Home Exam
Fys 3110

Quantum Mechanics

candidate number: 26

Matnat
UiO
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Contents

1 Problem 1: a spin-1/2 system

A single spin-1/2, orthonormal eigenstates of spin-squared operator $S^2 \equiv S_x^2 + S_y^2 + S_x^2$ and z-component spin S_z are denoted $|\uparrow\rangle \equiv |s = 1/2, m_s = 1/2\rangle$ and $|\downarrow\rangle \equiv |s = 1/2, m_s = -1/2\rangle$. This means that $S^2 |\uparrow\rangle = \hbar\frac{1}{2}(\frac{1}{2} + 1)$ and $S_z |\uparrow\rangle = +\frac{\hbar}{2} |\uparrow\rangle$. Simlilarly, $S^2 |\downarrow\rangle = \hbar\frac{1}{2}(\frac{1}{2} + 1)$ and $S_z |\downarrow\rangle = -\frac{\hbar}{2}$.
We also have the commutation relations
$[S^x, S^y] = i\hbar S^z, \quad [S^y, S^z] = i\hbar S^x, \text{ and } [S^z, S^x] = i\hbar S^y.$

1.1 Ladder operator and z-spin

Use commutation relations to show that $S^+ |\downarrow\rangle$ is an eigenstate of S^z. Also note the Eigenvalue.

$$\begin{aligned}
S^+ |\downarrow\rangle &\text{is eigenstate of } S_z. \\
S^z S^+ |\downarrow\rangle &= [S^z, S^+] |\downarrow\rangle + S^+ S^z |\downarrow\rangle \\
S^+ &= S^x + iS^y \\
[S^z, S^+] &= [S^z, S^x] + i[S^z, S^y] = i\hbar S^y - i^2\hbar S^x \\
&= \hbar(S^x + iSy) = \hbar S^+ \\
S^z S^+ |\downarrow\rangle &= \hbar S^+ |\downarrow\rangle - \frac{\hbar}{2} S^+ |\downarrow\rangle \\
&= (\frac{-\hbar}{2} + \hbar) S^+ |\downarrow\rangle
\end{aligned} \tag{1}$$

This is the eigenvalue of $S^z |\downarrow\rangle$ pluss $\hbar$. And we retain $S^+ |\downarrow\rangle$ as we should, given the fact that it is an eigenstate.

1.2 More Ladder operators

Express $S^- S^+$ in terms of S^z, S^2 and apropriate constants. Use this to compute norm of states $|\psi_1\rangle = S^+ |\downarrow\rangle$ and $|\psi_2\rangle = S^+ |\uparrow\rangle$

$$\begin{aligned}
S^{\pm} &= S^x \pm iS^y \\
S^- S^+ &= (s^x - iS^y) \cdot (S^x + iS^y) \\
&= S^{x2} + iS^x S^y - iS^y S^x + S^{y2} \\
&= S^2 - S^{z2} + i[S^x, S^y] \\
&= S^2 - S^{z2} - \hbar S^z \\
&\text{To find the norm:} \\
|\psi_1\rangle &= S^+ |\downarrow\rangle \to \langle\psi_1| = \langle\downarrow| S^- \\
|\psi_2\rangle &= S^+ |\uparrow\rangle \to \langle\psi_1| = \langle\uparrow| S^- \\
\langle\psi_1|\psi_1\rangle &= \langle\downarrow| S^- S^+ |\downarrow\rangle \\
&= \langle\downarrow| S^2 |\downarrow\rangle - \langle\downarrow| S^{z2} |\downarrow\rangle - \hbar \langle\downarrow| S^z |\downarrow\rangle \\
&= \hbar\frac{1}{2}(\frac{1}{2} + 1) - \frac{\hbar^2}{4} + \frac{\hbar^2}{2} \\
&= \hbar(\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2}) = \hbar \\
\langle\psi_2|\psi_2\rangle &= \langle\uparrow| S^- S^+ |\uparrow\rangle \\
&= \langle\uparrow| S^2 |\uparrow\rangle - \langle\uparrow| S^{z2} |\uparrow\rangle - \hbar \langle\uparrow| S^z |\uparrow\rangle \\
&= \hbar\frac{1}{2}(\frac{1}{2} + 1) - \frac{\hbar^2}{4} - \frac{\hbar^2}{2} \\
&= \hbar(\frac{1}{4} + \frac{1}{2} - \frac{1}{4} - \frac{1}{2}) = 0
\end{aligned} \tag{2}$$

The norms are thus $\hbar$ and 0, respectively. An explanaition for the 0, I would say is that we are applying a positive ladder operator on an up-spin, which is already in it's highest state. Thus, it anihilates.

