Unit-2
Number Theory

Permutation

An ordered arrangement of 'r' elements of a set containing 'n' elements.

$$nP_r = \frac{n!}{(n-r)!}$$

$$r = n \Rightarrow nP_n = n!$$

Combination

An unordered selection of 'r' elements of a set containing 'n' elements.

$$C(n,r) = nC_r = \frac{n!}{r!(n-r)!}$$

$$nC_n = 1$$

Sum rule

If 'r' activities can be performed in $n_1, n_2, \ldots, n_r$ ways and they are disjoint then any one of the 'r' activity can be performed in $n_1 + n_2 + \cdots + n_r$ ways.

Product rule

If an event can occur in 'm' ways and the another event can occur in 'n' ways and occurance of one does not depend on other then the two event can occur simultaneously in mn ways.

Q If there are 14 boys and 12 girls in the class, find the no. of ways of selecting one student as a CR.

Sol.

Either boy or girl (sum)

boys → 14 ways

girls → 12 ways

No. of ways → 14+12

→ 26 ways.

Q. Three person enter into the car where they are 5 seats. In how many ways will they take up their seats.

Soln.

choice for:

1st man = 5

2nd man = 4

3rd man = 3

Total ways = 3×4×5

= 60 ways.

Q. For a set of 6 True or False question find no. of ways of answering all question.

Total ways = 2×2×2×2×2×2

= 2^6

= 64 ways.

Q. Assuming the repetitions are not permitted.

a) How many 4 digit no. can be formed from the six digits 1, 2, 3, 5, 7, 8

b) How many of them are less than 4000.

c) How many even.
d) How many odd.
e. How many that contains both the digit 3 and 5.

Soln

a) $n = 6, r = 4$

No. of ways = $6P4 = \frac{6!}{(6-4)!} = 360$

b. __ __ __ __
1,2,3

1st digit can be filled in 3 ways.

Remaining = $5P3 = 60$

Total $= 3 \times 60$
$= 180$

c __ __ __ __
2 or 8

Last can be filled in 2 ways

Remaining = $5P3 = 60$
Total = 2×60
$= 120$

d. __ __ __ __
1, 3, 5, 7

Last can be filled in 4 ways
Remaining = $5P3 = 60$
Total = 4×60
$= 240$

e. 3 & 5 can occupy 2 of 4 places

$= 4P2 = 12$

Remaining $= 4P2$

$= 12$

$\therefore$ Total ways $= 12 \times 12$

$= 144$

Q. In how many ways the following words are arranged.

a) i) APPLE

ii) MATHEMATICS

iii) DAUGHTER

b) How many ways the vowel may never be separated.

soln.

a. (i) No. of ways $= \frac{5!}{2!} = 60$

ii) No. of ways $= \frac{11!}{2!\,2!\,2!}$

iii) No. of ways $= 8! = 40320$

b DAUGHTER

AUE → X

DXGHTR

No. of ways $= 6! \times 3!$

$= 4320$

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Q. How many +ve integer 'n' can be formed Using ~~using~~ the digits 3445567 if n has to exceed 50,00,000.

Soln.

1st Place : 5, 6, 7

when 5 occupies,

remaining: 3, 4, 4, 5, 6, 7

no. of ways $= \frac{6!}{2!} = 360$

When 6 occupies

remaining: 3 44557

no. of ways $= \frac{6!}{2!\,2!} = 180$

when 7 occupies

remaining: 344556

no. of ways $= \frac{6!}{2!\,2!}$

$= 180$

Total ways $= 180 + 180 + 360$

$= 720$

Q. How many permutation of the letters ABCDEFG contain

(i) String BCD

(ii) String BA and GF

(iii) String ABC and CDE

(iv) String CBA and BCD

soln.

(i) $5! = 120$ ways

(ii) BA GF CDE $= 5! = 120$ ways.

(iii) ABCDE $F_{(n)} = 3! = 6$

(iv) CBA and BCD = 0 [B in weird pos.]

Q (i) How many ways can 6 boys and 4 girls sit in a row.

(ii) How many ways can boys sit together and girls sit together.

(iii) How many ways ~~just~~ can girls are to sit together.

(iv) How many ways just the girls are to sit together.

Sol^n.

