\mathbb{R}^3)$ consisting of those bounded continuous functions $f(x)$ whose gradients $\nabla f(x)$ have entries in $C_b(\mathbb{R}^3)$. We put another norm on this space by defining*

$$\|f\|_{C_b^1(\mathbb{R}^3)} \doteq \|f\|_{C_b(\mathbb{R}^3)} + \|\nabla f\|_{C_b(\mathbb{R}^3)}. \tag{14.4}$$

We'll get comfortable with these norms by proving the following crucial estimate for solutions of the linear wave equation in $\mathbb{R}^3$; this estimate is one of the two main steps in constructing global-in-time solutions of (14.1).

Proposition 14.3.2 (Decay Rate for Wave Equation in $\mathbb{R}^3$)**.** *Suppose $F(x) \in C_b^1(\mathbb{R}^3)$ and $G(x) \in C_b(\mathbb{R}^3)$ are supported in the ball of radius $R > 1$ centred at the origin. Consider the initial value problem*

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \forall\ (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = F(x) \quad \forall\ x \in \mathbb{R}^3, \\ u_t|_{t=0} = G(x) \quad \forall\ x \in \mathbb{R}^3. \end{cases} \tag{14.5}$$

This problem admits a unique global-in-time solution $u(x,t) \in C^2(\mathbb{R}^3 \times \mathbb{R})$ satisfying the following decay property: for all $t \neq 0$,

$$\|u(\cdot,t)\|_{C_b(\mathbb{R}^3)} \leq \frac{R^2}{|t|}\left(\|F\|_{C_b^1(\mathbb{R}^3)} + \|G\|_{C_b(\mathbb{R}^3)}\right). \tag{14.6}$$

In particular, $u(x,t) \to 0$ as $|t| \to \infty$, uniformly in x.

Proof. You have already shown existence in class by deriving the 3D Kirchhoff formula. For this problem, we write Kirchhoff's formula in the following way (see exercise 14.3.2):

$$u(x,t) = \frac{1}{4\pi t^2} \oint_{\partial B(x,|t|)} F(y) + \nabla F(y) \cdot (y-x) + |t| G(y)\ \mathrm{d}S(y). \tag{14.7}$$

Uniqueness of the Kirchhoff solution then follows from a simple energy argument. So, the only hard part here is to obtain the temporal decay estimate (14.6). We split into two cases depending on $|t|$.

- **Case 1:** $|t| \geq 1$**.** By hypothesis and the Kirchhoff formula, $u(x,t)$ depends only on the values of F and G on $\partial B(x,|t|) \cap B(0,R)$: F, G vanish at all other points in $\overline{\partial B(x,|t|)}$. Now, notice that the largest sphere that can be inscribed inside the closure $\overline{B(0,R)}$ is simply $\partial B(0,R)$. Therefore,

$$\mathrm{Area}\left(\partial B(x,|t|) \cap B(0,R)\right) \leq \mathrm{Area}\left(\partial B(0,R)\right) = 4\pi R^2$$

  With this observation in mind, we have

$$\begin{aligned} |u(x,t)| &\leq \frac{1}{4\pi t^2} \oint_{\partial B(x,|t|)\cap B(0,R)} |F(y)| + |\nabla F(y)|\ |y-x| + |t|\ |G(y)|\ \mathrm{d}S(y) \\ &\leq \frac{1}{4\pi t^2}\ \left(\|F\|_{C_b} + |t|\ \|\nabla F\|_{C_b} + |t|\ \|G\|_{C_b}\right) \oint_{\partial B(x,|t|)\cap B(0,R)} \mathrm{d}S(y) \\ &\leq \frac{R^2}{t^2}\left(\|F\|_{C_b} + |t|\ \|\nabla F\|_{C_b} + |t|\ \|G\|_{C_b}\right) \\ &\leq \frac{R^2}{|t|}\left(\|F\|_{C_b} + \|\nabla F\|_{C_b} + \|G\|_{C_b}\right) \\ &= \frac{R^2}{|t|}\left(\|F\|_{C_b^1} + \|G\|_{C_b}\right), \end{aligned}$$

  so this case is all done.

- **Case 2:** $|t|< 1$**.** As in the previous case, we apply the Kirchhoff formula to estimate

$$\begin{aligned}
|u(x,t)| &\leq \frac{1}{4\pi t^2}\oint_{\partial B(x,|t|)} |F(y)|+|\nabla F(y)|\ |y-x|+|t|\ |G(y)|\ \mathrm{d}S(y) \\
&\leq \|F\|_{C_b}\frac{1}{4\pi|t|^2}\oint_{\partial B(x,|t|)}\mathrm{d}S(y) + \left(\|\nabla F\|_{C_b}+\|G\|_{C_b}\right)\frac{1}{4\pi|t|}\oint_{\partial B(x,|t|)}\mathrm{d}S(y) \\
&\leq \|F\|_{C_b} + \left(\|\nabla F\|_{C_b}+\|G\|_{C_b}\right)|t| \\
&< \|F\|_{C_b^1} + \|G\|_{C_b}\,, \qquad (14.8)
\end{aligned}$$

  where we have used $|t|< 1$ to obtain the last line. We conclude by observing that, for $|t|< 1$ and $R > 1$, we automatically have $1 < R^2/|t|$.

Putting both these cases together, the proof is finished. □

At long last, we're ready to show that the big bad IVP (14.1) always has at least one solution for small enough ICs.

Theorem 14.3.3 (Klainerman-Nirenberg, [17, pp. 45-46])**.** *Suppose* $f(x) \in C_b^1(\mathbb{R}^3)$ *and* $g(x) \in C_b(\mathbb{R}^3)$ *are supported in the ball of radius* $R > 1$ *centred at the origin. Then,* (14.1) *admits a global-in-time solution* $v(x,t) \in C^2(\mathbb{R}^3 \times \mathbb{R})$ *provided that* $\|f\|_{C_b^1(\mathbb{R}^3)}$ *and* $\|g\|_{C_b(\mathbb{R}^3)}$ *are sufficiently small.*

Proof. The proof is delightfully simple. First, we make a change of responding variable $v \mapsto u$ according to

$$u(x,t) = e^{v(x,t)} - 1. \qquad (14.9)$$

The inverse transformation is

$$v(x,t) = \log\left(u(x,t)+1\right). \qquad (14.10)$$

Note that this inverse is only well-defined if $u > -1$.

Let's pretend we have found a $v(x,t)$ solving (14.1). What IVP $u(x,t)$ does satisfy? A direct computation shows that

$$\begin{aligned}
u_t &= v_t e^v \\
u_{x_i} &= v_{x_i} e^v, \quad i = 1,2,3
\end{aligned}$$

so

$$\begin{aligned}
u_{tt} &= \left[v_{tt} + (v_t)^2\right] e^v \\
u_{x_i} &= \left[v_{x_i x_i} + (v_{x_i})^2\right] e^v, \quad i = 1,2,3.
\end{aligned}$$

Using the nonlinear wave equation satisfied by v then gives

$$\begin{aligned}
0 &= \left[u_{tt}e^{-v} - (v_t)^2\right] - \left((\Delta u)e^{-v} - |\nabla u|^2\right) + (v_t)^2 - |\nabla_x v|^2 \\
&= e^{-v}\left[u_{tt} - \Delta u\right].
\end{aligned}$$

Therefore, u satisfies the linear wave equation,

$$u_{tt} - \Delta u = 0 \quad \forall\ (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{14.11}$$

Now, we check what initial conditions u satisfies. By inspection of the ICs in (14.1), we see that

$$u|_{t=0} = F(x) \doteq e^{f(x)} - 1, \tag{14.12a}$$

$$u_t|_{t=0} = G(x) \doteq g(x)e^{f(x)}. \tag{14.12b}$$

Since f, g are supported in $B(0,R)$, we find that F, G are supported in this ball as well. In summary, we have shown that if $v(x,t)$ solving (14.1) exists then its corresponding $u(x,t)$ satisfies (14.5) with F, G supported inside $B(0,R)$. By reversing the computations performed above, we can likewise show that if

1. $u(x,t)$ solves (14.5) with the particular F, G defined in (14.12) and
2. $\|u(\cdot,t)\|_{C_b(\mathbb{R}^3)} < 1$,

then v defined by (14.10) solves (14.1). This gives us a recipe for constructing $v(x,t)$ explicitly by solving the linear wave equation and then bounding its solution!

So, let's define u to be the solution of (14.5) guaranteed to exist (and decay in time) by proposition 14.3.2. We now split into two cases depending on the size of $|t|$.

- **Case 1:** $|t| \geq 1$**.** In this case we see straight from (14.6) that $\|u(\cdot,t)\|_{C_b(\mathbb{R}^3)} < 1$ provided

$$\|F\|_{C_b^1(\mathbb{R}^3)} + \|G\|_{C_b(\mathbb{R}^3)} < \frac{1}{R^2}. \tag{14.13}$$

- **Case 2:** $|t| < 1$**.** Here we can't use (14.6) directly. Instead, we go back to the intermediary result (14.8) in the proof of proposition 14.3.2, which tells us that $\|u(\cdot,t)\|_{C_b(\mathbb{R}^3)} < 1$ provided

$$\|F\|_{C_b^1(\mathbb{R}^3)} + \|G\|_{C_b(\mathbb{R}^3)} < 1.$$

Putting both these cases together and using that $R > 1$, we find that $\|u(\cdot,t)\|_{C_b(\mathbb{R}^3)} < 1$ for all $t \neq 0$ *if we impose the smallness condition* (14.13). So, provided (14.13) holds, we can define $v(x,t)$ by (14.10) and get a global-in-time solution to (14.1) as outlined above. □

Notice again how the size of the initial data played into our analysis, but this time for the better. For most interesting nonlinear evolution equations, small initial data are unlikely to give rise to blowup, and they also tend to keep useful norms of the initial data "nice". For this reason, you'll often go to modern PDE talks where the speaker focuses entirely on small initial data.

Exercise 14.3.1. *Prove that* $C_b(\mathbb{R}^3)$ *is a Banach space. What about* $C_b^1(\mathbb{R}^3)$*?*

Exercise 14.3.2. *Obtain* (14.7) *using the form of the Kirchhoff formula given in Strauss* [27, p. 234].

Chapter 15

The Wave Equation in an Expanding Spacetime

15.1 Introduction

Today, we investigate the wave equation in an expanding spacetime. Our goal is to obtain a representation formula and time decay estimate that can be compared to those for the linear wave equation. The discussion here is based off the excellent paper [1], see also [9, Project 6.8]: I have only provided a little more physical intuition and added some computational details.