In the following, assume: $S^+ \left|\downarrow\right\rangle = \hbar \left|\uparrow\right\rangle, S^- \left|\uparrow\right\rangle = \hbar \left|\downarrow\right\rangle$

1.3 "Uncertainty product"

state $\left|\phi\right\rangle = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + e^{i\theta}\left|\downarrow\right\rangle)$, $\theta\epsilon\mathbb{R}$. Compute $\sigma^2_{S^x} \cdot \sigma^2_{S^y}$.
$\sigma^2_{S^x} = \langle\phi|(S^x - \langle\phi|S^x|\phi\rangle)^2|\phi\rangle$ and $\sigma^2_{S^y} = \langle\phi|(S^y - \langle\phi|S^y|\phi\rangle)^2|\phi\rangle$.
Find the values of θ for which $\sigma^2_{S^x} \cdot \sigma^2_{S^y} = 0$. Is the Heisenberg uncertainty relation violated for these θ?

$$\begin{aligned}
S^x \left|\phi\right\rangle &= \frac{1}{2}(S^+ + S^-) \cdot \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + e^{i\theta}\left|\downarrow\right\rangle) = \frac{\hbar}{2\sqrt{2}}(e^{i\theta}\left|\uparrow\right\rangle + \left|\downarrow\right\rangle) \\
S^y \left|\phi\right\rangle &= \frac{1}{2i}(S^+ - S^-) \cdot \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + e^{i\theta}\left|\downarrow\right\rangle) = \frac{\hbar}{2\sqrt{2}i}(e^{i\theta}\left|\uparrow\right\rangle - \left|\downarrow\right\rangle) \\
S^x(S^x \left|\phi\right\rangle) &= \frac{\hbar}{4\sqrt{2}}(S^+ + S^-)(e^{i\theta}\left|\uparrow\right\rangle + \left|\downarrow\right\rangle) = \frac{\hbar^2}{4\sqrt{2}}\left|\phi\right\rangle \\
S^y(S^y \left|\phi\right\rangle) &= \frac{-\hbar}{4\sqrt{2}}(S^+ - S^-)(e^{i\theta}\left|\uparrow\right\rangle - \left|\downarrow\right\rangle) = \frac{\hbar^2}{4\sqrt{2}}\left|\phi\right\rangle \\
\left\langle\phi\right| &= \frac{1}{\sqrt{2}}(\left\langle\uparrow\right| + e^{i\theta}\left\langle\downarrow\right|) \\
\langle\phi|\phi\rangle &= \frac{1}{2}(\left\langle\uparrow\right|\left|\uparrow\right\rangle + e^{-i\theta+i\theta}\left\langle\downarrow\right|\left|\downarrow\right\rangle) = 1 \\
\langle\phi|S^{i2}|\phi\rangle &= \frac{\hbar}{4} \quad | \quad i = x, y \\
\langle\phi|S^x|\phi\rangle &= \frac{\hbar}{4}(\left\langle\uparrow\right| + e^{-i\theta}\left\langle\downarrow\right|)(e^{i\theta}\left|\uparrow\right\rangle + \left|\downarrow\right\rangle) = \frac{\hbar}{4}(e^{i\theta} + e^{-i\theta}) = \frac{\hbar}{2}cos(\theta) \\
\langle\phi|S^x|\phi\rangle &= \frac{\hbar}{4}(\left\langle\uparrow\right| + e^{-i\theta}\left\langle\downarrow\right|)(e^{i\theta}\left|\uparrow\right\rangle - \left|\downarrow\right\rangle) = \frac{\hbar}{4}(e^{i\theta} - e^{-i\theta}) = \frac{\hbar}{2}sin(\theta) \\
\sigma^2_{S^x} &= \langle\phi|S^{x2}|\phi\rangle - \hbar cos(\theta)\langle\phi|S^x|\phi\rangle + \frac{\hbar^2}{4}cos^2(\theta)\langle\phi|\phi\rangle \\
&= \frac{\hbar^2}{4} - \frac{\hbar^2}{2}cos^2(\theta) + \frac{\hbar^2}{4}cos^2(\theta) = \frac{\hbar^2}{4}sin^2(\theta) \\
\sigma^2_{S^y} &= \langle\phi|S^{y2}|\phi\rangle - \hbar sin(\theta)\langle\phi|S^y|\phi\rangle + \frac{\hbar^2}{4}sin^2(\theta)\langle\phi|\phi\rangle \\
&= \frac{\hbar^2}{4} - \frac{\hbar^2}{2}sin^2(\theta) + \frac{\hbar^2}{4}sin^2(\theta) = \frac{\hbar^2}{4}cos^2(\theta)
\end{aligned} \tag{3}$$

