PROBABILITY THEORY

CLASSICAL DEFINITION OF PROBABILITY

Introduction

In everyday language, the word probability describes events that do not occur with certainty. When we look at the world around us, we have to conclude that our world functions more on uncertainty than on certainty. Thus we speak of the probability of rain tomorrow, the probability that an electric appliance will be defective, or even the probability of nuclear war. The concept of probability has been an object of debate among philosophers, logicians, mathematicians, statisticians, physicists and psychologists for the last couple of centuries and this debate is not likely to be over in the foreseeable future.

Probability is a number associated with an event, intented to represent its 'likelihood', 'chance of occurring', 'degree of uncertainity' and so on. The probability theory has its origin in '*Games of chance*'. Now it has become a fundamental tool of scientific thinking.

Classical Definition of Probability

Some Important Concepts

1. Random experiment

It is a physical phenomenon and at its completion we observe certain results. There are some experiments, called deterministic experiments, whose outcomes can be predicted. But in some cases, we can never predict the outcome before the experiment is performed. An experiment natural, conceptual, physical or hypothetical is called a random experiment if the exact outcome of the trails of the experiment is unpredictable. In other words by a random experiment, we mean

- It should be repeatable under uniform conditions.
- It should have several possible outcomes.
- One should not predict the outcome of a particular trail.

Example: Tossing a coin, rolling a die, life time of a machine, length of tables, weight of a new born baby, weather condition of a certain region etc.

2. Trial and Event

Trial is an attempt to produce an outcome of a random experiment. For example, if we toss a coin or throw a die, we are performing trails.

The outcomes in an experiment are termed as events or cases. For example, getting a head or a tail in tossing a coin is an event. Usually events are denoted by capital letters like A, B, C, etc…

3. Equally likely events

Events or cases are said to be equally likely when we have no reason to expect one rather than the other.

For example, in tossing an unbiased coin the two events head and tail are equally likely because we have no reason to expect head rather than tail. Similarly, when we throw a die the occurrence of the numbers 1 or 2 or 3 or 4 or 5 or 6 are equally likely events.

4. Exhaustive events

The set of all possible outcomes in a trial constitutes the set of exhaustive cases. In other words the totality of all possible outcomes of a random experiment will form the exhaustive cases. For example, in the case of tossing a coin there are two exhaustive cases head or tail. In throwing a die there are six exhaustive cases since any one of the six faces 1, 2, …, 6 may come upper most. In the random experiment of throwing two dice the number of exhaustive cases is $6^2 = 36$. In general, in throwing n dice, the exhaustive number of cases is 6^n.

5. Mutually exclusive events

Events are said to mutually exclusive or incompatible or disjoint if the happening of any one of them precludes or excludes the happening of all the others in a trail. That is, if no two or more of them can happen simultaneously in the same trial.

For example, the events of turning a head or a tail in tossing a coin are mutually exclusive. In throwing a die all the six faces numbered 1 to 6 are mutually exclusive since if any one of these faces comes, the possibility of others in the same trial, is ruled out.

6. Favourable cases

The cases which entail the occurrence of an event are said to be favourable to the events. For example, while throwing a die, the occurrence of 2 or 4 or 6 are the favourable events which entail the occurrence of an even number.

Classical Definition (Mathematical or ‘a priori’)

Classical definition is the oldest and simplest definition of probability. This is sometimes called equally-likely events approach. It is also known by the name Laplace definition. From a practical point of view it is the most useful definition of probability.

Definition

If a trial results in ‘n’ mutually exclusive, equally likely and exhaustive cases and ‘m’ of them are favourable (m < n) to the happening of an event A, then the probability of A, designated as P(A) is defined as

$$P(A) = \frac{m}{n} = \frac{\text{no of favourable cases}}{\text{Total number of cases}} \qquad (1)$$

Obviously, 0 £ P(A) £ 1

Note 1

If A is an impossible event, then P(A) = 0
If A a sure event, then P(A) = 1
If A is a random event, then 0 < P(A) < 1

Note 2

We can represent the probability given by (1) by saying that the odds in favour of A are m: (n – m) or the odds against A are (n – m): n.

Limitations of classical definition

The above definition of mathematical probability fails in the following cases.

- In the classical or a priori definition of probability only equally likely cases are taken into consideration. If the events cannot be considered equally likely classical definition fails to give a good account of the concept of probability.