Before we start, I'll introduce some notation that will help keep our inequalities uncluttered:

Definition 15.1.1. *Let $a(t), b(t) \geq 0$ be two given real-valued functions defined on a (possibly infinite) interval I. We say that*

$$a(t) \lesssim b(t) \quad \forall\ t \in I$$

if there exists a constant C independent of t so that

$$a(t) \leq Cb(t).$$

See exercise 15.1.1 for some basic practice with the order symbol $\lesssim$.

Exercise 15.1.1. *Define the **Japanese bracket** of $t \in \mathbb{R}$ by*

$$\langle t \rangle = \sqrt{1 + t^2}.$$

Show that, for $t > 1$,

$$\langle t \rangle \lesssim t \lesssim \langle t \rangle.$$

Are both bounding constants the same?

15.2 Executive Summary of the Physical Setup

Before we dive into the analysis, we need to go over some basic definitions and ideas from mathematical relativity. For a much more involved discussion on the underlying physics, see [35].

To describe a spacetime, we need two ingredients.

1. First, we need a manifold of the form

$$M = I \times \mathbb{R}^3$$

where $I \subseteq \mathbb{R}$ is a (possibly infinite) time interval. $\mathbb{R}^3$ represents our universe. In physics, any $(t, x) \in M$ is called an **event**, which is essentially short for "an event localized near spatial position x and time t". More interesting manifolds M appear in relativity theory, but for today we want to keep things simple.

2. Second, we must specify a **Lorentzian metric** on M. This is a matrix-valued function

$$g \colon M \to \mathrm{GL}\left(\mathbb{R}^4\right)$$

such that $g(t, x)$ is symmetric for all $(t, x) \in M$. Further, $g(t, x)$ must satisfy the $(0, 2)$-tensor transformation rule, but we won't go into details about this here.

What is the geometric meaning of the metric? Recall from multivariable calculus that the tangent space to $M = I \times \mathbb{R}^3$ at a particular point (t, x) is simply

$$\mathrm{T}_{(t,x)}M \doteq \left\{\mathbf{v} \in \mathbb{R}^4 \mid \text{s.t. } \mathbf{v} \text{ is tangent to } M \text{ at } (t, x)\right\} \simeq \mathbb{R}^4.$$

Thus g gives rise to an inner product

$$\langle \cdot, \cdot \rangle_g \colon \mathrm{T}_{(t,x)}M \times \mathrm{T}_{(t,x)}M \to \mathbb{R}, \quad \langle \mathbf{u}, \mathbf{v} \rangle_g \doteq \mathbf{u}^{\mathrm{T}} g(t, x) \mathbf{v}.$$

Crucially, since $g(t, x)$ is symmetric *but not positive-definite*, this inner product is indefinite: in principle, there may be nonzero $\mathbf{v} \in \mathrm{T}_{(t,x)}M$ such that

$$\langle \mathbf{v}, \mathbf{v} \rangle_g \leq 0.$$

Therefore, g does not give rise to a norm. However, the geometry induced by a Lorentzian metric is still very interesting.

It's useful to intuitively understand the "length" of $\mathbf{v} \in \mathrm{T}_{(t,x)}M$,

$$\langle \mathbf{v}, \mathbf{v} \rangle_g,$$

as the distance from (t, x) to the topologically nearby point $(t', x') \in M$ we'd reach if we travelled in the direction $\mathbf{v}$ for a short time. Thus if $\langle \mathbf{v}, \mathbf{v} \rangle_g \gg 1$ then we say (t, x) and (t', x') are very far away from each other. This understanding is admittedly very sloppy and self-referential, but it can be made quite precise using the full power of Riemannian geometry.

Example 15.2.1. Consider $M = \mathbb{R}^4$ with the constant Lorentzian metric

$$g(t,x) \equiv \eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15.1}$$

A quick computation reveals that η is invertible with $\eta^{-1} = \eta$, so η indeed defines a Lorentzian metric on $\mathbb{R}^4$ called the **Minkowski metric**. We often compactly write $\mathbb{R}^{1+3}$ to refer to $\mathbb{R}^4$ equipped with the Minkowski metric. $\mathbb{R}^{1+3}$ is usually called **Minkowski space**. On your current assignment, you are asked to show that the Minkowski metric can be used to characterize a rich symmetry group for the linear wave equation.

Example 15.2.2. Consider a fixed $t_0 > 0$, and let

$$M = (t_0, \infty) \times \mathbb{R}^3.$$

Let $S\colon \mathbb{R} \to [0,\infty)$ be a smooth, strictly increasing function satisfying $S(0) = 0$. Then,

$$g(t,x) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & S(t)^2 & 0 & 0 \\ 0 & 0 & S(t)^2 & 0 \\ 0 & 0 & 0 & S(t)^2 \end{bmatrix} \tag{15.2}$$

defines a Lorentzian metric: invertibility is guaranteed since we cut off before $t = 0$. g is an example of a **Robertson-Walker (RW)** metric. These appear in cosmology as models of an expanding universe.

To see how the expansion of space is captured by an RW metric g, we write $\mathbf{v} = (v_0, v_1, v_2, v_3)^{\mathrm{T}}$. Then,

$$\langle \mathbf{v}, \mathbf{v} \rangle_g = -v_0^2 + S(t)^2 \sum_{k=1}^{3} v_k^2.$$

Since $S(t)$ is increasing, we know that the contribution to $\langle \mathbf{v}, \mathbf{v} \rangle_g$ coming from the spatial components of $\mathbf{v}$ becomes larger with t. So, any $\mathbf{v}$ with nonzero spatial components has its length get larger over time! With our intuitive geometric understanding of the metric, this means that any points that are "close" to (t,x) for $t \approx t_0$ will eventually be "far" from (t,x). Thus RW metrics capture the gradual expansion of space as time elapses.

Formally, as $t \to 0$, any RW metric collapses and becomes degenerate: the spatial components of $\mathbf{v}$ now contribute *nothing*. This corresponds to all points in space being distance zero away from one another! Thus at time 0 our metric is singular. This emergence of an expanding universe from an initially singular metric is the mathematical basis for the **Big Bang hypothesis**. Thus our choice of starting our time interval at $t_0 > 0$ makes sense, as it allows us to avoid having to deal with the Big Bang singularity head-on.

In practice, we prefer to deal with RW metrics by changing the time variable

$$t \mapsto \tau$$

according to

$$S(\tau) = \frac{\mathrm{d}t}{\mathrm{d}\tau}.$$

Using the $(0,2)$-tensor transformation rule changes the metric (15.2) to

$$g(\tau, x) = S(\tau)^2 \eta, \tag{15.3}$$

where η is the Minkwoski metric. In these notes, we'll focus on the so-called **flat Robertson-Walker metric on** $(t_0, \infty) \times \mathbb{R}^3$. This corresponds to the choice

$$S(\tau) = \tau^2. \tag{15.4}$$

For posterity, we note that in the flat case

$$t = \frac{1}{3}\tau^3. \tag{15.5}$$

Before moving on, we briefly state the generalization of the wave equation to an arbitrary spacetime M of the form we're considering here. If $g(t,x)$ is a Lorentzian metric on M, then the **wave equation** for $u \in C^2(M)$ is given by

$$0 = \nabla_{t,x} \cdot \left(g^{-1} \sqrt{-\det g}\ \nabla_{t,x} u \right). \tag{15.6}$$

Loosely speaking, electromagnetic waves propagating through the spacetime (M, g) are described by (15.6).

Example 15.2.3. In Minkowski space, the wave equation is simply

$$0 = \partial_{tt} u - \Delta u,$$

the usual wave equation with unit speed.

Example 15.2.4. For a general RW metric in (τ, x) coordinates, the wave equation reads (see exercise 15.2.2)

$$0 = u_{\tau\tau} + \frac{2\dot{S}}{S} u_\tau - \Delta u. \tag{15.7}$$

For the flat metric $S = \tau^2$, this becomes

$$0 = u_{\tau\tau} + \frac{4}{\tau} u_\tau - \Delta u. \tag{15.8}$$

So, in an expanding spacetime, our wave equation picks up a sub-principal, variable-coefficient term. Note that the Big Bang singularity is tacitly reflected in the wave equation due to the $\frac{1}{\tau}$ coefficient.

Exercise 15.2.1. *Carefully prove that a Lorentzian metric g indeed defines an indefinite inner product on $T_{(t,x)}M$ for all $(t,x) \in M$.*

Exercise 15.2.2. *Verify* (15.7).

15.3 The Wave Equation in Flat RW Spacetime

Now, we turn towards understanding the behaviour of the wave equation (15.8) describing a special case of electromagnetic waves moving through an expanding universe. Our goal is to derive a representation formula and time decay estimate, as we have done for the usual (Minkowski space) wave equation. We'll also briefly talk about how our representation formula can be used to make sense of initial data prescribed at the Big Bang.

15.3.1 Review: Waves in Minkowski Spacetime

Theorem 15.3.1 (Existence-Uniqueness and Time Decay for the Wave Equation in Minkowski Spacetime)**.** *Suppose $F(x) \in C_b^1(\mathbb{R}^3)$ and $G(x) \in C_b(\mathbb{R}^3)$. Consider the initial value problem*

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \forall\ (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = F(x) \quad \forall\ x \in \mathbb{R}^3, \\ u_t|_{t=0} = G(x) \quad \forall\ x \in \mathbb{R}^3. \end{cases}$$

This problem admits a unique global-in-time solution $u(x,t) \in C^2(\mathbb{R}^3 \times \mathbb{R})$ given by the ***Kirchhoff formula***

$$u(x,t) = \frac{1}{4\pi t} \oint_{\partial B(x,|t|)} G(y)\ \mathrm{d}S(y) + \frac{\partial}{\partial t}\left[\frac{1}{4\pi t} \oint_{\partial B(x,|t|)} F(y)\ \mathrm{d}S(y)\right]. \tag{15.1}$$

Further, if F, G are compactly supported, then $u(t,x)$ satisfies the following decay property: for all $t \neq 0$,

$$\|u(\cdot,t)\|_{C_b(\mathbb{R}^3)} \lesssim \frac{1}{|t|}\left(\|F\|_{C_b^1(\mathbb{R}^3)} + \|G\|_{C_b(\mathbb{R}^3)}\right), \tag{15.2}$$

where the bounding constant depends only on the diameters of $\mathrm{supp}(F)$ *and* $\mathrm{supp}(G)$*.*

