

Practical Large Scale Optimization with illustrations in Matlab

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Chapter 0

Review: Linear Programs and the Revised Simplex Method

0.1 The simplex method

- Solves an LP in standard form

$$\begin{array}{lllllll} \min & c_1x_1 & +c_2x_2 & +\dots & +c_nx_n & & \\ \text{s.t.} & a_{11}x_1 & +a_{12}x_2 & +\dots & +a_{1n}x_n & = b_1 & \\ & a_{21}x_1 & +a_{22}x_2 & +\dots & +a_{2n}x_n & = b_2 & \\ & a_{m1}x_1 & +a_{m2}x_2 & +\dots & +a_{mn}x_n & = b_n & \\ & x_1, & x_2, & \dots & x_n & \geq 0 & \end{array}$$

- \leq constraints \rightarrow add a non-negative slack
- \geq constraint \rightarrow subtract a non-negative surplus
- free variables \rightarrow replace x free by $x = x' - x''$ where $x', x'' \geq 0$.

Example:

$$\begin{array}{llllll} \min & 2x_1 & -3x_2 & & & \\ \text{s.t.} & x_1 & -3x_2 & +2x_3 & \leq 3 & \\ & -x_1 & +2x_2 & & \geq 2 & \\ & x_1 \text{ free} & x_2 \geq 0 & x_3 \geq 0 & & \end{array}$$

$$\begin{array}{llllllll} \min & 2x_1' & -2x_1'' & -3x_2 & & & & \\ \text{s.t.} & x_1' & -x_1'' & -3x_2 & +2x_3 & +s_1 & & = 3 \\ & -x_1' & +x_1'' & +2x_2 & & & -s_2 & = 2 \\ & x_1' \geq 0, & x_1'' \geq 0, & x_2 \geq 0, & x_3 \geq 0, & s_1 \geq 0, & s_2 \geq 0 & \end{array}$$

0.2 The Simplex Method: Steps

- Find an initial basic feasible solution ([Two-phase](#) or the [big M](#) methods)
- Repeat until optimal solution is found:
 1. Test for optimality (using the reduced costs)

2. If not optimal, change the basis
3. Choose an entering variable.
4. Choose a leaving variable using the Minimum Ratio Test.
5. Update the solution

Example:

$$\begin{array}{lllll} \max & 5x_1 & +6x_2 & +2x_3 \\ \text{s.t.} & 4x_1 & +x_2 & & \leq 3 \\ & x_1 & -x_2 & +x_3 & \leq 2 \\ & x_1, & x_2, & x_3 & \geq 0 \end{array}$$

0.3 The Simplex Method:Example

Put in standard format and set-up the initial tableau

	z	x_1	x_2	x_3	s_2	s_3	rhs
Row 0	1	-5	-6	-2	0	0	0
Row 1	0	4	1	0	1	0	3
Row 2	0	1	-1	1	0	1	2

- **Check for optimality:** Is the current basic feasible solution optimal? Are all the reduced costs non-negative? **No**
- **Entering variable:** The variable with the most negative reduced cost enters (x_2)
- **Leaving variable:** Which variable leaves the basic? (s_2 or s_3) :
- **Ratio test:** take the minimum over the positive coefficients only: coefficient of s_2 is 1 while the coefficient of s_3 is -1: so s_2 leaves the basic
- **Make the column of the entering variable (x_2) an identity: Gauss-Jordan elimination:**

The second tableau

z	x_1	x_2	x_3	s_2	s_3	rhs
1	19	0	-2	6	0	18
0	4	1	0	1	0	3
0	5	0	1	1	1	5

- **Current bfs:** $x_2 = 3; s_3 = 0, z = 18;$
- **Check for optimality:** the reduced cost of x_3 is negative
- **Entering variable:** x_3
- **Leaving variable:** Which variable leaves the basic? (x_2 or s_3) :
- **Ratio test:** take the minimum over the positive coefficients only: coefficient of x_2 is 0 while the coefficient of s_3 is 1: so s_3 leaves the basic
- **make the column of the entering variable (x_3) an identity: Gauss-Jordan elimination:**

