A short introduction in Banach Spaces

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Vector Spaces

A vector space V is a set with two operations:

- An operation between elements called addition "$+$".
- An operation over a field $\mathbb{F}$, e.g. $\mathbb{R}$, called scalar multiplication "$\cdot$".

These two operations must obey the following rules:

1. **Closing property:** if $u, v \in V$ then $u + v \in V$.
2. **Commutative property:** $u + v = v + u$ for all $u, v \in V$.
3. **Associative property:** $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$

4. **Zero element property:** There exists an element $\mathbf{0} \in V$, such that $u + \mathbf{0} = \mathbf{0} + u$.
5. **Additive inverse property:** For each $u \in V$ there exists an element $v \in V$, such that $u + v = v + u = \mathbf{0}$.
6. **Closing property over scalar multiplication:** if $t \in \mathbb{F}$ and $u \in V$ then $t \cdot u \in V$.
7. **Distributive property 1:** $t \cdot (u + v) = t \cdot u + t \cdot v$ for each $t \in \mathbb{F}$ and all elements $u, v \in V$.
8. **Distributive property 2:** $(s + t) \cdot u = s \cdot u + t \cdot u$ for each $u \in V$ and all $s, t \in V$.
9. **Scalar multiplicative property:** $1 \cdot u = u$ for all $u \in V$, where 1 is the multiplicative identity of $\mathbb{F}$

Examples:

Examples 1: The n-dimensional Euclidean space $\mathbb{R}^n$ over the real field $\mathbb{R}$.

Examples 2: The set of all smooth functions over the reals $C(R)$.

Examples 3: The set of all periodic functions of a fixed period T.

Examples 4 The set of all convergent sequences.

Bases:

A finite subset $\{x_1, x_2, ..., x_n\}$ of V is said to be linearly independent if

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = \mathbf{0} \to \lambda_1 = \lambda_2 = ... = \lambda_n = 0.$$

An arbitrary subset $X \subset V$ is said to be **linearly independent** if every finite subset of X is linearly independent.

A **basis** is a subset $X \subset V$ if it is linearly independent and spans V, i.e., $\mathrm{span}(X) = V$.

Basic facts about bases:

1. Every element $x \in V$ admits a unique **basis decomposition**.
2. if $Y \subset V$ and $\mathrm{span}(Y) = V$, then Y contains a basis for V.
3. Every non-zero vector space has a basis.
4. Every linearly independent subset Y can be extended to form a basis for V.
5. A vector space is **finite dimensional** if it admits a basis with only finite many elements; otherwise, it is called **infinite dimensional**.

Normed linear spaces:

A vector space V is called a **normed linear space** if there exist a function called **norm**, denoted $||\cdot||$, from V into $\mathbb{R}$ such that:

1. $||x|| \geq 0$ for all $x \in V$ and $||x|| = 0$ if, and only if, $x = 0$.
2. $||\lambda x|| = |\lambda|||x||$ for all $x \in V$ and $\lambda \in \mathbb{F}$.
3. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in V$.

A norm gives V a topology induced by the following metric

$$\rho(x, y) = ||y - x||$$

A vector space V can have different norms

Also, for the rest of this slides our field is going to be $\mathbb{R}$.

Equivalent norms

So a vector space V may have different norms, that sounds exciting. Weeeeeeeeellllllll, sometimes two norms behave "topologically" the same.

Let $||\cdot||_1$ and $||\cdot||_2$ be two different norms on a vector space V. We say that they are **equivalent** if, and only if, there exists real number $0 < m \leq M < \infty$ such that $m||v||_1 \leq ||v||_2 \leq M||v||_1$ for all $v \in V$.

Theorem (Fundamental Theorem of Finite Dimensional Normed Linear Spaces)

ALL norms on a FINITE dimensional vector space are EQUIVALENT.

We can talk about continuity

Since we have a metric induced by $||\cdot||$, we can talk about continuity of a function between normed linear spaces.