Thus:

$$\sigma^2_{S^x}\sigma^2_{S^y} = (\ \frac{\hbar^2}{4}cos(\theta)sin(\theta)\ )^2$$

In order, for this expression to be 0, either sin or cos has to be 0. sin is 0 when $\theta = k\pi$ and cos is 0, when $\theta = (\frac{1}{2} + k)\pi$ k being a natural number.

p. 113 in Griffiths, tells us that

$$\sigma_A^2 \sigma_B^2 \geq (\frac{1}{2i} \langle [A, B] \rangle)^2$$

For our case:

$$\sigma_{S^x}^2 \sigma_{S^y}^2 \geq (\frac{1}{2i} \langle [S^x, S^y] \rangle)^2$$

which is

$$\sigma_{S^x}^2 \sigma_{S^y}^2 \geq (\frac{\hbar}{2} \langle \phi | S^z | \phi \rangle)^2$$

$$\sigma_{S^x} \sigma_{S^y} \geq (\frac{\hbar}{2}\frac{1}{2}(\langle \uparrow | + e^{-i\theta} \langle \downarrow |)\frac{\hbar}{2}(| \uparrow \rangle - e^{i\theta} | \downarrow \rangle))^2$$

$$\sigma_{S^x}^2 \sigma_{S^y}^2 \geq (\frac{\hbar^2}{8}(1-1))^2 = 0$$

$$(\frac{\hbar^2}{4} cos(\theta) sin(\theta))^2 \geq 0$$

The only variables on the left hand side, are sin and cos. Given the fact that they are squared, they cannot be negative and the relation therefore holds true.

Now: system of 3 interacting spin degrees of freedom. With Hamiltonian:

$$H = \frac{J}{\hbar^2}(\vec{S_1} \cdot \vec{S_2} + \vec{S_2} \cdot \vec{S_3} + \vec{S_3} \cdot \vec{S_1})$$

Here J is a positive number with unit energy. The spin operators are: $\vec{S_1} \equiv \vec{S} \otimes I \otimes I$, $\vec{S_2} \equiv I \otimes \vec{S} \otimes I$, $\vec{S_3} \equiv I \otimes I \otimes \vec{S}$. $\vec{S} = (S^x, S^y, S^z)$ S^α is the spin-operator in α direction. A general state of 3-spin system is a linear combination of the 3 spin-z states m_s:

$$|m_{S_1} m_{S_2} m_{S_3}\rangle \equiv |m_{S_1}\rangle \otimes |m_{S_2}\rangle \otimes |m_{S_3}\rangle$$

m being either up ($+\frac{1}{2}$ or down ($-\frac{1}{2}$) an example:

$$|\uparrow\downarrow\uparrow\rangle = |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle$$

.

