Deep Learning Course

Lesson 2 — Feedforward Neural Networks (ANNs)

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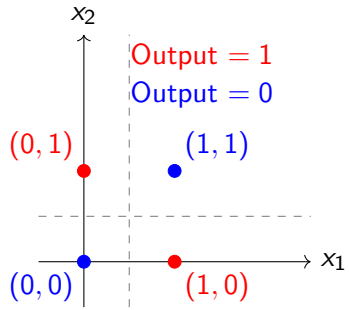
Linear Separability and the Perceptron

- A single perceptron learns **linear** decision boundaries.
- Works when classes are linearly separable.
- Fails when no hyperplane can separate the classes.

The XOR Problem

- XOR truth table: outputs 1 when inputs differ, 0 otherwise.
- The positive and negative points are **not** linearly separable.
- No single line/hyperplane separates XOR classes in $\mathbb{R}^2$.

XOR Problem Visualization



XOR (exclusive OR): The output is 1 if *exactly one* of x_1 or x_2 is 1.

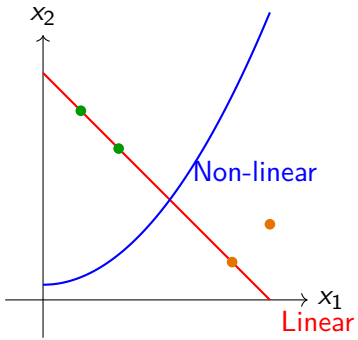
$$y = x_1 \oplus x_2$$

$\rightarrow$ *Not linearly separable with a single perceptron.*

Understanding Non-Linearity

- A **linear model** can only draw straight lines (or hyperplanes) to separate data.
- Some problems (like XOR) **cannot be separated** by a straight line.
- Adding **non-linear activation functions** (e.g. ReLU, sigmoid, tanh) allows the network to:
  - combine multiple linear regions,
  - create curved decision boundaries,
  - and learn complex relationships.

Without non-linearity, a neural network behaves like a single linear model.



Why Linear + Linear = Linear

Composing linear functions does not increase expressiveness.

$$\text{Let } f(\mathbf{x}) = W_2(W_1\mathbf{x} + \mathbf{b}_1) + \mathbf{b}_2$$

then

$$f(\mathbf{x}) = (W_2W_1)\mathbf{x} + (W_2\mathbf{b}_1 + \mathbf{b}_2)$$

Observation:

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{c} \text{ where } A = W_2W_1 \text{ and } \mathbf{c} = W_2\mathbf{b}_1 + \mathbf{b}_2.$$

The result is still linear

Therefore, stacking linear layers without a non-linear activation is equivalent to a single linear transformation.

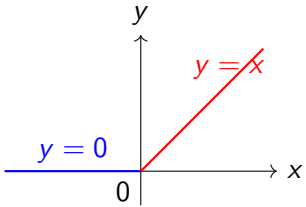
Non-Linear Activation: ReLU

Rectified Linear Unit (ReLU):

$$\text{ReLU}(x) = \max(0, x)$$

- Introduces **non-linearity** while keeping computation simple.
- Outputs 0 for negative inputs, and passes positive inputs unchanged.
- Allows neural networks to learn **non-linear decision boundaries**.

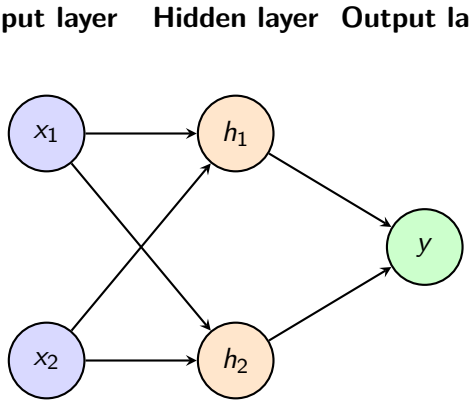
Without ReLU (or any non-linearity), multiple layers would collapse into a single linear transformation.



Two-Layer Network Solves XOR

- Hidden layer splits input space into regions.
- Step1: $h = f(W_1 x + b_1)$
- Step2: $y = W_2 h + b_2$.
- With suitable W_1, b_1, W_2, b_2 and f (e.g., ReLU/sigmoid), XOR is representable.

Neural Network with 2–2–1 Architecture



A simple feedforward neural network with two inputs, two hidden units, and one output.

