2023 Weston Calc 3 Pretest

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1. Let $\vec{u} = \langle 2, 0, -1\rangle$ and $\vec{v} = \langle 0, 1, 3\rangle$. Find the result of each of the following, and indicate whether each is a vector or a scalar (1 pt for each result, 1 pt for vector/scalar) :

   (a) $3\vec{u}$ $\langle 6, 0, -3\rangle$ vector

   (b) $\vec{u} + \vec{v}$ $\langle 2, 1, 2\rangle$ vector

   (c) $\vec{u} \cdot \vec{v}$

   -3, scalar

   (d) $\vec{u} \times \vec{v}$

   $\begin{matrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{matrix}$ $2\hat{k} + \hat{i} - 6\hat{j}$ $\langle 1, -6, 2\rangle$ vector

   (e) $\vec{v} \times \vec{u}$ $\langle -1, 6, -2\rangle$ vector

   (f) $\|\vec{u}\|$

   $\sqrt{5}$ scalar

   (g) A unit vector parallel to $\vec{u}$

   $\langle 2/\sqrt{5}, 0, -1/\sqrt{5}\rangle$, vector

   (h) A unit vector perpendicular to $\vec{v}$

   e.g. $\langle 1, 0, 0\rangle$, vector

2. (10 pts) Let $f(x)$ be the vector valued function $\begin{bmatrix} \cos(\ln(x)) \\ e^{2x} \end{bmatrix}$

   Find $\frac{df}{dx}$

   $\begin{bmatrix} -\sin(\ln(x))/x \\ 2e^{2x} \end{bmatrix}$

3. Let $f(x, y, z) = 2x^2 + x\sin(y) + \cos(y) + \ln(z)$
   Find each of the following partial derivatives (5 pts each):

   (a) $\frac{\partial f}{\partial x}$

   $4x + \sin y$

   (b) $\frac{\partial f}{\partial y}$

   $x\cos y - \sin y$

   (c) $\frac{\partial f}{\partial z}$

   $\frac{1}{z}$

4. (10 pts) Let $f(x, y, z) = 2x^2 + x\sin(y) + \cos(y) + \ln(z)$ and point $p = (1, \frac{\pi}{2}, e)$
   Find the gradient of f at point p

   $\langle 5, -1, \frac{1}{e} \rangle$

5. Consider the surface S in $\mathbb{R}^3$ described by the function $z = \sin(x) + y^{\frac{1}{3}}$

(a) (8 pts) Find an equation of the normal (perpendicular) line to S at the point $(\pi, -8)$.

$x: \cos(\pi)\quad \langle 1, 0, -1\rangle$

$y: \frac{1}{3}(-8)^{-\frac{2}{3}}\quad \langle 0, 1, \frac{1}{12}\rangle$

$\hat{i}\ \hat{j}\ \hat{k}$
$1\ 0\ {-1}$
$0\ 1\ \frac{1}{12}$

$\vec{r} = \langle 1, \frac{-1}{12}, 1\rangle t + (\pi, -8, -2)$

$x - \pi = 12(y+8) = -z+2$

(b) (8 pts) Find an equation of the tangent plane to S at the same point $(\pi, -8)$.

$$-(x-\pi) - \frac{1}{12}(y+8) = z+2$$

6. (10 pts) Again, consider the surface S in $\mathbb{R}^3$ described by $z = \sin(x) + y^{\frac{1}{3}}$ Find the volume between S and the xy plane within the box with corners at the points $(0,0), (\pi,0), (0,1), (\pi,1)$.

$$\int_0^{\pi}\int_0^1 \sin(x) + y^{\frac{1}{3}}\, dy\, dx = \int_0^{\pi} y\sin(x) + \frac{3}{4}y^{\frac{4}{3}}\Big|_0^1 dx$$

$$\int_0^{\pi} \sin(x) + \frac{3}{4}\, dx = -\cos(x) + \frac{3}{4}x\Big|_0^{\pi} = 2 + \frac{3\pi}{4}$$

7. Consider the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} 3x+y \\ y^2 \\ x\sqrt{z} \end{bmatrix}$$

(a) (8 pts) Find the function for the divergence of $\vec{F}$

$$3 + 2y - \frac{x}{2\sqrt{z}}$$

$\hat{k}\ 0\ \sqrt{z}\,\hat{j}$

(b) (8 pts) Find the function for the curl of $\vec{F}$

$\hat{i}\ \hat{j}\ \hat{k}$
$\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}$
$3x+y\ \ y^2\ \ x\sqrt{z}$

$$\langle 0, -\sqrt{z}, -1\rangle$$

8. (7 pts) Again, consider the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} 3x+y \\ y^2 \\ x\sqrt{z} \end{bmatrix}$$

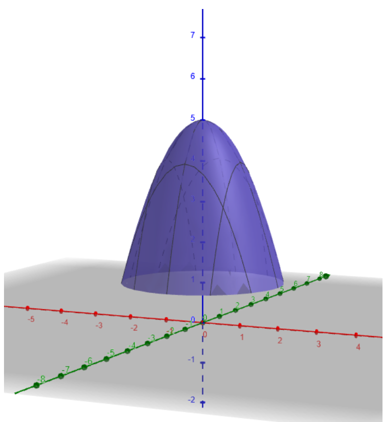
Let the surface S be the shape (circular paraboloid) defined by

$$z = -(x^2+y^2)+5, z \geq 1$$

Using Stokes' theorem and the curl function from question 7,[1] find the sum of the curl of $\vec{F}$ across S.[2]

You may find the following 3D graph of S helpful:

integral way:



total curl on S

= total circulation around ring

= total curl on disk contained in ring

disk has curl of -1, area of 4π

$\boxed{-4\pi}$

$\vec{r} = \langle 2\cos(T), 2\sin(T), 1\rangle$

$\int_0^{2\pi} \vec{F}(\vec{r})\cdot\vec{r}'\, dT$

$$\begin{bmatrix} 6\cos T + 2\sin T \\ 4\cos^2 T + 4\sin^2 T \\ 2\cos T \end{bmatrix} \cdot \begin{bmatrix} -2\sin T \\ 2\cos T \\ 0 \end{bmatrix} = -12\cos T\sin T - 4\sin^2 T + 4\cos T$$

$$\int_0^{2\pi} -12\cos T\sin T - 4\sin^2 T + 4\cos T\, dT$$

[1]This is a hint. How can you get Stokes' theorem to help even more than usual since you already know the curl function? If you find yourself taking the integral of $\sin^2 x$, you are not taking full advantage of the hint, but if you want to continue down that path it's dangerous to go alone, take this: $\sin^2(x) = \frac{1-\cos(2x)}{2}$, $\sin(2x) = 2\sin(x)\cos(x)$

[2]Technically it's the sum of the *flux* of the curl to make it a scalar, but don't worry about the distinction. Stokes' theorem is the last topic we plan to cover in the course.

$$\int_0^{2\pi} -6\sin(2T) - 2 + 2\cos(2T) + 4\cos T\, dt = 3\cos 2T - 2T + \sin(2T) - 4\sin(T)\Big|_0^{2\pi}$$

$$= \boxed{-4\pi}$$