Problem Set 8 **STAT 394/ MATH 394**

Probability I

Due Date: 10:00 PM, May 31, 2022 Prof. Alexander Giessing

Last Update: May 24, Spring Quarter, 2022

Study Group: ________________ ________________ ________________

Question 1. **[3 pts]** Let X and Y be random variables with density functions f and g, respectively, and ξ be a Bernoulli distributed random variable with success probability p, which is independent of X and Y. Compute the probability density function of $\xi X + (1-\xi)Y$.

Question 2. **[3 + 3 + 3 pts]** Let $(X_n)_{n\geq 1}$ be a sequence of independent Bernoulli random variables with success probability p. Denote by S_1 the number of failures until the first success, by S_2 the number of failures between the first and second success, and, in general, by S_k the number of failures between the $(k-1)$th and the kth success.

(a) Compute the joint probability mass function of $S_1, \ldots, S_n$.

(b) Are the random variables $S_1, \ldots, S_n$ independent? Prove or disprove.

(c) Compute the cdf of $U = \max\{S_1, \ldots, S_n\}$.

Question 3. **[3 pts]** Let X and Y have the joint probability density function $f(x,y) = e^{-x-y}\mathbf{1}_{(0,\infty)}(x)\mathbf{1}_{(0,\infty)}(y)$. Compute the density of $Z := Y - X$.

Question 4. **[3 + 3 + 3 pts]** Let (X,Y) be the coordinates of points distributed uniformly over $B = \{(x,y) : x, y > 0, x^2 + y^2 \leq 1\}$.

(a) Compute the densities of X and Y.

(b) Compute the expected value of the area of the rectangle with corners $(0,0)$ and (X,Y).

(c) Compute the covariance between X and Y.

Question 5. **[3 + 3 pts]** More on the uniform distribution.

(a) Let (X,Y) be uniformly distributed over $(0,1)^2$. Compute $\mathbb{E}[\sqrt{Y/X}]$.

(b) Let $U_1, \ldots, U_n$ be independent and uniformly distributed over $(0,1)$. Compute $\mathbb{P}(U_1 \leq \ldots \leq U_n)$.

Question 6. **[3 + 3 pts]** On covariances.

(a) You roll a die two times. Denote by X the outcome of the first roll and by Y the sum of both outcomes. Compute $\mathrm{Cov}(X,Y)$.

(b) Let X be a discrete random variable with symmetric distribution, i.e. $\mathbb{P}(X = x) = \mathbb{P}(X = -x)$ for all $x \in X(\Omega)$. Show that X and $Y := X^2$ are uncorrelated but not independent.