Back to school: Matrix Multiplication 1/2

Definition:

$$C = A \times B$$

If

$$A \in \mathbb{R}^{m \times n}, \quad B \in \mathbb{R}^{n \times p}$$

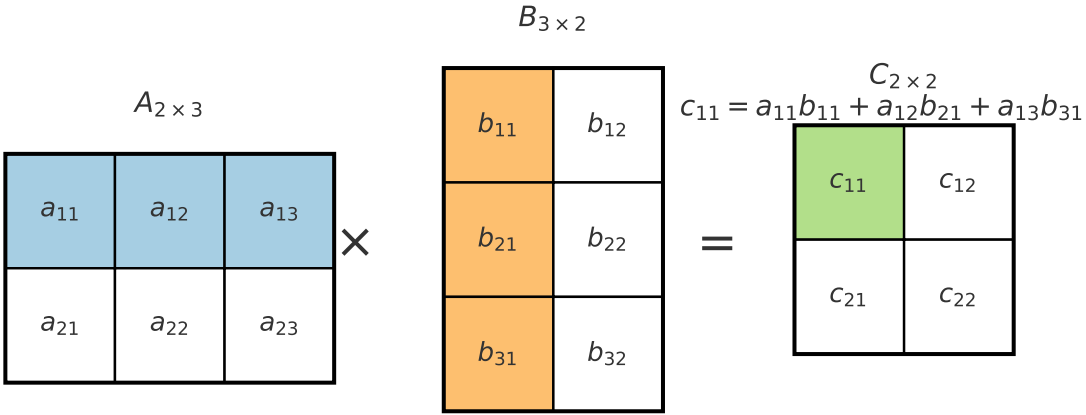
then

$$C \in \mathbb{R}^{m \times p}, \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Example: $A_{2\times 3} \times B_{3\times 2} = C_{2\times 2}$

- Each element c_{ij} is the **dot product** of row i of A and column j of B.
- Order matters: $A \times B \neq B \times A$ (in general).
- In neural networks: $\mathbf{y} = W\mathbf{x}$ combines inputs linearly.

Back to school: Matrix Multiplication 2/2



Quiz time: XOR Network, what are the shapes?

We use a simple neural network to solve the XOR problem:

$$\mathbf{x} \in \mathbb{R}^2 \xrightarrow{W_1, b_1} \text{Hidden layer (2 units)} \xrightarrow{W_2, b_2} \text{Output (1 unit)}$$

Question: Given this 2–2–1 architecture, what are the shapes of each matrix and bias?

Input: $\mathbf{x}$ (?) W_1 (?), b_1 (?), W_2 (?), b_2 (?)

XOR Network: Parameter Shapes (Answer)

For a 2–2–1 network:

$$\mathbf{x} \in \mathbb{R}^2 \xrightarrow{W_1, b_1} \text{Hidden layer } \mathbf{h} \in \mathbb{R}^2 \xrightarrow{W_2, b_2} \text{Output } y \in \mathbb{R}^1$$

$$W_1 \in \mathbb{R}^{2\times 2} \quad \text{(2 hidden units, 2 inputs)}$$

$$b_1 \in \mathbb{R}^2 \quad \text{(1 bias per hidden unit)}$$

$$W_2 \in \mathbb{R}^{2\times 1} \quad \text{(2 hidden units, 1 output)}$$

$$b_2 \in \mathbb{R}^1 \quad \text{(1 bias for the output unit)}$$

Quiz time 1/3

Question

x is a single sample, and W has shape equals to (output-size,input-size). Is x in the formula, $y = Wx$ a column vector or a row vector?

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Answer

A column vector: it must have shape equals to (input-size, 1), otherwise the matrix multiplication wouldn't work. y has shape (output-size,1)

Quiz time 2/3

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Answer

A row vector: it must have shape equals to (1,input-size), because W^T has shape (input-size,output-size). y has shape (1, output-size)

Quiz time 3/3

Question

If X is a batch of N samples with shape (N, input-size), what is the right formula to use? $y = WX$ or $y = XW^T$?

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Answer

$y = XW^T$ because X has shape (N, input-size) and W^T has shape (input-size,output-size). y has shape (N,output-size), so, it has one "row" for each input in the batch.

