$$\iint_\Omega f(x,y)\,dx\,dy$$

$\boxed{\text{Def}}$ Insieme $\Omega \subset \mathbb{R}^2$ si dice <u>y-semplice</u> se

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \ x \in [a,b], \ g_1(x) \le y \le g_2(x)\}$$

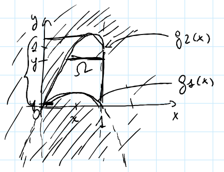
Insieme $\Omega \subset \mathbb{R}^2$ si dice <u>x-semplice</u> se

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \ y \in [c,d], \ h_1(y) \le x \le h_2(y)\}$$

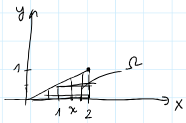
In generale, Ω è dominio <u>semplice</u> se è x-semplice o y-semplice.

Esempio

1) $\Omega = \{(x,y) \in \mathbb{R}^2 : \ \underset{x \in [0,1]}{0 \le x \le 1}, \ \underbrace{x(1-x)}_{g_1(x)} \le y \le \underbrace{x(2-x^2)}_{g_2(x)}\}$ <u>y-semplice</u> <u>non x-semplice</u>



2) <u>Sia x-semplice, sia y-semplice</u>



<u>Teo di riduzione per domini semplici</u>

(a) $f : \Omega \to \mathbb{R}$ continua, Ω dominio x-semplice

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \ y \in [c,d], \ h_1(y) \le x \le h_2(y)\}$$

continue

$$\Rightarrow \iint_\Omega f(x,y)\,dx\,dy = \int_c^d \underbrace{\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right)}_{\varphi(y)} dy$$

(b) $f : \Omega \to \mathbb{R}$ continua, Ω dominio y-semplice

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \ x \in [a,b], \ g_1(x) \le y \le g_2(x)\}$$

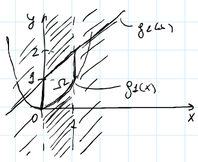
continue

$$\Rightarrow \iint_\Omega f(x,y)\,dx\,dy = \int_a^b \underbrace{\left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right)}_{\phi(x)} dx$$

<u>Esempio 1</u>

$$\iint_\Omega xy\,dx\,dy, \quad \Omega = \{(x,y) \in \mathbb{R}^2 : \ x \in [0,1], \ \underbrace{x^2}_{g_1(x)} \le y \le \underbrace{x+1}_{g_2(x)}\}$$

$\hookrightarrow$ <u>y-semplice</u>



$y = g_1(x)$
$y = g_2(x) = x+1$

(Ω è anche x-semplice)

$$\iint_\Omega xy\,dx\,dy = \int_0^1 \left(\int_{g_1(x)}^{g_2(x)} xy\,dy \right) dx = \int_0^1 \left(\int_{x^2}^{x+1} xy\,dy \right) dx =$$

$$= \int_0^1 x \cdot \frac{y^2}{2}\Big|_{x^2}^{x+1} dx = \int_0^1 x \cdot \left(\frac{(x+1)^2}{2} - \frac{x^4}{2} \right) dx =$$

$$= \int_0^1 \frac{1}{2} x \cdot (x^2 + 1 + 2x - x^4)\,dx = \frac{1}{2} \int_0^1 x^3 + x + 2x^2 - x^5\,dx =$$

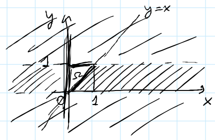
$$= \frac{1}{2}\left(\frac{x^4}{4} + \frac{x^2}{2} + 2\cdot\frac{x^3}{3} - \frac{x^6}{6}\Big|_0^1\right) = \frac{1}{2}\left(\frac{1}{4}+\frac{1}{2}+\frac{2}{3}-\frac{1}{6} - \emptyset\right) =$$

$$= \frac{1}{2}\cdot\frac{3+6+8-2}{12} = \frac{5}{8}$$

Esempio 2

$$\iint_\Omega e^{y^2}\,dxdy \quad , \quad \Omega = \{(x,y)\in\mathbb{R}^2 : \underbrace{0 \le y \le 1}_{y\in[0,1]} \;,\; \underbrace{0 \le x \le y}_{h_1(y) \le x \le h_2(y)}\}$$