1.4 Dot products and Hamiltonian

Express $\vec{S_1} \cdot \vec{S_2}$ with $S_1^+, S_1^-, S_2^+, S_2^-, S_1^z, S_2^z$. Use this and similar expressions to compute $H\,|\uparrow\downarrow\downarrow\rangle$

with $S^x = \frac{1}{2}(S^+ + S^-)$ and $S^y = \frac{1}{2}i(S^- - S^+)$ we can write

$$\vec{S} = (\frac{1}{2}(S^+ + S^-),\ \frac{1}{2}i(S^- - S^+),\ S^z)$$
$$\vec{S_1} \cdot \vec{S_2} = [\frac{1}{2}(S_1^+ + S_1^-),\ \frac{1}{2}i(S_1^- - S_1^+),\ S_1^z] \cdot [\frac{1}{2}(S_2^+ + S_2^-),\ \frac{1}{2}i(S_2^- - S_2^+),\ S_2^z]$$
$$= \frac{1}{4}[S_1^+ S_2^+ + S_1^+ S_2^- + S_1^- S_2^+ S_1^- S_2^-] - \frac{1}{4}[S_1^+ S_2^+ - S_1^+ S_2^- - S_1^- S_2^+ S_1^- S_2^-] + S_1^z S_2^z$$
$$= \frac{1}{2}(S_1^+ S_2^- + S_1^- S_2^+) + S_1^z S_2^z$$

Similarly for the remaining 2 parts of H:

$$S_2 S_3 = \frac{1}{2}(S_2^+ S_3^- + S_2^- S_3^+) + S_2^z S_3^z$$
$$S_3 S_1 = \frac{1}{2}(S_3^+ S_1^- + S_3^- S_1^+) + S_3^z S_1^z$$

$$H = \frac{J}{\hbar^2}(S_1 S_2 + S_2 S_3 + S_3 S_1)$$
$$= \frac{J}{\hbar^2}(\frac{1}{2}(S_1^+ S_2^- + S_1^- S_2^+) + S_1^z S_2^z + \frac{1}{2}(S_2^+ S_3^- + S_2^- S_3^+) + S_2^z S_3^z + \frac{1}{2}(S_3^+ S_1^- + S_3^- S_1^+) + S_3^z S_1^z)$$

With some legwork, we can write out H as follows, having stripped away the anihilations

$$H\left|\uparrow\downarrow\downarrow\right\rangle = \frac{J}{\hbar^2}(\frac{1}{2}(S_1^- S_2^+ \left|\uparrow\downarrow\downarrow\right\rangle + S_1^+ S_2^- \left|\uparrow\downarrow\downarrow\right\rangle) + S_1^z S_2^z \left|\uparrow\downarrow\downarrow\right\rangle + S_2^z S_3^z \left|\uparrow\downarrow\downarrow\right\rangle + S_2^z S_1^z \left|\uparrow\downarrow\downarrow\right\rangle$$
$$= \frac{J}{2}(\left|\downarrow\uparrow\downarrow\right\rangle + \left|\downarrow\downarrow\uparrow\right\rangle - \frac{1}{2}\left|\uparrow\downarrow\downarrow\right\rangle) \tag{4}$$

Define: total spin:

$$\vec{S_{tot}} \equiv S_1 + S_2 + S_3$$

and

$$S_{tot}^2 \equiv \vec{S_{tot}} \cdot \vec{S_{tot}}$$

1.5 Commutator, eigenstates and eigenvalues

compute the commutator $[H, S_{tot}^z]$ and write down the 8 eigenstates and their corresponding eigenvalues.

First off, we note that we can rewrite this to be

$$[H, S_{tot}^z] = [S_1 S_2, S_{tot}^z] + [S_1 S_2, S_{tot}^z] + [S_1 S_2, S_{tot}^z]$$

Starting off with the first of the 3 resulting commutators, we also note that the results are analogue between the 3, as the 1, 2 and 3 subscripts merely denote which one of 3 dimesnsions we will be acting in. The z-component works in all 3.

$$\begin{aligned}
[S_1 S_2, S_{tot}^z] &= [S_1^x S_2^x, S_1^z] + [S_1^x S_2^x, S_2^z] + [S_1^x S_2^x, S_3^z] \\
&+ [S_1^y S_2^y, S_1^z] + [S_1^y S_2^y, S_2^z] + [S_1^y S_2^y, S_3^z]
\end{aligned}$$

Z-operators commute between themselves and we can omit them. Also, we can remove both the parts with 1, 2 and 3 as they each work in a different degree of freedom and therefore do not interfere with each other.

