1 Derivate parziali e differenziabilità di funzioni $f : \mathbb{R}^n \to \mathbb{R}$

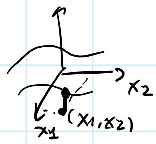
$$f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$f(x_1, \dots, x_n) \in \mathbb{R}$$

- $n = 1$, $f : \mathbb{R} \to \mathbb{R}$

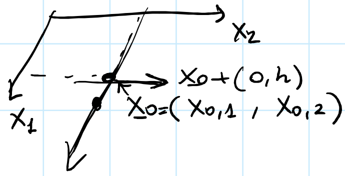
<u>la derivata in $x_0 \in A$</u> è definita come:

$$\frac{df}{dx}(x_0) = f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(se esiste)}$$

- $n = 2$, $f : \underset{(x_1, x_2)}{\mathbb{R}^2} \longrightarrow \mathbb{R}$



> $$\frac{\partial f}{\partial x_1}(\underline{x}_0) = \lim_{h \to 0} \frac{f(x_{0,1} + h, x_{0,2}) - f(x_{0,1}, x_{0,2})}{h}$$
> $$\underline{x}_0 = (x_{0,1}, x_{0,2})$$



> $$\frac{\partial f}{\partial x_2}(\underline{x}_0) = \lim_{h \to 0} \frac{f(x_{0,1}, x_{0,2} + h) - f(x_{0,1}, x_{0,2})}{h}$$

- n generico, $f : \underset{(x_1, \dots, x_n)}{A \subseteq \mathbb{R}^n} \longrightarrow \mathbb{R}$ $\qquad \underline{x}_0 = (x_{0,1}, \dots, x_{0,n})$

> $$\frac{\partial f}{\partial x_1}(\underline{x}_0), \quad \dots \dots, \quad \frac{\partial f}{\partial x_n}(\underline{x}_0)$$

$$\rightarrow \quad \frac{\partial f}{\partial x_i}(\underline{x_0}) := \lim_{h \to 0} \frac{f(\underline{x_0} + h\underline{e_i}) - f(\underline{x_0})}{h}$$

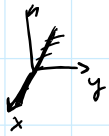
dove $\underline{e_i} = (0, 0, \dots, \underset{\substack{\uparrow \\ i\text{-esima posizione}}}{1}, 0 \dots 0)$

$\boxed{\text{Def}}$

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ si dice <u>DERIVABILE in $x_0 \in A$</u> se in $\underline{x_0}$ esistono tutte le derivate parziali.

Si dice che <u>f è DERIVABILE in A</u> se è derivabile in ogni $\underline{x_0} \in A$.

<u>Esempio 1</u> $\quad n = 2 \quad f : \mathbb{R}^2 \to \mathbb{R}$



$$f(x,y) = x^2 + \frac{5}{y}, \quad (x_0, y_0) = (1, \sqrt{5})$$

$> \frac{\partial f}{\partial x}(x,y) = 2x \quad \rightarrow \quad \frac{\partial f}{\partial x}(x_0,y_0) = 2 \cdot 1 = 2$

$> \frac{\partial f}{\partial y}(x,y) = -\frac{5}{y^2} \quad \rightarrow \quad \frac{\partial f}{\partial y}(x_0,y_0) = -\frac{5}{y_0^2} = -1$

<u>Esempio 2</u> $\quad f(x,y) = \frac{x-y}{x+y}$.

$> \frac{\partial f}{\partial x}(x,y) = \frac{1 \cdot (x+y) - (x-y) \cdot 1}{(x+y)^2} = \frac{\cancel{x} + y - \cancel{x} + y}{(x+y)^2} = \frac{2y}{(x+y)^2}$

$> \frac{\partial f}{\partial y}(x,y) = \frac{(-1) \cdot (x+y) - (x-y) \cdot 1}{(x+y)^2} = \frac{-x - \cancel{y} - x + \cancel{y}}{(x+y)^2} = -\frac{2x}{(x+y)^2}$

$\hookrightarrow \nabla f(x,y) = \left(\frac{2y}{(x+y)^2}, -\frac{2x}{(x+y)^2} \right)$

<u>Esempio 3</u>

$f(x,y) = y \underset{\substack{\shortparallel \\ x^{\frac{1}{3}}}}{\sqrt[3]{x}}$. Calcolare la derivata parziale di f rispetto a x in $(0,0) = \underline{x_0}$.