$\Rightarrow$ Ω è x-semplice



(Ω è anche y-semplice)

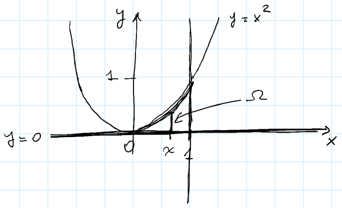
$$\iint_\Omega e^{y^2}\,dxdy = \int_0^1\left(\int_{h_1(y)}^{h_2(y)} e^{y^2}dx\right)dy = \int_0^1\left(\int_0^y e^{y^2}dx\right)dy =$$

$$= \int_0^1 e^{y^2}\, x\Big|_0^y\, dy = \int_0^1 e^{y^2}\cdot(y - \cancel{0})\,dy = \frac{1}{2}\int_0^1 \underbrace{2y}_{f'(y)}\, e^{\overbrace{y^2}^{=f(y)}}dy =$$

$$= \frac{1}{2}\cdot e^{y^2}\Big|_0^1 = \frac{1}{2}\cdot(e-1)$$

Esempio 3

$$\iint_\Omega x\cos y\,dxdy \quad , \quad \Omega = \text{regione limitata da } y=0,\ y=x^2,\ x=1$$



Ω è sia x-semplice che y-semplice

$$\Rightarrow \Omega = \{(x,y)\in\mathbb{R}^2 : x\in[0,1] \;,\; \underbrace{0}_{g_1(x)} \le y \le \underbrace{x^2}_{g_2(x)}\}$$

$$\iint_\Omega x\cos y\,dxdy = \int_0^1\left(\int_0^{x^2} x\cdot\cos y\,dy\right)dx = \int_0^1 x\cdot\sin y\Big|_0^{x^2} dx =$$

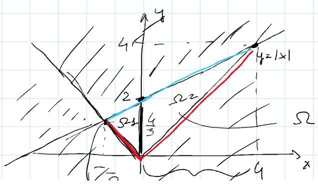
$$= \frac{1}{2}\int_0^1 \underbrace{2x}_{f'(x)}\cdot\sin\underbrace{x^2}_{f(x)}\,dx = \frac{1}{2}\left(-\cos x^2\right)\Big|_0^1 =$$

$$= -\frac{1}{2}\cdot\cos x^2\Big|_0^1 = -\frac{1}{2}\cdot\left(\cos 1 - \cos(0)\right) =$$

$$= -\frac{1}{2}\left(\cos 1 - 1\right)$$

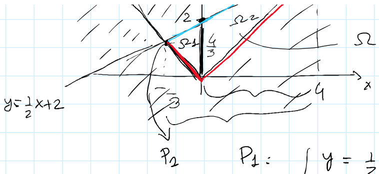
Esempio 4

$$\iint_\Omega (1+x)\,dxdy \quad , \quad \Omega = \{(x,y)\in\mathbb{R}^2 : y > |x| \;,\; y < \frac{1}{2}x + 2\}$$



$\Omega = \Omega_1 \cup \Omega_2$

$\Omega_1 \cap \Omega_2 = \emptyset$



$\Omega = \Omega_1 \cup \Omega_2$

$\Omega_1 \cap \Omega_2 = \emptyset$

$P_1: \begin{cases} y = \frac{1}{2}x + 2 \\ y = -x \end{cases} \longrightarrow \frac{1}{2}x + 2 = -x$

