STAT 390 A
Statistical Methods in Engineering and Science

Week 4 Lectures – Part 2 – Spring 2023

Continuous Random Variables and Probability Distributions

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April 17, 2023

Outline

1. t-Distribution and heavy-tailed data

2. How does the computer generate random variables?

t-Distribution

t-Distribution

A continuous random variable has a t-distribution with parameter $\nu \geq 1$, if its probability density function f is given by

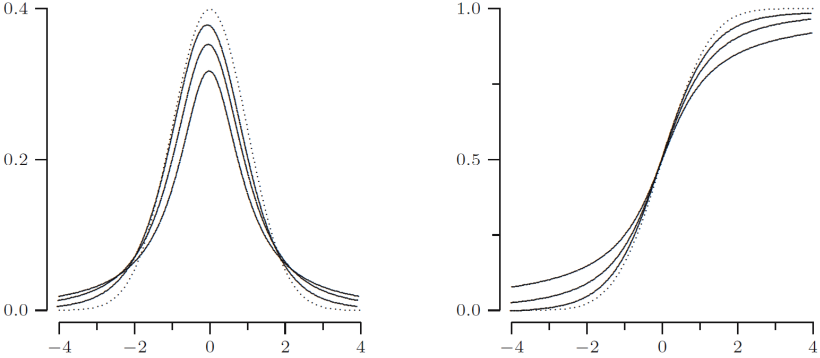
$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad \text{for } -\infty < x < \infty.$$

We denote this distribution by $t(\nu)$ and refer to it as the t-distribution with ν degrees of freedom.

- If $X \sim t(\nu)$, then $\mathrm{E}[X] = 0$ for $\nu > 1$ and
$$\mathrm{Var}(X) = \begin{cases} \frac{\nu}{\nu-2} & \text{if } \nu \geq 3 \\ \infty & \text{if } 1 < \nu \leq 2. \end{cases}$$
- If $X \sim t(1)$, then X is said to follow the Cauchy distribution and $\mathrm{E}[X]$ and $\mathrm{Var}(X)$ are not defined/ do not exist.

t-Distribution (Cont.)

(Plots of df and cdf of $t(1), t(2), t(5)$. Dotted lines df and cdf of $N(0,1)$.)



Facts about the t-Distribution

- The normalizing constant of the t-distribution is given in terms of the Gamma function (see Problem Set 5 for its definition).
- Densities of t-distributions look like that of the standard normal distribution: symmetric around 0 and bell-shaped.
- For large ν the $t(\nu)$ distribution can be well approxiamted by the standard normal distribution: $t(\nu) \to \phi(x)$ as $\nu \to \infty$.
- t-distributions have heavier tails than the standard normal distribution: $f(x) \to 0$ as $|x| \to \infty$, but more slowly than the density $\phi(x)$ of the standard normal distribution.

Example with R Code: SP500 index and IBM stock

- First, read data

```
> IBM <- read.csv("IBM.csv", header=T)      # read data
> SP500 <-  read.csv("SP500.csv", header=T)
```

- Get adjusted closing prices and convert them into log-returns.

```
> pSP500 <- SP500[,7] #take adj close price column
> pSP500 <- rev(pSP500) #reverse the time order
> rSP500 <- diff(log(pSP500))*100 #percentage of returns
> pIBM <- rev(IBM[,7]) #Closing prices of IBM
> rIBM <- diff(log(pIBM))*100 #percentage of log-returns
```

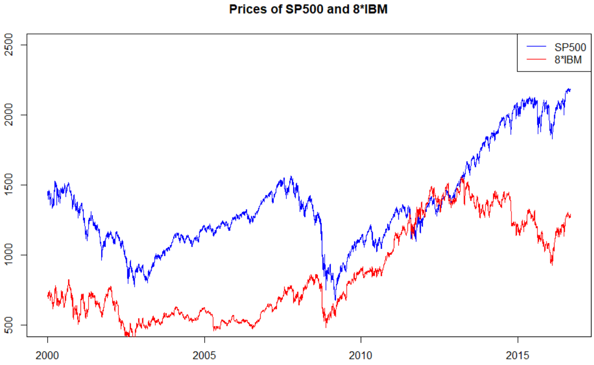
- Re-order data so that oldest observation is on top

```
> Dates <- as.vector(SP500[,1]) # dates of Data
> Dates <- strptime(Dates, "%m/%d/%Y") # convert to POSIXlt
# (a date class)
> Dates <- rev(Dates) # time from past to future
```