- When the total number of possible outcomes 'n' become infinite or countably infinite, this definition fails to give a measure for probability.
- If we are deviating from the games of chances like tossing a coin, throwing a die etc., this definition cannot be applied.
- Another limitation is that it does not contribute much to the growth of the probability theory.

Frequency Definition of Probability

Let the trials be repeated over a large number of times under essentially homogeneous conditions. The limit of the ratio of the number of times an event A happens (m) to the total number of trials (n), as the number of trials tends to infinity is called the probability of the event A. It is, however, assumed that the limit is unique as well as finite.

$$\text{Symbolically, } P(A) = \lim_{n\to\infty} \frac{m}{n}$$

Remark 1. The application of this definition to many problems cannot be extensive since n is usually finite and the limit of the ratio cannot normally be taken as it leads to mathematical difficulties. Besides, the definition of probability thus proposed by Von Mises would involve a mixture of empirical and theoretical concepts, which is usually avoided in modern axiomatic approach.

Remark 2. The two definitions of probability" are apparently different. The mathematical definition is the relative frequency of favourable cases to the total number of cases while in the statistical definition it is the limit of the relative frequency of the happening of the event.

Set Theory

Set: A Set is a collection of well defined objects. The following arc typical examples of sets.

- The students Minu, Jithu, Hari and Devi.
- The odd numbers 1, 3. 5, 7, 9
- The rivers in South India
- The metropolitan cities of India

Sets are usually denoted by capital letters A, B, C, X, Y. Z etc. The items which are included in a set are called the *elements* of the set.

If A = {3, 5, 12, 14, 21} is a set, then '3' is an element of set A, and it is written as '$3 \in A$'. This is read as 'element 3 belongs to set A. Thus the symbol '$\in$' denotes 'belongs to'. On the other hand, 8 does not belong to set A in the above case. Then the symbol '$\notin$' is used to indicate 'does not belong to'. ie., $8 \notin A$ implies element 8 is not a member of set A.

There are two methods of representing sets viz.

Roster method. 2. Rule method

I. Roster Method

Here each and every element of the set is listed or mentioned.

Example: i. A = {a. e, i, o, u} ii. B ={2, 3, 5, 7}

iii. Y = {6, 1, 5, 2, 4, 3}

Note

The flower brackets { } are used for denoting a set. The order in which the elements of a set are listed in the { } brackets is immaterial.

2. Rule Method

Here a rule is stated by which all the elements of the set intended to be identified.

Example: A: {x/x is a vowel among English alphabets}

This is read as set A. Set of all x such that x is a vowel among English alphabets.

Types of Sets

We have different kinds of sets, Consider the following

I. Finite Set

A Set which contains a finite or a fixed number of elements is called a 'Finite Set'. Example:

- Set A has only five elements i,e,, A = {1, 2. 6. 8. 10}
- B = {x/x is a composite number between 12 and 18}
  i.e., B = {14, 15, 16}

Y = {x/x shows a number on a die}

This is same as Y = {1, 2, 3, 4, 5, 6}

- **Infinite Set**

A set which contains infinite number of elements is called an 'infinite set'.

Example:

- X = {x/x is a natural number} i.e.. X = {1, 2, 3. 4...}
- Y = {…, –2, –1, 0, +1, +2. …}

3. Singleton Set

A set containing only one element is called a 'Singleton Set'.

Example

- A = {0}
- B = {x/x is an even number between 3 and 51 i.e., B = {4}

4. Null Set

A set which does not contain any element is called an "empty set or 'void set' or 'Null Set'.

Example:

- Set A denotes names of boys in a girls college.

  i.e., A = { } since nobody is admitted to a girl's college.
- T = {x/x is a perfect square between 10 and 15}

  i.e.. T = { } since no number which is a perfect square exists between 10 and 15.

A null a set is denoted by the greek letter ϕ (read as phi)

Example:

- = ϕ implies 'set T is a null Set'.

But T = {ϕ} implies 'set T is a singleton set with ϕ as an element'

5. Universal Set

A Universal Set is a set of all elements which are taken into consideration in a discussion. It is usually denoted by the capital letter U or otherwise defined in the context. In this text we shall use S to indicate a universal set since it is more convenient for application to probability.

For instance. Let S = {1, 2, 3, 4. 5, 6} be a universal set, showing possible numbers on a die.