(i) $10!$

(ii) $\underbrace{6!}_{\text{Boys}} \times \underbrace{4!}_{\text{Girls}} \times \underbrace{2!}_{\text{order}}$

(iii) Girls → 1 unit = $4!$

Total = $4! \times 7!$

(iv) (iii) − (ii)

Permutation with repetitions

1. When repeatition of 'n' elements contained in a set is permitted in 'r' permutation then the no. of ways is n^r.

2. The no. of different permutation of n objects with identical element n_1, identical element n_2, identical element n_k is,

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

where,

$n_1 + n_2 + \cdots + n_k = n$

Circular Permutation

1. The no. of different circular arrangement of 'n' object is $(n-1)!$

2. If no. difference between clockwise and anticlockwise then $\frac{(n-1)!}{2}$

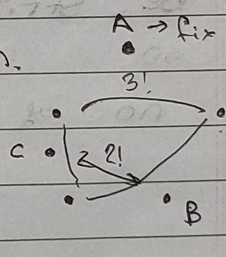
Q. If 6 people A, B, C, D, E, F are seated around a table,
(i) How many different circular arrangement.
(ii) If A, B, C are females and others are males, how many ways the sexes alternate.

soln.

$n = 6$

(i) Circular arrangement $= (n-1)!$
$= 5!$
$= 120$

(ii) A is seated in top position.
B and C are seated in 2!



Remaining = 3!

Total ways = $2! \times 3!$
= 12 ways.

Q(i) In how many ways can 7 people arranged in circular table.
ii. If two of them insist on sitting next to each other, find the ways.

soln.

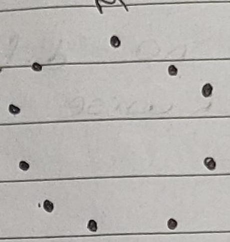
(i) $6!$ [∵ $(7-1)!$]

(ii) consider 2 as 1
∴ $2! \times 5!$ = 240 ways.

Q. There are 6 men and 4 ladies. How many ways can they be seated in a round table so that no ladies are together.

soln.

1 male sits in first position.



For male: $5!$

$= 120$

For female: $^6P_4$

$= \frac{6!}{2!}$

$= \frac{6 \times 5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}}$

$= 360$

Total $= 120 \times 360$

$= 43200$

Q. How many ways can 7 identical beads be string into a ~~be~~ ring.

soln.

no. of ways $= \frac{(7-1)!}{2}$

$= \frac{6!}{2}$

$= 6 \times 5 \times 4 \times 3 \times 1$

$= 360$ ways.

Combinations.

Q. From 6 men and 7 women, how many ways to select a committee of:
(i) 3 mean and 4 women
(ii) 4 person that has person of both sexes

Sol^n

(i) $6C3 \times 7C4$

(ii)

Men(6)	Women(7)	Combination
3	1	$6C3 \times 7C1 = 140$
2	2	$6C2 \times 7C2 = 315$
1	3	$6C1 \times 7C3 = 210$

+

Total = 665

Q. In how many ways can 20 out of 30 students be selected
(i) If A refuse to select.
(ii) If B and C insist on select.

Sol^n.

(i) $29C20 = 10015005$

(ii) $28C18 = 13123110$

Note

The no. of 'r' combination of 'n' kinds of objects, if repetition of object is allowed is,

$C(n+r-1, r)$

Q. There are 3 piles of identical red, blue and green balls where each pile contains at least 10 balls. In how many ways 10 balls be selected,
(i) If there is no restriction,

(ii) If atleast 1 red ball is to be selected.

(iii) atleast 1 red, 2 blue, 3 green ball is to be selected.

(iv) Exactly 1 red ball is selected

Soln,

$n = 3$ kinds

$r = 10$

(i) $C(3+10-1, 10)$

$= C(13-1, 10)$

$= C(12, 10)$

$= 12C10$

$= 66$

(ii) $n = 3$

$r = 9$

$C(3+9-1, 9)$

$= C(11, 9)$

$= 11C9$

$= 55$

(iii) $r = 4$

$n = 3$

$C(3+4-1, 4)$

$= C(6, 4)$

$= 6C4 = 15$

(iv) $n = 2, r = 9$

$= C(2+9-1, 9)$

$= C(10, 9)$

$= 10$

Q. How many binary string of length 10 contain
a) exactly four 1s
b) At most 4 1s
c) at least 4 1s.
d) equal no. of 0^s and 1^s

Sol^n.