Example with R Code: SP500 index and IBM stock

- Plot time series of adj. closing prices of SP500 index (and IBM stock).

```
> plot(Dates, pSP500, ylim=c(500, 2500), col=4, type="l") # creat plot
> lines(Dates, 8*pIBM, col=2) # add lines
> title("Prices of SP500 and 8*IBM") # add title
> legend(x = "topright", legend = c("SP500", "8*IBM"),
+ lty = c(1, 1), col = c("blue", "red"), lwd = 1) # add legend
```



Example with R Code: SP500 index and IBM stock

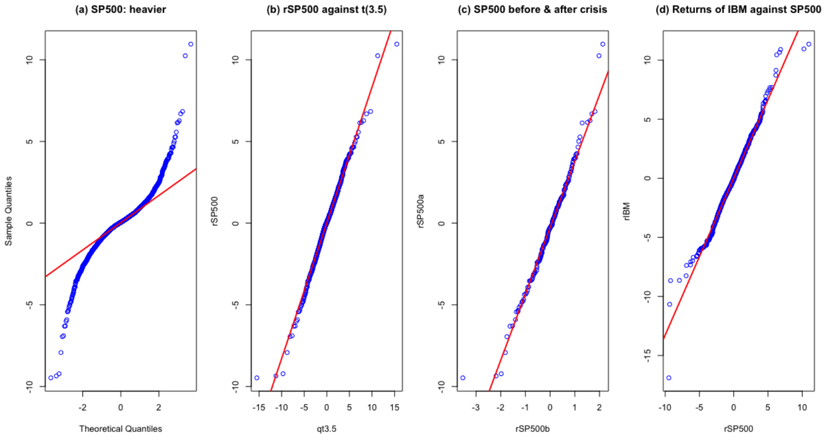
a. Are the log returns of the SP500 approximately normally distributed?

b. Which distribution provides a better fit than the normal distribution?

c. Did the distribution of log returns of the SP500 change before/ after the 2008 financial crises?

d. Are the distributions of log returns of the SP500 and the IBM stock approximately the same?

Example with R Code: SP500 index and IBM stock



Example with R Code: SP500 index and IBM stock

a. Are the log returns of the SP500 approximately normally distributed?
   - No, the log returns of the SP500 have heavier tails than the normal distribution.

b. Which distribution provides a better fit than the normal distribution?
   - The t-distr with 3.5 degrees of freem appear to be a better fit (vindicating the heavy-tailedness of the log returns).

c. Did the distribution of log returns of the SP500 change before/ after the 2008 financial crises?
   - Not much ...

d. Are the distributions of log returns of the SP500 and the IBM stock approximately the same?
   - Not really, but the difference is not too pronounced. You may want to consult a boxplot, too.

Example with R Code: SP500 index and IBM stock

```
> ## create plotting region for 4 side-by-side plots
> par(mfrow = c(1,4), mar=c(4,4,4,1)+0.1, cex=0.8)
>
> ## check normality of the data
> qqnorm(rSP500, col="blue", main="")
> qqline(rSP500, lwd=2, col="red") #Add a line
> title("(a) SP500: heavier") #Add a title
>
> ## check fit of t-distr with 3.5 degrees of freedom
> x <- (1:length(rSP500)-0.5)/length(rSP500) #compute percentiles
> qt3.5 <- qt(x,3.5) #compute theoretical quantiles of t(3.5)
> res <- qqplot(qt3.5, rSP500, col="blue", main="") #draw the QQplot
> x <- res$x #extract x component of the results
> y <- res$y #extract y component
> abline(lsfit(x,y), col="red", lwd=2)  # draw a regression line
> title("(b) rSP500 against t(3.5)") #add title
```