15.3.2 Representation Formula and Time Decay for Flat FLRW

Fix $\tau_0 > 0$. We study the IVP

$$\begin{cases} u_{\tau\tau} + \dfrac{4}{\tau} u_\tau - \Delta u = 0 \quad \forall\ (\tau,x) \in (\tau_0,\infty) \times \mathbb{R}^3, \\ u|_{\tau=\tau_0} = F(x) \quad \forall\ x \in \mathbb{R}^3, \\ u_\tau|_{\tau=\tau_0} = G(x) \quad \forall\ x \in \mathbb{R}^3. \end{cases} \tag{15.3}$$

Lemma 15.3.2. *Suppose $u(t,x)$ solves (15.3) on $(\tau_0,\infty) \times \mathbb{R}^3$. Then,*

$$v(\tau,x) = \tau^{-1}\partial_\tau\left(\tau^3 u\right) \tag{15.4}$$

satisfies the constant-coefficient problem

$$\begin{cases} v_{\tau\tau} - \Delta v = 0 \quad \forall\ (\tau,x) \in (\tau_0,\infty) \times \mathbb{R}^3, \\ v|_{\tau=\tau_0} = \phi(x) \doteq 3\tau_0 F(x) + \tau_0^2 G(x) \quad \forall\ x \in \mathbb{R}^3, \\ v_\tau|_{\tau=\tau_0} = \psi(x) \doteq \left(3 + \tau_0^2 \Delta\right) F(x) + \tau_0 G(x) \quad \forall\ x \in \mathbb{R}^3. \end{cases} \tag{15.5}$$

Proof. We directly compute

$$\begin{aligned} v &= 3\tau u + \tau^2 u_\tau, && \text{(15.6a)} \\ v_\tau &= 3u + 5\tau u_\tau + \tau^2 u_{\tau\tau}, && \text{(15.6b)} \\ v_{\tau\tau} &= 8u_\tau + y\tau u_{\tau\tau} + \tau^2 u_{\tau\tau\tau}, \\ \Delta v &= 3\tau \Delta u + \tau^2 \Delta u_\tau. \end{aligned}$$

Then, we have

$$v_{\tau\tau} - \Delta v = 3\tau \left(u_{\tau\tau} + \frac{4}{\tau} - \Delta u \right) - 4u_\tau + 4\tau u_{\tau\tau} + \tau^2 u_{\tau\tau\tau} - \tau^2 \Delta u_\tau.$$

Using the PDE in (15.3), the above becomes

$$v_{\tau\tau} - \Delta v = -4u_\tau + 4\tau u_{\tau\tau} + \tau^2 u_{\tau\tau\tau} - \tau^2 \Delta u_\tau. \tag{15.7}$$

However, by differentiating the PDE in (15.3), we find

$$\tau \Delta u - \tau u_{\tau\tau} = \tau^2 u_{\tau\tau} - \tau^2 \Delta u_\tau + 4\tau u_{\tau\tau}.$$

Plugging this into (15.7) and using (15.3) one more time gives

$$v_{\tau\tau} - \Delta v = -\tau u_{\tau\tau} - 4u_\tau + \tau \Delta u = 0.$$

To find the initial conditions, we use (15.6a) and (15.6b) to find

$$\begin{aligned} v|_{\tau=\tau_0} &= 3\tau F + \tau_0^2 G \\ v_\tau|_{\tau=\tau_0} &= 3F + 5\tau G(x) + \tau_0^2 u_{\tau\tau}|_{\tau=\tau_0}. \end{aligned}$$

The proof is complete once we use the PDE in (15.3) to write

$$u_{\tau\tau}|_{\tau=\tau_0} = \Delta F - \frac{4}{\tau_0} G(x).$$

□

To keep the presentation streamlined, from now on we'll only present the details for $G \equiv 0$ and leave the general case to the exercises.

Theorem 15.3.3 (Representation Formula, $G = 0$ case)**.** *If $G(x) = 0$ and $F \in C_b^1(\mathbb{R}^3)$, then there exists $u \in C^2((\tau_0, \infty) \times \mathbb{R}^3)$ solving* (15.3)*. Further, $u(\tau, x)$ is given explicitly by the Kirchhoff-type formula*

$$\begin{aligned} u(\tau, x) = \frac{1}{\tau^3} \Bigg\{ \frac{3}{4\pi} \int_{B(x,\tau-\tau_0)} F(y) \, \mathrm{d}y - \frac{\tau^3 - (\tau - \tau_0)^3}{(\tau - \tau_0)^2} \oint_{\partial B(x,\tau-\tau_0)} F(y) \, \mathrm{d}S(y) \\ + \frac{\tau \tau_0^2}{4\pi(\tau - \tau_0)} \oint_{\partial B(x,\tau-\tau_0)} \nabla F(y) \cdot \left(\frac{y - x}{|y - x|} \right) \, \mathrm{d}S(y) \Bigg\}. \end{aligned}$$

Proof. Applying Kirchhoff's formula from theorem 15.3.1 to the transformed system (15.5), we can write $v(\tau, x)$ explicitly in terms of spherical means:

$$v(\tau,x) = \frac{1}{4\pi(\tau-\tau_0)} \oint_{\partial B(x,\tau-\tau_0)} \psi(y)\ \mathrm{d}S(y) + \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\oint_{\partial B(x,t)} \phi(y)\ \mathrm{d}S(y)\right]_{t=\tau-\tau_0} \tag{15.8}$$

By multiplying both sides of (15.4) by τ and integrating the result on $[\tau_0, \tau]$, we discover

$$\begin{aligned} \tau^3 u(\tau,x) &= \tau_0^3 F(x) + \int_{\tau_0}^{\tau} \sigma\ v(\sigma,x)\ \mathrm{d}\sigma \\ &= \tau_0^3 F(x) + \int_0^{\tau-\tau_0} (r+\tau_0)\ v(r+\tau_0,x)\ \mathrm{d}r. \end{aligned} \tag{15.9}$$

Combining (15.8) and (15.9) and using the definitions of ϕ and ψ, we have

$$\begin{aligned} \tau^3 u(\tau,x) - \tau_0^3 F(x) &= \int_0^{\tau-\tau_0} (r+\tau_0)\ v(r+\tau_0,x)\ \mathrm{d}r \\ &= \int_0^{\tau-\tau_0} (r+\tau_0)\left\{\frac{1}{4\pi r}\oint_{\partial B(x,r)} \psi(y)\ \mathrm{d}S(y) + \frac{\partial}{\partial r}\left[\frac{1}{4\pi r}\oint_{\partial B(x,r)} \phi(y)\ \mathrm{d}S(y)\right]\right\}\ \mathrm{d}r. \end{aligned}$$

In the special case $G = 0$, the above reduces to

$$\tau^3 u(\tau,x) - \tau_0^3 F(x) = \int_0^{\tau-\tau_0} (r+\tau_0)\left\{\frac{1}{4\pi r}\oint_{\partial B(x,r)} (3+\tau_0^2\Delta)F(y)\ \mathrm{d}S(y) + \frac{\partial}{\partial r}\left[\frac{1}{4\pi r}\oint_{\partial B(x,r)} 3\tau_0 F(y)\ \mathrm{d}S(y)\right]\right\}\ \mathrm{d}r.$$

Integrating by parts in r in the last term gives

$$\begin{aligned} \tau^3 u(\tau,x) - \tau_0^3 F(x) &= \int_0^{\tau-\tau_0} \oint_{\partial B(x,r)} \frac{3(r+\tau_0)}{4\pi r} F(y)\ \mathrm{d}S(y)\ \mathrm{d}r \\ &\quad + \tau_0^2 \int_0^{\tau-\tau_0} \oint_{\partial B(x,r)} \frac{r+\tau_0}{4\pi r}\Delta F(y)\ \mathrm{d}S(y)\ \mathrm{d}r \\ &\quad + \frac{3\tau_0\tau}{4\pi(\tau-\tau_0)}\oint_{\partial B(x,\tau-\tau_0)} F(y)\ \mathrm{d}S(y) - \int_0^{\tau-\tau_0}\oint_{\partial B(x,r)} \frac{3\tau_0}{4\pi r}\ F(y)\ \mathrm{d}S(y)\ \mathrm{d}r \\ &= \frac{3}{4\pi}\int_{B(x,\tau-\tau_0)} F(y)\ \mathrm{d}y + \frac{3\tau_0\tau}{4\pi(\tau-\tau_0)}\oint_{\partial B(x,\tau-\tau_0)} F(y)\ \mathrm{d}S(y) \\ &\quad + \tau_0^2 \int_{B(x,\tau-\tau_0)} \left(\frac{1}{4\pi} + \frac{\tau_0}{4\pi r}\right)\Delta F(y)\ \mathrm{d}S(y)\ \mathrm{d}r \end{aligned} \tag{15.10}$$

Let's take an aside to compute the last term in (15.10). Using the divergence theorem, Green's second identity, and

$$\Delta\left(-\frac{1}{4\pi r}\right)(x) = \delta(x) \quad \forall\ x \in \mathbb{R}^3,$$

we have

$$\begin{aligned}\int_{B(x,\tau-\tau_0)}\left(\frac{1}{4\pi}+\frac{\tau_0}{4\pi r}\right)\Delta F(y)\,\mathrm{d}S(y)\,\mathrm{d}r &= \oint_{\partial B(x,\tau-\tau_0)}\frac{1}{4\pi}\nabla F(y)\cdot\left(\frac{y-x}{|y-x|}\right)\,\mathrm{d}S(y)\\ &\quad+\oint_{\partial B(x,\tau-\tau_0)}\frac{\tau_0}{4\pi r}\nabla F(y)\cdot\left(\frac{y-x}{|y-x|}\right)+\frac{\tau_0}{4\pi r^2}\,F(y)\,\mathrm{d}S(y)\Big|_{r=\tau-\tau_0}\\ &\quad-\tau_0F(x)\\ &=-\tau_0F(x)+\frac{\tau}{4\pi(\tau-\tau_0)}\oint_{\partial B(x,\tau-\tau_0)}\nabla F(y)\cdot\left(\frac{y-x}{|y-x|}\right)\,\mathrm{d}S(y)\\ &\quad-\frac{\tau_0}{4\pi\left(\tau-\tau_0\right)^2}\oint_{\partial B(x,\tau-\tau_0)}F(y)\,\mathrm{d}S(y).\end{aligned}$$

