BSE 3090 Assignment 3 (written part)

a) forward kinematics.

for this problem, we are working with a simple planar two-link arm system. each link has a fixed length ($r_1$ = 47 mm, $r_2$ = 63 mm). the position of the end point P depends on the joint angles $\theta_1$ and $\theta_2$.

to write the forward kinematic equations, I started by thinking about how each link contributes to the final x/y position. the first link rotates by $\theta_1$ and the second link rotates by the combined angle $\theta_1 + \theta_2$ because it is mounted on the end of the first link.

so the final position of P is:

$$x = r_1 \cdot \cos(\theta_1) + r_2 \cos(\theta_1 + \theta_2)$$
$$y = r_1 \cdot \sin(\theta_1) + r_2 \sin(\theta_1 + \theta_2).$$

this is exactly what I implemented in Matlab as an inline function fk($t_1$, $t_2$) so I could evaluate the tip position quickly throughout the assignment, especially during the animation part.

b) workspace of the arm

here I needed to figure out what points the arm can reach if $0 < \theta < \pi/2$ $0 < \theta_2 < \pi$; and to visualize this, I made a grid of angles:

- 100 evenly spaced values for $\theta_1$ between 0 and $\pi/2$.
- 100 evenly spaced values for $\theta_2$ between 0 and $\pi$.

then, for every pair, I used the forward kinematic equations to compute (x/y).

plotting all the (x,y) values as a scatter plot creates the „workspace cloud" the region of the plane where point P can actually go. this helped me later when choosing points for the letter P, because its very easy to accidentally pick coordinates outside of the reachable (coordinates) region.

c) solve IK for the point (   72) mm.

this part was about checking whether the specific point (20, 72) is actually reachable and if so, finding the joint angles that place the tip at that point.

to solve this I coded up the two-link inverse kinematics. the idea was:

1. start with the distance from the origin to the point:

$$d^2 = x^2 + y^2$$

2. use the cosine law to solve for $\theta_2$:

$$\cos\theta_2 = \frac{d^2 - r_1^2 - r_2^2}{2 r_1 r_2}$$

3. check if this value is valid (must be between -1 and 1), if not the point is unreachable.
4. solve for the two possible values of $\theta_2$
   - positive square root → elbow up
   - negative square root → elbow down
5. for each one, compute $\theta_1$ from geometry.

When I ran this for (20, 72) the solver returned two valid solutions, meaning the point is reachable. I displayed both angle sets in degrees.

d) choosing four points for the letter P.

so we need to break the letter P into four anchor points. I tried to pick coordinates that visually match the letter but still stay inside the arm's reachable workspace.

$P_1 = (40, 20)$ bottom of the stem
$P_2 = (40, 50)$ middle of the stem
$P_3 = (40, 80)$ top of the stem
$P_4 = (60, 80)$ rightmost point of the upper bowl.

for each point I ran the IK function and extracted the elbow-up solution so the arm stayed consistent while animating. I printed all the joint angles in degrees in a table. this confirmed that all four points were reachable after slight adjustments.

e) drawing the letter P without lifting the pen.

this was the bigger part of the problem,... my logic was:

1. build a set of points in cartesian coordinates.
   (I divided the letter into four curve segments. — 1. vertical stem (bottom → top) 2. top horizontal segment (left → right) 3. curved bowl (using cosine and sine with a radius to approximate a smooth arc). 4. diagonal line back to the stem.)
2. solve IK for every single point
   for each point along the path:
   - I solved IK
   - picked the elbow-up solution so the animation looks smooth.
   - stored the joint angles.
3. animate
   inside a loop:
   - convert the angles to link endpoint coordinates using forward kinematics
   - plot the arm position
   - plot the red trace of the letter P.
   - clear and redraw the arm at each step so the movement looks clean.