$\frac{3}{2}x = -2$

$\Longrightarrow x = -\frac{4}{3}, \; y = \frac{4}{3}$

$\Longrightarrow P_1 = \left(-\frac{4}{3}, \frac{4}{3}\right)$

[Ω è sia x-semplice sia y-semplice]

$$\iint_\Omega (1+x)\,dx\,dy = \underbrace{\iint_{\Omega_1} (1+x)\,dx\,dy}_{(a)} + \underbrace{\iint_{\Omega_2} (1+x)\,dx\,dy}_{(b)}$$

(a) $$\iint_{\Omega_1} (1+x)\,dx\,dy = \int_{-\frac{4}{3}}^{0} \left(\int_{-x}^{\frac{1}{2}x+2} (1+x)\,dy \right) dx = \int_{-\frac{4}{3}}^{0} (1+x)\cdot y \Big|_{-x}^{\frac{1}{2}x+2} dx =$$

$$= \int_{-\frac{4}{3}}^{0} (1+x)\cdot\left(\frac{1}{2}x + 2 + x\right) dx = \int_{-\frac{4}{3}}^{0} (1+x)\left(\frac{3}{2}x+2\right) dx =$$

$$= \int_{-\frac{4}{3}}^{0} \frac{3}{2}x + 2 + \frac{3}{2}x^2 + 2x \quad dx = \int_{-\frac{4}{3}}^{0} 2 + \frac{7}{2}x + \frac{3}{2}x^2 \quad dx =$$

$$= 2x + \frac{7}{4}x^2 + \frac{1}{2}x^3 \Big|_{-\frac{4}{3}}^{0} = -\left(-\frac{8}{3} + \frac{7}{4}\cdot\frac{16}{9} - \frac{1}{2}\cdot\frac{64}{27}\right) =$$

$$= \frac{20}{27}$$

(b) $$\iint_{\Omega_2} (1+x)\,dx\,dy = \int_0^4 \left(\int_x^{\frac{1}{2}x+2} (1+x)\,dy \right) dx = \int_0^4 (1+x)\cdot y \Big|_x^{\frac{1}{2}x+2} dx =$$

$$= \int_0^4 (1+x)\cdot\left(\frac{1}{2}x+2-x\right) dx = \int_0^4 (1+x)\left(-\frac{1}{2}x+2\right) dx =$$

$$= \int_0^4 -\frac{1}{2}x + 2 - \frac{1}{2}x^2 + 2x \quad dx = \int_0^4 2 + \frac{3}{2}x - \frac{1}{2}x^2 \quad dx =$$

$$= 2x + \frac{3}{4}x^2 - \frac{1}{6}x^3 \Big|_0^4 = 8 + \frac{3}{4}16 - \frac{1}{6}\cdot 64 = \frac{28}{3}$$

$$\Longrightarrow \iint_\Omega (1+x)\,dx\,dy = \frac{20}{27} + \frac{28}{3} = \frac{272}{27} \quad \blacksquare$$

Cambiamento di variabili / sostituzioni

Vogliamo calcolare $\iint_\Omega f(x,y)\,dx\,dy$, f continua $: \Omega \to \mathbb{R}$

Ω <u>regolare</u> [Unione finita di insiemi semplici]

Consideriamo $T: \Omega' \subseteq \mathbb{R}^2 \longrightarrow \Omega$ $\quad (\mathbb{R}^2 \to \mathbb{R}^2)$

$$(u,v) \longmapsto \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{array}{l} x = g(u,v) \\ y = h(u,v) \end{array}$$

Allora,

$$\iint_{\Omega} f(x,y)\, dx\, dy = \iint_{\Omega'} f(\underbrace{g(u,v)}_{x}, \underbrace{h(u,v)}_{y}) \cdot |\det D_T(u,v)|\, du\, dv$$

Perché è utile?

