Problem Set 7

Statistical Methods
In Engineering And Science

Due Date: 10:00 PM, Feb 26, 2023

Last Update: February 20, 2023

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Study Group: ____________________ ____________________ ____________________

Please upload your solution in a single pdf file on Canvas. Include all calculations, R-code, and figures (if applicable). All data sets are available on Canvas https://canvas.uw.edu/courses/1614615.

Question 1. Given a random sample $X_1, \dots X_n$ from $Exp(\lambda)$ with unknown $\lambda > 0$ you want to estimate the mean $1/\lambda$. From the lecture you know that the sample average $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ is an unbiased estimator for $1/\lambda$. Now, consider a more general estimator of the form $T_c = c \times (\sum_{i=1}^n X_i)$ for $c \in \mathbb{R}$. You are interested in the MSE of this estimator and you would like to know whether there are choices for $c \in \mathbb{R}$ such that T_c has smaller MSE than $\bar{X}_n$.

(a) Compute the $\text{MSE}(T_c)$ for arbitrary $c \in \mathbb{R}$.

(b) For which $c \in \mathbb{R}$ does the estimator T_c perform best in the MSE sense? Compare the MSE of this "best" estimator with the MSE of the unbiased estimator $\bar{X}_n$ (which is just the estimator T_c with $c = 1/n$).

Question 2. Suppose that you have two unbiased estimators U and V with the same variance $\text{Var}(U) = \text{Var}(V)$. Based on the MSE criterion you cannot choose one estimator over the other. Now, consider $W = (U + V)/2$.

(a) Show that W is unbiased.

(b) Show that

$$\frac{\text{Var}\big((U+V)/2\big)}{\text{Var}(U)} = \frac{1}{2} + \frac{1}{2}\rho(U,V),$$

where $\rho(U,V)$ is the correlation coefficient between U and V.

(c) Why do the results for part (a) and (b) imply that we should use W instead of U or V?

Question 3. Consider the following data set consisting of 12 observations

.92 .79 .90 .65 .86 .47 .73 .97 .94 .77 .79 .91

Suppose that the data are the realization of a random sample $X_1, \dots, X_{12}$ from a distribution with pdf

$$f(x) = \begin{cases} (\theta+1)x^\theta & \text{if } 0 \le x \le 1, \\ 0 & o/w. \end{cases}$$

(a) What is the method of moment estimator for θ? Compute an estimate based on the data set.

(b) What is the maximum likelihood estimator for θ? Compute an estimate based on the data set.

Question 4. Let $X_1, \dots, X_n$ be a random sample from $Ber(p)$. Show that the method of moment estimator and the maximum likelihood estimator, both, are equal to the sample proportion $\hat{p} = n^{-1} \sum_{i=1}^{n} X_i$.

Question 5. Let $X_1, \dots, X_n$ be a random sample from a $Bin(N, p)$.

(a) Suppose that N is known and only success probability p is unknown. Compute the method of moment estimator and the maximum likelihood estimator for p.

(b) Suppose that N and p are unknown. Compute the method of moment estimators for N and p.

Question 6. Let $X_1, \dots, X_n$ be a random sample from $Gamma(\alpha, \beta)$ with pdf

$$f(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{if } x \in (0, \infty); \\ 0 & o/w, \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$. Assume that the shape parameter $\alpha > 0$ is known and compute the maximum likelihood estimator of the rate parameter $\beta > 0$.