$> \frac{\partial f}{\partial x}(x,y) = y \frac{1}{3} \frac{1}{x^{2/3}} = \frac{y}{3x^{2/3}} \leftarrow$ <u>$x = 0$ non è definita</u>

$$> \frac{\partial f}{\partial x}(x,y) = y\,\frac{1}{3}\,\frac{1}{x^{\frac{2}{3}}} = \frac{y}{3x^{\frac{2}{3}}} \leftarrow \quad \underline{x=0 \text{ non è definita}}$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\boxed{\text{Def}}$ Se $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ è <u>derivabile in $\underline{x}_0 \in A$</u>, si chiama GRADIENTE di f in $\underline{x}_0$ il vettore:

$$\nabla f(\underline{x}_0) = \left(\frac{\partial f}{\partial x_1}(\underline{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\underline{x}_0) \right)$$

$$\nabla f : \overset{A \subseteq}{\mathbb{R}^n} \to \mathbb{R}^n$$

<u>Esempio 4</u>

$$f: \mathbb{R}^n \to \mathbb{R}, \qquad f(\underline{x}) = e^{|\underline{x}|} \in \mathbb{R}$$

$$f(\underline{x}) \in \mathbb{R} \qquad \text{dove } |\underline{x}| = \sqrt{x_1^2 + \ldots + x_n^2}$$

Calcolo le der. parziali in $\underline{x}$.

- $$\frac{\partial f}{\partial x_i}(\underline{x}) = \frac{\partial}{\partial x_i}\left(e^{\sqrt{x_1^2 + \cancel{x_i^2} + x_n^2}}\right)(\underline{x}) = \frac{1}{\cancel{2}} (x_1^2 + \ldots + x_n^2)^{-\frac{1}{2}} \cdot \cancel{2}x_i \cdot e^{|\underline{x}|} =$$

$$= \frac{x_i}{|\underline{x}|} \cdot e^{|\underline{x}|}, \quad \text{per } |\underline{x}| = 0 \text{ non ha senso!}$$

$$\Updownarrow$$

$$\underline{x} = \underline{0} = (0, \ldots, 0)$$

- $\underline{x} = \underline{0}$,

$$\frac{\partial f}{\partial x_i}(\underline{0}) = \lim_{h\to 0} \frac{f(h\underline{e}_i) - f(\underline{0})}{h} = \lim_{h\to 0} \frac{e^{|h\underline{e}_i|} - e^{|\underline{0}|}}{h}$$

$$\underline{e}_i = (0, \ldots, 0, 1, 0, \ldots 0) \qquad = \lim_{h\to 0} \frac{e^{|h|} - 1}{h} \quad \nexists$$

$\underline{e}_i = (0, \dots, 0, 1, 0, \dots 0)$ ← i-esima posizion

$$= \lim_{h \to 0} \frac{e^{|h|} - 1}{h} \quad \nexists$$

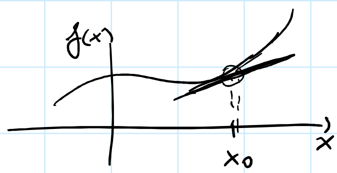
$$\lim_{h \to 0^+} \frac{e^{|h|} - 1}{h} = 1$$

$$\lim_{h \to 0^-} \frac{e^{|h|} - 1}{h} = -1$$

$\Rightarrow$ f non è derivabile in $\underline{0}$.

