
Exercise 1. *Prove that the only holomorphic form on $\mathbb{P}^1$ is the trivial one (identically zero). Prove that the only holomorphic form on a torus $\mathbb{C}/\Lambda$ is the one induced by scalar multiples of dz.*

Consider the meromorphic function $f\colon \mathbb{P}^1 \to \mathbb{P}^1$, $[z:w] \mapsto [z:w]$. Let $\omega = df$. By writing in the two charts of the domain $\mathbb{P}^1$, show that $\operatorname{div}(\omega) = -2[\infty]$. Deduce that it degree is -2.

Exercise 2 (Riemann–Hurwitz formula on divisors)**.** *Let $f\colon X \to Y$ be a holomorphic map between compact Riemann surfaces. Let K_X and K_Y be any canonical divisor on X and Y respectively, and define the ramification divisor as*

$$R_f \coloneqq \sum_{x \in \mathsf{Ram}_f} \big(\mu_x(f) - 1\big)[x] \in \operatorname{Div}(X)\,.$$

Prove that

$$K_X \sim f^* K_Y + R_f$$

and deduce the Riemann–Hurwitz formula by taking degrees.

Exercise 3 (Holomorphic forms on projective curves)**.** *Let $X = Z(F) \subset \mathbb{P}^2$ be a smooth projective plane curve defined by a homogeneous polynomial $F(x,y,z)$ of degree $d \geq 3$. Let $f(u,v) = F(u,v,1)$ define the associated affine curve.*

- *Prove that du and dv define meromorphic forms on X.*
- *Show that $f_u\,du = -f_v\,dv$ as meromorphic forms on X (here $f_u = \frac{\partial f}{\partial u}$ and $f_v = \frac{\partial f}{\partial v}$).*
- *Show that for any polynomial $p(u,v)$ of degree $\leq d-3$, the form*

  $$\omega_p = p(u,v)\frac{du}{f_v}$$

  is holomoprhic. How many polynomials of degree $\leq d-3$ in two variables there are? How does it relate to the genus of X?

Exercise 4 (Complete linear systems and projective embeddings 💀)**.** *Given a divisor D on a compact Riemann surface X, define the map*

$$\varphi_D\colon X \longrightarrow \mathbb{P}^n\,, \qquad x \longmapsto [f_0(x) : f_1(x) : \cdots : f_n(x)]\,,$$

where $f_0, \ldots, f_n$ form a basis of $\mathcal{L}(D)$. A basic question is: when is φ_D and embedding?

- *We say that the complete linear system $|D|$ is free iff $\ell(D) - l(D - [x_0]) = 1$ for every x_0 in the support of D. Fix a point $x \in X$. Show that if $|D|$ is free, then there exists a basis $f_0, f_1, \ldots, f_n$ for $\mathcal{L}(D)$ such that $\operatorname{ord}_x(f_0) = -n_x$ and $\operatorname{ord}_x(f_i) > -n_x$ for $i \geq 1$.*
- *Let $|D|$ be free. Show that $\varphi_D(x) = \varphi_D(y)$ iff $\mathcal{L}(D - [x] - [y]) = \mathcal{L}(D - [x]) = \mathcal{L}(D - [y])$. Conclude that φ_D is injective iff $\ell(D - [x] - [y]) = \ell(D) - 2$ for all distinct $x, y \in X$.*
- *Conclude that, for $|D|$ be free, φ_D is an embedding iff $\ell(D - [x] - [y]) = \ell(D) - 2$ for all $x, y \in X$ (including coinciding points). In this case, $|D|$ is called very ample.*