① Semplificare l'integrale

② Semplificare il dominio di integrazione

Esempio

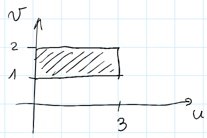
$$\iint_{\Omega} \frac{(x+y)^2}{x-y}\, dx\, dy, \quad \Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le \underbrace{x+y}_{u} \le 3,\ 1 \le \underbrace{x-y}_{v} \le 2\}$$

$$\left.\begin{array}{l} x+y = u \\ x-y = v \end{array}\right\} \to x = u - y \qquad x = \frac{u+v}{2} = g(u,v)$$

$$u - y - y = v \to y = \frac{u-v}{2} = h(u,v)$$

$$T: (u,v) \longmapsto \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{array}{l} x = g(u,v) \\ y = h(u,v) \end{array}$$

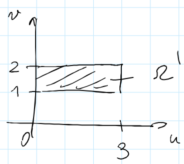
$$\Omega' = \{(u,v) \in \mathbb{R}^2 : 0 \le u \le 3,\ 1 \le v \le 2\}$$



$$\to D_T(u,v) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad \det D_T(u,v) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\hookrightarrow \iint_{\Omega} f(x,y)\, dx\, dy = \iint_{\Omega'} f(g(u,v), h(u,v)) \cdot |\det D_T(u,v)|\, du\, dv$$

$$\iint_{\Omega} \frac{(x-y)^2}{x-y}\, dx\, dy = \iint_{\Omega'} \frac{u^2}{v} \cdot |\det D_T(u,v)|\, du\, dv = \iint_{\Omega'} \frac{u^2}{v} \cdot \frac{1}{2}\, du\, dv =$$



$$= \frac{1}{2} \int_0^3 \left(\int_1^2 \frac{u^2}{v}\, dv \right) du =$$

$$= \frac{1}{2} \int_0^3 u^2 \log|v| \Big|_1^2\, du = \frac{1}{2} \int_0^3 u^2 \log 2\, du = \frac{\log 2}{2} \frac{u^3}{3} \Big|_0^3 = \frac{\log 2}{2} \cdot \frac{27}{3} = \frac{9}{2} \log 2$$

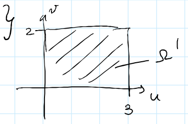
Esempio

$$\iint_\Omega 4x^2+2y^2-6xy \; dxdy \; , \quad \Omega=\{(x,y)\in\mathbb{R}^2 : \; 0\le \underbrace{y+x}_{u}\le 3, \; 0\le \underbrace{y-x}_{v}\le 2\}$$

$$\begin{cases} y+x=u \\ y-x=v \end{cases} \;\rightarrow\; x=u-y \;\hookrightarrow\; y-u+y=v \;\rightarrow\; \begin{cases} x=\frac{u-v}{2} & = g(u,v) \\ y=\frac{v+u}{2} & = h(u,v) \end{cases}$$

$$T:(u,v)\mapsto \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{matrix} x=g(u,v) \\ y=h(u,v) \end{matrix}$$

$$\Omega'=\{(u,v)\in\mathbb{R}^2 : \; 0\le u\le 3, \; 0\le v\le 2\}$$



- $$D_T(u,v)=\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det D_T(u,v)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\hookrightarrow \iint_\Omega 4x^2+2y^2-6xy \; dxdy = \iint_{\Omega'} \left(\cancel{4}\cdot\frac{(u-v)^2}{\cancel{4}} + \cancel{2}\,\frac{(u+v)^2}{2\cancel{4}} - \overset{3}{\cancel{6}}\,\frac{(u+v)(u-v)}{2\cancel{4}} \right)\cdot\frac{1}{2} \; dudv$$

$$= \frac{1}{2}\iint_{\Omega'} u^2+v^2-2uv+\frac{1}{2}(u^2+v^2+2uv)-\frac{3}{2}(u^2-v^2) \; dudv =$$

$$= \frac{1}{2}\iint_{\Omega'} \cancel{u^2}+\boxed{v^2}-2uv+\cancel{\frac{1}{2}u^2}+\boxed{\frac{1}{2}v^2}+uv-\cancel{\frac{3}{2}u^2}+\boxed{\frac{3}{2}v^2} \; dudv =$$

$$= \frac{1}{2}\iint_{\Omega'} 3v^2-uv \; dudv =$$

$$= \frac{1}{2}\int_0^3\left(\int_0^2 3v^2-uv \; dv\right) du =$$

$$= \frac{1}{2}\int_0^3 \cancel{3}\frac{v^3}{\cancel{3}}-u\frac{v^2}{2}\Big|_0^2 \; du =$$