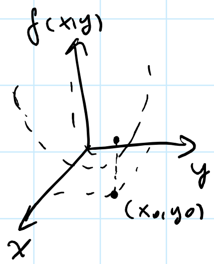
Problema

- $n = 1$, se f è derivabile in $x_0 \in A$ $\Longrightarrow$ / $\Longleftarrow$ f è continua in x_0 ed esiste la retta tangente a f in x_0



- $n > 1$,

f è derivabile in $x_0 \in A$ $\not\Longleftarrow$ f è continua in $\underline{x}_0$ ed esiste il piano tangente in $\underline{x}_0$



$\boxed{\text{Def}}$ $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, $\underline{x}_0 \in A$.

f è DIFFERENZIABILE in $\underline{x}_0$ se esiste un vettore $\underline{a} \in \mathbb{R}^n$ tale che

$$f(\underline{x}_0 + \underline{h}) - f(\underline{x}_0) = \underline{a} \cdot \underline{h} + o(|\underline{h}|) \quad \text{per } \underline{h} \to \underline{0}$$

i.e.

$$\lim f(\underline{x}_0 + \underline{h}) - f(\underline{x}_0) - \underline{a} \cdot \underline{h} = 0$$

i.e.

$$\lim_{\underline{h} \to \underline{0}} \frac{f(\underline{x_0}+\underline{h}) - f(\underline{x_0}) - \underline{a}\cdot\underline{h}}{|\underline{h}|} = 0$$

Prop 0

se f è DIFFERENZ in $\underline{x_0} \in A$ $\iff$ f è continua in $\underline{x_0}$ ed esiste il piano tangente in $\underline{x_0}$

Come sono collegate la derivab e differenz?

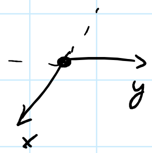
Prop 1 f DIFFERENZ in $\underline{x_0}$ $\implies$ f DERIVAB. in $\underline{x_0}$

e $\underset{\in \mathbb{R}^n}{\underline{a}} = \underset{\in \mathbb{R}^n}{\nabla f(\underline{x_0})}$

Coroll 1 se f non è derivabile in $\underline{x_0}$ $\implies$ f non è differenz in $\underline{x_0}$

Esempio 5

$$f(x,y) = \begin{cases} \dfrac{x^2 y^3}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$



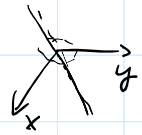
f è differenziab. in $(0,0)$?

- Calcola le derivate parziali in $(0,0)$:

> $$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

> $$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\Rightarrow \nabla f(0,0) = (0,0)$$



$\approx$

f è differenz in $(0,0)$ $\Leftrightarrow \lim_{\substack{h\to 0\\ k\to 0}} \dfrac{f(h,k) - f(0,0) - 0}{\sqrt{h^2+k^2}} = 0$

$\underline{h} = (h,k)$

$$\Leftrightarrow \lim_{\substack{h\to 0\\ k\to 0}} \frac{h^2k^3}{h^4+k^4} \cdot \frac{1}{\sqrt{h^2+k^2}} = 0$$

scelgo $h = k$ $\quad \underline{h} = (h,h)$

$$\lim_{h\to 0} \frac{h^2h^3}{\underbrace{h^4+h^4}_{2h^4}} \cdot \frac{1}{\underbrace{\sqrt{h^2+h^2}}_{\sqrt{2}|h|}} = \lim_{h\to 0} \frac{h^5}{2h^4\sqrt{2}\,|h|} =$$

$$= \lim_{h\to 0} \frac{1}{2\sqrt{2}} \cdot \frac{h}{|h|}$$

$\lim_{h\to 0^+} \dots = \dfrac{1}{2\sqrt{2}}$

$\lim_{h\to 0^-} \dots = -\dfrac{1}{2\sqrt{2}}$

$\nexists$

$\Rightarrow$ f NON è differenz in $(0,0)$

$\approx$

Prop 2

- Se le derivate parziali esistono in un intorno di $\underline{x}_0$ e sono <u>continue</u> in $\underline{x}_0$

$\Rightarrow$ f è differenziabile in $\underline{x}_0$

- Se le derivate parziali esistono e sono continue in A, $(f \in C^1(A))$
  $\Rightarrow$ f è differenziabile in A

$\sim$

<u>Esempio 6</u>

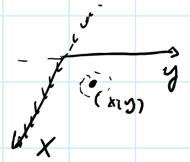
$f(x,y) = x\sqrt[3]{y}$ $\quad$ definita su $\mathbb{R}^2$

In quali punti è differenziabile?

