

# Practical Large Scale Optimization with illustrations in Matlab

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# Chapter 0

## Review: Linear Programs and the Revised Simplex Method

### 0.1 The simplex method

- Solves an LP in standard form

$$\begin{array}{llllll}
 \min & c_1x_1 & +c_2x_2 & +\dots & +c_nx_n & \\
 \text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\dots & +a_{1n}x_n & = b_1 \\
 & a_{21}x_1 & +a_{22}x_2 & +\dots & +a_{2n}x_n & = b_2 \\
 & & & & & \\
 & a_{m1}x_1 & +a_{m2}x_2 & +\dots & +a_{mn}x_n & = b_n \\
 & x_1, & x_2, & \dots & x_n & \geq 0
 \end{array}$$

- $\leq$  constraints  $\rightarrow$  add a non-negative slack
- $\geq$  constraint  $\rightarrow$  subtract a non-negative surplus
- free variables  $\rightarrow$  replace  $x$  free by  $x = x' - x''$  where  $x', x'' \geq 0$ .

Example:

$$\begin{array}{llllll}
 \min & 2x_1 & -3x_2 & & & \\
 \text{s.t.} & x_1 & -3x_2 & +2x_3 & \leq & 3 \\
 & -x_1 & +2x_2 & & \geq & 2 \\
 & x_1 \text{ free} & x_2 \geq 0 & x_3 \geq 0 & & 
 \end{array}$$

$$\begin{array}{llllllllll}
 \min & 2x_1' & -2x_1'' & -3x_2 & & & & & & \\
 \text{s.t.} & x_1' & -x_1'' & -3x_2 & +2x_3 & +s_1 & & & = & 3 \\
 & -x_1' & +x_1'' & +2x_2 & & & -s_2 & & = & 2 \\
 x_1' \geq 0, & x_1'' \geq 0, & x_2 \geq 0, & x_3 \geq 0, & s_1 \geq 0, & s_2 \geq 0 & & & & 
 \end{array}$$

### 0.2 The Simplex Method:Steps

- Find an initial basic feasible solution ([Two-phase](#) or the [big M](#) methods)
- Repeat until optimal solution is found:
  1. Test for optimality (using the reduced costs)

2. If not optimal, change the basis
3. Choose an entering variable.
4. Choose a leaving variable using the Minimum Ratio Test.
5. Update the solution

Example:

$$\begin{array}{llll}
 \max & 5x_1 & +6x_2 & +2x_3 \\
 \text{s.t.} & 4x_1 & +x_2 & \leq 3 \\
 & x_1 & -x_2 & +x_3 \leq 2 \\
 & x_1, & x_2, & x_3 \geq 0
 \end{array}$$

### 0.3 The Simplex Method:Example

Put in standard format and set-up the initial tableau

	$z$	$x_1$	$x_2$	$x_3$	$s_2$	$s_3$	rhs
Row 0	1	-5	-6	-2	0	0	0
Row 1	0	4	1	0	1	0	3
Row 2	0	1	-1	1	0	1	2

- **Check for optimality:** Is the current basic feasible solution optimal? Are all the reduced costs non-negative? **No**
- **Entering variable:** The variable with the most negative reduced cost enters ( $x_2$ )
- **Leaving variable:** Which variable leaves the basic? ( $s_2$  or  $s_3$ ) :
- **Ratio test:** take the minimum over the positive coefficients only: coefficient of  $s_2$  is 1 while the coefficient of  $s_3$  is  $-1$ : so  $s_2$  leaves the basic
- **Make the column of the entering variable ( $x_2$ ) an identity: Gauss-Jordan elimination:**

The second tableau

$z$	$x_1$	$x_2$	$x_3$	$s_2$	$s_3$	rhs
1	19	0	-2	6	0	18
0	4	1	0	1	0	3
0	5	0	1	1	1	5

- **Current bfs:**  $x_2 = 3; s_3 = 0, z = 18$ ;
- **Check for optimality:** the reduced cost of  $x_3$  is negative
- **Entering variable:**  $x_3$
- **Leaving variable:** Which variable leaves the basic? ( $x_2$  or  $s_3$ ) :
- **Ratio test:** take the minimum over the positive coefficients only: coefficient of  $x_2$  is 0 while the coefficient of  $s_3$  is 1: so  $s_3$  leaves the basic
- **make the column of the entering variable ( $x_3$ ) an identity: Gauss-Jordan elimination:**

