Problem Set 3 **STAT 394/ MATH 394**

Probability I

Due Date: 10:00 PM, April 19, 2022 Prof. Alexander Giessing

Last Update: April 16, 2022 Spring Quarter, 2022

Study Group: ____________________ ____________________ ____________________

Question 1. [3 + 3 + 3 + 3 pts] In this exercise you will generalize the binomial distribution.

(a) Show that the number of possible ways in which a set A with cardinality $|A| = n$ can be partitioned into r subsets $A_1, \ldots, A_r$ with cardinalities $n_1, \ldots, n_r$ such that $n_1 + \ldots + n_r = n$ is equal to

$$\frac{n!}{n_1! \cdots n_r!}.$$

(b) Consider an urn with N balls which are labeled $1, \ldots, N$ with the first N_1 balls of color 1, the second N_2 balls of color 2, ..., and the last N_r balls of color r. You draw n balls with replacement from this urn. Denote the event that out of the n balls exactly n_1 balls are of color 1, n_2 balls are of color 2, ..., and n_r balls are of color r by

$$E_{n_1,\ldots,n_r} = \left\{ (a_1, \ldots, a_n) : \left| \left\{ i : a_i \in \left\{ \sum_{j=0}^{k-1} N_j + 1, \ldots, \sum_{j=0}^{k} N_j \right\} \right\} \right| = n_k,\ k \in \{1, \ldots, r\} \right\},$$

where $N_0 = 0$, $N_1 + \ldots + N_r = N$, and $n_1 + \ldots + n_r = n$. Use part (a) to find a collection of disjoint sets $C_1, C_2, \ldots, C_m$ (for some m that you need to specify) such that

$$E_{n_1,\ldots,n_r} = \bigcup_{j=1}^{m} C_j \quad \text{and} \quad |C_j| = N_1^{n_1} N_2^{n_2} \cdots N_r^{n_r} \quad \text{for all} \quad j = 1, \ldots, m.$$

(Hint: Take a hard look at the proof of Lemma 3.5.)

(c) Use part (b) to show that

$$\mathbb{P}(E_{n_1,\ldots,n_r}) = \frac{n!}{n_1! \cdots n_r!} \left(\frac{N_1}{N}\right)^{n_1} \left(\frac{N_2}{N}\right)^{n_2} \cdots \left(\frac{N_r}{N}\right)^{n_r}.$$

(Hint: Argue via Definition 1.5 of Laplace experiments.)

(d) Let $\Omega = \{(n_1, \ldots, n_r) : n_1 + \ldots + n_r = n, n_k \in \mathbb{N}_0, 1 \leq k \leq r\}$. Show that the function $p : \Omega \mapsto [0, 1]$ defined by

$$p(n_1, \ldots, n_r) := \mathbb{P}(E_{n_1,\ldots,n_r})$$

is a discrete probability measure on $(\Omega, 2^{\Omega})$. Also, show that this probability measure does not describe a Laplace experiment. *(Hint: First, check the properties of a discrete probability measure in Definition 3.2. Second, direct calculation.)*

Question 2. **[3 + 3 + 3 + 3 pts]**. Suppose that A and B are events with $0 < P(A) < 1$ and $0 < P(B) < 1$. Prove the following statements.

(a) If A and B are disjoint, then they are not independent.

(b) If A and B are independent, then they are not disjoint.

(c) If $A \subset B$, then A and B are not independent.

(d) A and $A \cup B$ are not independent.

(Hint: Draw Venn-Diagrams for your intuition; however, points are only given for rigorous proofs.)

Question 3. **[3 pts]**. Let $A_1, \ldots, A_n$ be independent events. Show that

$$\mathbb{P}\left(A_1 \cup \ldots \cup A_n\right) = 1 - \prod_{k=1}^{n} \big(1 - \mathbb{P}(A_k)\big) \geq 1 - \exp\left(-\sum_{k=1}^{n} \mathbb{P}(A_k)\right).$$

(Hint: $e^{-x} \geq 1 - x$ for all $x \in \mathbb{R}$.)

Question 4. **[3 pts]** You and one of your study group members toss a fair coin n times. Show that the probability that both of you toss the same number of heads is $\binom{2n}{n} 2^{-2n}$. *(Note: There are several ways to derive this probability; in particular, there is one way which does not require any calculation at all!)*

Question 5. **[3 + 3 + 3 + 3 pts]** More urn problems.

(a) Consider an urn with 3 red and 6 white balls. Compute all probabilities of all possible draws of 3 balls without replacement.

(b) Consider an urn with 2 red, 2 black, and 2 white balls. What is the probability of drawing exactly 1 ball from each color when you draw 3 balls with replacement?

(c) Consider an urn with R red and W white balls. You draw balls from the urn without replacement. What is the probability that the first red ball is drawn at the kth draw for $k = 1, 2, \ldots$?

(d) Consider 2 urns with 5 balls each. The first urn contains 3 white and 2 red balls, the second urn contains 2 white and 3 red balls. You randomly choose the first urn with probability $1/4$ (the second with probability $3/4$) and draw 2 times without replacement. Compute the conditional probability that you have drawn balls from the first urn, given that you have drawn two red balls.