Problem Set 2 **STAT 394/ MATH 394**

Probability I

Due Date: 10:00 PM, March 12, 2022 Prof. Alexander Giessing

Last Update: April 15, 2022 Spring Quarter, 2022

Study Group: ______________________ ______________________ ______________________

Question 1. **[3 + 3 + 3 pts]** Let $A_k \in \mathcal{A}$, $k = 1, 2, \ldots$, be arbitrary events.

(a) Show that $\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mathbb{P}(A_k)$. *Hint: First, prove the inequality for $A_1, \ldots, A_n$, see Lemma 1.17. Then, consider $n \to \infty$ and use the continuity of the probability measure.*

(b) How useful is above inequality? Answer this question by considering the following two cases: (i) the A_k's are such that $\mathbb{P}(A_k) = 3^{-k}$, and (ii) the A_k's are such that $\mathbb{P}(A_k) = k^{-2}$.

(c) Show that $\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) = 1$ if $\mathbb{P}(A_k) = k^{-2}$ for $k = 1, 2, \ldots$ *Hint: An exercise from PS 1 might contain a useful idea.*

Question 2. **[3 pts]** Suppose that you toss a fair coin repeatedly. Show that, with probability one, you will toss a head eventually. *Hint: Introduce the events $A_n = \{$"no head in the first n tosses"$\}$, $n = 1, 2, \ldots$.*

Question 3. **[3 + 3 pts]** Consider tossing a fair coin n times. For $k = 1, \ldots, n$, define the events

$$A_k = \{\text{“the first } k \text{ tosses yield only head”}\}.$$

(a) You want to model this experiment as a Laplace experiment. Find a suitable sample space and compute $\mathbb{P}(A_k)$.

(b) Describe in words the event $A_{k-1} \setminus A_k$. What is the probability of this event?

Question 4. **[3 + 3 pts]** You roll a fair die 6 times.

(a) What is the probability that you roll at least one 6?

(b) What is the probability of rolling 5 different numbers?

Question 5. **[3 + 3 pts]** Consider an urn with 10 balls labeled $1, \ldots, 10$. You draw four times without replacement from this urn.

(a) What is the probability of only drawing balls with odd numbers?

(b) What is the probability that the smallest drawn number is equal to k for $k = 1, \ldots, 10$?

Question 6. **[3 pts]** McMahon Hall on UW's North Campus has 11 floors. You observe 7 people entering the elevator on the ground floor. In the absence of additional information you assume that any distribution of 7 people over 11 floors is equally likely. What is the probability that on each floor at most 1 person leaves the elevator?

Question 7. [0 + 0 + 0 pts] [Optional] Another birthday problem.

(a) Show that the probability that one of the 121 students in STAT/ MATH 394 has the same birthday as their professor is $1 - (1 - 1/365)^{121}$.

(b) Let $q(n, k) := 1 - (1 - 1/n)^k$. Show that for $k = \lceil \alpha n \rceil$ and $\alpha \in (0, \infty)$,

$$\lim_{n \to \infty} q(n, k) = 1 - e^{-\alpha}.$$

Remark: In fact, $\lim_{n \to \infty} q(n, k) = 0$ *for every fixed* $k > 0$ *and* $\lim_{k \to \infty} q(n, k) = 1$ *for all fixed* $n \geq 1$*. Choosing* $k = \lceil \alpha n \rceil$ *ensures that we obtain a non-trivial probability in the limit.*

(c) The result in (b) differs markedly from the result obtained for the birthday problem discussed in the lecture. In plain English, what is the main difference between the two birthday problems? Can you make sense of how this difference results in the very different asymptotic behaviors?