Problem Set 4 | **STAT 394/ MATH 394**

Probability I

Due Date: 10:00 PM, April 27, 2022 | Prof. Alexander Giessing

Last Update: April 22, 2022 | Spring Quarter, 2022

Study Group: ______________________ ______________________ ______________________

Question 1. **[3 pts]** Consider 2 urns with 5 balls each. The first urn contains 3 white and 2 red balls, the second urn contains 2 white and 3 red balls. You randomly choose the first urn with probability 1/4 (the second with probability 3/4) and draw 2 times without replacement. Compute the conditional probability that you have drawn balls from the first urn, given that you have drawn two red balls.

Question 2. **[3 + 3 pts]** There are $N+1$ urns with N balls each. The ith urn contains $i-1$ red balls and $N+1-i$ white balls. We randomly select an urn and then keep drawing balls from this selected urn with replacement.

(a) Compute the probability that the $(N+1)$th ball is red given that the first N balls were red. Compute the limit as $N \to \infty$.

(b) What is the probability that the first ball is red? What is the probability that the second ball is red?

(Historical note: Pierre Laplace considered this toy model to study the probability that the sun will rise again tomorrow morning. Can you make the connection?)

Question 3. **[3 + 3 pts]** You throw a fair die n times. Denote by p_n the probability of throwing an even number of sixes in n throws.

(a) Prove the following difference equation

$$p_n = \frac{1}{6}(1 - p_{n-1}) + \frac{5}{6}p_{n-1}.$$

(b) Solve above difference equation to obtain an explicit formula for p_n.

Question 4. **[3 pts]** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $A \subseteq \Omega$ be an event with $\mathbb{P}(A) > 0$. Show that the conditional probability given A is a probability measure. *(Hint: Show that the function $Q : \mathcal{A} \mapsto [0,1]$ defined by $Q(B) = \mathbb{P}(B \mid A)$ for $B \in \mathcal{A}$ satisfies the definition of a probability measure.)*

Question 5. **[3 + 3 + 3 + 3 pts]** Roughly 28% of the smartphones (in the US) run on iOS, 70% on Android, and 2% on another operating system. Suppose that 45% of the smartphones running on iOS, 63% of those running on Android, and 25% of those running on another operating system are infected by a virus.

(a) Draw the (probability) tree diagram for this problem.

(b) What is the probability that a randomly selected smartphone is infected by a virus?

(c) Given that a randomly selected smartphone is infected by the virus, what is the probability that it runs on Android?

(d) Given that a randomly selected smartphone is not infected by the virus, what is the probability that it does not run on iOS?

Question 6. [3 + 3 + 3 pts] The Seattle Police Department reaches out to you to help them calibrate their breath analyzer used to test whether drivers exceed the legal limit set for blood alcohol content. Let A be the event "breath analyzer indicates that the driver's blood alcohol content exceeds the legal limit" and B the event "driver's blood alcohol content exceeds legal limit". The breath analyzer can be calibrated by choosing $0 < p < 1$ where $p = P(A \mid B)$. Due to technical constraints, $P(A \mid B) = P(A^c \mid B^c)$. On a typical Saturday night about 3% of the drivers are known to exceed the legal limit.

(a) Draw the (probability) tree diagram for this problem.

(b) Describe in words the meaning of $P(B^c \mid A)$ and determine $P(B^c \mid A)$ for $p = 0.9$.

(c) What value of p should the police use so that $P(B \mid A) = 0.8$?