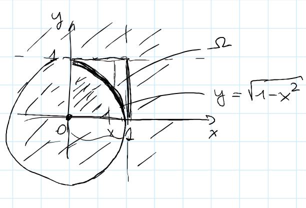
$$= \frac{1}{2}\int_0^3 8-u\cdot 2 \; du = \frac{1}{2}\cdot\left(8u-\cancel{2}\frac{u^2}{\cancel{2}}\Big|_0^3\right) =$$

$$= \frac{1}{2}(24-9) = \frac{15}{2}$$

Esercizio

$$\iint_\Omega 2y\sqrt{x} \; dxdy \; , \quad \Omega=\left\{(x,y)\in\mathbb{R}^2 : \; \begin{matrix} x^2+y^2 \ge 1, & 0\le x\le 1 \\ 0+0\ge 1 & 0\le y\le 1 \end{matrix}\right\}$$

1° modo $\quad x^2+y^2=1 \quad$ equazione della circonferenza di raggio 1



2° modo

$$x^2+y^2=1 \;\rightarrow\; y^2=1-x^2 \;\rightarrow\; \begin{matrix} 1) \; y=\sqrt{1-x^2} \\ 2) \; \cancel{y=-\sqrt{1-x^2}} \end{matrix}$$

$$\Rightarrow \Omega=\{(x,y)\in\mathbb{R}^2 : \; x\in[0,1], \; \sqrt{1-x^2}\le y\le 1\}$$, y-semplice

$$\Rightarrow \quad \Omega = \{ (x,y) \in \mathbb{R}^2 : \quad x \in [0,1], \quad \sqrt{1-x^2} \le y \le 1 \} \quad , \quad \underline{y\text{-semplice}}$$

$$\iint_\Omega 2y\sqrt{x} \, dx dy = \int_0^1 \left(\int_{\sqrt{1-x^2}}^1 2y\sqrt{x} \, dy \right) dx = \int_0^1 \cancel{2}\sqrt{x} \cdot \frac{y^2}{\cancel{2}} \Big|_{\sqrt{1-x^2}}^1 dx =$$

$$= \int_0^1 \underbrace{\sqrt{x}}_{x^{\frac{1}{2}}} \cdot \underbrace{\left(1 - (1-x^2) \right)}_{x^2} dx = \int_0^1 x^{\frac{5}{2}} dx = \frac{2}{7} x^{\frac{7}{2}} \Big|_0^1 = \frac{2}{7}$$

Esercizio ①

$$f(x,y) = \begin{cases} \dfrac{(x-1)^2(x-y)}{\sqrt{(x-1)^2+y^2}} & , \quad (x,y) \neq (1,0) \\ 0 & , \quad (x,y) = (1,0) \end{cases}$$

In quali (x,y) è differenziabile ?

- $\dfrac{\partial}{\partial x}\left[(x-1)^2(x-y) \right](x,y) =$

  $= 2(x-1)(x-y) + (x-1)^2 \cdot 1$

- $\dfrac{\partial}{\partial x}\left[\left[(x-1)^2+y^2 \right]^{\frac{1}{2}} \right](x,y) =$

  $= \dfrac{1}{2\sqrt{(x-1)^2+y^2}} \cdot 2(x-1)$

Per $(x,y) \neq (1,0)$;

