Lecture 11 – Quantum simulation 1

physical

Operational theories

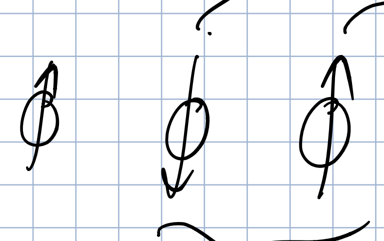
Hilbert space $\leftarrow$ 1) State representation S

Unitary evolution $\leftarrow$ 2) Relationship between states (dynamics)

Tensor product $\leftarrow$ 3) Composition

Born rule $\leftarrow$ 4) Observation

$|\phi\rangle \in H_1$ $\quad |\psi\rangle \in H_1$



$|\phi\rangle \otimes |\psi\rangle \in H_2$

Hermitian operators

$$O = O^\dagger$$

$\downarrow$

- eigenvalues are real numbers
- eigenvectors form ONB

Hermitian	Non hermitian
$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$
Y	$\downarrow$
Z	$T^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$
CNOT, H	

$$P_0 = |0\rangle\langle 0| \quad , \quad P_1 = |1\rangle\langle 1|$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{evals} = \{1, -1\}$$

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \text{evals} = \{+1, -1\}$$

$$X|+\rangle = X \cdot \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$$

$$X|-\rangle = X \cdot \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = -|-\rangle$$

$$|\psi\rangle \; , \quad \{P_0, P_1\} \qquad \left(P_0 + P_1 = I\right)$$

$$\langle\psi|P_0|\psi\rangle = Pr(0)$$

$$\langle\psi|P_1|\psi\rangle = Pr(1)$$

Observables

$$O = O^\dagger$$

Let $\{|v_i\rangle\}_i$ be the eigenvectors of O

and $\{\lambda_i\}_i$ be the eigenvalues

(O is non-degenerate)

→ measurement with proj $P_i = |v_i\rangle\langle v_i|$

$$\left(\sum P_i = I\right)$$

(measurement outcomes are eigenvalues λ_i

on outcome i we say meas result was λ_i)

E.g. measuring $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\big(\underset{+1}{|+\rangle}, \underset{-1}{|-\rangle}\big)$$

on same qubit $|\psi\rangle$ is equivalent to meas

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$Pr(+1) = \langle\psi|\underbrace{+\rangle\langle+}_{P_0}|\psi\rangle = |\langle\psi|+\rangle|^2 =$$

$$= \frac{1}{2}|a+b|^2$$

$$Pr(-1) = \langle\psi|-\rangle\langle-|\psi\rangle = \frac{1}{2}|a-b|^2$$

n-qubit state $|\psi\rangle$

Obs $\quad \boxed{Z \otimes \underbrace{I \otimes I \otimes \_ \_ \otimes I}_{n-1}}$

Degenerate case let $\{\lambda_i\}_i$ be distinct evals

$$P_i = \sum_i |v_i\rangle\langle v_i|$$

$\downarrow$ have eval λ_i

$$Z \otimes Z = \begin{pmatrix} \boxed{Z} & \\ & \boxed{-Z} \end{pmatrix} = \begin{array}{c} \begin{matrix} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{matrix} \\ \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \end{array}$$

$$P_{+1} = |00\rangle\langle 00| + |11\rangle\langle 11| \rightarrow \text{span}(|00\rangle, |11\rangle)$$

$$P_{-1} = |01\rangle\langle 01| + |10\rangle\langle 10| \rightarrow \text{span}(|01\rangle, |10\rangle)$$

Tomography - measurements to characterize unknown q-state.

O

Define $V = \lambda_i$ with prob $\langle\psi|P_i|\psi\rangle$

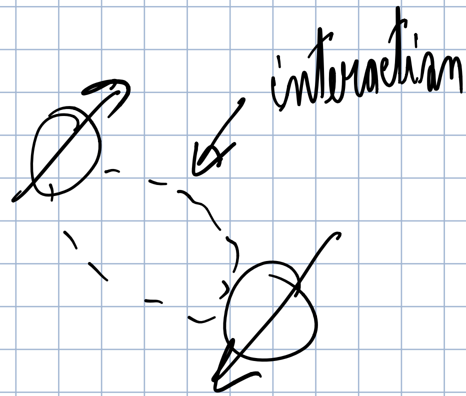
$$E(V) = \sum_i \lambda_i\, p(\lambda_i) = \sum_i \lambda_i \langle\psi|P_i|\psi\rangle$$

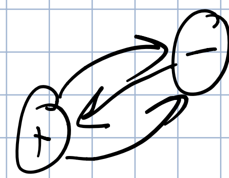
$$\langle O\rangle_\psi = \langle\psi|O|\psi\rangle = \sum_i \lambda_i \langle\psi|P_i|\psi\rangle$$

$$O = \sum_i \lambda_i P_i$$

Hamiltonian - observable for energy
(evals = energies)
(evector - energy states)