Plugging this back into (15.10), we find

$$\begin{aligned}\tau^3u(\tau,x) &= \frac{3}{4\pi}\int_{B(x,\tau-\tau_0)}F(y)\,\mathrm{d}y+\frac{3\tau_0\tau}{4\pi(\tau-\tau_0)}\oint_{\partial B(x,\tau-\tau_0)}F(y)\,\mathrm{d}S(y)\\ &\quad+\frac{\tau\tau_0^2}{4\pi(\tau-\tau_0)}\oint_{\partial B(x,\tau-\tau_0)}\nabla F(y)\cdot\left(\frac{y-x}{|y-x|}\right)\,\mathrm{d}S(y)-\frac{\tau_0^3}{4\pi\left(\tau-\tau_0\right)^2}\oint_{\partial B(x,\tau-\tau_0)}F(y)\,\mathrm{d}S(y)\\ &=\frac{3}{4\pi}\int_{B(x,\tau-\tau_0)}F(y)\,\mathrm{d}y+\left[\frac{3\tau_0\tau}{4\pi(\tau-\tau_0)}-\frac{\tau_0^3}{4\pi\left(\tau-\tau_0\right)^2}\right]\oint_{\partial B(x,\tau-\tau_0)}F(y)\,\mathrm{d}S(y)\\ &\quad+\frac{\tau\tau_0^2}{4\pi(\tau-\tau_0)}\oint_{\partial B(x,\tau-\tau_0)}\nabla F(y)\cdot\left(\frac{y-x}{|y-x|}\right)\,\mathrm{d}S(y)\end{aligned}$$

A bit of algebra shows that

$$-\left[\frac{3\tau_0\tau}{\tau-\tau_0}-\frac{\tau_0^3}{\left(\tau-\tau_0\right)^2}\right]=\frac{\tau^3-(\tau-\tau_0)^3}{(\tau-\tau_0)^2},$$

and the claim is proven. □

Corollary 15.3.4 (Decay Estimate in Flat RW Spacetime)**.** *If $F\in C_b^1(\mathbb{R}^3)$ is compactly supported, then for all $\tau>\tau_0$ sufficiently large we have*

$$\|u(\tau,\cdot)\|_{C_b(\mathbb{R}^3)}\lesssim\tau^{-3}\|F\|_{C_b^1(\mathbb{R}^3)},$$

where the bounding constant depends only on τ_0 and the diameter of $\mathrm{supp}(F)$*. In terms of the original time variable t, the decay estimate takes the form*

$$\|u(t,\cdot)\|_{C_b(\mathbb{R}^3)}\lesssim t^{-1}\|F\|_{C_b^1(\mathbb{R}^3)}\quad\forall\ t\gg t_0.$$

Proof. We proceed the same way we proved the decay estimate in theorem 15.3.1 last week, using that the size of the support of each integration domain in the representation formula from theorem 15.3.3 can be controlled in terms of the diameter of $\mathrm{supp}(F)$. Suppose that $R>0$ is such that

$$\mathrm{supp}(F)\subseteq B(0,R).$$

Then, using our representation formula,

$$\begin{aligned}\tau^3|u| &\lesssim \int_{B(x,\tau-\tau_0)} |F(y)| \ \mathrm{d}y + \left|\frac{\tau^3-(\tau-\tau_0)^3}{(\tau-\tau_0)^2}\right| \oint_{\partial B(x,\tau-\tau_0)} |F(y)| \ \mathrm{d}S(y) + \left|\frac{\tau}{\tau-\tau_0}\right| \oint_{\partial B(x,\tau-\tau_0)} |\nabla F(y)| \ \mathrm{d}S(y) \\ &\lesssim \left[R^3 + \left|\frac{\tau^3-(\tau-\tau_0)^3}{(\tau-\tau_0)^2}\right| R^2\right] \|F\|_{C_b} + R^2 \left|\frac{\tau}{\tau-\tau_0}\right| \ \|\nabla F\|_{C_b}\end{aligned}$$

If $\tau > \tau_0 + 1$, a quick check with a plotting software tells us that

$$\left|\frac{\tau}{\tau-\tau_0}\right|, \ \left|\frac{\tau^3-(\tau-\tau_0)^3}{(\tau-\tau_0)^2}\right| \lesssim 1,$$

whence the above becomes

$$\tau^3|u| \lesssim (R^3+R^2)\|F\|_{C_b} + R^2\|\nabla F\|_{C_b} \lesssim \|F\|_{C_b^1}.$$

This proves the decay estimate in terms of the variable τ. By definition of τ, however,

$$t = \frac{1}{3}\tau^3,$$

so we're all done. □

Therefore, a solution to the wave equation in flat RW spacetime has the same decay rate as a solution to the wave equation in Minkowski spacetime! However, the strong Huygens principle fails, as our representation formula requires integration over solid balls rather than just spheres. So, in an expanding universe we can use some, but not all, of our intuitive understanding of the usual wave equation to help make physical predictions. Indeed, by studying how solutions of the flat RW wave equation deviate from solutions of the Minkowski wave equation, scientists may be able to develop novel experiments in cosmology; see [1, §5] for some excellent discussion on this point.

15.3.3 Approaching the Big Bang Singularity

To conclude our brief venture into Einstein's mind-palace, let's study how solutions of the wave equation in flat RW spacetime behave near the Big Bang. We do this by fixing the future time τ and taking the initial time τ_0 to 0. Formally taking the limit $\tau_0 \to 0^+$ in the representation formula from theorem 15.3.3 gives

$$u(\tau,x) = \frac{3}{4\pi\tau^3}\int_{B(x,\tau)} F(y) \ \mathrm{d}y. \tag{15.11}$$

At first glance, the Big Bang singularity still appears to screw this formula up due to the blowup of the τ^{-3} out from. We now prove this does not happen!

Proposition 15.3.5 (Existence for Initial Data At the Big Bang)**.** *Let $F \in C_b^1(\mathbb{R}^3)$ be compactly supported. Then, the function $u(\tau, x)$ defined by* (15.11) *satisfies*

$$\lim_{\tau \to 0^+} u(\tau, x) = F(y), \quad \lim_{\tau \to 0^+} u(\tau, x) = 0. \tag{15.12}$$

In particular, the problem

$$\begin{cases} u_{\tau\tau} + \dfrac{4}{\tau} u_\tau - \Delta u = 0 \quad \forall\ (\tau, x) \in (0, \infty) \times \mathbb{R}^3, \\ u|_{\tau=0} = F(x) \quad \forall\ x \in \mathbb{R}^3, \\ u_\tau|_{\tau=0} = 0 \quad \forall\ x \in \mathbb{R}^3. \end{cases} \tag{15.13}$$

admits a solution given by (15.11)*, provided we only demand the initial conditions hold in the relaxed sense* (15.12)*.*

Proof. First notice that the limit actually makes sense since

$$|u| \lesssim \tau^{-3} \mathrm{Vol}\left(B(x, \tau)\right) \|F|_{C_b} \lesssim \|F\|_{C_b}.$$

However, notice that we can write

$$u(\tau, x) = \frac{1}{\mathrm{Vol}\left(B(x, \tau)\right.} \int_{B(x,\tau)} F(y)\ \mathrm{d}y,$$

which is the mean of $F(y)$ on $B(x, \tau)$. Since the means of F on small balls about x converge to $F(x)$, the first limit is proved. To get the second limit, re-write u using polar coordinates as

$$u(\tau, x) = \frac{3}{4\pi\tau^3} \int_0^\tau \int_{\partial B(x,r)} F(y)\ \mathrm{d}S(y)\ \mathrm{d}r.$$

We then get the claim using the fundamental theorem of calculus and another argument with convergence of averages. □

Thus we have discovered the interesting fact that electromagnetic radiation present at the Big Bang can sensibly propagate into the future: it won't remain stuck at the singularity. This "antediluvian" radiation does not satisfy the strong Huygens principle. Indeed, since the non-Huygens term is the only one that survives the limit $\tau_0 \to 0^+$, we can (maybe? I'm no expert) say that the Big Bang singularity somehow breaks the strong Huygens principle: since all points in the universe are touching one another in our chosen metric geometry at time 0, then it makes sense that interior points also contribute to our solution.

Exercise 15.3.1. *Generalize theorem 15.3.3 and corollary 15.3.4 to the case $G \neq 0$. See* [1] *if you get stuck.*

Exercise 15.3.2. *Fill in the details missing from the proof of proposition 15.3.5.*

Chapter 16

Practice with Distributions and the Fourier Transform

16.1 Introduction/Notation

You have just finished learning about the basics of rigorous distribution theory. This should make you very happy: from the first month of the class when we were dealing with the heat kernel to our communal struggles with Green's functions, we've all desperately wanted to be able to justifiably use δ-functions in our computations! Since the physical motivation and utility for distributions should be clear by now, today I'll focus on doing lots of smaller practice problems with distributions (as opposed to my usual practice of doing an extended case study of a problem where they appear). We'll also briefly touch on an important connection between distributions and the Fourier transform. For slightly more advanced reading on these topics, see [25], [28].

Here is the notation I will use today:

- The space of smooth, compactly supported, complex-valued functions on $\mathbb{R}^n$ will be denoted $\mathcal{D}(\mathbb{R}^n)$.
- The space of distributions on $\mathbb{R}^n$ is denoted by $\mathcal{D}'(\mathbb{R}^n)$.
- The action of $T \in \mathcal{D}'(\mathbb{R}^n)$ on $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is denoted by $\langle T, \varphi \rangle$.
- For any fixed $x_0 \in \mathbb{R}^n$, we let δ_{x_0} denote the distribution

$$\langle \delta_{x_0}, \varphi \rangle = \varphi(x_0).$$

- Given an absolutely integrable function $f : \mathbb{R}^n \to \mathbb{C}$, we define its Fourier transform by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\ \mathrm{e}^{-ix\cdot\xi}\ \mathrm{d}x.$$

16.2 Practice with Distributions

These examples will be solved in tutorial. Two of the three examples here are modified from problems in [6].

Example 16.2.1. *We say $f\colon \mathbb{R}^n \to \mathbb{C}$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$ if it is Lebesgue-measurable and, for any compact $K \subseteq \mathbb{R}^n$, we have*

$$\int_K |f(x)| \, \mathrm{d}x < \infty.$$

a) Prove that any bounded, Lebesgue-measurable function is in $L^1_{\text{loc}}(\mathbb{R}^n)$.

b) Find a Lebesgue-measurable function f that is not in $L^1_{\text{loc}}(\mathbb{R}^n)$. Hint: focus on $n = 1$.

c) Construct an injective map $L^1_{\text{loc}}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$.

If you don't know what Lebesgue-measurable means, ignore this phrase in the above exercises.