$$\begin{aligned}
[S_1^x S_2^x, S_1^z] &= S_1^x S_2^x S_1^z - S_1^z S_1^x S_2^x \\
&= (S^x S^z \otimes S^x \otimes I) - (S^z S^x \otimes S^x \otimes I) \\
&= [S^x, S^z] \otimes S^x \otimes I = -i\hbar(S^y \otimes S^x \otimes I)
\end{aligned}$$

This can bereplicated for remaining elements:

$$\begin{aligned}
[S_1 S_2, S_{tot}^z] &= -i\hbar(S^y \otimes S^x \otimes I) + i\hbar(S^y \otimes S^x \otimes I) \\
&+ i\hbar(S^y \otimes S^x \otimes I) - i\hbar(S^y \otimes S^x \otimes I) = 0
\end{aligned} \tag{5}$$

This can be done similarly for the remaining commutators and we therefore
see that H and S^z_{tot} commute. As for Eigenstates and eigen vectors:

$$I)\,|\uparrow\uparrow\uparrow\rangle\,,II)\,|\downarrow\uparrow\uparrow\rangle\,,III)\,|\uparrow\downarrow\uparrow\rangle\,,IV)\,|\uparrow\uparrow\downarrow\rangle\,,$$
$$V)\,|\downarrow\uparrow\downarrow\rangle\,VI)\,|\downarrow\downarrow\uparrow\rangle\,,VII)\,|\uparrow\downarrow\downarrow\rangle\,,VIII)\,|\downarrow\downarrow\downarrow\rangle$$

$$\begin{aligned} &\text{As for Eigenvalues:} \\ I)\ &\frac{3}{2}\hbar, \\ II,III,IV)\ &\frac{\hbar}{2} \\ V,VI,VII)\ &\frac{-\hbar}{2} \\ VIII)\ &-\frac{3}{2}\hbar \end{aligned} \tag{6}$$

1.6 Energy eigenvalues

Find energy eigenvalues of H.

$H = \frac{J}{\hbar^2}(\vec{S_1}\cdot\vec{S_2}+\vec{S_2}\cdot\vec{S_3}+\vec{S_3}\cdot\vec{S_1})$. As per the suggestion, We wish to express it in terms of S^2_{tot}

$$\begin{aligned} S^2_{tot} &= (S_1+S_2+S_3)(S_1+S_2+S_3) \\ &= S_1S_1+S_1S_2+S_1S_3+S_2S_1+S_2S_2+S_2S_3+S_3S_1+S_3S_1+S_3S_3 \\ [S,S] &= 0: \\ S^2_{tot} &= S_1S_1+S_2S_2+S_3S_3+2(S_1S_2+S_1S_3+S_2S_3) \\ &= S_1^2+S_2^2+S_3^2+\frac{2\hbar^2}{J}H \end{aligned}$$

$$H = \frac{J}{2\hbar^2}(S^2_{tot}-S_1^2-S_2^2-S_3^2) \tag{7}$$

S^2_{tot} works on the total spin of the composite system, while S_i^2 contends itself with the relevant single spin-state. $-(S_1^2+S_2^2-S_3^2) = -3(\hbar^2\frac{1}{2}(\frac{1}{2}+1)) = -\frac{9}{4}\hbar$ is a constant.

$S^2_{tot}\left|\lambda\right> = \hbar^2 s(s+1)\left|\lambda\right>$. With our composite system, we have a total spin possible of either $\frac{3}{2}$ or $\frac{1}{2}$ keeping in mind that total spin cannot be negative. H therefore has 2 eigenstates:

$$\begin{aligned}
&\text{for } s_{tot} = \frac{1}{2}: \\
&\qquad \lambda_{\frac{1}{2}} = \hbar^2 \frac{1}{2}(\frac{1}{2}+1) - \frac{9}{4}\hbar^2 \\
&\qquad\quad = \hbar^2(\frac{3-9}{4}) = -\frac{3}{2}\hbar^2 \text{for } s_{tot} \quad = \frac{3}{2}: \\
&\qquad \lambda_{\frac{3}{2}} = \hbar^2(\frac{3}{2}(\frac{3}{2}+1) - \frac{9}{4}) \\
&\qquad\quad = \frac{3}{2}\hbar^2
\end{aligned} \tag{8}$$