6. Sub-sets and Super-sets

Let A and B be two sets. If every element of B is present in A, then B is a 'Sub Set' of A. ie., $B \subset A$. In other words. A is a 'Super Set' of B i.e., $A \supset B$

Example:

i. If A {a. b, c, d, e} and B {a, d}

then $B \subset A$ or $A \supset B$

If A = {2, 4} and B = {I, 2, 3, 4, 5}

then $A \subset B$ or $B \supset A$

- **Equal Sets**

Two sets A and B are said to be equal if $A \subset B$ and $B \supset A$ and is denoted by A = B

Example:

i. Let A = {3, 2. 5. 6} and B = {2, 5. 6. 3}

Here all the elements of A are elements of B{ie. $A \subset B$} and all the elements of B are elements of A(ie., $B \subset A$). Hence

A = B

8. Equivalent Sets

Two sets A and B are said to be equivalent if they have equal number of elements and is denoted by $A \equiv B$ For example

Let $A = \{X, Y, Z\}$ and $B = \{1, 2, 3\}$ Then A and B are said to be equivalent sets and are denoted by $A \equiv B$

9. Power Set

The power set is defined as the collection of all subsets of a given set. It is also called Master set. Example:

The powerset of a given set {a, b, c} is {ϕ, {a}, {b}, {c}, {a,b}, {b, c}, {c, a}, {a, b, c}}.

The number of elements in a set is called *cardinality* of the set, Thus the cardinality of powerset of a given set having 3 elements is 2^3. Generally the cardinality of power-set of given set having n elements is 2^n.

Venn Diagrams

Sets can be represented diagrammatically using Venn diagrams. These were introduced by John Venn, an English logician.

Here, the Universal set is represented by a rectangle and all other sub-sets by circles or triangles etc. Venn diagrams are especially useful for representing various set operations. Hence we first learn about set operations and employ Venn diagrammatic approach to represent the same.

Set Operations

The basic set operations are (i) union (ii) intersection [iii) compliment and (iv) difference.

i. Union of sets

If A and B are two sets, then the 'union' of sets A and B is the set of all elements which belong to either A or B or both (i.e., which belongs to at least one). It is denoted by $A \cup B$.

That is, $\mathbf{x \in A \cup B}$ **implies** $\mathbf{x \in A}$ **or** $\mathbf{x \in B}$

Example:

If $A = \{3, 8, 5\}$ and $B = \{3, 6, 8\}$

then $A \cup B = \{3, 8, 5\} \cup \{3, 6, 8\} = \{3, 8, 5, 6\}$

2. Intersection of Sets

If A and B are two sets, then the 'intersection" of A and B is the set of all elements which are common to both of them. Intersection of sets A and B is denoted by $A \cap B$

That is, $\mathbf{x \in A \cap B}$ **implies** $\mathbf{x \in A}$ **and** $\mathbf{x \in B}$

Example:

If $A = \{2, 5\}$ and $B = \{5, 7, 9\}$

then $A \cap B = \{2, 5\} \cap \{5, 7, 9\} = \{5\}$

Disjoint Sets

Two sets are said to be 'disjoint' or 'mutually exclusive' if they do not have any common element between them

$(A \cap B) = \phi$ or $(A \cap B) = \{\ \}$, a null set

Example:

If $\mathbf{A = \{1, 2\}}$ **and** $\mathbf{B = \{a, b, c\}}$**.** $\mathbf{(A \cap B) = \phi}$

3. Difference of Sets

If A and B are two sets, A – B is the difference of two sets A and B which contains all elements which belong to A but not to B,

That is $x \in A–B$ implies $x \in A$ and $x \notin B$

Example:

If $A = \{0, 1, 2, 3\}$ and $B = \{2. 3. 5, 7\}$ then

$A–B = \{0, 1\}$

4. Complement of Sets

Suppose A is a sub set of some Universal set S. Its complementary set is the set of all elements of the Universal set S which does not belong to the set A. The complementary set of A is denoted by A' {A dash) or A ($\bar{A}$ bar) or A^C (*A* complement).

That is $x \in A^C$ implies $x \notin A$ but $x \in S$

Note

A complement set A cannot be developed without the elements of the Universal set S being known.

Example

If $S = \{1, 2, 3, 4, 5\}$ and $A = \{2, 4\}$

then $A^C = \{1, 3, 5\}$

Algebra of Sets

The following results are very useful is the context of probability theory. We can easily verify the results by choosing the sets appropriately.