$n = 10$

$r = 4$

(a) $10C4$
$= 210$

(b)

1^s	0^s	combination.	
4	6	$10C4$	$= 210$
3	7	$10C3$	$= 120$
2	8	$10C2$	$= 45$
1	9	$10C1$	$= 10$
0	10	$10C0$	$= 1$

Total ways $= 210 + 120 + 45 + 10 + 1$
$= 386$

(c)

1^s	0^s	Combination	
4	6	$10C4$	$= 210$
5	5	$10C5$	$= 252$
6	4	$10C6$	$= 210$
7	3	$10C7$	$= 120$
8	2	$10C8$	$= 45$
9	1	$10C9$	$= 10$
10	0	$10C10$	1

Total $= 210 + 252 + 210 + 120 + 45 + 10 + 1 = 848$

d $10C5$
$= 252$

Principle of Inclusion-Exclusion

If A and B are finite subsets of universal set U then,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

where $|A|$ denotes the cardinality of set A.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

In general,

$$|A_1 \cup A_2 \cup A_3 \ldots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \ldots \cap A_n|$$

Q. Find the no. of integers between 1 and 250 both inclusive that are not divisible by any of the +ve integers 2, 3, 5 and 7.

Solⁿ.

Let A, B, C, D be the set of integers that are divisible by 2, 3, 5 and 7 between 1 and 250.

$$|A| = \left\lfloor \frac{250}{2} \right\rfloor = 125$$

$$|B| = \left\lfloor \frac{250}{3} \right\rfloor = 83$$

$$|C| = \left\lfloor \frac{250}{5} \right\rfloor = 50$$

$$|D| = \left\lfloor \frac{250}{7} \right\rfloor = 35$$

$$|A \cap B| = \left\lfloor \frac{250}{6} \right\rfloor = 41$$

$$|A \cap C| = \left\lfloor \frac{250}{10} \right\rfloor = 25$$

$$|A \cap D| = \left\lfloor \frac{250}{14} \right\rfloor = 17$$

$$|B \cap C| = \left\lfloor \frac{250}{15} \right\rfloor = 16$$

$$|B \cap D| = \left\lfloor \frac{250}{21} \right\rfloor = 11$$

$$|C \cap D| = \left\lfloor \frac{250}{35} \right\rfloor = 7$$

$$|A \cap B \cap C| = \left\lfloor \frac{250}{30} \right\rfloor = 8$$

$$|B \cap C \cap D| = \left\lfloor \frac{250}{105} \right\rfloor = 2$$

$$|C \cap D \cap A| = \left\lfloor \frac{250}{70} \right\rfloor = 3$$

$$|D \cap A \cap B| = \left\lfloor \frac{250}{42} \right\rfloor = 5$$

$$|A \cap B \cap C \cap D| = \left\lfloor \frac{250}{250} \right\rfloor = 1$$

By the principle of inclusion-exclusion, the no. of integers between 1 and 250 that are divisible by 2, 3, 5, 7 are

$$\begin{aligned}|A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| \\ &\quad - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + \\ &\quad |A \cap B \cap C| + |B \cap C \cap D| + |C \cap D \cap A| + \\ &\quad |D \cap A \cap B| - |A \cap B \cap C \cap D| \\ &= 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - \\ &\quad 11 - 7 + 8 + 2 + 3 + 5 - 1 \\ &= 193\end{aligned}$$

Total no. not divisible $= 250 - 193$

$= 57$

Q. In a survey of 120 passengers, an airline found that :-

52 enjoyed juice

75 enjoyed mixed drinks

62 enjoyed ice tea

35 enjoyed any given pair of these

20 enjoyed all of them.