Example 16.2.2. *Given a smooth function $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ and a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$, we define their* ***product*** *$gT \in \mathcal{D}'(\mathbb{R}^n)$ by*

$$\langle gT, \varphi \rangle \doteq \langle T, g\varphi \rangle.$$

a) Prove that gT is actually a distribution.

b) Fill in the blanks below with words from your algebra class to make the statement true:

$$\text{“}\mathcal{D}'(\mathbb{R}^n) \text{ is a } \_\_\_\_ \text{ over the } \_\_\_\_ \mathcal{C}^\infty(\mathbb{R}^n)\text{”}.$$

c) Show that the distributions

$$x\delta_0$$
$$x\left(\delta_0'\right)$$
$$x^n\delta_0^{(n)}$$

are all scalar multiples of δ_0.

d) Rigorously formulate and justify the formal statement

$$f(x)\ \delta(x - x_0) = f(x_0)\ \delta(x - x_0).$$

Example 16.2.3.

a) Consider the ***Dirac comb*** *defined by the infinite sum*

$$T \doteq \sum_{n=-\infty}^{\infty} \delta_n.$$

Is T a distribution? If so, construct a function $f(x)$ so that

$$f'(x) = T \quad \text{in the sense of distributions.}$$

b) Repeat the above exercise, but replace T with

$$\widetilde{T} = \sum_{q \in [0,1] \cap \mathbb{Q}} \delta_q.$$

16.3 Why Not all Distributions can be Fourier-Transformed

Next, I'd like to ask the question "can a general distribution be Fourier-transformed"? The short answer is no, and I'll convince you why in the next few pages. However, if we restrict to the class of *tempered* distributions, then all is well; we'll discuss tempered distributions more in future tutorials.

16.3.1 The Fourier Transform Screws Up Supports

Before discussing why the Fourier transform does not make sense on $\mathcal{D}'(\mathbb{R}^n)$, however, we have to understand how the Fourier transform affects the support of its inputs. We begin with an easy and instructive example.

Example 16.3.1. *Let $\mathbf{1}_{[0,1]}$ denote the indicator function of the unit interval. Compute its Fourier transform $\widehat{\mathbf{1}_{[0,1]}}$, and draw its real part. Is $\widehat{\mathbf{1}_{[0,1]}}$ compactly supported?*

Solution:
We directly compute

$$\begin{aligned}
\widehat{\mathbf{1}_{[0,1]}}(\xi) &= (2\pi)^{-1/2} \int_{\mathbb{R}} \mathbf{1}_{[0,1]}(x)\ \mathrm{e}^{-ix\cdot\xi}\ \mathrm{d}x \\
&= (2\pi)^{-1/2} \int_0^1 \mathrm{e}^{-ix\cdot\xi}\ \mathrm{d}x \\
&= (2\pi)^{-1/2} \left[\frac{e^{-ix\xi}}{\xi}\right]_0^1 \\
&= \frac{i}{\sqrt{2\pi}\xi}\left(e^{-i\xi} - 1\right).
\end{aligned}$$

Then,

$$\mathrm{Re}\left(\widehat{\mathbf{1}_{[0,1]}}\right)(\xi) = \mathrm{sinc}(\xi) \doteq \frac{\sin(\xi)}{\sqrt{2\pi}\xi}$$

since sine is odd (you may see different normalization constants for sinc in other contexts). See figure 16.1 for a plot of this function. $\mathrm{sinc}(\xi)$ does decay as $|\xi| \to \infty$, but it is certainly not compactly supported! Therefore, the Fourier transform $\widehat{\mathbf{1}_{[0,1]}}(\xi)$ is not compactly supported.

This example tells us that the Fourier transform can, in general, really screw up the support of its inputs. In other words, the Fourier transform of a compactly supported function may not be compactly supported.

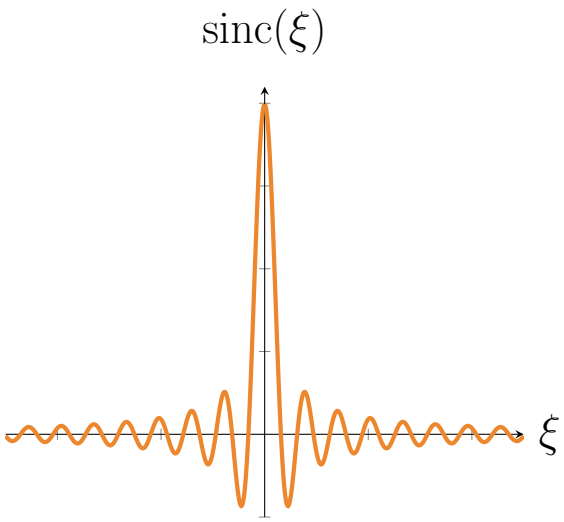

Figure 16.1: Plot of $\operatorname{sinc}(\xi) = \operatorname{Re}\widehat{\mathbf{1}_{[0,1]}}(\xi)$ on $\xi \in [-50, 50]$.

Perhaps this example has not quite convinced you of how badly the Fourier transform plays with supports. You may say the example doesn't count because $\mathbf{1}_{[0,1]}$ was discontinuous, and in particular not smooth. Perhaps, you say, if we are only interested in (smooth) test functions φ, then we can control the size of the support of $\hat{\varphi}$. Unfortunately, it turns out that Fourier transform of a (nonzero) compactly supported function is *never* compactly supported!

Theorem 16.3.2 (Paley-Wiener)**.** *Suppose $\varphi \in \mathcal{D}(R)$. Then, $\hat{\varphi}(\xi)$ can be analytically continued to an entire function on $\mathbb{C}$. In particular, $\hat{\varphi}(\xi)$ cannot vanish on an open set unless $\varphi \equiv 0$.*

This is technically only a small part of the Paley-Wiener theorem; see [28, Ch. 7] for a complete discussion. So, in general we can't say much at all about the support a Fourier transform, even for test functions. With a bit more thought, this actually makes sense: $\hat{\varphi}(\xi)$ tells us "how much of $\varphi(x)$ has frequency ξ", and if $\varphi(x)$ is compactly supported it must have a small but nonzero piece of very high frequency to force decay to zero in a finite interval.

16.3.2 Non-existence of a Distributional Fourier Transform (in general)

How would we define the Fourier transform of $T \in \mathcal{D}'(\mathbb{R}^n)$? Well, when we wanted to define derivatives of distributions or products of distributions with smooth functions, we always did so "weakly", or in other words by testing against $\varphi \in \mathcal{D}(\mathbb{R}^n)$: the operation in question is always brought onto the test function. This motivates us to "define" a distributional Fourier transform $\widehat{T}$ by

$$\langle \widehat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \quad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^n)\,. \tag{16.1}$$

Seems reasonable, right? The problem is this: by the discussion in the previous subsection, $\hat{\varphi}$ is not a test function hence the right-hand side of the above is actually not well-defined! Therefore, an arbitrary distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ cannot be Fourier-transformed!

The above conclusion should not sit well with you. Let's consider the example $n = 1$, $T = \delta_{x_0}$. Then, a formal computation using the definition of the Fourier transform gives

$$\widehat{\delta_{x_0}} = \frac{1}{\sqrt{2\pi}} e^{-ix_0\xi} \in L^1_{\text{loc}}(\mathbb{R}),$$

which makes perfect sense as a distribution by an earlier example. There must be something special about δ_{x_0} that allows it to be Fourier-transformed. Indeed, it turns out that δ_{x_0} is a **tempered distribution**, which is a particularly nice distribution that makes sense even when acting on functions that are *not compactly supported*. Once we learn more about tempered distributions and the **Schwartz space** of smooth functions they naturally act on, we'll see that tempered distributions can always be Fourier-transformed.

Chapter 17

Additional Problems

1. Consider the free dynamic Timoshenko beam equation for $w(x,t)$:

$$w_{xxxx} + w_{tt} - w_{xxtt} + w_{tttt} + w_{xx} = 0. \tag{17.1}$$

Give one-word answers to the questions below.

a) Is (17.1) linear or nonlinear?

b) Is (17.1) homogeneous or inhomogeneous?

c) What is the order of (17.1)?

2. Consider the Tricomi equation for $u(x,y)$

$$u_{xx} + xu_{yy} = 0. \tag{17.2}$$

Identify the regions in the xy plane where the Tricomi equation is elliptic, hyperbolic, or parabolic.

3. Consider the PDE

$$u_{xx} + 4u_{xy} + 3u_{yy} + y = 0. \tag{17.3}$$

a) Classify (17.3) as elliptic, hyperbolic, or parabolic.

b) Implement the change of variables

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} y - x \\ y - 3x \end{bmatrix}$$

to transform (17.3) to

$$4u_{\xi\eta} - u = 0.$$

4. In this problem we investigate the scaling symmetries of some equations modelling finite-amplitude waves in shallow water.

a) Suppose the function

$$u: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

solves the linear Korteweg-de Vries (KdV) equation:

$$u_t + u_{xxx} = 0 \quad \forall\ (x,t) \in \mathbb{R} \times \mathbb{R}. \tag{17.4}$$

Find a, b so that, for any $\lambda > 0$, the function

$$u_\lambda(x,t) = u\left(\lambda^a t, \lambda^b x\right)$$

also solves (17.4).

b) Now, pick $p \in \mathbb{N}$. Suppose instead that $u(x,t)$ solves the generalized KdV equation

$$u_t + u_{xxx} + u^{p-1}u_x = 0 \quad \forall\ (x,t) \in \mathbb{R} \times \mathbb{R}. \tag{17.5}$$

Find a, b, c so that, for any $\lambda > 0$, the function

$$u_\lambda(x,t) = \lambda^c u\left(\lambda^a t, \lambda^b x\right)$$

also solves (17.5).

5. Consider the nonlinear PDE for $u(x,t)$ below:

$$u_t + u_x = -u^2. \tag{17.6}$$

a) Use the characteristics of the linear PDE

$$v_t + v_x = 0$$

to determine a suitable change of variables that reduces (17.6) to an ODE.

b) Solve your ODE from part a to determine the general solution to (17.6).

c) Use part b to solve (17.6) with initial data

$$u(x, t=0) = x.$$

d) Does your particular solution from part c exist globally in the xt plane?

6. Consider the following Cauchy problem for an unknown real-valued function $u(x,y)$:

$$\begin{cases} xu_x + yu_y = 2u, \\ \qquad u|_{y=1} = u_0(x). \end{cases} \tag{17.7}$$

Use the method of characteristics to determine the largest region in the xy plane where a solution to this problem exists, and find this solution explicitly in terms of u_0. Make sure to sketch the region where the given data determines $u(x,y)$.

