UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: FYS3110, Quantum mechanics

Day of exam: Nov. 30., 2015

Exam hours: 09:00-13:00 (4 hours)

This examination paper consists of 3 pages.

Permitted material: Approved calculator, D.J. Griffiths: "Introduction to Quantum Mechanics", the printed notes: "Time evolution of states in quantum mechanics", "Symmetry and degeneracy" and "WKB connection formulae", one handwritten A4-sheet(2 pages) with your own notes, and K. Rottmann: "Matematisk formelsamling".

Check that the problem set is complete before you start working. Some of the subproblems have more than one question.

A particle with mass m is confined to move in a two-dimensional harmonic oscillator potential characterized by a frequency ω. The Hamiltonian expressed in polar coordinates, $x = r\cos\phi$, $y = r\sin\phi$ is denoted H^0 and is

$$H^0 = -\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right) + \frac{1}{2}m\omega^2 r^2.$$

a) The energy eigenfunctions can be written $\psi(r,\phi) = f(r)e^{ik\phi}$. Find the equation the function $f(r)$ must satisfy and determine the allowed values of k.

The angular momentum operator normal to the plane of motion is

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}.$$

b) Show that $\psi(r,\phi) = f(r)e^{ik\phi}$ are also eigenfunctions of L_z and determine their eigenvalues.

The Hamiltonian H^0 can also be written in cartesian coordinates

$$H^0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2$$

where p_i is the momentum operator in the i-direction ($i \in \{x, y\}$). H^0 can therefore be viewed as a composite system of two independent one-dimensional harmonic oscillators, each with ladder operators a_i, $a_i^\dagger$, $i \in \{x, y\}$. Use the basis states $|n_x n_y\rangle$ ($\equiv |n_x\rangle \otimes |n_y\rangle$) where n_i is the eigenvalue for $a_i^\dagger a_i$ on the one-dimensional oscillator in direction $i \in \{x, y\}$.

c) Write down the Hamiltonian H^0 using ladder operators for the two one-dimensional oscillators. Write also down the energies and energy eigenstates for the *three* lowest energy levels. Assume here that the particle has no spin.

d) The angular momentum operator can be expressed as $L_z = xp_y - yp_x$. Express L_z in terms of ladder operators. For the *two* lowest energy levels: Write down the set of normalized eigenstates that are eigenstates for both L_z and H^0 simultaneously (i.e. the common set of eigenfunctions).

A perturbation term H^1 (to be specified later) is added to H^0.

e) Show that if $H^1 = [H^0, A]$ for some operator A, then the perturbative first order energy correction is zero. Make sure that your proof holds both in the non-degenerate and the degenerate case. Can A be hermitian?

The perturbation term is $H^1 = g\sqrt{\frac{2m\omega}{\hbar}}x$ where g is a real positive constant with units of energy.

f) Explain, by finding the operator A, see problem e), that the first order energy correction due to the perturbation $H^1 = g\sqrt{\frac{2m\omega}{\hbar}}x$ is zero. Calculate also the second order perturbative correction to the ground state energy.

g) Use the variational method with trial state $|\psi\rangle = \cos\theta|00\rangle + \sin\theta|10\rangle$ to find a lowest possible upper bound on the ground state energy of $H^0 + H^1$.

In the following set $g = 0$ and assume that the particle in the potential has an electric charge q and is influenced by a weak time-varying electric field; a Gaussian pulse

$$\vec{E}(t) = E_0 \hat{n} e^{-t^2/\tau^2}.$$

Here $\hat{n}$ is a constant unit vector in the polarization direction of the electric field, E_0 is a constant with units of force divided by charge and τ, with the unit of time, indicates the duration of the pulse.

h) Calculate the probability that the Gaussian pulse excites the particle from the ground state to the first excited energy level. Use first-order time-dependent perturbation theory and assume that the polarization direction of the electric field is in the x-direction. Neglect the magnetic field.

In the following set $E_0 = 0$ and $q = 0$. Now consider *two* fermions, each with spin-1/2, in the same two-dimensional harmonic oscillator potential. The spins of the two fermions interact with each other according to an extra term

$$H_s = \frac{b}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2$$

added to H^0. Here b is a positive constant with units of energy and $\vec{S}_i$ is the spin-1/2 operator acting on the spin of particle i.

i) Assume $b \ll \hbar\omega$ and consider the *three* lowest energy levels for the system of the two interacting spin-1/2 fermions in the harmonic oscillator potential. Write down all the exact energy eigenstates. Find the energy and the degeneracy for each of the energy levels.

In the last subproblem set $b = 0$ and allow for an integer n number of fermions, each with spin-1/2, in the potential.

j) Make a plot that shows the lowest total energy per particle E_n/n as a function of n for n spin-1/2 fermions located in the two-dimensional harmonic oscillator potential. Make your plot such that E_n/n is in units of $\hbar\omega$ and plot values for $n \in \{1, 2, \ldots, 7\}$. What are the atomic numbers for the *three* first inert elements (noble gases) in a hypothetical two-dimensional world where electrons are fermions with spin-1/2 and the $1/r$-potential from the nucleus is replaced by a two dimensional r^2-potential? – THE END –