Lab Time: the xor network

Find the weights of the W_2 matrix.

```
import numpy as np

def relu(z): return np.maximum(z, 0.0)

X = np.array([[0, 0],
              [1, 0],
              [0, 1],
              [1, 1]])

y = np.array([0, 1, 1, 0]).reshape(-1,4)
W1 = np.ones((2, 2))
W2 = ???
b1 = np.array([0,-1])
b2 = 0
H = relu(X @ W1.T + b1)
y_hat = H @ W2
assert(y_hat.all() == y.all())
```

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Lab Time: the MLP class

Create a MLP class with only the *constructor* and the *forward* method. The constructor takes $\mathbf{W_1}, \mathbf{b_1}, \mathbf{W_2}, \mathbf{b_2}$ and the **activation function** as input. The forward method takes $\mathbf{X}$ as input and return the $\widehat{\mathbf{y}}$ vector. Test the class with inputs and weights from the xor network example.

My trivial implementation

```
#....
class MLP:
    def __init__(self,W1,b1,W2,b2,activaction):
        self.W1 = W1
        self.W2 = W2
        self.b1 = b1
        self.b2 = b2
        self.activation = activaction

    def forward(self,X):
        H = self.activation(X @ self.W1.T + self.b1)
        y_hat = H @ self.W2.T + self.b2
        return y_hat

mlp = MLP(W1,b1,W2,0,relu)
print(mlp.forward(X))
```

A more interesting problem: the MNIST Dataset

MNIST (Modified National Institute of Standards and Technology) is one of the most iconic datasets in machine learning and computer vision. It contains **70,000 grayscale images** of handwritten digits (0–9), each of size **28×28 pixels**, divided into **60,000 training** and **10,000 test** samples.

Historical background:

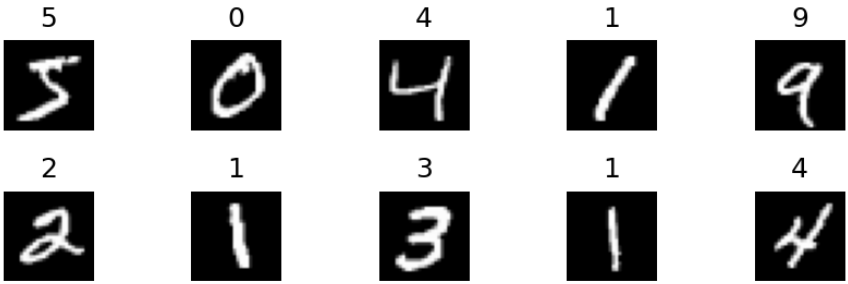
- Created in **1998** by **Yann LeCun**, **Corinna Cortes**, and **Christopher J.C. Burges**.
- Derived from the original **NIST handwritten digit database** from the late 1980s.
- The original dataset used digits written by two groups: U.S. Census Bureau employees and American high school students.
- MNIST combined and normalized samples from both sources, hence the "**Modified**" in its name.

Why it matters:

- Served as the **benchmark dataset** for early image recognition research.
- Used to test **Convolutional Neural Networks (CNNs)**, including LeCun's famous **LeNet-5**.
- Though now considered a "**solved problem**", MNIST remains a standard starting point for learning and testing new models.

Examples from the MNIST Dataset

Each image in MNIST is a **28×28 grayscale** handwritten digit. The examples below illustrate the natural variation in handwriting across individuals.



- Digits range from **0** to **9**.
- Pixel values range from **0 (white)** to **255 (black)**.

Recreating MNIST `.npy` Files 1/2

The original MNIST dataset is distributed as four compressed binary files in **IDX** format. We can convert them into easy-to-use **NumPy arrays** by following three main steps:

1. Download the Original Files

- Hosted on: `https://ossci-datasets.s3.amazonaws.com/mnist/`
- Files:
  - `train-images-idx3-ubyte.gz` — 60,000 images
  - `train-labels-idx1-ubyte.gz` — 60,000 labels
  - `t10k-images-idx3-ubyte.gz` — 10,000 images
  - `t10k-labels-idx1-ubyte.gz` — 10,000 labels

2. Decompress and Parse

- Use `gzip` to read binary files.
- Skip headers: `offset=16` (images), `offset=8` (labels).
- Reshape images to `(-1, 28×28)`.

Recreating MNIST .npy Files 2/2

3. Save as .npy

- Save the NumPy arrays:
  - `np.save("train_images.npy", np_train_images)`
  - `np.save("train_labels.npy", np_train_labels)`
- Load them later with `np.load(...)` for instant access.

Result: Four reusable files — `train/test_images.npy` and `train/test_labels.npy`.

How a Pixel is Represented in Color Images

In a **color image**, each pixel is composed of **three channels**:

- Red (R)
- Green (G)
- Blue (B)

Each channel stores an intensity value between **0 and 255**:

$$\text{Pixel} = [R,\ G,\ B] = [255,\ 128,\ 0]$$

A grayscale image (like MNIST) uses only one channel — a single value per pixel. 8-bit $\Rightarrow$ 256 levels (0–255).