Calcoliamo

> $\frac{\partial f}{\partial x}(x,y) = y^{\frac{1}{3}}$

> $\frac{\partial f}{\partial y}(x,y) = \frac{x}{3y^{\frac{2}{3}}}$



1° caso (x,y) con $y \neq 0$

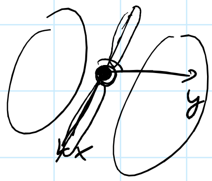
$\Rightarrow f$ è differenziabile in (x,y), $y \neq 0$

2° caso $(x,0)$

> $\frac{\partial f}{\partial y}(x,0) = \lim_{h \to 0} \frac{f(x,h) - f(x,0)}{h} =$

$= \lim_{h \to 0} \frac{x h^{\frac{1}{3}} - 0}{h} = \lim_{h \to 0} x \cdot \frac{1}{h^{\frac{2}{3}}}$

• se $x \neq 0 \Rightarrow$ il limite non esiste (infinito) $\Rightarrow f$ non è DIFFEREN in $(x,0)$, con $x \neq 0$



• $\frac{\partial f}{\partial y}(0,0) = 0$

$\Rightarrow (0,0)$ non si sa.

$\boxed{\text{Derivate di ordine superiore}}$ $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$

• $n = 2$, $f(x,y)$: $\frac{\partial f}{\partial x}(x,y)$, $\frac{\partial f}{\partial y}(x,y)$ sono funzioni $A \to \mathbb{R}$

• $n = 2$, $f(x,y)$: $\frac{\partial f}{\partial x}(x,y)$, $\frac{\partial f}{\partial y}(x,y)$ sono funzioni $A \to \mathbb{R}$

$$\longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y)$$

• $n > 1$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(\underline{x})$$

Prop

$$f \in C^2(A) \implies \frac{\partial^2 f}{\partial x_i \partial x_j} \equiv \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Def Matrice Hessiana di f in $\underline{x}_0$

$$H_f(\underline{x}_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\underline{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\underline{x}_0) \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\underline{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\underline{x}_0) \end{bmatrix}$$

(matrice $n \times n$)

≈

Esempio 7 $f(x,y) = x^3y^2 - 3xy^4$.
Calcolare $H_f(x,y)$.

$$\left(\begin{array}{l} \nabla f : \mathbb{R}^n \to \mathbb{R}^n \\ H_f : \mathbb{R}^n \to \mathbb{R}^{n \times n} \end{array} \right)$$

• $\frac{\partial f}{\partial x}(x,y) = 3x^2y^2 - 3y^4$, $\frac{\partial f}{\partial y}(x,y) = 2x^3y - 12xy^3$

• $\frac{\partial^2 f}{\partial x^2}(x,y) = \underline{\underline{6xy^2}}$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 2x^3 - 36xy^2$

$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 6 \cdot x^2 y - 12y^3$, $\frac{\partial^2 f}{\partial y \partial x}(x,y) = 6x^2 y - 12y^3$

$$\Rightarrow \quad H_f(x,y) = \begin{bmatrix} 6xy^2 & 6x^2y - 12y^3 \\ \dots & 2x^3 - 36xy^2 \end{bmatrix}$$

<u>Esempio 5</u> $f(x,y) = \frac{xy}{x+y}$. Calcolare Hessiana.

<u>Res:</u>

$$H_f(x,y) = \begin{bmatrix} -\frac{2y^2}{(x+y)^3} & \frac{2xy}{(x+y)^3} \\ \frac{2xy}{(x+y)^3} & -\frac{2x^2}{(x+y)^3} \end{bmatrix}$$