7. Let $\lambda, \mu \in [0,1]$ be fixed constants. The following IVP for a real-valued function $G(t,s)$ with analytic initial state $g(s)$ appears in the probabilistic modelling of telephone network usage:

$$\begin{cases} \partial_t G + \mu(s-1)\partial_s G - \lambda(s-1)G = 0, \\ G|_{t=0} = g(s). \end{cases} \tag{17.8}$$

Solve this IVP using the method of characteristics, making sure to sketch the characteristic curves along the way.

8. Let $H\colon \mathbb{R} \to [0,\infty)$ be a given rapidly decaying function and let $g \in \mathbb{R}$ be a fixed constant. Consider the following IVP for an unknown vector of functions $(u(x,t), \eta(x,t))$.

$$u_t(x,t) + g\eta_x(x,t) = 0 \quad \forall\ (x,t) \in \mathbb{R} \times (0,\infty), \tag{17.9a}$$

$$\eta_t(x,t) + \partial_x\left(H(x)\ u(x,t)\right) = 0 \quad \forall\ (x,t) \in \mathbb{R} \times (0,\infty), \tag{17.9b}$$

$$u(x,t=0) = u_0(x) \quad \forall\ x \in \mathbb{R}, \tag{17.9c}$$

$$\eta(x,t=0) = \eta_0(x) \quad \forall\ x \in \mathbb{R}. \tag{17.9d}$$

This IVP appears when modelling small-amplitudes waves in a thin layer of water above an uneven seafloor. In such an application, $H(x)$ represents the mean depth of the water in equilibrium. Let us suppose that we can find u and η solving the above system and **decaying rapidly in x for all t.**

a) Show that the mass, defined by

$$M(t) = \int_{-\infty}^{\infty} \eta\ \mathrm{d}x,$$

is constant in time.

b) Show that the energy, defined by

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} Hu^2 + g\eta^2\ \mathrm{d}x$$

is constant in time.

c) Using the energy defined in part b, prove that rapidly decaying solutions to (17.9) are unique.

d) In the case $H \equiv H_0 = $ constant, reduce (17.9) to an IVP for $\eta(x,t)$ alone.

e) Solve the IVP from part d.

f) Based on your answer from part e, explain why waves more faster in deep water than in shallow water.

9. Suppose $\kappa > 0$ and $u(x,t)$ solves

$$\begin{cases} u_t - \kappa u_{xx} = 0 \quad \forall\ (x,t) \in \mathbb{R} \times (0,\infty) \\ \qquad u|_{t=0} = u_0(x) \end{cases} \tag{17.10}$$

Then, assume that

$$M = \int_{-\infty}^{\infty} |u_0(x)|\ \mathrm{d}x < \infty.$$

a) Prove that

$$\sup_{x\in\mathbb{R}} |u(x,t)| \leq \frac{M}{2\sqrt{\pi\kappa t}} \quad \forall\ t > 0. \tag{17.11}$$

b) Conclude that

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t)| = 0.$$

10. Define a function $\phi(x)$ by

$$\phi(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 0 & \text{else.} \end{cases}$$

Solve the IVP

$$\begin{cases} u_t - u_{xx} = 0 \quad \forall\ (x,t) \in \mathbb{R} \times (0,\infty) \\ \qquad u|_{t=0} = \phi(x). \end{cases} \tag{17.12}$$

Write your answer in terms of the error function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2}\ \mathrm{d}y.$$

11. Consider the IBVP

$$\begin{cases} u_t + x u_x = 0 \quad \forall\ (x,t) \in (1,\infty) \times (0,\infty), \\ \qquad u|_{t=0} = f(x) \quad \forall\ x \in (1,\infty), \\ \qquad u|_{x=1} = h(t) \quad \forall\ t > 0 \end{cases} \tag{17.13}$$

subject to the compatibility condition $h(0) = f(1)$. Does the solution $u(x,t)$ ever reflect off the spatial boundary $x = 1$? Explain.

12. Define the Dirac δ-function $\delta(x)$ by demanding that, for any smooth and rapidly decaying function $\phi\colon \mathbb{R} \to \mathbb{R}$, we have

$$\phi(0) = \int_{-\infty}^{\infty} \phi(x)\ \delta(x)\ \mathrm{d}x.$$

Solve the inhomogeneous IVP

$$\begin{cases} u_t - u_{xx} = \delta(x) \quad \forall\ (x,t) \in \mathbb{R} \times (0,\infty), \\ \qquad u|_{t=0} = f(x) \quad \forall\ x \in \mathbb{R}. \end{cases} \tag{17.14}$$

Leave your answer in terms of an integral involving *only* the heat kernel.

13. Assume that $f(x)$ is twice continuously differentiable, 2π-periodic, and bounded. Further, assume the derivatives of $f(x)$ are also bounded.

a) Prove that $f'(x)$ is also periodic.

b) Let $\hat{f}(n)$ denote the n^{th} coefficient of the complex Fourier series of $f(x)$. Prove that there is a constant $C > 0$ (depending on f and/or its derivatives but not on n) so that

$$\left|\hat{f}(n)\right| \leq Cn^{-2}.$$

Hint: try to use integration by parts.

14. In this problem we investigate an IBVP appearing in the study of transistors. For a real-valued function $p(x,t)$ and constants K, D, η, L, the problem reads

$$\begin{cases} p_t - D\left(p_{xx} - \dfrac{\eta}{L}p_x\right) = 0 \quad \forall\ (x,t) \in (0,L)\times(0,\infty) \\ \\ p(0,t) = p(L,t) = 0 \quad \forall\ t > 0, \\ \\ p(x,0) = \varphi(x) \doteq \dfrac{KL}{D}\left[\dfrac{1-\exp\left(-\eta\left(1-\frac{x}{L}\right)\right)}{\eta}\right] \quad \forall\ x \in (0,L). \end{cases} \tag{17.15}$$

a) Implement the substitution

$$p(x,t) = \exp\left(\frac{\eta x}{2L} - \frac{D\eta^2 t}{4L^2}\right)\ u(x,t)$$

to change (17.15) into an IBVP involving a heat equation for $u(x,t)$. You should find that

$$u(x,0) = \phi(x) \doteq \exp\left(-\frac{\eta x}{2L}\right)\ \varphi(x).$$

b) Use separation of variables to solve the IBVP for $u(x,t)$. Make sure to justify why all eigenvalues have the same sign. You may use without proof that the sine series of $\phi(x)$ is given by

$$\phi(x) = \frac{2KL}{D}\frac{1-e^{-\eta}}{\eta}\sum_{n=1}^{\infty}\frac{n\pi}{n^2\pi^2 + \frac{1}{4}\eta^2}\ \sin\left(\frac{n\pi x}{L}\right).$$

c) Introduce another constant I_0. Define the **reclaimable charge** in our transistor by

$$Q \doteq \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty}\frac{I_0 D}{K}p_x(x=0,t)\ \mathrm{d}t.$$

Find an expression for Q as an infinite series. You may assume without proof that all Fourier series appearing in your computation are term-by-term integrable. *Hint: be careful not to mix up u and p!*

d) Apply Parseval's identity to $\phi(x)$ to show that Q admits the closed-form expression

$$Q = \frac{I_0 L^2}{D} \frac{1 - e^{-\eta}}{\eta} \left[\frac{\frac{\sinh \eta}{\eta} - 1}{\cosh \eta - 1} \right].$$

15. Solve the IBVP

$$\begin{cases} u_t - u_{xx} = 0 \quad \forall\ (x,t) \in (0,\pi) \times (0,\infty) \\ \\ u(0,t) = 0 \quad \forall\ t > 0, \\ \\ u_x(\pi,t) = 0 \quad \forall\ t > 0, \\ \\ u(x,0) = \dfrac{4}{5} \sin\left(\dfrac{1}{2}x\right) + \dfrac{1}{5} \sin\left(\dfrac{5}{2}x\right) \quad \forall\ x \in (0,\pi). \end{cases} \tag{17.16}$$

You may take for granted the following orthogonality relation: for any $\ell > 0$ and $m, n = 1, 3, 5, ...$, we have

$$\int_0^\ell \sin \frac{n\pi x}{2\ell} \sin \frac{m\pi x}{2\ell}\ \mathrm{d}x = \frac{\ell}{2} \delta_{mn}. \tag{17.17}$$

16. In this problem we investigate the scaling symmetries of some model equations arising in quantum theory and condensed matter physics.

a) Suppose the function

$$u \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$$

solves the free-particle Schrödinger equation:

$$iu_t + \frac{1}{2} \Delta u = 0 \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{17.18}$$

Find a, b so that, for any $\lambda > 0$, the function

$$u_\lambda(t,x) = u\left(\lambda^a t, \lambda^b x\right)$$

also solves (17.18).

b) Now, pick $p \in \mathbb{N}$ and $\mu = \pm 1$. Suppose instead that $u(t,x)$ solves the nonlinear Schrödinger equation

$$iu_t + \frac{1}{2} \Delta u + \mu |u|^{p-1} u = 0 \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{17.19}$$

Find a, b, c so that, for any $\lambda > 0$, the function

$$u_\lambda(t,x) = \lambda^c u\left(\lambda^a t, \lambda^b x\right)$$

also solves (17.19).

17. Fix a bounded function $V\colon \mathbb{R}^n \to \mathbb{R}$. Suppose $u\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ solves the Schrödinger equation with potential $V(x)$:

$$iu_t = -\frac{1}{2}\Delta u + Vu. \tag{17.20}$$

Further, suppose that for all $t \in \mathbb{R}$, the function

$$x \mapsto u(t, x)$$

is rapidly decaying.

a) Prove that

$$\partial_t |u|^2 = 2\mathrm{Re}\left(\overline{u}u_t\right).$$

Hint: remember that, for $z \in \mathbb{C}$, $|z|^2 = z\overline{z}$. Use the product rule.

b) Prove that the total probability functional

$$M[u] = \frac{1}{2}\int_{\mathbb{R}^n} |u(x)|^2 \ \mathrm{d}x$$

is constant in time.

c) Prove that the energy functional

$$E[u] = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2 + 2V(x)|u(x)|^2 \ \mathrm{d}x$$

is constant in time.