Final tableau

$z$	$x_1$	$x_2$	$x_3$	$s_2$	$s_3$	rhs
1	29	0	0	8	2	28
0	4	1	0	1	0	3
0	5	0	1	1	1	5

- **Current bfs:**  $x_2 = 3; x_3 = 5, z = 28;$
- **Check for optimality:** all reduced costs are non-negative:

optimal solution:  $x_1^* = 0; x_2^* = 3; x_3^* = 5; s_2^* = 0; s_3^* = 0$  with objective value 28.

## 0.4 The Revised Simplex Method

$$\begin{aligned} \min \quad & z = c_B x_B + c_N x_N \\ \text{s.t.} \quad & Bx_B + Nx_N = b \\ & x_N, x_B \geq 0 \end{aligned}$$

$$\begin{aligned} x_B &= B^{-1}b - B^{-1}Nx_N \\ z &= c_B B^{-1}b + [c_N - c_B B^{-1}N]x_N \end{aligned}$$

Basic Feasible Solution:  $x_B = B^{-1}b; x_N = 0; z = c_B B^{-1}b$

- If reduced costs  $c_N - c_B B^{-1}N \geq 0$  then stop the current basis is optimal
- Else choose an entering variable, one with a negative reduced cost, say  $x_{\tilde{j}}$
- Choose the leaving variable through a ratio test:  $x_B = B^{-1}b - B^{-1}\tilde{N}x_N - B^{-1}a_{\tilde{j}}x_{\tilde{j}}$  where  $\tilde{N}$  is the collection of nonbasic columns other than that corresponding to  $x_{\tilde{j}}$ .
- setting  $x_{\tilde{j}} = t, x_B = B^{-1}b - tB^{-1}a_{\tilde{j}}$ .

To prevent  $x_B$  from becoming negative,

$$x_B = B^{-1}b - tB^{-1}a_{\tilde{j}} \geq 0 \text{ implies that}$$

$$B^{-1}b - tB^{-1}a_{\tilde{j}} \geq 0 \text{ which means}$$

$$t = \min \left\{ \frac{(B^{-1}b)_i}{(B^{-1}a_{\tilde{j}})_i}, (B^{-1}a_{\tilde{j}})_i > 0 \right\}$$

The leaving variable is  $x_s$  where

$$s = \operatorname{argmin} \left\{ \frac{(B^{-1}b)_i}{(B^{-1}a_j)_i}, (B^{-1}a_j)_i > 0 \right\}$$

i.e., the one corresponding to the minimum ratio test.

## 0.5 Initializing the Simplex Method

$$\begin{aligned} \min \quad & z = c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Need an initial basic feasible solution
- Add artificial variables  $x^a$ , so that

$$\begin{aligned} \text{s.t. } & Ax + x^a = b \\ & x, x^a \geq 0 \end{aligned}$$

- We have changed the problem so we need to make sure that the artificial variables are pushed to 0
- [The two-phase method](#)
- Phase I:

$$\begin{aligned} \min & \sum_i x_i^a \\ \text{s.t. } & Ax + x^a = b \\ & x, x^a \geq 0 \end{aligned}$$

- Suppose the optimal solution is  $x^*, x^{a*}$
- if at optimality  $x_{a*} \neq 0$ , the problem is infeasible
- else  $Ax^* = b$ ,  $x^*$  is used to start the Simplex (Phase II)
- [The Big M method](#)

$$\begin{aligned} \min & c^T x + M \sum_i x_i^a \\ \text{s.t. } & Ax + x^a = b \\ & x, x^a \geq 0 \end{aligned}$$

- M is a very large positive number
- Suppose the optimal solution is  $x^*, x^{a*}$
- if at optimality  $x_{a*} \neq 0$ , the problem is infeasible.
- else  $x_{a*} = 0$ , and  $x^*$  is optimal.

