Problem Set 5 **STAT 394/ MATH 394**

Probability I

Due Date: 10:00 PM, May 9, 2022 Prof. Alexander Giessing

Last Update: May 3, Spring Quarter, 2022

Study Group: ______________________ ______________________ ______________________

Question 1. **[3 pts]** Let X_N, $N = 1, 2, \ldots$ be random variables with hypergeometric distribution with parameters $(N, \lfloor pN \rfloor, n)$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x. Show that

$$\lim_{N\to\infty} \mathbb{P}(X_N = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

i.e. the hypergeometric distribution converges to the binomial distribution (pointwise for any fixed k).

Question 2. **[3 pts]** Let X be a random variable with values in $\mathbb{N}$ and the "memory-less property"

$$\mathbb{P}(X > k + j \mid X > k) = \mathbb{P}(X > j) \quad \text{for all } j, k \in \mathbb{N}.$$

Show that X is geometrically distributed with some parameter $p \in (0,1)$. What is the parameter $p \in (0,1)$?

(As an aside: In the lecture we have shown that if a random variable is geometrically distributed then it satisfies the "memoryless property". Here, you show the converse.)

Question 3. **[3 + 3 + 3 pts]** A produce wholesaler applies the following simple rule to decide whether or not to accept a produce delivery: For every delivery of 100 items, the wholesaler draws a sub-sample of size 10. If one or more items are defective, the delivery is rejected.

(a) What is the probability that a delivery of 100 items of which 10% are defective is rejected by the wholesaler? Use the appropriate random variable to answer this question.

(b) If items are drawn one after another, occasionally a decision can be reached before the 10th draw. Compute the probability that a decision is made before or at the kth draw. Use the appropriate random variable to answer this question.

(c) Provide a formula to approximate the probability of part (a). What is the approximate probability?

(Hint: "drawing a sub-sample of size 10" means to draw 10 items at once, i.e. without replacement and without order.)

Question 4. **[3 + 3 pts]** Rolling an unbiased die.

(a) You roll a die 12 times and denote by X the number of sixes that you throw. What is the distribution of X? Compute $\mathbb{P}(X \leq 4)$.

(b) Let X be the number of the throw on which you roll a six for the first time. What is the distribution of X? Compute $\mathbb{P}(X > 12)$ and describe this event in plain English.

Question 5. **[0 pts] [Optional]** Suppose that the number of children in a household is Poisson distributed with parameter $\lambda > 0$. Each child has equal probability of being a boy or a girl. What is the distribution of the random variable which describes the number of girls in a household? Below questions will guide you to the answer.

(a) Let G be the number of girls and B the number of boys in a household. Show that

$$\mathbb{P}(G = k) = e^{-\lambda} \sum_{n=k}^{\infty} \mathbb{P}(G = k \mid G + B = n) \frac{\lambda^n}{n!} \quad \text{for all } k = 0, 1, 2 \ldots$$

(b) Argue (via an urn model) that

$$\mathbb{P}(G = k \mid G + B = n) = \binom{n}{k} \left(\frac{1}{2}\right)^n \quad \text{for all } k \in \{0, 1, \ldots, n\}.$$

(c) Combine parts (a) and (b) to conclude that $G \sim \mathit{Poisson}(\lambda/2)$.

(d) Show that if the probability of a child being a girl is $\alpha \in (0, 1)$, then $G \sim \mathit{Poisson}(\alpha\lambda)$ and $B \sim \mathit{Poisson}\big((1 - \alpha)\lambda\big)$.

(Comment: This phenomenon is also known as "thinning a Poisson process" and has applications in logistics, operations research, biology, genetics, etc.)