Example with R Code: SP500 index and IBM stock

```
> ## compare distr of two data sets
>
> rSP500a <- rSP500[2136:2387] # returns from 7/1/08 -- 6/30/09
> # (after financial crisis)
> rSP500b <- rSP500[1509:1904] # returns from 01/03/06 -- 07/31/07
> # (before financial crisis)
>
> ## rSP500 before versus after finanacial crisis
> res <- qqplot(rSP500b, rSP500a, col="blue", main="")
> x <- res$x
> y <- res$y
> abline(lsfit(x,y), col="red", lwd=2) # draw a regression line
> title("(c) SP500 before & after crisis")
>
> ## rSP500 versus rIBM
> res <- qqplot(rSP500, rIBM, col="blue", main="")
> x <- res$x
> y <- res$y
> abline(lsfit(x,y), col="red", lwd=2)
> title("(d) Returns of IBM against SP500")
```

example financial market Geometric brownian motion.

Outline

Tossing coins and rolling dice without coins and dice

1. How would you simulate a coin toss when instead of a coin you only had a die?
   - Idea: Define two events $H, T \subseteq \Omega = \{1, 2, 3, 4, 5, 6\}$ that both have probability $1/2$ and are mutually exclusive. For example, set $H = \{1, 2, 3\}$ and $T = \{4, 5, 6\}$.
2. Can you also simulate the roll of a die if you only have a single coin?
3. You are handed an unfair coin and you do not know $P(H) = p$ for this coin. Can you simulate a fair coin, and how many tosses do you need for each fair coin toss?

Tossing coins and rolling dice without coins and dice

2. Can you also simulate the roll of a die if you only have a single coin?
   - Idea: We need to define six events that each have probability 1/6 and are mutually exclusive.
   - A single coin toss does not have a sample space rich enough to encode 6 different outcomes. However, the sample space of three coin flips is large enough: $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
   - Define "1" as HHH, "2" as HHT, ..., "6" as THT. If you toss the sequences TTH or TTT, ignore the outcome and flip the coin another three times.
   - Let's check that the probabilities associated with these events are indeed 1/6 (and not 1/8). Ignoring the sequences TTH and TTT amounts to conditioning on tossing anything but TTH and TTT, i.e.

$$\begin{aligned} P(\text{"1"}) &= P(HHH \mid \text{any sequence but } TTH \text{ and } TTT) \\ &= P(HHH)/P(HHH \cup HHT \cup HTH \cup \ldots \cup THT) \\ &= \frac{1/8}{6/8} = 1/6. \end{aligned}$$

   Probabilities of "2", ..., "6" can be computed in the same way.

Tossing coins and rolling dice without coins and dice

3. You are handed an unfair coin and you do not know $P(H) = p$ for this coin. Can you simulate a fair coin, and how many tosses do you need for each fair coin toss?

   - We need to define two mutually exclusive events both with probability 1/2.
   - Consider flipping the unfair coin twice. Then $P(HT) = P(TH) = p(1-p)$. These events are mutually exclusive and have equal probabilities. However, $p(1-p) \neq 1/2$.
   - Idea: Toss a the unfair coin twice. Call it "heads" if HT and "tails" if TH. If you toss the sequences HH or TT, flip the coin two more times.
   - As above, ignoring the sequences HH and TT amounts to conditioning on tossing anything but HH and TT, i.e.

     $$\begin{aligned} P(\text{“heads”}) &= P(HT \mid HT \cup TH) \\ &= P(HT)/P(HT \cup TH) = p(1-p)/\big(2p(1-p)\big) \\ &= 1/2. \end{aligned}$$

     The probability for "tails" can be computed similarly.

Constructing a Bernoulli RV from a Uniform RV

Construct a Bernoulli random variable with parameter $0 < p < 1$ *using the uniform random variable* $U \sim Unif(0, 1)$.

Define

$$X = \begin{cases} 1 & \text{if } U < p, \\ 0 & \text{if } U \geq p. \end{cases}$$

Then,

$$P(X = 1) = P(U < p) = p \quad \text{and} \quad P(X = 0) = P(U \geq p) = 1 - p.$$

Thus, $X \sim Ber(p)$, i.e. X has Bernoulli distribution with parameter p.

A slightly more complicated example

Consider a discrete random variable Y with outcomes $1, 3, 4$ and associated probabilities $P(Y = 1) = 3/5$, $P(Y = 3) = 1/5$, and $P(Y = 4) = 1/5$. Describe how to construct Y from a $Unif(0,1)$ random variable.