- $A \cup B = B \cup A$, $A \cap B = B \cap A$ - Commutative property
- $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$

  Associative property
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ – Distributive property
- $(A \cup S) = S$, $A \cap S = A$

  $A \cup \phi = A$, $A \cap \phi = \phi$
- $A \cup A^C = S$, $A \cap A^C = \phi$
- $(A \cup B)^C = A^C \cap B^C$

  $(A \cap B)^C = A^C \cup B^C$

More generally,

$$\left(\bigcup_{i=1}^{n} A_i\right)^C = \bigcap_{i=1}^{n} A_i^C$$

$$\left(\bigcap_{i=1}^{n} A_i\right)^C = \bigcup_{i=1}^{n} A_i^C \quad \text{— De' Morgan's Laws.}$$

Set terminology

The following terminologies are verbally used for calculating the probability of occurrence of events where the events are represented by sets.

1. For one event A

i. Occurrence of an event is represented by – A

ii. No occurrence of an event – A^C

2. For two events A and B

i. Occurrence of none – $A^C \cap B^C$

- Occurrence of both A and B – $A \cap B$
- Occurrence of exactly one – $(A \cap B^C) \cup (A^C \cap B)$

iv. Occurrence of at least one – $A \cup B$

3. For three events A, B, and C

i. Occurrence of all – $A \cap B \cap C$

ii. Occurrence of None – $A^C \cap B^C \cap C^C$

- Occurrence exactly of one

  – $(A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C)(A^C \cap B^C \cap C)$
- Occurrence of exactly two

  – $(A \cap B \cap C^C)(A \cap B^C \cap C) \cup (A^C \cap B \cap C)$

v. Occurrence of at least one – $A \cup B \cup C$

Permutations and Combinations

Fundamental Principle:

If an event 'A' can happen in 'n ' ways and another event 'B' can happen in 'n_2' ways, then the number [1]of ways in which both the events A and B can happen in a specified order is '$n_1 \times n_2$'.

If there are three routes from X to Y: two routes from Y to Z then the destination Z can be reached from X in $3 \times 2 = 6$ ways.

Permutation

Definition: Permutation refers to the *arrangement* which can be made by taking some (say r) of things at a time or all of 'n' things at a time with attention given to the order of arrangement of the selected objects.

Mathematicians use a neat notation for permutation (i.e., arrangement) of 'n' objects taking 'r' objects at a time by writing this statement as ${}_nP_r$ or nPr. Here, letter 'P' stands for 'permutation' (i.e., a rule for arrangement).

Suppose we want to arrange 3 students A, B and C by choosing 2 of them at a time. This arrangement can be done in the following ways.

AB, BC, CA, BA, CB and AC

The arrangement of 3 things taken 2 at a time is denoted by $3P_2$.

Therefore, $3P_2 = 6 = 3 \times 2$.

In general, suppose there are 'n' objects to be permuted in a row taking all at a time. This can be done in ${}^nP_n$ different ways. It is given by

$${}_nP_n = n(n-1)(n-2)\ldots 3.\ 2.\ 1$$

Example

$$4P_4 = 4.3.2.1 = 24$$

The permutation of n things taken r at a time (r < n) is given by

$${}_nP_r = n(n-1)\ldots(n-r+1)$$

eg: $7P_5 = 7\times6\times5\times4\times3 = 2520$

Factorial notation

We have a compact notation for the full expression given by the product n (n –1) (n – 2) … 3. 2. 1. This is written as n! read as 'n factorial'.

So, $${}^nP_n = n! = n(n-1)(n-2)\ldots 3.\ 2.\ 1.$$

$$6P_6 = 6! = 6.\ 5.\ 4.\ 3.\ 2.\ 1 = 720$$

By, definition, 0! =1

We have, ${}^nPr = n(n-1)\ldots(n-r+1)$

$$= n(n-1)(n-2)\ldots(n-r+1)\left[\frac{(n-r)(n-r-1)\ldots3.2.1}{(n-r)(n-r-1)\ldots3.2.1}\right]$$

$$np_r = \frac{n!}{(n-r)!}$$

Results

- The number of permutations of n objects when r objects taken at a time when repetition allowed = n^r.
- The number of permutations of n objects when all the n taken at a time when repetition allowed = n^n.
- The number of permutations of n objects of which, n_1 are of one kind, n_2 are of another kind, n_3 are of another kind etc., taking all the ntogetherisgivenby $\dfrac{n!}{n_1!n_2!n_3!\ldots n_k!}$ where $n_1 + n_2 + \ldots + n_k = n$.