Coordinate Conventions

- Image indices: (row $= y$, column $= x$), origin at top-left.
- Zero-based indexing; bounds checking matters.

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Important

No learning occurs during inference — the model only performs a **forward pass**.

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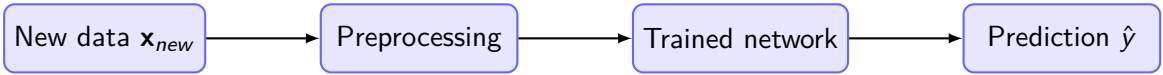
4. **Output interpretation:**
   - Regression $\rightarrow \hat{y}$ is a numeric value
   - Classification $\rightarrow$ choose $\arg\max(\hat{y})$

Inference Pipeline Overview



The network applies the same learned transformations to the new input without modifying its parameters.

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Model summary

$$\mathbf{x} \in \mathbb{R}^{784} \xrightarrow{\mathbf{W_1}, \mathbf{b_1}} \mathbf{h} \in \mathbb{R}^{32} \xrightarrow{\mathbf{W_2}, \mathbf{b_2}} \hat{\mathbf{y}} \in \mathbb{R}^{10}$$

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Training complete

Weights $\mathbf{W_1}, \mathbf{W_2}$ and biases $\mathbf{b_1}, \mathbf{b_2}$ are now **frozen** and ready for inference.

What the Network Has Learned

Hidden layer representation

Each hidden neuron combines pixel patterns to detect simple features, like edges, curves, or parts of digits.

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The 10 output neurons represent the probability of each class:

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Example

For an image of "3":

$$\hat{\mathbf{y}} = [0.01, 0.02, 0.05, \mathbf{0.90}, 0.01, 0.00, \ldots]$$

The network predicts **digit 3**.

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Ensure that all input features have comparable ranges, so that the neural network trains stably and converges faster.

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Procedure

1. The input values has been scaled, so the range is not between $0 - 255$ but between $0 - 1$.
2. Compute the **mean** and **standard deviation** of each feature using only the training data.
3. If a standard deviation is extremely small (std $< 10^{-6}$), replace it with 10^{-6} to avoid division by zero.
4. Normalize all datasets (training, validation, test) using: $X' = \frac{X - \text{mean}}{\text{std}}$

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Key point

The same mean and std values computed on the **training set** must be reused during inference - never recomputed on new data.

Lab Time

1. Load **W1.npy, b1,npy, W2.npy, b2.npy, mean.npy** and **std.npy**
2. Let X the **test_images.npy** matrix
3. Scale X so that all values are between 0 and 1
4. Normalize X with the formula $\frac{X - \text{mean}}{\text{std}}$
5. Use the previous created MLP class and the loaded weights to predict the labels of X ($\hat{Y}$)
6. Let Y the **test_labels.npy** vector and calculate the model accurancy, where

$$\text{Accuracy} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}(\hat{y}_i = y_i)$$

My trivial implementation

```python
import numpy as np

def relu(x): return np.maximum(0, x)

class MLP:
    def __init__(self, W1, b1, W2, b2, activation):
        self.W1 = W1.astype(np.float32)
        self.b1 = b1.astype(np.float32).reshape(-1)
        self.W2 = W2.astype(np.float32)
        self.b2 = b2.astype(np.float32).reshape(-1)
        self.activation = activation

    def forward(self, X):
        H = self.activation(X @ self.W1.T + self.b1)
        y_hat = H @ self.W2.T + self.b2
        return y_hat

W1 = np.load("W1.npy")
b1 = np.load("b1.npy")
```

My trivial implementation

```
W2 = np.load("W2.npy")
b2 = np.load("b2.npy")
mean = np.load("mean.npy").astype(np.float32)
std  = np.load("std.npy").astype(np.float32)


X_test = np.load("test_images.npy").astype(np.float32)
y_test = np.load("test_labels.npy")

X_test /= 255.0
X_test = (X_test - mean) / std


model = MLP(W1, b1, W2, b2, activation=relu)
logits = model.forward(X_test)

y_pred = np.argmax(logits, axis=1)
num_correct = int((y_pred == y_test).sum())
total = y_test.shape[0]
accuracy = num_correct / total

print(f"Correct predictions: {num_correct}/{total}  (accuracy = {accuracy:.4%})")
```

Thanks!