- $$\frac{\partial f}{\partial x}(x,y) = \frac{\left[2(x-1)(x-y) + (x-1)^2 \right] \cdot \sqrt{(x-1)^2+y^2} - (x-1)^2(x-y) \cdot \frac{1}{2} \frac{2(x-1)}{\sqrt{(x-1)^2+y^2}}}{(x-1)^2+y^2} \quad , \quad (x,y) \neq (1,0)$$

- $\dfrac{\partial f}{\partial y}(x,y) = \dots$

$$\Rightarrow \quad f \text{ è differenziabile in } (x,y) \neq (1,0).$$

- Per $(x,y) = (1,0)$,

$$\frac{\partial f}{\partial x}(1,0) = \lim_{h \to 0} \frac{f(1+h,0) - f(1,0)}{h} = \lim_{h \to 0} \frac{(\cancel{1}+h-\cancel{1})^2(1+h-0)}{\sqrt{(\cancel{1}+h-\cancel{1})^2+0^2}} \cdot \frac{1}{h} =$$

$$= \lim_{h \to 0} \frac{h^{\cancel{2}}(1+h)}{|h| \cdot \cancel{h}} = \lim_{h \to 0} \frac{h + h^{\cancel{2}}}{|h|} = \lim_{h \to 0} \frac{h}{|h|} = \begin{cases} \lim_{h \to 0^+} \dfrac{h}{h} = 1 \\ \lim_{h \to 0^-} \dfrac{h}{-h} = -1 \end{cases}$$

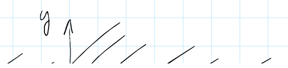
$$\nexists$$

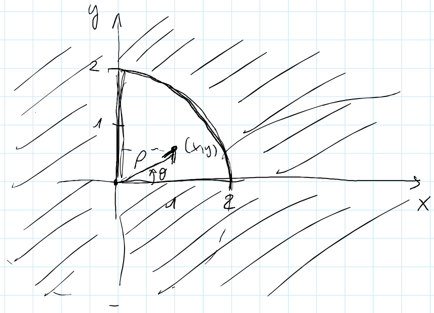
$$\Rightarrow \quad f \text{ non è derivabile in } (1,0)$$

$$\Rightarrow \quad f \text{ non è differenz. in } (1,0)$$

Esercizio ②

$$\iint_\Omega \frac{xy^2}{x^2+y^2} \, dx dy \quad , \quad \Omega = \{ (x,y) \in \mathbb{R}^2 : \quad x^2+y^2 < 4 \quad , \quad x > 0, \quad y > 0 \}$$





$x^2+y^2=4$ eq circonferenza di raggio 2

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases} \iff \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

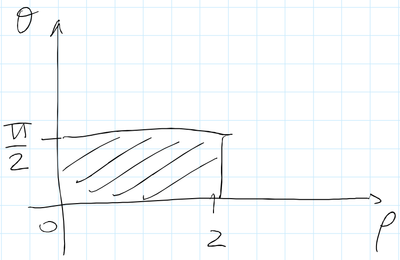
$T: \Omega' \to \Omega$

$(\rho,\theta) \longmapsto \begin{pmatrix} x \\ y \end{pmatrix}$, $\quad x = \rho\cos\theta = g(\rho,\theta)$, $\quad y = \rho\sin\theta = h(\rho,\theta)$

- $$D_T(\rho,\theta) = \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix}$$

$$\Rightarrow \det D_T(\rho,\theta) = \rho(\cos\theta)^2 + \rho(\sin\theta)^2 = \rho\left[\overbrace{\cos^2\theta + \sin^2\theta}^{=1}\right] = \rho$$