1.7 Eigenstates of half-spinn. Clebsch Gordan makes an entrance

Write all normalized eigenstates of S^2_{tot} with a total quantum spin number $s_{tot} = \frac{1}{2} \rightarrow$ has eigenvalue $\hbar^2 \frac{1}{2}(\frac{1}{2}+1)$. $S_{tot} = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$ There is still the combination of the 2 spin-1/2 totaling 0 spin and the final spin totaling 1/2. $|0,0\rangle \otimes |\frac{1}{2}, \pm\frac{1}{2}\rangle$ Reading off from the C-G table:

$$
\begin{aligned}
I)\ |\frac{1}{2},\frac{1}{2}\rangle &= \sqrt{\frac{2}{3}}\,|1,1\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle - \frac{1}{\sqrt{3}}\,|1,0\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \\
II)\ |\frac{1}{2},-\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}\,|1,0\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|1,-1\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \\
III)\ &|0,0\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \\
IV)\ &|0,0\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\text{C-G of } |1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle \\
&|0,0\rangle = \frac{1}{\sqrt{2}}\,|\frac{1}{2},-\frac{1}{2}\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle - \frac{1}{\sqrt{2}}\,|\frac{1}{2},-\frac{1}{2}\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \\
&|1,0\rangle = \frac{1}{\sqrt{2}}\,|\frac{1}{2},\frac{1}{2}\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle + \frac{1}{\sqrt{2}}\,|\frac{1}{2},-\frac{1}{2}\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \\
&|1,-1\rangle = |\frac{1}{2},-\frac{1}{2}\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle \\
&|1,1\rangle = |\frac{1}{2},\frac{1}{2}\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \\
&\text{using } |\frac{1}{2},-\frac{1}{2}\rangle = |\downarrow\rangle \text{ and } |\frac{1}{2},\frac{1}{2}\rangle = |\uparrow\rangle \\
&I)\ \sqrt{\frac{2}{3}}\,|\uparrow\uparrow\downarrow\rangle - \frac{1}{\sqrt{6}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle) \\
&II)\ \frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle) - \sqrt{\frac{2}{3}}\,|\downarrow\downarrow\uparrow\rangle \\
&III)\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle) \\
&IV)\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle)
\end{aligned}
\tag{9}
$$

1.8 time evolution of spin system.

At time t=0 the spin system is in state $|\uparrow\downarrow\downarrow\rangle$. Find analytical expression for probability of finding the system in the same state at a time t. Plot the probability over time.

$|\psi(t=0)\rangle = |0\rangle \equiv H\,|\uparrow\downarrow\downarrow\rangle = \sum_n c_n\,|E_n\rangle$
$|E_n\rangle$ are energy eigenstates. Coefficients are found through Fourier's trick. $c_n = \langle E_n|0\rangle$. We know we have

$$H = \frac{J}{2\hbar^2}[S_{tot}^2 - S_1^2 - S_2^2 - S_3^2]$$

S_1^2, S_2^2 and S_3^2 are all constants totaling $\frac{3}{4}\hbar^2$. The eigenstates of H thus predicate on S_{tot}^2. We already know that these are for $s_{tot} = \frac{1}{2}, \frac{3}{2}$. Having already written out the first 4 eigenstates, we now need to write out the 4 pertaining to spin $\frac{3}{2}$

$$\begin{aligned}
V)\ |\frac{3}{2},\frac{3}{2}\rangle &= |1,1\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \\
VI)\ |\frac{3}{2},-\frac{3}{2}\rangle &= |1,-1\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle \\
VII)\ |\frac{3}{2},\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}\,|1,1\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|1,0\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \\
VIII)\ |\frac{3}{2},-\frac{1}{2}\rangle &= \sqrt{\frac{2}{3}}\,|1,0\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle + \frac{1}{\sqrt{3}}\,|1,-1\rangle \otimes |\frac{1}{2},\frac{1}{2}\rangle \\
V)\ |\frac{3}{2},\frac{3}{2}\rangle &= |\uparrow\uparrow\uparrow\rangle \\
VI)\ |\frac{3}{2},-\frac{3}{2}\rangle &= |\downarrow\downarrow\uparrow\rangle \\
VII)\ |\frac{3}{2},\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}(|\downarrow\downarrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) \\
VIII)\ |\frac{3}{2},-\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle)
\end{aligned} \tag{10}$$