18. Let the radial function $v\colon \mathbb{R}^2 - \{0\} \to \mathbb{R}$ be defined by

$$v(r) = \frac{1}{8\pi} r^2 \log r. \tag{17.21}$$

a) Prove that $\Delta^2 v = 0$. In other words, v is **biharmonic** on $\mathbb{R}^2 - \{0\}$. Make sure to explain why v is not biharmonic at the origin.

b) Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, connected C^1 domain with unit outward normal $\mathbf{n}$. For any $x \in \Omega$ let

$$v^x(y) \doteq v(|y - x|) \quad \forall\ y \in \overline{\Omega} - \{x\}.$$

Prove that, for any $u \in C^4\left(\overline{\Omega}\right)$ such that $\Delta^2 u = 0$, we have

$$u(x) = \int_{\partial\Omega} u\frac{\partial \Delta v^x}{\partial \mathbf{n}} - \Delta v^x \frac{\partial u}{\partial \mathbf{n}} - v^x \frac{\partial \Delta u}{\partial \mathbf{n}} + \Delta u \frac{\partial v^x}{\partial \mathbf{n}} \ \mathrm{d}S(y).$$

19. Define a domain Ω by

$$\Omega \doteq \left\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\right\}.$$

Suppose $g \in C(\partial\Omega)$ is a given continuous function and $u_0 \colon \Omega \to \mathbb{R}$ is a given bounded function. Solve the IBVP

$$\begin{cases} u_t(t,x) - \Delta u(t,x) = 0 & \forall\ x \in \Omega, \quad \forall\ t > 0, \\ u|_{t=0}(x) = u_0(x) & \forall\ x \in \Omega, \quad \text{and} \\ u|_{\partial\Omega}(t,x) = 0 & \forall\ t > 0. \end{cases} \tag{17.22}$$

20. Define $\Omega \subseteq \mathbb{R}^4$ by

$$\Omega \doteq \{\mathbf{x} = (x_1, x_2, x_3, x_4) \mid x_4 > 0\}.$$

Suppose that $B \subseteq \Omega$ is an open ball of unit measure lying entirely inside Ω. Show that the inhomogenous Neumann BVP

$$\begin{cases} \Delta u(\mathbf{x}) = \begin{cases} 2 & \mathbf{x} \in B \\ 0 & \text{otherwise} \end{cases} & \forall\ \mathbf{x} \in \Omega, \\ \left.\dfrac{\partial u}{\partial \mathbf{n}}\right|_{\partial\Omega} = \dfrac{1}{\pi^2 |x_1^2 + x_2^2 + x_3^2 + 1|^2} & \forall\ \mathbf{x} \in \partial\Omega \end{cases} \tag{17.23}$$

has no solution.

21. Given a fixed weight function $\rho(x) \colon \mathbb{R} \to [0, \infty)$, consider the Euler-Bernoulli beam equation for a real-valued function $u(t, x)$:

$$u_{tt} + \partial_x^2 \left(\rho(x) u_{xx}\right) = 0.$$

Find the conserved energy functional $E[u]$ for this PDE.

22. Consider the Rayleigh beam equation for a real-valued function $u(t, x)$:

$$0 = \left(1 - \partial_x^2\right) u_{tt} + u_{xxxx}.$$

Find the conserved energy functional $E[u]$ for this PDE.

23. Consider the Benjamin-Bona-Mahony equation for a real-valued function $u(t, x)$:

$$u_t - u_{xxt} + u_x + 2uu_x = 0.$$

a) Find the conserved energy functional $E[u]$ for this PDE.

b) Show also that the modified energy

$$H[u] = \int_{-\infty}^{+\infty} \frac{u^2(t,x)}{2} + \frac{u^3(t,x)}{3}\, \mathrm{d}x$$

is conserved. *Hint: this one is a bit tricky, maybe do it after finishing the other problems.*

24. Solve the Cauchy problem

$$\begin{cases} u_x - 2xu_y = x & \forall\ (x,y) \in \mathbb{R}^2 \\ u|_{y=0} = \dfrac{3}{2}x^2 & \forall\ x \in \mathbb{R}. \end{cases}$$

You must also sketch the characteristic curves of the PDE in your solution.

25. For some $T > 0$, consider the Cauchy problem

$$\begin{cases} u_t + u^2 u_x = 0 & \forall\ (t,x) \in [0,T) \times \mathbb{R} \\ u|_{t=0} = u_0(x) & \forall\ x \in \mathbb{R}. \end{cases}$$

a) Assuming the solution $u(t,x)$ is rapidly decaying in space for all $t \in [0,T]$, establish the conservation law

$$\int_{-\infty}^{\infty} u(t,x)\ \mathrm{d}x = \int_{-\infty}^{\infty} u_0(x)\ \mathrm{d}x.$$

b) Show that the solution can be expressed implicitly as

$$u(x,t) = u_0\left(x - u^2(x,t)t\right).$$

c) Suppose that $u_0(x) \geq 0$ for all $x \in \mathbb{R}$. Prove the weak minimum principle

$$u(x,t) \geq 0 \quad \forall\ (t,x) \in [0,T) \times \mathbb{R}.$$

26. Suppose we know that $u(x,t)$, $v(x,t)$ satisfy

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & \forall\ (x,t) \in \mathbb{R}^2 \\ v_{tt} - 4v_{xx} = 0 & \forall\ (x,t) \in \mathbb{R}^2 \\ u|_{t=0} = v|_{t=0} & \forall\ x \in [0,1], \\ u_t|_{t=0} = v_t|_{t=0} & \forall\ x \in [0,1]. \end{cases} \tag{17.24}$$

Draw the largest subset of the xt plane $\mathbb{R}^2$ where we can be **certain** that $u(x,t) = v(x,t)$.

27. Determine whether or not solutions of the IBVP

$$\begin{cases} u_{tt} - u_{xx} = 0 & \forall\ (t,x) \in (0,\infty) \times (0,1), \\ u'(t,0) + u(t,0) = 0, \\ u'(t,1) + 2u(t,1) = 0, \\ u(0,x) = u_0(x), \\ u_t(0,x) = v_0(x) \end{cases}$$

are necessarily bounded on $[0,1]$ *uniformly in time.*

28. Consider the eigenvalue problem

$$\begin{aligned} X''(x) &= -\lambda X(x) \quad \forall\ x \in (0,\pi), \\ X(0) &= 0, \\ X'(\pi) + X(\pi) &= 0. \end{aligned}$$

a) Is $\lambda = 0$ an eigenvalue?

b) Use a picture to prove that there exist infinitely many positive eigenvalues $\lambda > 0$. Find a nonzero lower bound for the smallest positive eigenvalue.

c) Are there any negative eigenvalues λ? Use a picture to explain why or why not.

29. Consider the rectangle

$$\Omega \doteq \left\{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1,\ -2 \leq y \leq 2\right\}.$$

Suppose $u(x, y)$ solves the BVP

$$\begin{cases} \Delta u = 0 \quad \forall\ (x, y) \in \Omega, \\ u|_{x=-1} = 0, \\ u|_{x=1} = y, \\ u|_{y=-2} = 1 - x^2, \\ u|_{y=2} = -3. \end{cases}$$

Find $m, M \in \mathbb{R}$ so that

$$m \leq u(x, y) \leq M \quad \forall\ (x, y) \in \overline{\Omega}.$$

30. Let $R_1, R_2 > 0$ and consider the spherical shell

$$\Omega \doteq \left\{(x, y, z) \in \mathbb{R}^3 \mid R_1^2 \leq x^2 + y^2 + z^2 \leq R_2^2\right\}.$$

Solve the inhomogeneous BVP

$$\begin{aligned} \Delta u &= 6 - \frac{2(R_1 + R_2)}{r}, \\ u|_{r=R_1} &= 0, \\ u|_{r=R_2} &= 0. \end{aligned}$$

31. Define

$$\Omega \doteq \left\{(x, y) \in \mathbb{R}^2 \mid r \leq 2\right\}.$$

a) Solve the BVP

$$\begin{cases} \Delta u = 0 & \forall\ (x, y) \in \Omega, \\ |u(r, \theta)| < \infty, & \\ u|_{\partial\Omega} = 1 + 4\cos\theta & \forall\ \theta \in [0, 2\pi]. \end{cases}$$

b) Explain why

$$u(0, 0) = \frac{1}{4\pi} \int_0^2 \int_0^{2\pi} r u(r, \theta)\ \mathrm{d}\theta\ \mathrm{d}r,$$

and manually verify this for the solution you computed in part a.

32. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open domain with smooth boundary. Let $p, q: \Omega \to (0, \infty)$ and $V: \mathbb{R} \to \mathbb{R}$ be fixed functions, with p continuously differentiable on $\overline{\Omega}$, q continuous on $\overline{\Omega}$, and V continuously differentiable on all of $\mathbb{R}$. Define the functional

$$F[u] = \frac{1}{2} \int_\Omega p(x)|\nabla u(x)|^2 + q(x)u^2(x) + V\left(u^2(x)\right) \, \mathrm{d}x$$

and consider the minimization problem

$$\min_{u \in C^2(\Omega)} F[u] \quad \text{such that} \quad u|_{\partial\Omega} = 0.$$

Assuming a solution to this minimization problem exists, show that it satisfies the nonlinear elliptic BVP

$$\begin{cases} -\nabla \cdot (p(x)\nabla u(x)) + q(x)u(x) + V'\left(u^2(x)\right) u(x) = 0 & \forall x \in \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

33. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, open, connected set with smooth boundary. Recall that a real-valued function $u: \Omega \to \mathbb{R}$ defines a surface given by

$$\Sigma_u = \left\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \Omega, \ z = u(x, y)\right\}.$$

The surface area of Σ_u is

$$A[u] = \int_\Omega \sqrt{1 + |\nabla u(x)|^2} \, \mathrm{d}x.$$

Now, let $g \in C(\partial\Omega)$ be fixed. Suppose we pretend the curve

$$\Gamma = \left\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \partial\Omega, \ z = g(x, y)\right\}$$

is a thin wire we dip in a soap solution. The function $u_*(x, y)$ giving rise to the shape of the corresponding soap bubble then satisfies the minimization problem

$$\min_{u \in C^2(\overline{\Omega})} A[u] \quad \text{such that} \quad u|_{\partial\Omega} = g.$$

In geometric language, this means Σ_{u_*} is a **minimal surface**.

Show that, if a solution u_* to this minimization problem exists, then the mean curvature of Σ_{u_*} vanishes. *Hint: first review, or figure out, a simple way to compute the mean curvature of such a surface using only the outward unit normal* $\mathbf{n}$.