- $0 \le \rho \le 2$, $\quad 0 \le \theta \le \frac{\pi}{2}$, $\quad \Omega' = \{(\rho,\theta) \in \mathbb{R}^2 : \rho \in [0,2], \ \theta \in [0,$



$$\iint_\Omega \underbrace{\frac{xy^2}{x^2+y^2}}_{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}} dx\,dy = \iint_{\Omega'} \frac{\rho^3\cos\theta\sin^2\theta}{\rho^2} \cdot \rho \ d\rho\, d\theta =$$

$$= \iint_{\Omega'} \rho^2\cos\theta\sin^2\theta \ d\rho\, d\theta =$$

$$= \int_0^2 \left(\int_0^{\frac{\pi}{2}} \rho^2 \cos\theta \sin^2\theta \ d\theta \right) d\rho =$$

$$= \int_0^2 \left(\rho^2 \int_0^{\frac{\pi}{2}} \cos\theta [\sin\theta]^2 d\theta\right) d\rho = \int_0^2 \rho^2 \cdot \frac{(\sin\theta)^3}{3}\Bigg|_0^{\frac{\pi}{2}} d\rho =$$

$\frac{\pi}{2}\big]\big\}$

$$= \int_0^2 \Big(\rho^2 \int_0^2 \underbrace{\cos\theta}_{f'(\theta)} \underbrace{[\sin\theta]}_{f(\theta)}^2 \, d\theta \Big) \, d\rho = \int_0^2 \rho^2 \cdot \frac{(\sin\theta)^3}{3} \Big|_0^2 \, d\rho =$$

$$= \int_0^2 \rho^2 \cdot \frac{1}{3} \, d\rho = \frac{1}{3} \frac{\rho^3}{3} \Big|_0^2 = \frac{1}{3}\left(\frac{8}{3}\right) = \boxed{\frac{8}{9}}$$

Esercizio

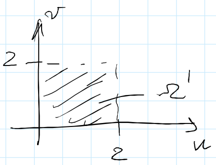
$$\iint_\Omega \frac{(x-y)^2}{1+(x-y)^2} \, dx\,dy \quad , \quad \Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le \underbrace{x}_{v} \le 2 \ , \ 0 \le \underbrace{x-y}_{u} \le 2\}$$

$$\begin{cases} x - y = u \\ x = v \end{cases} \Rightarrow \quad x = v \quad y = x - u = v - u \qquad \begin{cases} x = v \\ y = v - u \end{cases}$$

$$T : \Omega' \to \Omega$$
$$(u,v) \mapsto \begin{pmatrix} x \\ y \end{pmatrix} , \quad \begin{matrix} x = v = g(u,v) \\ y = v - u = h(u,v) \end{matrix}$$

- $D_T(u,v) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad , \quad \det D_T(u,v) = 0 + 1 = 1$

- $\Omega' = \{ (u,v) \in \mathbb{R}^2 : \ 0 \le v \le 2 \ , \ 0 \le u \le 2 \}$



$$\iint_\Omega \frac{(x-y)^2}{1+(x-y)^2} \, dx\,dy = \iint_{\Omega'} \frac{u^2}{1+u^2} \, du\,dv = \int_0^2 \left(\int_0^2 \frac{u^2}{1+u^2} \, du \right) dv =$$

$$= \int_0^2 \left(\int_0^2 1 - \frac{1}{1+u^2} \, du \right) dv =$$

$$= \int_0^2 u - \arctan u \Big|_0^2 \, dv = \int_0^2 2 - \arctan 2 \, dv =$$

$$= 2 - \arctan 2 \cdot v \Big|_0^2 = \big(2 - \arctan(2)\big) \cdot 2$$

Esercizio

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \quad , \quad f(x,y) = (xy \quad x+y)$$

Esercizio

$$f: \mathbb{R}^2 \to \mathbb{R}^2 \quad , \quad f(x,y) = (xy \quad x+y)$$

$$g: \mathbb{R}^2 \to \mathbb{R}^2 \quad , \quad g(z,t) = \left(\frac{z}{t} \, , \, z-t\right)$$

$$h = f \circ g \quad , \quad h(z,t) = f(g(z,t))$$

$$\begin{tikzcd} \underset{(z,t)}{\mathbb{R}^2} \arrow[r, "g"'] \arrow[rr, bend right, "h"'] & \underset{g(z,t)}{\mathbb{R}^2} \arrow[r, "f"'] & \mathbb{R}^2 \end{tikzcd} \qquad h: \mathbb{R}^2 \to \mathbb{R}^2$$

