2024 Weston Calculus & Proofs Study Guide

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This study guide has a total of 100 points - 50 for calculus and 50 for vectors and proofs. No points off for incorrect answers, so ***feel free to guess***. A calculator should not be necessary, but you may use one if you have one.

The point of this study guide is to help you prepare for the posttest on Wednesday. If you were with us only for the vectors and proofs section of class, only do the vectors proofs section of the study guide, otherwise do both sections. If you are confused about anything, please ask me.

1 Calculus

1. (6 pts) Find the following limit, either by cancelling or by l'Hospital's rule: $\lim_{x\to 1} \dfrac{2x^2-2}{2x^2+x-3}$

$\dfrac{4x}{4x+1} = \dfrac{4}{5}$ $\qquad \dfrac{2(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(2x+3)}$ $\qquad \dfrac{4}{5}$

2. (6 pts) Find the derivative of the function $f(x) = 6(x-3)^2 - x + 4$

$6(2(x-3)\cdot 1) - 1 \qquad 12(x-3) - 1$

$6(x^2-6x+9) - x + 4 \qquad 6(2x-6) - 1$

3. Let $f(x) = -\sin(x) + x^3$

   (a) (4 pts) Find the indefinite integral (antiderivative) $\int f(x)\, dx$

   $\cos x + \dfrac{x^4}{4} + C$

   (b) (4 pts) Find the definite integral $\int_0^\pi f(x)\, dx$

   $\cos x + \dfrac{x^4}{4} + C \Big|_0^\pi$

   $-1 + \dfrac{\pi^4}{4} + C$

   $-(1 + 0 + C)$

   $-2 + \dfrac{\pi^4}{4}$

4. (8 pts) Let $f(x)$ be the vector valued function $\begin{bmatrix} \cos(\ln(x)) \\ e^{2x} \end{bmatrix}$

   Find the derivative $\frac{df}{dx}$

$$\begin{bmatrix} -\sin(\ln x)\cdot\frac{1}{x} \\ 2e^{2x} \end{bmatrix}$$

$\langle -\sin(\ln x)\cdot\frac{1}{x}, 2e^{2x}\rangle$

5. Let $f(x, y, z) = 2x^2 + x\sin(y) + \cos(y) + 3z + 1$
   Find each of the following partial derivatives (3 pts each):

   (a) $\frac{\partial f}{\partial x}$

   $4x + \sin(y)$

   (b) $\frac{\partial f}{\partial y}$ $x(\cos(y)) - \sin(y)$

   (c) $\frac{\partial f}{\partial z}$ 3

   (d) (4 pts) Find the gradient of f at the point $(1, \frac{\pi}{2}, 5)$ by plugging it in to the partial derivatives

$\langle 5, -1, 3\rangle$

$\int 6\,dx = 6x + C$

6. Consider the surface S in 3D space described by $z = \sin(x) + y^2$
   Use a double integral to find the volume under the surface S within the box with corners at the points $(0, 0), (\pi, 0), (0, 1), (\pi, 1)$. (9 pts)

$-\cos x + xy^2 + C \Big|_0^{\pi}$

$$\int_0^1 \int_0^{\pi} \sin(x) + y^2 \, dx\, dy$$

$[1 + \pi y^2 + C) - (-1 + 0 + C)$

$\int_0^1 2 + \pi y^2 \, dy$ $\quad 2y + \frac{\pi}{3}y^3 + C \Big|_0^1$

$2 + \frac{\pi}{3} \cancel{+C} \cancel{-C}$

$2 + \frac{\pi}{3}$

2 Vectors & Proofs

7. Consider vectors $\vec{u} = \langle 3, 0, -1 \rangle$ and $\vec{v} = \langle 0, 2, 1 \rangle$. Find the result of each of the following, ***and*** say whether each result is a vector or a scalar. [A scalar is a regular, non-vector number.]
(1 pt for each result, 1 pt for vector/scalar):

   (a) $3\vec{u}$ $\langle 9, 0, -3 \rangle$ V

   (b) $\vec{u} + \vec{v}$ $\langle 3, 2, 0 \rangle$ V

   (c) $\vec{u} \cdot \vec{v}$

   $0 + 0 + -1 = -1$ S

   (d) $\vec{u} \times \vec{v}$

   xyzzy

   $0\hat{k} - 2\hat{i} + 3\hat{j}$

   $$\begin{matrix} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 3 & 0 & -1 & 3 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{matrix}$$

