# CONDITIONAL PROBABILITY

### Definition

Let A and B be any two events. The probability of the event A given that the event B has already occured or the conditional probability of A given B, denoted by P(A | B) is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

Similarly the conditional probability of B given A is defined as

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

### Remarks:

(i) For P(B) > 0 $\quad P(A \mid B) \leq P(A)$

- P(A | B) is not defined if P(B) = 0
- P(B | B) = 1

### Theorem

For a fixed B with P(B) > 0, P(A | B) is a probability function (or probability measure).

### Proof

Here we have to show that conditional probability satisfies all the axioms of probability.

(i) $P(A \mid B) = \dfrac{P(A \cap B)}{} \geq 0$, by axiom (1)

(ii) $P(S \mid B) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(iii) For any two adjoint events A and C



$$P(A \cup C \mid B) = \frac{P[(A \cup C) \cap B]}{P(B)}$$

$$= \frac{P[(A \cap B) \cup (C \cap B)]}{P(B)} \quad \text{by associative property}$$

$$= \frac{P(A \cap B) + P(C \cap B)}{P(B)} \quad \text{since } A \cap B \text{ and } C \cap B \text{ are disjoint}$$

$$= \frac{P(A \cap B)}{P(B)} + \frac{P(C \cap B)}{P(B)} = P(A|B) + P(C|B)$$

That is, conditional probability satisfies all the axioms of probability. Therefore P(A|B) is a probability function or probability measure.

## Multiplication law of probability

### Theorem

For any two events A and B

$P(A \cap B) = P(A)\, P(B|A),\ P(A) > 0$

$\qquad\quad\ = P(B)\,.\,P(A|B),\ P(B) > 0$

where P(A|B) and P(B|A) are the conditional probabilities of A and B respectively.

## Independent Events

### Definition

Two or more events are said to be *independent* if the probability of any one them is not affected by the supplementary knowledge concerning the materialisation of any number of the remaining events. Otherwise they are said to be *dependent.*

### Independence of two events A and B

An event A is said to be independent (statistically independent) of event B, if the conditional probability of A given B, i.e., $P(A|B)$ is equal to the unconditional probability of A.

In symbols, $P(A \mid B) = P(A)$

Similarly if the event B is independent of A, we must have

$P(B \mid A) = P(B)$

Since $P(A \cap B) = P(A)\, P(B|A)$ and since $P(B|A) = P(B)$ when B is independent of A, we must have, $P(A \cap B\} = P(A) \,.\, P(B)$

Hence, the events A and B are independent if

$P(A \cap B) = P(A)\, P(B)$

## Pairwise and Mutual independence

### Definition

A set of events $A_1, A_2, \ldots, A_n$ are said to be pairwise independent if every pair of different events are independent.

That is, $P(A_i \cap A_j) = P(A_i)\, P(A_j)$ for all i and j, $i \neq j$.

### Definition

A set of events $A_1, A_2, \ldots, A_n$ are said to be mutually independent if

$P(A_i \cap A_j \cap \ldots \cap A_r) = P(A_i)\, P(A_j) \ldots P(A_r)$ for every subset $(A_i, A_j, \ldots, A_r)$ of $A_1, A_2, \ldots, A_n$

That is the probabilities of every two, every three…, every n of the events are the products of the respective probabilities.

For example, three events A, B and C are said to be mutually independent if

| | |
|---|---|
| $P(A \cap B)$ | $= P(A)\, P(B)$ |
| $P(B \cap C)$ | $= P(B)\, P(C)$ |
| $P(A \cap C)$ | $= P(A)\, P(C)$ and |
| $P(A \cap B \cap C)$ | $= P(A)\, P(B)\, P(C)$ |



**Note. 1**

For the mutal independence of n events, $A_1, A_2, \ldots, A_n$ the total number of conditions to be satisfied is $2^n - 1 - n$. In particular, for three events we have $4 = (2_3 - 1 - 3)$ conditions for their mutual independence.

**Note. 2**

We can note that pairwise or mutual independence of events $A_1, A_2, \ldots$

 $A_n$ is defined only when $P(A_i) \neq 0$ , for $i = 1, 2, \ldots, n$.

**Note 3**

Pairwise independence does not imply mutual independence.

