Problem Set 6 | **STAT 394/ MATH 394**
Probability I

Due Date: 10:00 PM, May 16, 2022 | Prof. Alexander Giessing

Last Update: May 10, Spring Quarter, 2022

Study Group: ______________________ ______________________ ______________________

Question 1. **[3 pts]** Let $X \sim HyperGeo(N, M, n)$. Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Question 2. **[3 pts]** Let X be a random variable with $\mathrm{Var}(X) = 0$. Show that there exists a constant $c \in \mathbb{R}$ such that $\mathbb{P}(X = c) = 1$.
(As an aside: In this case, we also say that X is almost surely constant.)

Question 3. **[3 + 3 pts]** Let X be a random variable with values in $\mathbb{N} \cup \{0\}$.

(a) Show that $\mathbb{P}(X = 0)\mathbb{E}[X]^2 \leq \mathrm{Var}(X)$.

(b) Show that $\mathbb{E}[X] = \sum_{k=0}^{\infty} \mathbb{P}(X > k)$.

(Note: The proofs of both statements are unrelated, i.e. part (b) does not require part (a) and vice versa.)

Question 4. **[0 pts]** **[Will be part of Problem Set 7.]** The Gamma distribution with shape parameter α and scale parameter λ, denoted by $Gamma(\alpha, \lambda)$. Let $\Gamma(\alpha) := \int_0^{\infty} x^{\alpha-1}e^{-x}dx$ be the Gamma function.

(a) Consider a density of the form

$$f(x) = \begin{cases} cx^{\alpha-1}e^{-x/\lambda}, & x \in (0, \infty) \\ 0, & o/w. \end{cases}$$

where $\alpha, \lambda > 0$ are two parameters and $c > 0$ a positive constant. Determine the value of the constant $c > 0$ for which $f(x)$ is a legitimate probability density function (pdf).

(b) Show that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for all $\alpha \in (0, \infty)$.

(c) Suppose $X \sim Gamma(\alpha, \lambda)$. Use part (b) to compute $\mathrm{E}[X]$ and $\mathrm{Var}(X)$.

(d) Let $Y \sim Exp(1)$. Use your results from parts (a) and (c) to find $\mathrm{E}[Y]$ and $\mathrm{Var}(Y)$.

Question 5. **[3 pts]** Consider the function

$$F(x) = \begin{cases} 1 - \frac{1}{2}e^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that F is a cumulative distribution function (cdf).

Question 6. **[3 + 3 + 3 pts]** Let $X \sim Unif(0, 1)$. Compute the probability density functions (pdfs) and cumulative distribution functions (cdfs) of

(a) X^2,

(b) $\sqrt{X}$,

(c) $-\log X$.

Question 7. [0 pts] [Optional] The upper bound in Chebychev's inequality can be larger than one and may thus be meaningless. Prove the following (one-sided) improvement of Chebychev's inequality: For $a > 0$ arbitrary,

$$\mathbb{P}(X - \mathbb{E}[X] \geq a) \leq \frac{\mathrm{Var}(X)}{\mathrm{Var}(X) + a^2}.$$