## 0.6 Example: Unbounded problem

$$\begin{aligned} \max & \quad x_1 \quad +x_2 \\ \text{s.t. } & \quad -2x_1 \quad +x_2 \leq 2 \\ & \quad x_1 \quad -2x_2 \leq 2 \\ & \quad x_1, \quad x_2 \geq 0 \end{aligned} = \begin{aligned} \min & \quad -x_1 \quad -x_2 \\ \text{s.t. } & \quad -2x_1 \quad +x_2 \quad +x_3 = 2 \\ & \quad x_1 \quad -2x_2 \quad +x_4 = 2 \\ & \quad x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0 \end{aligned}$$

- Iteration 1:
  - Initial basis:  $\{x_3, x_4\}$ ; Nonbasic:  $\{x_1, x_2\}$
  - $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $N = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ ,  $B^{-1} = B$ .
  - Optimality Check: Reduced costs:
 
$$c_N - c_B B^{-1} N = [-1, -1] - [0, 0] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = [-1, -1]$$
  - Entering variable:  $x_1$
  - Leaving variable: ratio test;  $B^{-1}a_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ;  $B^{-1}b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ;  $\min\{\frac{2}{1}\} = 2 \Rightarrow x_4$  leaves

## 0.7 Example

Unbounded problem

- Iteration 2

- New basis:  $\{x_1, x_3\}$ ; Nonbasic:  $\{x_2, x_4\}$
- $B = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$ ,  $N = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ ,  $B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ ;  $c_B = [-1, 0]$ ;  $c_N = [-1, 0]$ ;
- Optimality Check: Reduced costs:  
 $c_N - c_B B^{-1} N = [-1, 0] - [-1, 0] \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = [-1, 1]$
- Entering variable:  $x_2$
- Leaving variable: ratio test;  $B^{-1} a_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ ; All entries are  $< 0$ ; ratio test fails, problem unbound

## 0.8 Duality in LP: Primal-dual relationship

Primal	Minimization LP	Maximization LP	Dual
Constraints	$\geq$ $\leq$ $=$	$\geq 0$ $\leq 0$ free	Variables
Variables	$\geq 0$ $\leq 0$ free	$\leq$ $\geq$ $=$	Constraints

Example:

$$\begin{array}{l|l}
 \text{[Primal LP:]} & \min \quad c^T x \\
 & s.t. \quad Ax = b \\
 & \quad \quad x \geq 0 \\
 \hline
 \text{[Dual LP:]} & \max \quad b^T y \\
 & s.t. \quad A^T y \leq c \\
 & \quad \quad y \text{ free}
 \end{array}$$

## 0.9 Duality in LP: Primal-dual Theory:

Consider the pair of duals  $[P]$  and  $[D]$

$$\begin{array}{l|l}
 [P:] & \min \quad c^T x \\
 & s.t. \quad Ax = b \\
 & \quad \quad x \geq 0 \\
 \hline
 [D] & \max \quad b^T y \\
 & s.t. \quad A^T y \leq c \\
 & \quad \quad y \text{ free}
 \end{array}$$

For any  $\bar{x}$  and  $\bar{y}$  feasible to  $[P]$  and  $[D]$  respectively, then

$$b^T \bar{y} \leq c^T \bar{x}$$

If

$$c^T \bar{x} = b^T \bar{y} \quad : \quad \text{Strong duality}$$

OR

$$\begin{array}{ll}
 \bar{x}_i (a_i^T \bar{y} - c_i) = 0 & : \text{complementarity slackness} \\
 (\text{Also } \bar{y}_j (a_j^T \bar{x} - b_j) = 0 & : \text{complementarity slackness, but redundant in this case})
 \end{array}$$

Then  $\bar{x}$  and  $\bar{y}$  are optimal to  $[P]$  and  $[D]$  respectively.

## 0.10 Sensitivity Analysis in LP

- **Change in  $c$ :** check if the current basis remains optimal by checking optimality (the reduced costs).
- Solve for  $c_N - c_B B^{-1} N \geq 0$
- **Change in  $b$ :** check if the current basis remains optimal by checking feasibility (rhs).
- Solve for  $B^{-1} b \geq 0$

**For more details, consult**

- Winston W.L. and M. Venkataramanan. *Introduction to mathematical programming*, Duxbury Press, 2003.
- Hillier F.S. and G.J. Lieberman. *Introduction to Operations Research*, McGraw Hill, 2005.