### Theorem

Mutual independence of events implies pairwise independence of events. The converse is not true.

### Proof

From the definition of mutual independence, it is clear that mutual independence implies pair-wise independence. We shall prove that the converse is not necessarily true. i.e., pair-wise independence does not imply mutual independence. We can illustrate it by means of an example due to S.N. Bernstein.

Let $S = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ where $P(\omega_i) = 1/4$ for for i = 1, 2, 3, 4.

Let $A = \{\omega_1, \omega_2\}$, $B = \{\omega_1, \omega_3\}$ and $C = \{\omega_1, \omega_4\}$

Then $P(A)= P(B) = P(C) = 1/2$

and consider the collection of events A.B,C. These events are pairwise independent but not mutually independent.

Since they are pairwise independent we have,

$P(A \cap B) = 1/4 = P(A)P(B)$

$P(B \cap C) = 1/4 = P(B)P(C)$

$P(A \cap C) = 1/4 = P(A)P(C)$

But $P(A \cap B \cap C) = P(\omega_1) = 1/4$

$$P(A).P(B).P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Thus $P(A \cap B \cap C)\ ^{1}\ P(A)\,P(B)\,P(C)$

Hence they are not mutually independent.

## Multiplication Theorem (independent events)

If A and B are two independent events,

$P(A \cap B) = P(A).P(B)$

**Proof**

We have, for any two events A and B

$P(A\cap B) = P(A)\,P(B|A)$

Since A and B are independent, we have $P(B|A) = P(B)$,

♣ $P(A \cap B) = P(A)\,P(B)$.

**Note**

If A and B are independent the addition theorem can be stated as $P(A \cup B) = P(A) + P(B) - P(A).\,P(B)$

**Theorem**

If A and B are two independent events

then (i) A and $B^C$ are independent

□ $A^C$ and B are independent

□ $A^C$ and $B^C$ are independent

**Proof**

Since A and B are independent, we have

$P(A|B) = P(A).\ P(B|A) = P(B)$ and $P(A\cap B) = P(A).P(B)$

(i) Now, $P(A\cap B^C) = P(A)\,P(B^C|A)$

□ $P(A)\,[1-P(B|A)]$

□ $P(A)\,[1-P(B)]$

□ $P(A)\,P(B^C)$

ie., A and $B^C$ are independent



(ii) $P(A^C \cap B) = P(B)\,P(A^C|B)$

□ $P(B)\,[1 - P(A \mid B)]$

□ $P(B)\,[1 - P(A)]$

□ $P(A^C)\,P(B)$

ie., $A^C$ and B are independent

(iii) $P(A^C \cap B^C) = P(A \cup B)^C = 1 - P(A \cup B)$

□ $1 - \{P(A) + P(B) - P(A \cap B)\}$

□ $1 - P(A) - P(B) + P(A)\,P(B)$

since $P(A\cap B) = P(A)\,P(B)$

□ $[1 - P(A)] - P(B)\,[1 - P(A)]$

□ $[1 - P(A)]\,[1 - P(B)] = P(A_C)\,P(B_C)$

i.e., $A^C$ and $B^C$ are independent

**Baye's Theorem**

$$P(B_i / A) = \frac{P(B_i)\ P(A\ |}{\sum_{i=1}^{n} P(B_i)\ P(A\ |}$$

**Note 1**

Here the probabilities $P(B_i \mid A)$ for i = 1, 2, …, n are the probabilities determined after observing the event A and $P(B_i)$ for i = 1, 2, ....., n are the probabilities given before hand. Hence P(Bi) for i = 1, 2, ......, n are called 'a priori' probabilities and $P(B_i \mid A)$ for i =1, 2, ....., n are called "a posteriori" probabilities. The probabilities $P(A|B_i)$, i = 1, 2, ....., n are called 'likely hoods' because they indicate how likely the event A under consideration is to occur, given each and every, 'a priori' probability. Baye's theorem gives a relationship between $P(B_i \mid A)$ and $P(A \mid B_i)$ and thus it involves a type of inverse reasoning. Baye's theorem plays an important role in applications. This theorem is due to Thomas A Baye's.

