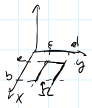
⑤ Integrali funzioni in due variabili $f: \mathbb{R}^2 \to \mathbb{R}$

$$\iint_\Omega f(x,y)\,dxdy$$

Teo riduzione per rettangoli



$f: [a,b]\times[c,d] \to \mathbb{R}$ continua, allora:

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dxdy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

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(a) $\varphi(x) = \int_c^d f(x,y)\,dy \in \mathbb{R}$ | (a) $\tilde{\varphi}(y) = \int_a^b f(x,y)\,dx$

(b) $\int_a^b \varphi(x)\,dx \in \mathbb{R}$ | (b) $\int_c^d \tilde{\varphi}(y)\,dy$

Proprietà $\Omega \subset \mathbb{R}^2$ limitato e misurabile (suff. regolare)

① Linearità

$$\iint_\Omega (f(x,y) + g(x,y))\,dxdy = \iint_\Omega f(x,y)\,dxdy + \iint_\Omega g(x,y)\,dxdy$$

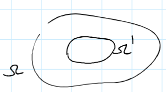
$$\iint_\Omega c\cdot f(x,y)\,dxdy = c\cdot \iint_\Omega f(x,y)\,dxdy$$

② Positività e monotonia rispetto all'integranda

- $f(x,y) \geq 0$ in Ω $\Rightarrow \iint_\Omega f(x,y)\,dxdy \geq 0$
- $f(x,y) \geq g(x,y)$ in Ω $\Rightarrow \iint_\Omega f(x,y)\,dxdy \geq \iint_\Omega g(x,y)\,dxdy$

③ Monotonia rispetto al dominio di integrazione

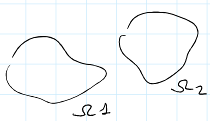
$\Omega' \subseteq \Omega$, $f \geq 0$ in Ω



$$\Rightarrow \iint_{\Omega'} f(x,y)\,dxdy \leq \iint_\Omega f(x,y)\,dxdy$$

④ Additività dell'integrale rispetto al dominio di integrazione

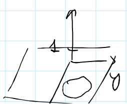
Ω_1, Ω_2, $\Omega_1 \cap \Omega_2 = \emptyset$, f integrabile su $\Omega_1 \cup \Omega_2$

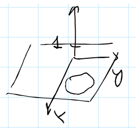


$$\Rightarrow \iint_{\Omega_1 \cup \Omega_2} f(x,y)\,dxdy = \iint_{\Omega_1} f(x,y)\,dxdy + \iint_{\Omega_2} f(x,y)\,dxdy$$

⑤ $|\Omega| = 0$ $\Rightarrow \iint_\Omega f(x,y)\,dxdy = 0$

$$|\Omega| = \iint_\Omega 1\cdot dxdy$$





Eserc ① $f(x,y) = x^3 + y^3 + xy$. Trovare punti estremi liberi.

(Step 1)

$\nabla f(x,y) = (0,0)$

- $\frac{\partial f}{\partial x}(x,y) = 3x^2 + y$, $\frac{\partial f}{\partial y}(x,y) = 3y^2 + x$

$$\begin{cases} 3x^2 + y = 0 & \longrightarrow y = -3x^2 \\ 3y^2 + x = 0 & \longrightarrow 3 \cdot 9x^4 + x = 0 \Rightarrow 27x^4 + x = 0 \end{cases}$$

$$x(27x^3 + 1) = 0$$

$x = 0$, $27x^3 + 1 = 0$

$x^3 = -\frac{1}{27}$, $x = -\frac{1}{3}$

- $x = 0$, $y = 0$: $P_1 = (0,0)$
- $x = -\frac{1}{3}$, $y = -\frac{1}{3}$: $P_2 = \left(-\frac{1}{3}, -\frac{1}{3}\right)$

Step 2

- $\frac{\partial^2 f}{\partial x^2}(x,y) = 6x$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 6y$, $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 1$

$$\Rightarrow H_f(x,y) = \begin{bmatrix} 6x & 1 \\ 1 & 6y \end{bmatrix}$$

$\boxed{P_1}$ $H_f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\det H_f(0,0) = 0 - 1 = -1 < 0 \Rightarrow P_1$ sella

$\boxed{P_2}$ $H_f\left(-\frac{1}{3}, -\frac{1}{3}\right) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ $\det H_f\left(-\frac{1}{3}\ \frac{1}{3}\right) = 4 - 1 = 3 > 0$

$\frac{\partial^2 f}{\partial x^2}\left(-\frac{1}{3},\ \right) = -2 < 0 \Rightarrow P_2$ MASSIMO (locale)

ES ② $f(x,y) = x + e^y$. Calcolare le ∇ con la definizione in $(1,0)$

- $\frac{\partial f}{\partial x}(1,0) = \lim_{h\to 0} \frac{f(1+h,0)-f(1,0)}{h} = \lim_{h\to 0} \frac{\cancel{1}+h\cancel{+1}-(\cancel{1}\cancel{+1})}{h} = 1$

- $\frac{\partial f}{\partial y}(1,0) = \lim_{h\to 0} \frac{f(1,h)-f(1,0)}{h} = \lim_{h\to 0} \frac{1+e^h-2}{h} =$

$$= \lim_{h\to 0} \frac{e^h-1}{h} = 1$$

$$\Rightarrow \nabla f(1,0) = (1,1)$$

Es 3 $\quad f:\mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = e^x \cdot y \cdot \cos z$