Final tableau

z	x_1	x_2	x_3	s_2	s_3	rhs
1	29	0	0	8	2	28
0	4	1	0	1	0	3
0	5	0	1	1	1	5

- **Current bfs:** $x_2 = 3; x_3 = 5, z = 28;$
- **Check for optimality:** all reduced costs are non-negative:

optimal solution: $x_1^* = 0; x_2^* = 3; x_3^* = 5; s_2^* = 0; s_3^* = 0$ with objective value 28.

0.4 The Revised Simplex Method

$$\begin{aligned} \min \quad & z = c_B x_B + c_N x_N \\ \text{s.t.} \quad & B x_B + N x_N = b \\ & x_N, x_B \geq 0 \\ x_B &= B^{-1}b - B^{-1}N x_N \\ z &= c_B B^{-1}b + [c_N - c_B B^{-1}N] x_N \end{aligned}$$

Basic Feasible Solution: $x_B = B^{-1}b; x_N = 0; z = c_B B^{-1}b$

- If reduced costs $c_N - c_B B^{-1}N \geq 0$ then stop the current basis is optimal
- Else choose an entering variable, one with a negative reduced cost, say $x_{\tilde{j}}$
- Choose the leaving variable through a ratio test: $x_B = B^{-1}b - B^{-1}\tilde{N}x_N - B^{-1}a_{\tilde{j}}x_{\tilde{j}}$ where \tilde{N} is the collection of nonbasic columns other than that corresponding to $x_{\tilde{j}}$.
- setting $x_{\tilde{j}} = t, x_B = B^{-1}b - tB^{-1}a_{\tilde{j}}$.

To prevent x_B from becoming negative,

$$x_B = B^{-1}b - tB^{-1}a_{\tilde{j}} \geq 0 \text{ implies that}$$

$$B^{-1}b - tB^{-1}a_{\tilde{j}} \geq 0 \text{ which means}$$

$$t = \min \left\{ \frac{(B^{-1}b)_i}{(B^{-1}a_{\tilde{j}})_i}, (B^{-1}a_{\tilde{j}})_i > 0 \right\}$$

The leaving variable is x_s where

$$s = \arg \min \left\{ \frac{(B^{-1}b)_i}{(B^{-1}a_j)_i}, (B^{-1}a_j)_i > 0 \right\}$$

i.e., the one corresponding to the minimum ratio test.

0.5 Initializing the Simplex Method

$$\begin{aligned} \min \quad & z = c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Need an initial basic feasible solution
- Add artificial variables x^a , so that

$$\begin{aligned} s.t. \quad & Ax + x^a = b \\ & x, x^a \geq 0 \end{aligned}$$

- We have changed the problem so we need to make sure that the artificial variables are pushed to 0
- [The two-phase method](#)
- Phase I:

$$\begin{aligned} \min \quad & \sum_i x_i^a \\ s.t. \quad & Ax + x^a = b \\ & x, x^a \geq 0 \end{aligned}$$

- Suppose the optimal solution is x^*, x^{a*}
- if at optimality $x_{a*} \neq 0$, the problem is infeasible
- else $Ax^* = b$, x^* is used to start the Simplex (Phase II)
- [The Big M method](#)

$$\begin{aligned} \min \quad & c^T x + M \sum_i x_i^a \\ s.t. \quad & Ax + x^a = b \\ & x, x^a \geq 0 \end{aligned}$$

- M is a very large positive number
- Suppose the optimal solution is x^*, x^{a*}
- if at optimality $x_{a*} \neq 0$, the problem is infeasible.
- else $x_{a*} = 0$, and x^* is optimal.