## Note 2

In the case of two events A and B satisfying the assumption P(B) > 0 and 0 < P(B) < 1 we have,

$$P(B \mid A) = \frac{P(B)P(A \mid B)}{P(B)P(A \mid B) + P(B^C)P(A \mid B^C)}$$

## Example 1

Let A and B be two events associated with an experiment and suppose P(A) = 0.5 while P(A or B) = 0.8. Let P(B) = p. For what values of p are (a) A and B mutually exclusive (b) A and B independent.

## Solution

Given P(A) = 0.5, $P(A \cup B) = 0.8$, P(B) = p

(a) If A and B are mutually exclusive

$P(A \cup B) = P(A) + P(B)$

i.e., 0.8 = 0.5 + p

**p = 0.3**

 If A and B are independent, we have

$P(A \cup B) = P(A) + P(B) - P(A)P(B)$

i.e., 0.8 = 0.5 + p – .5p

♣ .5p = 0.3 ∴ **p = 3/5**

## Example 2

If A and B are two events such that P(A) = 1/3, P(B) = 1/4 and P(AÇB) = 1/8. Find P(A|B) and $P(A|B^C)$

## Solution

Given P(A) = 1/3, P(B) = 1/4, $P(A \cap B) = 1/8$

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{1/4} = 4/8 = \mathbf{1/2}$$



$$P(A|B^C) = \frac{P(A \cap B^C)}{P(B^C)} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{1/3 - 1/8}{1 - 1/4} = \frac{5/24}{3/4} = \mathbf{5/18}$$

## Example 3

The odds that A speaks the truth are 3:2 and the odds that B speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an identical point?

## Solution

Define the events,

A - A speaks the truth

B - B speaks the truth

♣ P(A) = 3/5, $P(A^C) = 2/5$

P(B) = 5/8, $P(B^C) = 3/8$

They will contradict each other on an identical point means that when A speaks the truth, B will tell a lie and conversely.

$\therefore$ P(They will contradict each other) = $[P(A \cap B^C) \cup (A^C \cap B)]$

$= P(A \cap B^C) + P(A^C \cap B)$, since the events are m.e.

$= P(A)\,P(B^C) + P(A^C)\,P(B)$

$$= \frac{3}{5} \cdot \frac{3}{8} + \frac{2}{5} \cdot \frac{5}{8} = \mathbf{19/40}$$

ie,, In 47.5% of the cases, A and B contradict each other.

## Example 4

A husband and wife appear in an interview for two vacancies in a firm. The probability of husbands selection is 1/7 and that of wife's selection is 1/5. What is the probability that

 both of them will be selected.

 only one of them will be selected. (c) none of them will be selected.

## Solution

Let us define the events as

A - The husband get selection.

B - The wife get selection.

♣ $P(A) = 1/7, P(B) = 1/5; P(A^C) = 6/7; P(B^C) = 4/5$

(a) P(both of them will be selected) = $P(A \cap B)$ =

$P(A) . P(B)$, since A and B are independent

$$= \frac{1}{7} \cdot \frac{1}{5} = \frac{1}{35}$$

P(only one of them will be selected)

$$P[(A \cap B^C) \cup (A^C \cap B)]$$

$$P(A \cap B^C) + P(A^C \cap B)$$

$$= P(A) \quad P(B^C) + P(A^C) \quad P(B)$$

$$\frac{1}{7} \times \frac{4}{5} + \frac{6}{7} \times \frac{1}{5}$$

**10/35**

(c) P(none of them will be selected) = $P(A^C \cap B^C)$

$$= P(A^C)\,P(B^C) = \frac{6}{7} \cdot \frac{4}{5} = \mathbf{24/35}$$

## Example 5

If A, B and C are independent, show that $A \cup B$ and C are independent.

## Solution

Since, A, B and C are independent, we have

$P(A \cap B) = P(A)P(B)$, $P(B \cap C) = P(B)P(C)$

$P(A \cap C) = P(A)P(C)$ and $P(A \cap B \cap C) = P(A)P(B)P(C)$

We have to show that



$$P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$$

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

$$= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)$$

$$= P(C)\,[P(A) + P(B) - P(A)\,P(B)]$$

$$= P(A \cup B) . P(C)$$

i.e., $A \cup B$ and C are independent.

## Example 6

A problem in statistics is given to 3 students A, B and C whose chances of solving it are 1/2, 3/4 and 1/4 respectively. What is the probability that the problem will be solved?