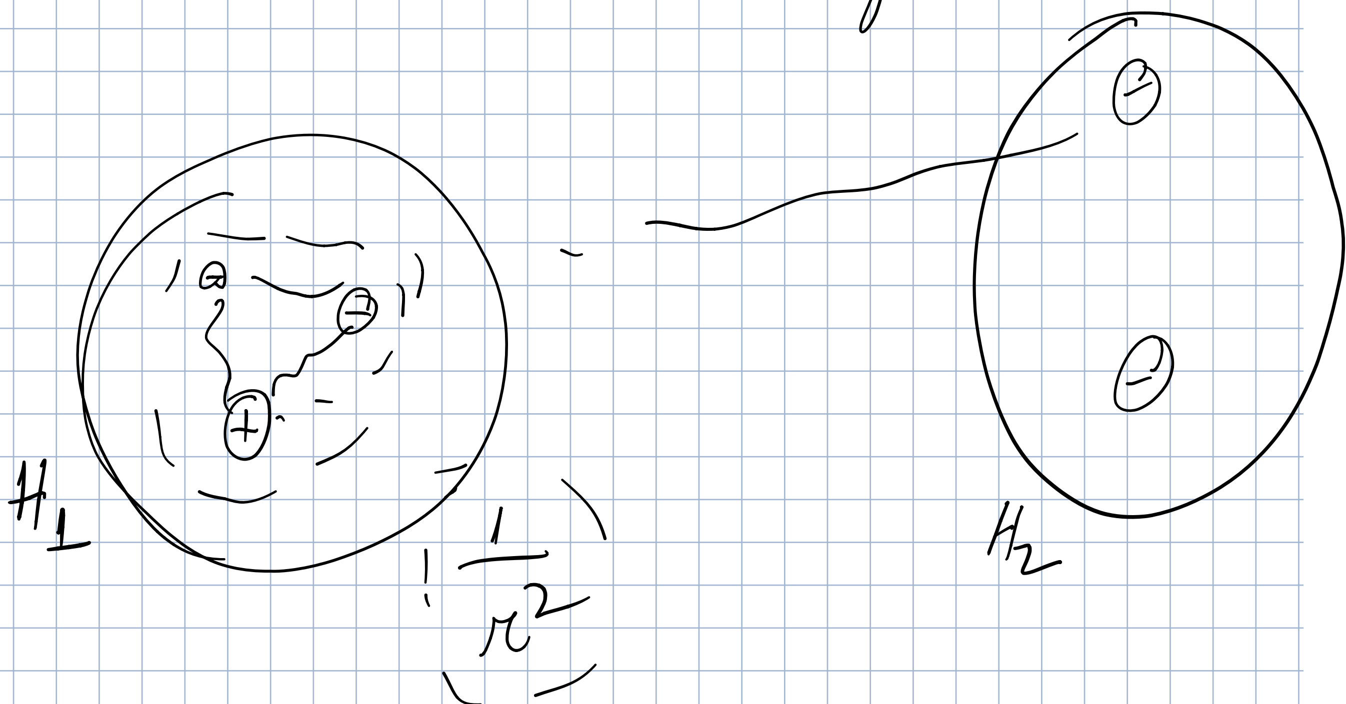
Dynamics





$$H = \sum_i H_i$$

local hermitian operators



$$H = H_1 + H_2$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 2^n \times 2^n & 2^n \times 2^n & 2^n \times 2^{-n} \end{array}$$

5 qubit systems

$\uparrow_1 \downarrow_2 \uparrow_3 \uparrow_4 \uparrow_5 \quad \uparrow z$

$$H = Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_4 + Z_4 Z_5$$

2-local Hamiltonian

k-local Hams $\quad k = \text{const.}$ (log)

time of interaction

$$U(t) = \underbrace{e^{-iH\cdot t}}$$

(Schrödinger's equation)

$$\underbrace{i\hbar \frac{d|\psi(t)\rangle}{dt} = H(t)\cdot|\psi(t)\rangle}$$

$$|\psi(0)\rangle \longrightarrow |\psi(t)\rangle$$

$H, n, t \longrightarrow$ q circuit of size (# gates) poly(n, t)

$\downarrow$
local

Trotter-Suzuki (first order)

$|\psi_0\rangle$ is determined — local terms act on k qubits

$$e^{-iHt} = e^{-i\cdot t\cdot \sum_j H_j}$$

$$\cancel{\approx}$$

$$\prod_j e^{-it\cdot H_j}$$

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \longrightarrow \exp(A+B) = I + (A+B) + \frac{(A+B)^2}{2} + \dots$$

$$\frac{(A+B)^2}{2} = \frac{1}{2}\left(A^2 + B^2 + \underbrace{AB + BA}\right)$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$[A,B] \neq 0$

$$\exp(A)\cdot\exp(B) =$$

$$= \left(I + A + \frac{A^2}{2} + \dots\right)$$

$$\cdot \left(I + B + \frac{B^2}{2} + \dots\right)$$

$$= \underbrace{I + A + B + \frac{A^2}{2} + \frac{B^2}{2} + AB} + \dots$$

$$\lim_{n\to\infty}\left(e^{\frac{A}{n}}\cdot e^{\frac{B}{n}}\right)^n = e^{A+B}$$

$$e^{\frac{A}{n}}\cdot e^{\frac{B}{n}} \neq e^{\frac{A+B}{n}}$$

$$e^{-iHt} = e^{-ti\cdot\sum_j H_j} \simeq \left(\prod_j e^{-it H_j/\Delta}\right)^{\Delta} + \text{error}$$

$$\frac{t}{\Delta}$$

$$\left(\prod_j e^{-it H_j/\Delta}\right)^{\Delta} = \Big(\underbrace{e^{-itH_1/\Delta}}\cdot e^{-itH_2/\Delta}\cdot \text{---}\Big)^{\Delta}$$

local unitary acts on k qubits

(Solovay-Kitaev can decompose this to a universal)

$$\Delta = \text{poly}(n, t)$$

↳ poly-size circuit

↳ error can be made arbitrarily small $\frac{1}{\text{poly}(n,t)}$

$$e^{-it\cdot(X\otimes X)}$$

$$e^{-\theta X/2}$$ rotation around X axis of Bloch sphere