   $0\hat{i} + 0\hat{j} + 6\hat{k}$

   $6\hat{k} + 2\hat{i} - 3\hat{j}$ $\langle 2, -3, 6 \rangle$ V

   (e) $\vec{v} \times \vec{u}$

   $\langle -2, 3, -6 \rangle$ V

   (f) $\|\vec{u}\|$ $\sqrt{3^2 + 0^2 + (-1)^2} = \sqrt{10}$ S

   (g) A unit vector parallel to $\vec{u}$

   $\vec{u} \cdot \frac{1}{\|\vec{u}\|}$ $\left\langle \frac{3}{\sqrt{10}}, \frac{0}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \right\rangle$ V

   (h) A unit vector perpendicular to $\vec{v}$

   $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$

   $\langle 1, 0, 0 \rangle$

   $\vec{v} = \langle 0, 2, 1 \rangle$

   $0x + 2y + 1z = 0$

8. Mark which of the phrases below are mathematical statements - that is, that are mathematically either true or false. You do not need to say whether each specific one is true or false. (1 pt each)

   (a) $2 + 2 = 0$

   (b) $2 + 2 = 4$

   (c) $\frac{3\pi}{2}$

   (d) x is even.

   (e) There exists an integer x that is even.

   (f) There exists an odd perfect number.

   (g) Consider a polyhedron that can be flattened onto a plane.

   (h) For a given line ℓ, and a given point p not on ℓ, there is exactly one line parallel to ℓ that goes through p.

9. Which one of the following would be a valid counterexample to the conjecture that "For every prime number p, $2^p - 1$ is also a prime number"? (5 pts)

   (a) $2^7 - 1 = 127$ and 127 is prime

   (b) $2^4 - 1 = 15$ and 15 is 3×5 and thus not prime

   (c) 19 is prime, and cannot be written as $2^p - 1$ for a prime p

   (d) $2^{11} - 1 = 2047$ and 2047 is 23×89 and thus not prime

10. In class, we proved that $\sqrt{2}$ is irrational. Here is a proof of this:

Assume for the sake of contradiction that $\sqrt{2}$ were rational. Let a, b be the smallest positive integers such that $\frac{a}{b} = \sqrt{2}$ - that is, $\frac{a}{b}$ is a fully simplified fraction. Then we have the following:

$$\frac{a}{b} = \sqrt{2} \tag{1}$$
$$\frac{a^2}{b^2} = 2 \tag{2}$$
$$a^2 = 2b^2 \tag{3}$$

Now, if a is odd (not a multiple of 2), then a^2 is also odd, which would be a contradiction since a^2 is 2 times b^2.

On the other hand, if a is even (a multiple of 2), then a^2 is a multiple of 2^2. This would mean that b^2 must also be a multiple of 2, so b must be a multiple of 2. But if a and b are both multiples of 2, then $\frac{a}{b}$ is not a fully simplified fraction, which would also be a contradiction.

Thus, we know that $\sqrt{2}$ must be irrational. QED

(a) What kind of proof is this? (e.g. Induction, Contradiction, Demonstration, etc.) (3 pts)

Contradiction

(b) Write another true statement that follows from the fact that $\sqrt{2}$ is irrational. (3 pts)

$\sqrt{8}$ is irrational $\sqrt{8} = 2\sqrt{2}$

(c) Write 3 assumptions that the above proof makes, other than the assumption for the sake of contradiction. (2 pts per assumption)

- if a is even, then a^2 is even

an even number is a number that is 2 times an integer

- if a^2 is even, and a is an integer, then a is even

(d) Using the above proof as a starting point, write a similar proof that $\sqrt{3}$ is irrational. (9 pts)

Assume for contradiction that $\sqrt{3}$ is rational. $\sqrt{3} = \frac{a}{b}$ with smallest $a, b \in \mathbb{Z}^+$

then: $\left(\frac{a}{b}\right)^2 = 3$ $\frac{a^2}{b^2} = 3$ $a^2 = 3b^2$. If a is a multiple of 3, a^2 is a multiple of 9, so b^2 is a multiple of 3, so b is also. then $\frac{a}{b}$ is not in simplest terms, Contradiction.

If a is not a multiple of 3, a^2 is not either, but $a^2 = 3b^2$, also a contradiction.

Thus $\sqrt{3}$ is irrational QED