## Solution

Let us define the events as

A – the problem is solved by the student A

B – the problem is solved by the student B

C – the problem is solved by the student C

♣ $P(A) = 1/2, P(B) = 3/4$ and $P(C) = 1/4$

The problem will be solved if at least one of them solves the problem. That means we have to find $P(A \cup B \cup C)$.

Now $P(A \cup B \cup C)$

$$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$P(A) + P(B) + P(C) - P(A)P(B) - P(B)P(C) - P(A)P(C) + P(A)P(B)P(C)$$

$$\frac{1}{2} + \frac{3}{4} + \frac{1}{4} - \frac{1}{2} \times \frac{3}{4} - \frac{3}{4} \times \frac{1}{4} - \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{4} \times \frac{1}{4}$$

**29/32**

### Aliter

$$P(A \cup B \cup C) = 1 - P(A \cup B \cup C)^c$$
$$= 1 - P(A^c \cap B^c \cap C^c)$$
$$= 1 - P(A^C)\,P(B^C)\,P(C^C)$$
$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{3}{4}\right)\left(1 - \frac{1}{4}\right)$$

= **29/32**

### Example 7

A purse contains 2 silver coins and 4 copper coins and a second purse contains 4 silver coins and 3 copper coins. If a coin is selected at random from one of the purse. What is the probability that it is a silver coin?

### Solution

Define the events

$B_1$ – selection of 1st purse

$B_2$ – selection of 2nd purse

$A$ – selection of silver coin

$P(B_1) = P(B_2) = 1/2$

$P(A|B_1) = 2/6$, $P(A|B_2) = 4/7$

By theorem on total probabilities

$$P(A) = P(A \cap B_1) + P(A \cap B_2)$$
$$= P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2)$$
$$= \frac{1}{2} \times \frac{2}{6} + \frac{1}{2} \times \frac{4}{7}$$
$$= \frac{1}{6} + \frac{2}{7} = \frac{7+12}{42} = \frac{19}{42}$$



### Example 8

Suppose that there is a chance for a newly constructed house to collapse wether the design is faulty or not. The chance that the design is faulty is 10%. The chance that the house collapse if the design is faulty is 95% and otherwise it is 45%. It is seen that the house collapsed. What is the probability that it is due to faulty design?

### Solution

Let $B_1$ and $B_2$ denote the events that the design is faulty and the design is good respectively. Let A denote the event that the house collapse. Then we are interested in the event $(B_1|A)$, that is, the event that the design is faulty given that the house collapsed. We are given,

$P(B_1) = 0.1$ and $P(B_2) = 0.9$

$P(A|B_1) = 0.95$ and $P(A|B_2) = 0.45$

Hence

$$P(B_1|A) = \frac{P(B_1).P(A \mid B_1)}{P(B_1).P(A \mid B_1) + P(B_2).P(A \mid B_2)}$$
$$= \frac{(0.1)(0.95)}{(0.1)(0.95) + (0.9)(0.45)}$$
$$= \mathbf{0.19}$$

### Example 9

Two urns I and II contain respectively 3 white and 2 black bails, 2 white and 4 black balls. One ball is transferred from urn I to urn II and then one is drawn from the latter. It happens to be white. What is the probability that the transferred ball was white.

### Solution

Define

$B_1$ - Transfer a white ball from Urn I to Urn II

$B_2$ - Transfer a black ball from Urn I to Urn II.

A - Select a white ball from Urn II.

Here, $P(B_1) = 3/5$, $P(B_2) = 2/5$

$P(A|B_1) = 3/7$, $P(A|B_2) = 2/7$

We have to find $P(B_1|A)$,

By Baye's theore,

$$P(B_1|A) = \frac{P(B_1).P(A \mid B_1)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2)}$$

$$= \frac{3/5 \times 3/7}{3/5 \times 3/7 + 2/5 \times 2/7} = \frac{9/35}{13/35} = \frac{9}{13}$$