Let f be a function between the normed space $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$, i.g. $f : V \to W$. Then f is continuous at $v \in V$ if

$$\forall x \in V \quad \forall \epsilon > 0 \quad \exists \delta > 0 : \quad ||v - x||_V < \delta \to ||f(v) - f(x)||_W < \epsilon$$

Corollary

Let $(V, ||\cdot||_V)$ be a normed linear space and $(W, ||\cdot||_W)$ be a **finite dimensional** *normed linear space. If f is a continuous function between V and W then f is also continuous function if $||\cdot||_W$ is replaced by any other norm of W*

Banach spaces... Finally!. Please stop talking Andrus

A normed linear space V is called a Banach space if it is a **complete metric space** under the metric induced by the norm.

Intuitively, if a set is complete it means that it does not have **holes**.

More formally, it means that any Cauchy sequence in V converges in V... hmmm...what is a Cauchy sequence?

A sequence $\{x_n\}_{n=1}^{\infty}$ in V is a Cauchy sequence if it fullfils:

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N} : \quad m, n > N \to ||x_m - x_n||_V < \epsilon$$

Again, finite dimensional spaces

Corollary

Every finite dimensional normed linear space over the fields $\mathbb{R}$ or $\mathbb{C}$ is a Banach space

Corollary

Let $(Y, ||\cdot||)$ be a finite dimensional subspace of $(X, ||\cdot||)$. Then Y is a closed subspace of X.

Wait, what it means to be closed? well, we called a set A **closed** if the limit point of any convergent sequence lies in A (we can go more abstract but this is enough for us).

Linear Operators

A linear operator T between vector spaces V and W is a function that satisfies the following conditions:

1. $T(u+v) = T(u) + T(v)$ for all $u, v \in V$
2. $T(\lambda u) = \lambda T(u)$ for all $u \in V$ and $\lambda \in \mathbb{F}$.

Additionally, if $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ are normed linear spaces. Then a linear operator **might** be **bounded linear operator** which means that there exist an $M \in \mathbb{R}$, such that

$$||T(x)||_w \leq M||x||_V \qquad \text{for all } x \in V$$

Bounded linear operators and continuity

Theorem (Continuity for the people)

Let $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ be normed linear space and $(W, ||\cdot||_W)$ and let $T : V \to W$ be a linear operator. Then the following are equivalent:
T is continuous $\leftrightarrow$ T is a bounded operator.

The space of linear operators

Let $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ be normed linear spaces. Then by $B(V, W)$ we denote the space of all bounded linear operators from V to W.

Interestingly, $B(V, W)$ is a vector space and we can define an operator norm for $B(V, W)$ in the following form:

$$||T|| := \sup\{||T(x)||_W : ||x||_V = 1$$

Theorem

Let $(V, ||\cdot||_V)$ be a normed linear space and $(W, ||\cdot||_W)$ be normed linear (Banach) space and $(W, ||\cdot||_W)$. Then $B(V, W)$ is a normed linear (Banach) space with the operator norm as defined above.

Finite dimension are useful and boring

Theorem

All linear operators defined on finite dimensional normed linear spaces are continuous.

Corollary

Any two n-dimensional normed linear space over the same field $\mathbb{F}$ *are isomorphic.*

Corollary

A normed linear space $(X, ||\cdot||)$ *is finite dimensional if, and only if, every linear functional on* X *is continuous.*

Basically, finite dimensional normed linear spaces largely reduces to linear algebra and matrices.

What about differentiability?

There exist two different notion of differentiability

- Frechet differentiability related to the Jacobian in finite dimensions.
- Gâteaux differentiability related to directional derivatives in finite dimensions.

What about differentiability 2?

Let V and W be normed vector spaces, and let $U \subset V$ be an open set of V. A function $f : U \to W$ is called **Frechet differentiable** if there exist a bounded linear operator $T \in B(V, W)$ such that:

$$\lim_{t \to 0} \frac{||f(x+h) - f(x) - Th||_W}{||h||_V} = 0$$

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