Let $U \sim Unif(0,1)$ and define

$$Y = \begin{cases} 1 & \text{if } U < \frac{3}{5}, \\ 3 & \text{if } \frac{3}{5} \leq U < \frac{4}{5}, \\ 4 & \text{if } U \geq \frac{4}{5}. \end{cases}$$

Then, we compute

$$\begin{aligned} P(Y = 1) &= P\left(U < \frac{3}{5}\right) = \frac{3}{5}, \\ P(Y = 3) &= P\left(\frac{3}{5} \leq U < \frac{4}{5}\right) = \frac{1}{5}, \\ P(Y = 4) &= P\left(U \geq \frac{4}{5}\right) = 1 - P\left(U < \frac{4}{5}\right) = \frac{1}{5}. \end{aligned}$$

A general approach to simulating continuous RV

Simulating continuous RVs from Uniform RVs

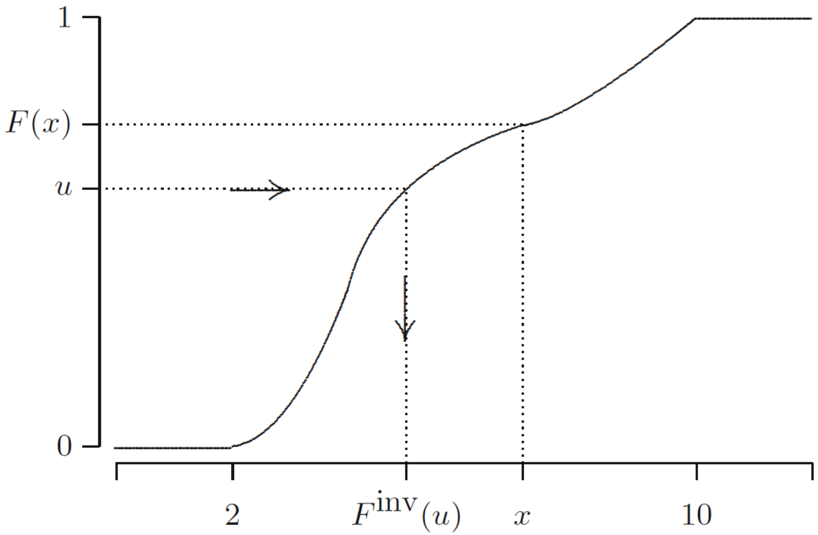
- Purpose: Generate RVs that have distribution F.
- Suppose F is strictly increasing from 0 to 1 and denote by F^{-1} (or F^{inv}) the inverse of F.
- Let $U \sim Unif(0,1)$. Then, for all $x \in \mathbb{R}$,

$$P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x),$$

i.e. the random variable $X := F^{-1}(U)$ has distribution F.

- Computers only need to be able generate uniform random variables on the interval $(0,1)$.
- This is still a difficult problem/ generally not possible/ equires so-called "pseudo random number generators" (not a topic of this class).
- Approach can be modified to handle distributions F that are not strictly increasing (e.g. discrete RVs).

A general approach to simulating continuous RV (Cont.)



Example: generating exponential RVs

On the interval $(0,\infty)$ *the* $Exp(\lambda)$ *distribution function is strictly increasing and given by* $F(x) = 1 - e^{\lambda x}$*. Use a uniform RV to generate RV with distribution function* F*.*

- We need to find the inverse function F^{-1}:

$$\begin{aligned} F(x) = u &\Leftrightarrow 1 - e^{-\lambda x} = u \\ &\Leftrightarrow e^{-\lambda x} = 1 - u \\ &\Leftrightarrow -\lambda x = \ln(1-u) \\ &\Leftrightarrow x = -\frac{1}{\lambda}\ln(1-u). \end{aligned}$$

- Thus, if $U \sim Unif(0,1)$, then

$$X := -\frac{1}{\lambda}\ln(1-U)$$

  has an $Exp(\lambda)$ distribution.

- Note: Since U and $1-U$ both have the $Unif(0,1)$ distribution,

$$Y := -\frac{1}{\lambda}\ln(U)$$

  has also an $Exp(\lambda)$ distribution.