Now, we can use these states to find c_n through Fourier's trick

$$c_n = \langle E_n|0\rangle$$

$$\begin{aligned}
c1 &= \langle I|\,|\uparrow\downarrow\downarrow\rangle\rangle = 0, & c2 &= \langle II|\,|\uparrow\downarrow\downarrow\rangle\rangle = \frac{1}{\sqrt{6}} \\
c3 &= \langle III|\,|\uparrow\downarrow\downarrow\rangle\rangle = 0, & c4 &= \langle IV|\,|\uparrow\downarrow\downarrow\rangle\rangle = \frac{1}{\sqrt{2}} \\
c5 &= \langle V|\,|\uparrow\downarrow\downarrow\rangle\rangle = 0, & c6 &= \langle VI|\,|\uparrow\downarrow\downarrow\rangle\rangle = 0 \\
c7 &= \langle VII|\,|\uparrow\downarrow\downarrow\rangle\rangle = 0, & c8 &= \langle VIII|\,|\uparrow\downarrow\downarrow\rangle\rangle = \frac{1}{\sqrt{3}}
\end{aligned} \tag{11}$$

Setting up the trick:

$$\begin{aligned}
|\psi(t)\rangle &= \sum_n c_n e^{-i\frac{E_n t}{\hbar}} |E_n\rangle \\
|\psi(t)\rangle &= \frac{1}{\sqrt{6}} e^{i\frac{3Jt}{4\hbar}} \left(\frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle) - \sqrt{\frac{2}{3}}\,|\downarrow\downarrow\uparrow\rangle\right) \\
&+ \frac{1}{2} e^{i\frac{3Jt}{4\hbar}} (|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle) \\
&+ \frac{1}{3} e^{-i\frac{3Jt}{4\hbar}} (|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle)
\end{aligned} \tag{12}$$

For the probability, we have $P_0(t) = |\,\langle 0|\psi(t)\rangle\,|^2$ and we need to figure out $\langle\uparrow\downarrow\downarrow|\,|\psi(t)\rangle$.. We notice that any ket that is not $|\uparrow\downarrow\downarrow\rangle$ is anihilated by the kroenicker delta. Thus, we can write it as:

$$\begin{aligned}
\langle\uparrow\downarrow\downarrow|\,|\psi(t)\rangle &= \frac{1}{\sqrt{6}} e^{i\frac{3Jt}{4\hbar}} + \frac{1}{2} e^{i\frac{3Jt}{4\hbar}} + \frac{1}{3} e^{-i\frac{3Jt}{4\hbar}} \\
&= \left(\frac{1}{\sqrt{6}} + \frac{1}{2}\right) e^{i\frac{3Jt}{4\hbar}} + \frac{1}{3} e^{-i\frac{3Jt}{4\hbar}}
\end{aligned}$$

For the probability, we then get: (13)

$$\begin{aligned}
P_0 &= \langle\psi(t)|\,|\uparrow\downarrow\downarrow\rangle\,\langle\uparrow\downarrow\downarrow|\,|\psi(t)\rangle \\
&= \left(\frac{4+1}{9}\right) e^{(1-1)\frac{3Jt}{2\hbar}} + \frac{2}{9}\left(e^{-i\frac{3Jt}{2\hbar}} + e^{i\frac{3Jt}{2\hbar}}\right) \\
&= \frac{5}{9} + \frac{4}{9} cos\left(\frac{3Jt}{2\hbar}\right)
\end{aligned}$$

Which is the equation for our probability. and the visualization can be seen in figure 1.