$r_1 = 47\,mm$ $\quad r_2 = 63\,mm$ $\quad P(x,y) = (20, 72)\,mm$

$\theta_1$ and $\theta_2$ $\rightarrow$

$$x = r_1 \cos\theta_1 + r_2 \cos(\theta_1 + \theta_2)$$
$$y = r_1 \cdot \sin\theta_1 + r_2 \cdot \sin(\theta_1 + \theta_2)$$

$d^2 = x^2 + y^2 = 20^2 + 72^2 = 5584$ $\quad$ ($d = \sqrt{5584}$ but we need $d^2$)

1) $\cos\theta_2 = \dfrac{d^2 - r_1^2 - r_2^2}{2r_1r_2} = \dfrac{-594}{5922} = -0.1003$ $\quad$ since $|\cos\theta_2| < 1$ the point is reachable.

now $\theta_2 = \cos^{-1}(0.1003)$

- $\theta_{2a} \approx 95.8^\circ$ or $1.67\,rad$ (elbow up)
- $\theta_{2b} \approx -95.8^\circ$ or $-1.67\,rad$ (elbow down)

$\sin\theta_2 = \pm\sqrt{1 - \cos^2\theta_2} \approx 0.9949$

- for elbow up $\quad \sin\theta_{2a} \approx +0.9949$
- for elbow down $\quad \sin\theta_{2b} \approx -0.9949$.

2) $\theta_1 = \text{atan2}(y, x) - \text{atan2}(r_2 \sin\theta_2, r_1 + r_2\cos\theta_2)$.

1. $\text{atan2}(72, 20) = \arctan(3.6) \approx 74.5^\circ \approx 1.30\,rad$

elbow up solution

$\theta_{2a} = 1.67\,rad$ $\quad \sin\theta_{2a} = 0.9949$ $\quad \cos\theta_{2a} = -0.1003$

$k_1 = r_1 + r_2\cos\theta_{2a} = 47 + 63(-0.1003) \approx 40.68$

$k_2 = r_2 \sin(\theta_{2a}) = 62.68$

$\text{atan2}(k_2, k_1) \approx \text{atan2}(62.68, 40.68) \approx 0.99\,rad$

$\theta_{1a} \approx 0.31 \cdot \frac{180}{\pi} = 17.5^\circ$

$\theta_{2a} \approx 1.67 \cdot \frac{180}{\pi} = 95.8^\circ$

IK $\rightarrow (\theta_1, \theta_2) = (0.31\,rad, 1.67\,rad)$ or $(17.5^\circ, 95.8^\circ)$

satisfies the workspace $0 < \theta_1 < \pi/2$ and $0 < \theta_2 < \pi$.

elbow down solution

$\theta_{2b} = -1.67\,rad$ $\quad \sin\theta_{2b} = -0.9949$ $\quad \cos\theta_{2b} = -0.1003$.

$k_1 = r_1 + r_2\cos\theta_{2b} = 40.68$ $\quad \text{atan2}(k_2, k_1) = -0.99\,rad$

$k_2 = r_2\sin\theta_{2b} = -62.68$

$\theta_{1b} = 1.30 + 0.99 = 2.29\,rad \approx 131.5^\circ$

$\theta_{2b} \approx 95.8^\circ$

so the second IK solution is $(\theta_1, \theta_2) \approx (2.29 \text{ rad}, -1.67 \text{ rad})$ or $(131.5^\circ, -95.8^\circ)$

this one violates the given range. it's mathematically valid but outside the specific range, so I did not use it in the animation. I implemented both in two link IK (both solutions) but for this assignment I only used the elbow up solution in the script (main), because that one satisfies the joint limits

3) plugging back into forward kinematics.

$\theta_1 \approx 0.31$ rad $\quad \theta_2 \approx 1.67$ rad $\quad \theta_1 + \theta_2 \approx 1.98$ rad

$\cos\theta_1 \approx 0.95 \qquad \sin\theta_1 \approx 0.30$

$\cos(\theta_1 + \theta_2) \approx -0.39 \qquad \sin(\theta_1 + \theta_2) \approx 0.92.$

$$x = r_1 \cos\theta_1 + r_2 \cos(\theta_1 + \theta_2) = 47 \cdot (0.95) + 63 \cdot (-0.39) = 20.1 \text{ mm}$$
$$y = r_1 \sin(\theta_1) + r_2 \sin(\theta_1 + \theta_2) = 72.1 \text{ mm}.$$

matches the target point $P = (20, 72)$ mm, IK math is consistent.