Combination

A combination is a grouping or a selection or a collection of all or a part of a given number of things without reference to their order of arrangement.

If three letters, a, b, c are given, ab, bc, ca are the only combinations of the three things a, b, c taken two at a time and it is denoted as $3C_2$. The other permutations ba, cb and ac are not new combinations. They are obtained by permuting each combination among themselves.

So $3P_2 = 3C_2 \times 2!$

or $3C_2 = \dfrac{3P_2}{2!} = \dfrac{3.2}{1.2} = 3$

Co mbi na tio n o f n d iff ere nt th ing s t aken r a t time (r < n)

The number of combinations of n different things taken r at a time is denoted as nCr or nC_r or $\binom{n}{r}$. It is given by

$$nC_r = \frac{nP_r}{r!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot2\cdot3\ \ r}$$

or $$nC_r = \frac{n!}{r!(n-r)!}$$

For example, $^{7}C_{3} = \frac{7!}{3!\,4!} = \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3} = 35$

$$^{10}C_{4} = \frac{10!}{4!\,6!} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = 210$$

Important results

- $nC_n = \frac{n!}{n!\,0!} = 1$. This is the combination of n things taken all at a time.
- $nC_0 = \frac{n!}{n!\,0!} = 1$. This is the combination of n things taken none at a time.
- $nC_r = nC_{n-r}$

This says that, $10C_8 = 10C_2 = \frac{10\cdot 9}{1\cdot 2}$

$$12C_9 = 12\,C_3 = \frac{12\cdot 11\cdot 10}{1\cdot 2\cdot 3} = 220$$

$$100C_{98} = 100\,C_2 = \frac{100\cdot 99}{1\cdot 2} = 4950$$

4. $nC_r + nC_{r-1} = (n+1)\,C_r$.

SOLVED PROBLEMS

Example 1

What is the probability that a leap year selected at random will contain 53 Sundays?

Solution

In a leap year there are 366 days consisting of 52 weeks plus 2 more days. The following are the possible combinations for these two days. (i) Sunday and Monday (ii) Monday and Tuesday (iii) Tuesday and Wednesday □ Wednesday and Thursday (v) Thursday and Friday (vi) Friday and Saturday (vii) Saturday and Sunday.

For getting 53 Sundays in a leap year, out of the two days so obtained one should be a Sunday. There are two cases favourable for getting a Sunday out of the 7 cases.

Required probability = **2/7.**

Example 2

Three coins are tossed. What is the probability of getting (i) all heads □ exactly one head (iii) exactly two heads (iv) atleast one head (v) atleast two heads (vi) at most one head (vii) at most two heads (viii) No head.

Solution

When three coins are tossed, the possible outcomes are given by [HHH, HHT, HTH, THH, HTT, THT, TTH, TTT]

i. P (all heads) = 1/8
ii. P (exactly one head) = 3/8
iii. P (exactly two heads) = 3/8
iv. P (atleast one head) = 7/8
v. P (atleast two heads) = 4/8
vi. P (at most one head) = 4/8
vii. P (at most two heads) = 7/8
viii. P (no head) = 1/8

Example 3

What is the probability of getting a spade or an ace from a pack or cards?

Solution

P (Spade or Ace) = **16/52**

Example 4

A box contains 8 red, 3 white and 9 blue balls. If 3 balls are drawn at random, determine the probability that (a) all three are blue (b) 2 are red and 1 is white (c) atleast one is white and (d) one of each colour is drawn.

Solution

Assume that the balls are dreawn from the urn one by one without replacement.

Example 5

What is the probability of getting 9 cards of the same suit in one hand at a game of bridge?

Solution

One hand in a game of bridge consists of 13 cards. Total number of possible cases = $52C_{13}$

The number of ways in which a particular player can have 9 cards of one suit are $13C_9$ and the number of ways in which the remaining 4 cards are of some other suit are $39C_4$. Since there are 4 suits in a pack of cards, the total number of favourable cases = $4 \times 13C_9 \times 39C_4$.

Required probability = $$\frac{4 \times 13C_9 \times 39C_4}{52C_{13}}$$