Calcolare $D_h(z,t)$.

$$\underset{(2\times 2)}{D_h(z,t)} = D_f(g(z,t)) \cdot D_g(z,t) \quad (*)$$

- $D_f(x,y) = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$

- $D_g(z,t) = \begin{bmatrix} \frac{1}{t} & -\frac{z}{t^2} \\ 1 & -1 \end{bmatrix}$

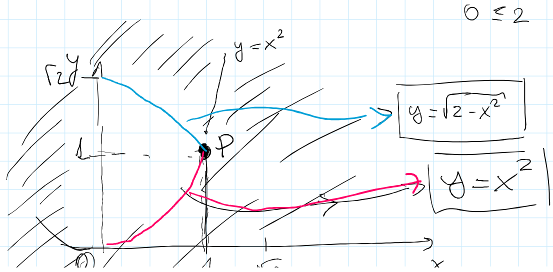
$$(*) = \begin{bmatrix} z-t & \frac{z}{t} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{t} & -\frac{z}{t^2} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{z-t}{t} + \frac{z}{t} & -\frac{z}{t^2}(z-t) - \\ \frac{1}{t} + 1 & -\frac{z}{t^2} - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot \frac{z}{t} - 1 & -\frac{z^2}{t^2} \\ \frac{1}{t} + 1 & -\frac{z}{t^2} - 1 \end{bmatrix}$$

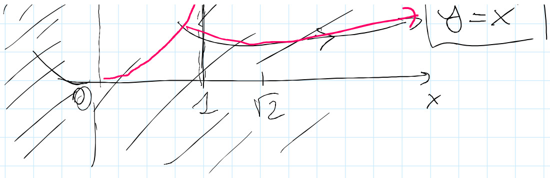
Esercizio

$$\iint_\Omega y \, dx\,dy \quad , \quad \Omega = \{(x,y) \in \mathbb{R}^2 : x \geq 0 \, , \, x^2+y^2 \leq 2 \, , \, y \geq x^2\}$$

$$0 \leq 2$$



$$\frac{z}{t}\Bigg]$$



• $x^2+y^2=2 \longrightarrow y^2 = 2-x^2 \Rightarrow y = \pm\sqrt{2-x^2}$ $\quad \left((2-x^2 \geq 0) \Leftrightarrow 0 < x < \sqrt{2} \right)$

$$y = \sqrt{2-x^2}$$
$$y = -\sqrt{2-x^2}$$

$$P: \begin{cases} x^2+y^2=2 \\ y = x^2 \end{cases} \longrightarrow y + y^2 = 2$$
$$y^2+y-2=0$$

$$\hookrightarrow y_{1,2} = \frac{-1\pm\sqrt{9}}{2} = \frac{-1\pm 3}{2} = \begin{cases} 1 \\ \cancel{-2} \end{cases}$$

$$x^2 = 1 \Rightarrow x = 1$$

$$P = (1,1)$$

$$\Rightarrow \iint_\Omega y \, dx dy = \int_0^1 \left(\int_{x^2}^{\sqrt{2-x^2}} y \, dy \right) dx = \int_0^1 \left. \frac{y^2}{2} \right|_{x^2}^{\sqrt{2-x^2}} dx =$$

$$= \int_0^1 \frac{1}{2}\left[2-x^2-x^4 \right] dx = \frac{1}{2}\left(2x - \frac{x^3}{3} - \left. \frac{x^5}{5} \right|_0^1 \right) =$$

$$= \frac{1}{2}\left(2 - \frac{1}{3} - \frac{1}{5} \right) = \frac{11}{15} \quad \blacksquare$$