# EXERCISES

**Multiple choice questions**

- Probability is a measure lying between
  - a) $-\infty$ to $+\infty$  b) $-\infty$ to $+1$
  - c) –1 to +1  d) 0 to 1
- Classical probability is also known as
  - a) Laplace's probability b) mathematical probability
  - c) a priori probability  d) all the above
- Each outcome of a random experiment is called
  - a) primary event  b) compound event
  - c) derived event  d) all the above
- If A and B are two events, the probability of occurance of either A or B is given by
  - a) P(A)+P(B)  b) $P(A\cup B)$
  - c) $P(A\cap B)$  d) P(A)P(B)
- The probability of intersection of two disjoint events is always
  - a) infinity  b) zero
  - c) one  d) none of the above



6. If $A \subset B$, the probability P(A|B) is equal to
   - a) zero  b) one
   - c) P(A)/P(B)  d) P(B)/P(A)

- The probability of two persons being borned on the same day (ignoring date) is
  - a) 1/49  b) 1/365
  - c) 1/7  d) none of the above

8. The probability of throwing an odd sum with two fair dice is
   - a) 1/4  b) 1/16  c) 1  d) 1/2
9. If P(A|B) = 1/4, P(B|A) = 1/3, then P(A)|P(B) is equal to
   - a) 3/4  b) 7/12
   - c) 4/3  d) 1/12

- If four whole numbers are taken at random and multiplied, the chance that the first digit is their product is 0, 3, 6 or 9 is
  - a) $(2/5)^3$  b) $(1/4)^3$  c) $(2/5)^4$  d) $(1/4)^4$

**Fill in the blanks**

- Classical definition of probability was given by ……….
- An event consisting of only one point is called ……….
- Mathematical probability cannot be calculated if the outcomes are ……….
- In statistical probability n is never ……….
- If A and B are two events, the $P(A \cap B)$ is ……….
- Axiomatic definition of probability is propounded by ……….
- Baye's rule is also known as ……….
- If an event is not simple, it is a ……….

## Very short answer questions

- Define a simple event.
- Define random experiment.
- Define equally likely cases.
- State statistical definition of probability.
- Define conditional probability
- State Baye's rule

## Short essay questions

- Define Sample space and Event When will you say that two events are mutually exclusive?
- Define random experiment, sample space and Event. A coin is repeatedly tossed till a head turns up. Write down the sample space.
- Give the classical and axiomatic definition of probability, Explain how axiomatic definition is more general than classical.
- Define (i) Mutually exclusive events: (ii) Equally likely events: and (iii) Independent events and give example of each.
- Give Von Mises definition of empirical probability, Compare this with the classical definition of probability.
- State and prove the addition theorem of probability.
- Define Conditional probability.
- State and prove addition and multiplication theorem of probability.
- Show that $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$
- State and prove Bayes' theorem.
- Define Conditional probability. Prove that if $P(A) > P(B)$ then $P(A|B) > P(B|A)$.
- Let A, B and C denote events. If $P(A \mid C) \geq P(B \mid C)$ and $P(A \mid C^{c}) \geq P(B \mid C^{C})$, then show that $P(A) \geq P(B)$

## Long essay questions

- Two unbiased dice are tossed. What is the probability that the sum of points scored on the two dice is 8?
- From a group consisting of 6 men and 4 women a committee of 3 is to be chosen by lot. What is the probability that all 3 are men?
- Two events A and B are statistically independent. P(A) = 0.39, P(B) = 0.21 and P(A or B) = 0.47. Find the probability that

  Neither A nor B will occur

  Both A and B will occur

  B will occur given that A has occurred

  A will occur given that B has occurred
- If P(A) = 0.3, P(B) = 0.2. $P(A \cup B) = 0.4$, find

  $P(A \cap B)$. Examine whether A and B are independent.
- The probability that A hits a target is 1/4 and the probability that B hits it is 2/5. What is the probability that the target will be hit if A and B each shoot at the target?
- A coin is tossed four times. Assuming that the coin is unbiased, find the probability that out of four times, two times result in head,
- Two urns each contain balls of different colours are stated below. urn I : 4 black; 3 red; 3 green.

  urn II : 3 black; 6 red: 1 green.

  An urn is chosen at random and two balls are drawn from it. What is the probability that one is green and the other is red.

- If two dice are rolled, what is the probability that the sum is 7 if we know that at least one die shows 4?
- There are three urns containing balls of different colours as stated below:

  Urn I : 4 red, 2 black, 4 green.

  Urn II : 3 red, 4 black, 5 green.

  Urn III: 2 red, 4 black, 2 green