34.

a) Let $x' \in \mathbb{R}$ be arbitrary. Prove that

$$\left(1 - \partial_x^2\right) e^{-|x - x'|} = 2\delta(x - x')$$

in the sense of distributions.

b) Let $f(x) \in \mathcal{S}(\mathbb{R})$. Use part a to derive a representation formula for the PDE

$$\begin{cases} \left(1 - \partial_x^2\right) u(x) = f(x) \\ |u(x)| \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{17.25}$$

c) Use part b to prove that there exists $C > 0$ such that the solution $u(x)$ to (17.25) satisfies

$$\sup_{x \in \mathbb{R}} |u(x)| \leq C \left\| f \right\|_{L^1(\mathbb{R})} \tag{17.26}$$

d) Again, let $u(x)$ denote the solution to (17.25). Use the Fourier transform to prove that there exists $C > 0$ such that

$$\left\| u \right\|_{L^2(\mathbb{R})} \leq C \left\| f \right\|_{L^2(\mathbb{R})}. \tag{17.27}$$

e) Use part a in conjunction with the method of reflection to find the Green's function

$$G(x, x') : [0, \infty)^2 \to \mathbb{R}.$$

of the elliptic operator $1 - \partial_x^2$ on the spatial domain $[0, \infty)$ subject to the Dirichlet boundary condition

$$G(0, x') = 0 \quad \forall \; x' \in [0, \infty).$$

f) Verify that your result from part b satisfies the reciprocity condition

$$G(x, x') = G(x', x) \quad \forall \; (x, x') \in [0, \infty)^2.$$

g) If f is the restriction of a Schwartz function on $\mathbb{R}$ to $(0, \infty)$, derive a representation formula for the solution of

$$\begin{cases} \left(1 - \partial_x^2\right) u(x) = f(x) \quad \forall \; x \in (0, \infty), \\ u(0) = 0, \\ |u(x)| \to 0 \quad \text{as} \quad x \to \infty. \end{cases} \tag{17.28}$$

h) Derive an analogue of (17.26) for the BVP (17.28).

i) Explain in one sentence why the proof from part b does not immediately translate to an analogue of (17.27) for the BVP (17.28). *Hint: this is meant to be really, really easy!*

35. Consider the functon

$$f(x) = \frac{1}{2} \text{sgn}\left(\log|x|\right) \in L^1_{\text{loc}}(\mathbb{R}).$$

a) Explain how $f(x)$ can be identified as a distribution.

b) Compute $f'(x)$ in the sense of distributions.

36. Let $\mathcal{D}(\mathbb{R}^n)$ denote the set of test functions on $\mathbb{R}^n$, and let $\mathcal{D}'(\mathbb{R}^n)$ denote the set of distributions on $\mathbb{R}^n$.

a) How would you define the **gradient** of $T \in \mathcal{D}'(\mathbb{R}^n)$, denoted ∇T? Make sure to address what space ∇T acts on. *Hint: explain first why ∇T does not naturally act on $\mathcal{D}(\mathbb{R}^n)$.*

b) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. We define the **indicator function** of Ω by

$$\mathbf{1}_\Omega(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega \\ 0 & \text{else.} \end{cases}$$

Explain why $\mathbf{1}_\Omega \in \mathcal{D}'(\mathbb{R}^n)$, then compute $\nabla \mathbf{1}_\Omega$ in the sense of distributions using your definition from part a. Express your answer in terms of the distribution

$$\delta_{\partial\Omega} \in \mathcal{D}'(\mathbb{R}^n)$$

defined by

$$\langle \delta_{\partial\Omega}, \varphi \rangle = \int_{\partial\Omega} \varphi(x)\, \mathrm{d}S(x) \quad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^n).$$

37. Consider three parameters $\omega_0 \in \mathbb{R}$, $\tau > 0$, $M \in \mathbb{N}$. Let $E_0(t)$ be a Schwartz function of time, then define

$$E_{\text{pulse}}(t) = E_0(t)e^{i\omega_0 t}$$

and

$$E_M(t) = \sum_{m=-M+1}^{M-1} E_{\text{pulse}}(t - m\tau).$$

In laser optics, $E_M(t)$ is used to represent a train of M pulses (wavepackets) propagating without dispersion.

a) Let ω denote the Fourier variable dual to t. Prove that

$$\widehat{E_M}(\omega) = \widehat{E_0}(\omega - \omega_0) \sum_{m=-M+1}^{M-1} e^{-im\omega\tau}$$

b) Consider the "Dirac comb" distribution T defined formally by the infinite sum

$$T = 2\pi\widehat{E_0}(\omega - \omega_0) \sum_{n=-\infty}^{\infty} \delta(\omega\tau - 2n\pi).$$

Prove that T is actually a distribution. In particular, show that

$$\langle T, \varphi \rangle$$

has finite modulus for any test function $\varphi \in \mathcal{D}(\mathbb{R})$.

c) Prove that

$$\widehat{E_M}(\omega) \rightharpoonup T \quad \text{(weak convergence)} \quad \text{as } M \to \infty.$$

Hint: use examples 5 and 9 from section 12.1 of Strauss. Try to think of how to extend the results of these examples from functions defined on $(-\pi, \pi)$ to functions defined on all of $\mathbb{R}$.

38. Solve the IVP

$$\begin{cases} u_t - u_{xx} + u_{xxx} = 0 & \forall\ (t,x) \in (0,\infty) \times \mathbb{R}. \\ u(t=0,x) = u_0(x) \end{cases}$$

where $u_0(x)$ is Schwartz. Leave your answer in the form of an inverse Fourier transform.

Bibliography

[1] Bilal Abbasi and Walter Craig, *On the initial value problem for the wave equation in Friedmann-Robertson-Walker space-times*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **470** (2014), no. 2169, 20140361, 13.

[2] Grigory I. Barenblatt, *Scaling, self-similarity, and intermediate asymptotics*, Cambridge Texts in Applied Mathematics, vol. 14, Cambridge University Press, Cambridge, 1996.

[3] Chuck A. Bennett, *Principles of physical optics*, John Wiley & Sons, 2008.

[4] David T. Blackstock, *Fundamentals of physical acoustics*, Wiley, New York, 2000.

[5] John P. Boyd, *Chebyshev and Fourier spectral methods*, second ed., Dover Publications, Inc., Mineola, NY, 2001.

[6] Rustum Choksi, *Partial differential equations: a first course*. To appear in print in summer 2022.

[7] David Colton and Rainer Kress, *Inverse acoustic and electromagnetic scattering theory*, fourth edition, Applied Mathematical Sciences, vol. 93, Springer, Cham, 2019.

[8] Richard Courant and Kurt O. Friedrichs, *Supersonic flow and shock waves*, Applied Mathematical Sciences, Vol. 21, Springer-Verlag, New York-Heidelberg, 1976. Reprinting of the 1948 original.

[9] Walter Craig, *A course on partial differential equations*, Graduate Studies in Mathematics, vol. 197, American Mathematical Society, Providence, RI, 2018.

[10] Lev Eppelbaum, Izzy Kutasov, and Arkady Pilchin, *Applied geothermics*, Springer, Berlin/Heidelberg, 2014.

[11] Lawrence C. Evans, *Partial differential equations*, second edition, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[12] Grant R. Fowles, *Introduction to modern optics*, Dover Publications, New York, 1989. Corrected reprint of the second edition (1975).

[13] David Gottlieb and Steven A. Orszag, *Numerical analysis of spectral methods: theory and applications*, CBMS-NSF Regional Conference Series in Applied Mathematics, No. 26, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1977.

[14] David J. Griffiths, *Introduction to electrodynamics*, fourth edition, Pearson Education, Inc., 2013.

[15] Sadri Hassani, *Mathematical physics: a modern introduction to its foundations*, second ed., Springer, Cham, 2013.

[16] Oliver D. Kellogg, *Foundations of potential theory*, Dover Publications, New York, 1953. Reprint of the 1929 edition.

[17] Sergiu Klainerman, *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math. **33** (1980), no. 1, 43–101.

[18] Thomas W. Körner, *Fourier analysis*, Cambridge University Press, Cambridge, 1988.

[19] Lev D. Landau and Evgeny M. Lifshitz, *Fluid mechanics*, second edition, Course of Theoretical Physics, Vol. 6, Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass., 1987. Translated from the Russian by J. B. Sykes and W. H. Reid.

[20] Nikolai N. Lebedev, *Special functions and their applications*, Dover Publications, Inc., New York, 1972. Unabridged and corrected republication of the revised edition, translated from the Russian and edited by Richard A. Silverman.

[21] Jennifer L. Mueller and Samuli Siltanen, *Linear and nonlinear inverse problems with practical applications*, Computational Science & Engineering, vol. 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2012.

[22] James D. Murray, *Mathematical biology I: An introduction*, third edition, Interdisciplinary Applied Mathematics, vol. 17, Springer-Verlag, New York, 2002.

[23] Joseph Pedlosky, *Geophysical fluid dynamics*, second edition, Springer-Verlag, 1987.

[24] Nanako Shigesada, *Spatial distribution of dispersing animals*, J. Math. Biol. **9** (1980), no. 1, 85–96.

[25] Ivar Stakgold and Michael Holst, *Green's functions and boundary value problems*, third edition, Pure and Applied Mathematics (Hoboken), John Wiley & Sons, Inc., Hoboken, NJ, 2011.

[26] Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, vol. 2, Princeton University Press, Princeton, NJ, 2003.

[27] Walter A. Strauss, *Partial differential equations: an introduction*, second edition, John Wiley & Sons, Ltd., Chichester, 2008.

[28] Robert S. Strichartz, *A guide to distribution theory and Fourier transforms*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003. Reprint of the 1994 original.

[29] Terence Tao, *Nonlinear dispersive equations: local and global analysis*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006.

[30] Andrey N. Tikhonov and Alexander A. Samarskiĭ, *Equations of mathematical physics*, Dover Publications, Inc., New York, 1990. Translated from the Russian by A. R. M. Robson and P. Basu, Reprint of the 1963 translation.

[31] Lloyd N. Trefethen, *Spectral methods in MATLAB*, Software, Environments, and Tools, vol. 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.

[32] ———, *Approximation theory and approximation practice*, revised ed., Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013.

[33] Lloyd N. Trefethen and David Bau III, *Numerical linear algebra*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[34] Naum J. Vilenkin, *Special functions and the theory of group representations*, Translations of Mathematical Monographs, Vol. 22, American Mathematical Society, Providence, R.I., 1968. Translated from the Russian by V. N. Singh.

[35] Robert M. Wald, *General relativity*, University of Chicago Press, Chicago, IL, 1984.