

*Notation:*  $Y = (Y_1, Y_2, \dots, Y_n)^T \in \mathbb{R}^N$  and  $U = (U_1, U_2, \dots, U_q)^T \in \mathbb{R}^q$  are random vectors (response and random/mixed effects); their realizations are denoted  $y = (y_1, y_2, \dots, y_n)^T$  and  $u = (u_1, u_2, \dots, u_q)^T$ ;  $\beta \in \mathbb{R}^p$  is the vector of fixed effects;  $X \in \mathbb{R}^{n \times p}$  and  $Z \in \mathbb{R}^{n \times q}$  are model matrices, where  $x_i^T$  or  $z_i^T$  denote the  $i$ th row of  $X$  or  $Z$  respectively.

Generalized linear mixed model (GLMM):

$$\begin{aligned} (Y_i|U = u) &\sim \text{indep.} f_{Y_i|U}(y_i|u) \text{ for } i \in \{1, 2, \dots, n\}, U \sim f_U(u). \\ f_{Y_i|U}(y_i|u) &= h(y_i) \exp(\langle T(y_i), \eta \rangle - A(\eta)), \\ \mu_i &= E(Y_i|U = u) = f(\eta), \\ g(\mu_i) &= x_i^T \beta + z_i^T u. \end{aligned}$$

Normality

$Z = 0$

Linear mixed model (LMM):

$$(Y|U = u) \sim \mathcal{N}(X\beta + Zu, \sigma^2 I), \\ U \sim \mathcal{N}(0, \Sigma_\theta).$$

*Solution:*

$$\hat{\beta}_{\text{ML}} = (X^T \hat{V}^{-1} X)^{-1} X^T \hat{V}^{-1} y \text{ is MLE, where } \hat{V} = Z \Sigma_{\hat{\theta}_{\text{ML}}} Z^T + \hat{\sigma}_{\text{ML}}^2 I \text{ is MLE of Var}(Y).$$

Generalized linear model (GLM):

$$\begin{aligned} Y_i &\sim \text{indep.} f_{Y_i}(y_i) \text{ for } i \in \{1, 2, \dots, n\}, \\ f_{Y_i}(y_i) &= h(y_i) \exp(\langle T(y_i), \eta \rangle - A(\eta)), \\ \mu_i &= E(Y_i) = f(\eta), \\ g(\mu_i) &= x_i^T \beta. \end{aligned}$$

$Z = 0$

Normality

Linear model (LM):

$$\begin{aligned} Y &\sim \mathcal{N}(X\beta, \sigma^2 I). \\ \text{Closed form solution:} \\ \hat{\beta} &= (X^T X)^{-1} X^T y \text{ is MLE,} \\ &\text{UMVU and BLUE.} \end{aligned}$$

Normality,  $\lambda = 0$

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LASSO:

$$\begin{aligned} E(Y) &= X\beta, \text{ tuning parameter } \lambda \geq 0, \\ \hat{\beta} &= \text{argmin}_{b \in \mathbb{R}^p} \left\{ \frac{1}{2n} \|y - Xb\|_2^2 + \lambda \|b\|_1 \right\}. \end{aligned}$$

Ridge regression:

$$\begin{aligned} E(Y) &= X\beta, \text{ tuning parameter } \lambda \geq 0, \\ \hat{\beta} &= \text{argmin}_{b \in \mathbb{R}^p} \left\{ \frac{1}{2n} \|y - Xb\|_2^2 + \lambda \|b\|_2^2 \right\}. \\ \text{Closed form solution:} \\ \hat{\beta} &= (X^T X + \lambda I)^{-1} X^T y. \end{aligned}$$

$\alpha = 1$

$\alpha = 0$

Elastic net:

$$\begin{aligned} E(Y) &= X\beta, \text{ tuning parameters } \lambda \geq 0 \text{ and } \alpha \in [0, 1], \\ \hat{\beta} &= \text{argmin}_{b \in \mathbb{R}^p} \left\{ \frac{1}{2} \|y - Xb\|_2^2 + \lambda \left[ \frac{1}{2} (1 - \alpha) \|b\|_2^2 + \alpha \|b\|_1 \right] \right\}. \end{aligned}$$