0.6 Example:Unbounded problem

$$\begin{array}{lll} \max & x_1 & +x_2 \\ s.t. & -2x_1 & +x_2 \leq 2 \\ & x_1 & -2x_2 \leq 2 \\ & x_1, & x_2 \geq 0 \end{array} = \begin{array}{lll} \min & -x_1 & -x_2 \\ s.t. & -2x_1 & +x_2 +x_3 = 2 \\ & x_1 & -2x_2 +x_4 = 2 \\ & x_1, & x_2, & x_3, & x_4 \geq 0 \end{array}$$

- Iteration 1:
 - Initial basis: $\{x_3, x_4\}$; Nonbasic: $\{x_1, x_2\}$
 - $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, N = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, B^{-1} = B.$
 - Optimality Check: Reduced costs: $c_N - c_B B^{-1}N = [-1, -1] - [0, 0] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = [-1, -1]$
 - Entering variable: x_1
 - Leaving variable: ratio test; $B^{-1}a_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}; B^{-1}b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}; \min\{\frac{2}{-1}\} = 2 \Rightarrow x_4$ leaves

0.7 Example

Unbounded problem

- Iteration 2

- New basis: $\{x_1, x_3\}$; Nonbasic: $\{x_2, x_4\}$
- $B = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}; c_B = [-1, 0]; c_N = [-1, 0];$
- Optimality Check: Reduced costs:
 $c_N - c_B B^{-1} N = [-1, 0] - [-1, 0] \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = [-1, 1]$
- Entering variable: x_2
- Leaving variable: ratio test; $B^{-1} a_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$; All entries are < 0 ; ratio test fails, problem unbound

0.8 Duality in LP: Primal-dual relationship

Primal Minimization LP		Maximization LP	Dual
Constraints	\geq \leq $=$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	\leq \geq =	Constraints

Example:

$$\begin{array}{ll} [\text{Primal LP:}] & \min \quad c^T x \\ & \text{s.t.} \quad Ax = b \\ & \quad x \geq 0 \end{array} \quad \left| \quad \begin{array}{ll} [\text{Dual LP:}] & \max \quad b^T y \\ & \text{s.t.} \quad A^T y \leq c \\ & \quad y \text{free} \end{array} \right.$$

0.9 Duality in LP: Primal-dual Theory:

Consider the pair of duals $[P]$ and $[D]$

$$\begin{array}{ll} [P:] & \min \quad c^T x \\ & \text{s.t.} \quad Ax = b \\ & \quad x \geq 0 \end{array} \quad \left| \quad \begin{array}{ll} [D] & \max \quad b^T y \\ & \text{s.t.} \quad A^T y \leq c \\ & \quad y \text{free} \end{array} \right.$$

For any \bar{x} and \bar{y} feasible to $[P]$ and $[D]$ respectively, then

$$b^T \bar{y} \leq c^T \bar{x}$$

If

$$c^T \bar{x} = b^T \bar{y} \quad : \quad \text{Strong duality}$$

OR

$$\begin{aligned} \bar{x}_i (a_i^T \bar{y} - c_i) &= 0 & : & \text{complementarity slackness} \\ (\text{Also } \bar{y}_j (a_j^T \bar{x} - b_j) &= 0 & : & \text{complementarity slackness, but redundant in this case}) \end{aligned}$$

Then \bar{x} and \bar{y} are optimal to $[P]$ and $[D]$ respectively.

0.10 Sensitivity Analysis in LP

- **Change in c :** check if the current basis remains optimal by checking optimality (the reduced costs).
- Solve for $c_N - c_B B^{-1}N \geq 0$
- **Change in b :** check if the current basis remains optimal by checking feasibility (rhs).
- Solve for $B^{-1}b \geq 0$

For more details, consult

- Winston W.L. and M. Venkataramanan. *Introduction to mathematical programming*, Duxbury Press, 2003.
- Hillier F.S. and G.J. Lieberman. *Introduction to Operations Research*, McGraw Hill, 2005.