$g:\mathbb{R}\to\mathbb{R}^3, \quad g(u) = \begin{pmatrix} u \\ u^2 \\ u^3 \end{pmatrix} \begin{matrix} \leftarrow g_1(u) \\ \leftarrow g_2(u) \\ \leftarrow g_3(u) \end{matrix}$

$h = f \circ g \quad$ i.e. $\quad h(u) = f(g(u))$

$$\begin{tikzcd} \underset{u}{\mathbb{R}} \arrow[r, "g"'] \arrow[rr, bend left, "h"] & \underset{g(u)}{\mathbb{R}^3} \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

$h:\mathbb{R}\to\mathbb{R}$

Calcolare $h'(u) = \frac{d}{du} h(u)$

1° modo

$$h(u) = f(g(u)) = f\left(\begin{pmatrix} u \\ u^2 \\ u^3 \end{pmatrix}\right) = (e^u \cdot u^2) \cdot \cos(u^3)$$

$$\Rightarrow h'(u) = (e^u u^2 + e^u \cdot 2u) \cdot \cos u^3 + (e^u u^2) \cdot (-\sin(u^3)) \, 3u^2 =$$

$$= e^u \cdot u\,(u+2)\,\cos(u^3) - 3\,e^u u^4 \sin(u^3)$$

2° modo

$$h'(u) = D_f(g(u)) \cdot D_g(u)$$

$$= \nabla f(g(u)) \cdot D_g(u) \quad \underline{\circledast}$$

- $\nabla f(x,y,z) = \left(e^x y \cos z \,,\; e^x \cos z \,,\; -e^x y \sin z\right)$

- $D_g(u) = \begin{pmatrix} \frac{\partial g_1}{\partial u}(u) \\ \frac{\partial g_2}{\partial u}(u) \\ \frac{\partial g_3}{\partial u}(u) \end{pmatrix} = \begin{pmatrix} 1 \\ 2u \\ 3u^2 \end{pmatrix},$

$$(*) = \nabla f(u, u^2, u^3) \cdot D_g(u)$$

$$= \left(e^u \cdot u^2 \cos u^3,\ e^u \cos u^3,\ -e^u \cdot u^2 \cdot \sin u^3\right) \cdot \begin{pmatrix} 1 \\ 2u \\ 3u^2 \end{pmatrix}$$

$$= e^u \cdot u^2 \cos u^3 + 2u e^u \cos u^3 - 3u^4 e^u \sin u^3$$

$$= e^u u(u+2) \cos u^3 - 3u^4 e^u \sin u^3$$

Es (4) (estremi vincolati)

$$f(x,y,z) = x + y + z$$

vincolo $G = \{ (x,y,z) \in \mathbb{R}^3 : \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1 \}$

$$\hookrightarrow \underbrace{\frac{x^2}{4} + y^2 + \frac{z^2}{9} - 1}_{g(x,y,z)} = 0$$

(Step 1)

$$\nabla g(x,y,z) = \left(\frac{1}{2}x,\ 2y,\ \frac{2}{9}z\right) \stackrel{!}{=} (0,0,0)$$

$$\Rightarrow x=0,\ y=0,\ z=0 \notin G$$

(Step 2) $\mathcal{L}(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z)$

Bisogna risolvere $\nabla \mathcal{L}(x,y,z,\lambda) = \underline{0} = (0,0,0,0)$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) - \lambda \frac{\partial g}{\partial x}(x,y,z) = 0 \\ \frac{\partial f}{\partial y}(x,y,z) - \lambda \frac{\partial g}{\partial y}(x,y,z) = 0 \\ \frac{\partial f}{\partial z}(x,y,z) - \lambda \frac{\partial g}{\partial z}(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}$$

$$\begin{cases} 1 - \lambda \frac{x}{2} = 0 & \longrightarrow \lambda = \frac{2}{x} \quad (x \neq 0) \\ 1 - \lambda 2y = 0 & \longrightarrow 1 = 2\lambda y \\ & 1 = 2 \cdot \frac{2}{x} \cdot y \longrightarrow y = \frac{x}{4} \\ 1 - \lambda \frac{2}{9} z = 0 & \rightsquigarrow 1 = \frac{2}{9} \cdot \lambda \cdot z \\ & 1 = \frac{2}{9} \cdot \frac{2}{x} \cdot z \longrightarrow z = \frac{9}{4}x \\ \underbrace{\frac{x^2}{4} + y^2 + \frac{z^2}{9} - 1}_{g} = 0 \end{cases}$$

$$\frac{x^2}{4} + \frac{x^2}{16} + \frac{9^{\cancel{2}}}{4^2} x^2 \frac{1}{\cancel{9}} - 1 = 0$$

$$\frac{4+1+9}{16}x^2 = 1$$

$$\longrightarrow \frac{14}{16}x^2 = 1 \longrightarrow x^2 = \frac{8}{7} \longrightarrow x = \pm\sqrt{\frac{8}{7}} = \pm\frac{2\sqrt{2}}{\sqrt{7}}$$

- $x = \frac{2\sqrt{2}}{\sqrt{7}}$, $y = \frac{\sqrt{2}}{2\sqrt{7}}$, $z = \frac{3}{4} \cdot \frac{2\sqrt{2}}{\sqrt{7}} = \frac{3}{\sqrt{14}}$

$$\Rightarrow P_1 = \left(\frac{2\sqrt{2}}{\sqrt{7}}, \frac{\sqrt{2}}{2\sqrt{7}}, \frac{3}{\sqrt{14}}\right)$$

- $x = -\frac{2\sqrt{2}}{\sqrt{7}}$, - - -