Problem Set 7 **STAT 394/ MATH 394**

Probability I

Due Date: 10:00 PM, May 23, 2022 Prof. Alexander Giessing

Last Update: May 17, Spring Quarter, 2022

Study Group: ____________ ____________ ____________

Question 1. **[3 + 3 + 3 + 3 pts]** The Gamma distribution with shape parameter α and scale parameter λ, denoted by $Gamma(\alpha, \lambda)$. Let $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1}e^{-x}dx$ be the Gamma function.

(a) Consider a density of the form

$$f_c(x) = \begin{cases} cx^{\alpha-1}e^{-x/\lambda}, & x \in (0, \infty) \\ 0, & o/w. \end{cases}$$

where $\alpha, \lambda > 0$ and $c \in \mathbb{R}$. Determine the value $c^* \in \mathbb{R}$ such that f_{c^*} is a pdf.

(b) Show that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for all $\alpha \in (0, \infty)$.

(c) Suppose $X \sim Gamma(\alpha, \lambda)$. Use part (b) to compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

(d) Let $Y \sim Exp(1)$. Use your results from parts (a) and (c) to find $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

Question 2. **[3 + 3 + 3 + 3 pts]** Let $c, \lambda \in \mathbb{R}$ and consider

$$f_c(x) = \begin{cases} cx^{-\lambda}, & x \in (1, \infty) \\ 0, & o/w. \end{cases}$$

(a) Determine $c^* \in \mathbb{R}$ such that f_{c^*} is a pdf for any $\lambda > 1$.

(b) Compute the cdf associated with f_{c^*}.

(c) Compute $\mathbb{P}(2 \leq X \leq 5)$ and $\mathbb{P}(X \geq 4)$ for a random variable X with pdf f_{c^*} and $\lambda = 2$.

(d) For which values of $\lambda > 1$ do expected value and variance of a random variable with pdf f_{c^*} exist? Compute the expected value and variance for these $\lambda > 1$.

Question 3. **[3 + 3 + 3 pts]** Let X and Y be two independent and identically distributed random variables taking values with pmf

$$p(k) = \begin{cases} 2^{-k}, & k \in \mathbb{N}, \\ 0, & o/w. \end{cases}$$

Compute the following probabilities:

(a) $\mathbb{P}(\min(X, Y) \leq n)$.

(b) $\mathbb{P}(X = Y)$.

(c) $\mathbb{P}(X > Y)$.

Question 4. **[3 pts]** Let X and Y be two independent random variables Poisson distributed random variables with parameters λ and μ, respectively. Show that $X + Y \sim Poisson(\mu + \lambda)$.

Question 5. **[3 pts]** Let X and Y be two independent random variables. Show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ provided that the expected values $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ exist.
(You may assume that X and Y are either both discrete or both continuous; however, the results holds more general.)

Question 6. **[3 pts]** Let $X_1, \ldots, X_n$ be independent and identically distributed random variables for which $\mathbb{E}[X_1^{-1}]$ exists. Let $m \leq n$. Show that $\mathbb{E}[S_m/S_n] = \frac{m}{n}$, where $S_m = X_1 + X_2 + \ldots + X_n$.
(Hint: Linearity of expectation and show that $\mathbb{E}[X_i/S_n] = \frac{1}{n}$ for all $i = 1, \ldots, n$.)