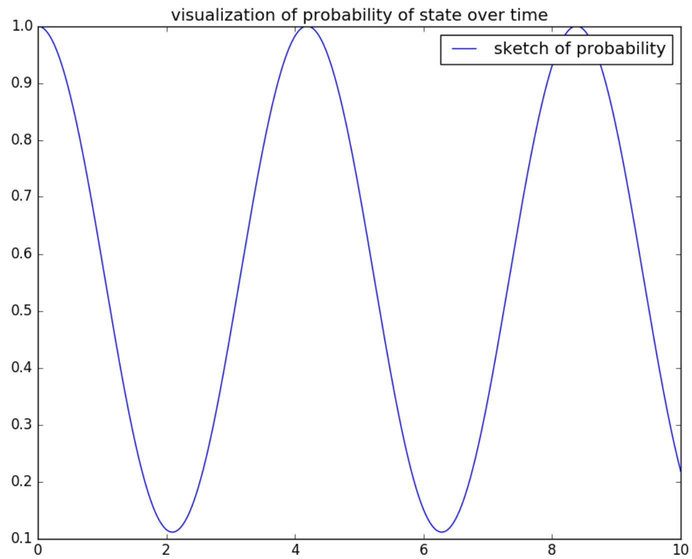

Figure 1: Visualization of probability over time. I have not included $\hbar$ or J in this calculation, as they seemed unnecessary to the direct visualization.

2 Problem 2: operator on state

we have an operator e^{-Hs}, where s is a positive, real number with units of inverse energy. H is the Hamiltonian.

2.1 Estimation of unknown ground state.

Hamiltonian is known, but not it's ground state $|E_0\rangle$. We can compute $|\psi(s)\rangle \equiv e^{-Hs}|\psi\rangle$ for any s and $|\psi\rangle$. Use this to calculate an approximation of the ground state expectation value. $\langle E_0|O|E_0\rangle$ for a given hermitian operator O. How does the error of the estimation depend on s? What are the requirements on $|\psi\rangle$ for this approximation to work?

We can write

$$|\psi(s)\rangle = \sum_n c_n e^{-E_n s}|E_n\rangle$$

The larger s becomes, the smaller our sum gets. As the ground state is the lowest, the greatest expression will be in the ground state. Thus (for large s):

$$|\psi(s)\rangle = c_0 e^{-E_0 s}|E_0\rangle + ...$$

where the latter is sums of higher energies and subsequently lower function value. We can approximate this:

$$|\psi(s)\rangle \approx c_0 \cdot e^{-E_0 s}|E_0\rangle$$

This lets us express:

$$\langle\psi(s)|\psi(s)\rangle = |c_0|^2 e^{-2E_0 s}$$

which we can then utilize later, after writing out:

$$\langle\psi(s)|O|\psi(s)\rangle = |c_0|^2 e^{-2E_0 s}\langle E_0|O|E_0\rangle$$

Since $|c_0|^2 e^{-2E_0 s} = \langle\psi(s)|\psi(s)\rangle$, we can factor this to the other side of the equality.

$$\frac{\langle\psi(s)|O|\psi(s)\rangle}{\langle\psi(s)|\psi(s)\rangle} = \langle E_0|O|E_0\rangle$$

This approximation becomes less acurate as s decreases, as per our assumption that it was large for the key approx of c_0.... In other words, for small s we have a large error. Due to the approximations here, there are a few

requirements on $|\psi(s)\rangle$
1. $c_0 \neq 0$. This is so that the final expression is valid.
2. We need discrete steps between $|\psi(s)\rangle' s$, otherwise it wouldn't make sense to chop it up as we have done.
3. $|\psi(s)\rangle$ cannot go to infinity, as this would invalidate our aproximations of $E_0 and c_0$ since no matter how large we make s, if there is an infinite and non-converging string after, there will still be infinity as a sum.

3 code

```
"""
This is a quick program to plot the probability curve of
a state with known initial state.
"""
# imports
import numpy as np
import matplotlib.pyplot as plt

#constants
#J = 1 # units of energy unknown, immaterial for now.
#h_ = 1.054e-34 #J*s, reduced planck constant (low accuracy)


#settign function
def P(t):
    return (5./9) + (4./9)*np.cos( 3*t/2)

# defining time array
t = np.linspace(0, 10, 1e4) # could be nanoseconds
p = P(t)

# plotting:

plt.plot(t, p, label=('sketch of probability'))
plt.legend()
plt.title('visualization of probability of state over time')
plt.show()
```