Find the no. of passenger who enjoyed

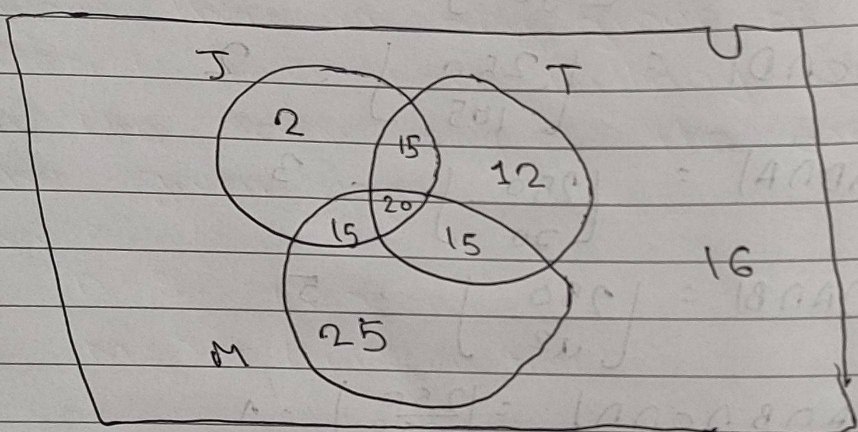
i) only tea

ii) only one of these

iii) None

Sol^n

Let T, M, J denotes the Tea, mixed drinks and Juice respectively.



(i) only tea = 62 - 50

= 12

(ii) only one of 3 juice = 2 + 12 + 25

= 39

iii) None = 120 - 12 - 2 - 25 - 15 - 20 - 15 - 15

= 16

Q. There are 250 students in an Engineering college 180 have taken course in fortran, 100 in C, 35 in Java. further 88 have taken course both fortran and C. 23 have taken both C & Java. 29 have taken both fortran and Java. If 19 students have taken all the course. find the no. of students who have not taken any course.

sol^n:

Let F, C and J be students who have taken Fortran, C and Java respectively.

Given,

$|F| = 180$

$|C| = 100$

$|J| = 35$

$|F \cap C| = 88$

$|C \cap J| = 23$

$|F \cap J| = 29$

$|F \cap C \cap J| = 19$

Then,

$|F \cup C \cup J| = |F| + |C| + |J| - |F \cap C| - |C \cap J| - |F \cap J| + |F \cap C \cap J|$

$= 180 + 100 + 35 - 88 - 23 - 29 + 19$

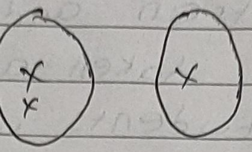
$= 194$

$|\overline{F \cup C \cup J}| = 250 - 194$

$= 56.$

Pigeon Hole Principle

In 'n' pigeons are accomodated in 'm' pigeon hole and $n > m$ then atleast one pigeon hole will contain two or more pigeons



$n = 3$

$m = 2$

Generalized Pigeon Hole principle

If $n > m$ then one pigeon hole must contain atleast $\left(\frac{n-1}{m}\right) + 1$ pigeons.

Q. Show that in any group of 8 people atleast two have birthday which falls on the same day of the week in the given year.

Given,

$n = 8$ (pigeon)

$m = 7$ (pigeon hole)

By principle of pigeon hole,

$$\left(\frac{n-1}{m}\right) + 1$$

$$: \left(\frac{7-1}{7}\right) + 1$$

$$= 2$$

classmate

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Q. In a group of 100 people, several will have birthdays in the same month. Atleast how many of them have birthday on same month?

Given,

$n = 100$

$m = 12$

By principle of pigeon hole;

$= \left(\frac{100-1}{12}\right) + 1$

$= \frac{99}{12} + 1$

$= 9$

∴ Atleast 9 of them will have birthdays on same month.

Q. How many people must ~~be~~ you have to guarentee that atleast 9 of then will have birthday in the same day of week

Given

$n \rightarrow$ pigeon (people)

$m \rightarrow 7$ (pigeon hole)

By principle.

$\frac{n-1}{m} + 1 = 9$

$\frac{n-1}{m} = 8$

$n-1 = 56$

$\therefore n = 57.$

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Q What is the most no. of students required in a class so that atleast will receive the same grade if there are 5 grades A, B, C, D, E.

Given,

$n = ?$

$m = 5$

Also,

$$\left(\frac{n-1}{m}\right) + 1 = 6$$

$$\frac{n-1}{m} = 5$$

$$n-1 = 25$$

$$n = 26$$ //