More on …

Set Packing/ Set Covering/Column Generation formulations

Assignment-type formulations

$$z_k = \begin{cases} 1 & \text{if } bin\ k\ is\ used \\ 0 & otherwise. \end{cases}$$

$k = 1, \dots, K = 5.$

$$y_{kl} = \begin{cases} 1 & \text{if } item\ l\ is\ assigend\ to\ bin\ k \\ 0 & otherwise. \end{cases}$$

$k = 1, \dots, K = 5; l = 1, \dots L = 5.$

$$
\begin{aligned}
\min \quad & \sum_{k=1}^{5} z_k = z_1 + z_2 + z_3 + z_4 + z_5 \\
\text{s.t.} \quad & y_{11} + y_{21} + y_{31} + y_{41} + y_{51} = 1; \\
& y_{12} + y_{22} + y_{23} + y_{24} + y_{25} = 1; \\
& y_{13} + y_{23} + y_{33} + y_{43} + y_{53} = 1; \\
& y_{14} + y_{24} + y_{34} + y_{44} + y_{54} = 1; \\
& y_{15} + y_{25} + y_{35} + y_{45} + y_{55} = 1; \\
& 20y_{11} + 50y_{12} + 50y_{13} + 50y_{14} + 70y_{15} \leq 100; \\
& 20y_{21} + 50y_{22} + 50y_{23} + 50y_{24} + 70y_{25} \leq 100; \\
& 20y_{31} + 50y_{32} + 50y_{33} + 50y_{34} + 70y_{35} \leq 100; \\
& 20y_{41} + 50y_{42} + 50y_{43} + 50y_{44} + 70y_{45} \leq 100; \\
& 20y_{51} + 50y_{52} + 50y_{53} + 50y_{54} + 70y_{55} \leq 100; \\
& y_{11} \leq z_1; y_{12} \leq z_1; y_{13} \leq z_1; y_{14} \leq z_1; y_{15} \leq z_1; \\
& y_{21} \leq z_2; y_{22} \leq z_2; y_{23} \leq z_2; y_{24} \leq z_2; y_{25} \leq z_2; \\
& y_{31} \leq z_3; y_{32} \leq z_3; y_{33} \leq z_3; y_{34} \leq z_3; y_{35} \leq z_3; \\
& y_{41} \leq z_4; y_{42} \leq z_4; y_{43} \leq z_4; y_{44} \leq z_4; y_{45} \leq z_4; \\
& y_{51} \leq z_5; y_{52} \leq z_5; y_{53} \leq z_5; y_{54} \leq z_5; y_{55} \leq z_5; \\
& y_{11}, y_{12}, \dots, y_{55}, z_1, z_2, z_3, z_4, z_5 \in \{0, 1\}
\end{aligned}
$$

MIP Formulation

$$z_k = \begin{cases} 1 & \text{if } bin\ k\ is\ used \\ 0 & otherwise. \end{cases}$$

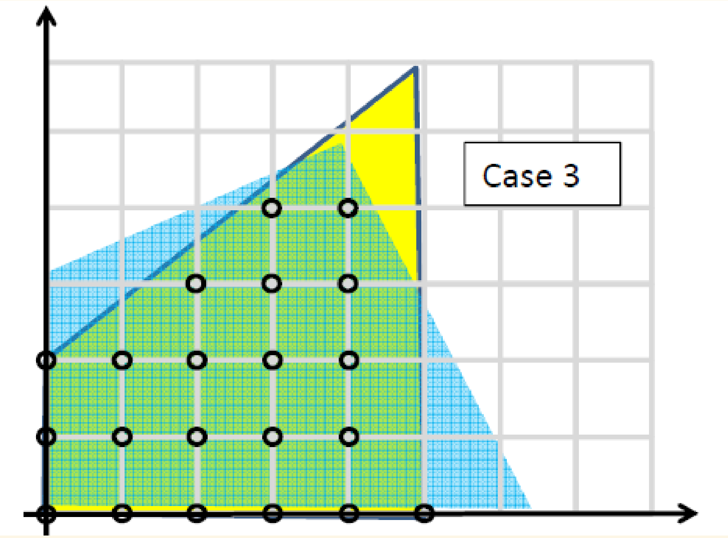
$k = 1, \ldots, K = 5.$

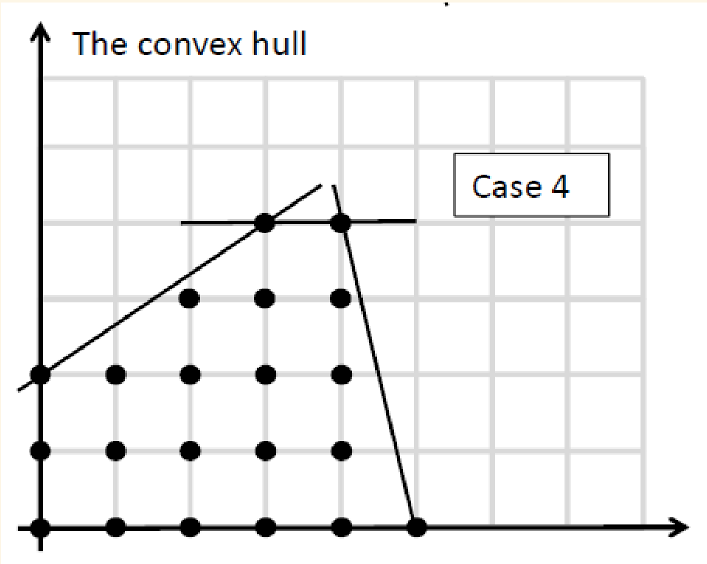
$$y_{kl} = \begin{cases} 1 & \text{if } item\ l\ is\ assigend\ to\ bin\ k \\ 0 & otherwise. \end{cases}$$

$k = 1, \ldots, K = 5; l = 1, \ldots L = 5.$

$$
\begin{array}{ll}
\min & \sum_{k=1}^{5} z_k = z_1 + z_2 + z_3 + z_4 + z_5 \\
\text{s.t.} & y_{11} + y_{21} + y_{31} + y_{41} + y_{51} = 1; \\
& y_{12} + y_{22} + y_{23} + y_{24} + y_{25} = 1; \\
& y_{13} + y_{23} + y_{33} + y_{43} + y_{53} = 1; \\
& y_{14} + y_{24} + y_{34} + y_{44} + y_{54} = 1; \\
& y_{15} + y_{25} + y_{35} + y_{45} + y_{55} = 1; \\
& 20y_{11} + 50y_{12} + 50y_{13} + 50y_{14} + 70y_{15} - 100z_1 \leq 0; \\
& 20y_{21} + 50y_{22} + 50y_{23} + 50y_{24} + 70y_{25} - 100z_2 \leq 0; \\
& 20y_{31} + 50y_{32} + 50y_{33} + 50y_{34} + 70y_{35} - 100z_3 \leq 0; \\
& 20y_{41} + 50y_{42} + 50y_{43} + 50y_{44} + 70y_{45} - 100z_4 \leq 0; \\
& 20y_{51} + 50y_{52} + 50y_{53} + 50y_{54} + 70y_{55} - 100z_5 \leq 0; \\
& y_{11}, y_{12}, \ldots, y_{55}, z_1, z_2, z_3, z_4, z_5 \in \{0, 1\}
\end{array}
$$

Strength of Formulations. Which one is better?





- The tightest feasible region is given by the convex hull of integer points.
- The formulation that corresponds to the convex hull will solve the MIP at the root node as every extreme poir of the convex hull is an integer solution.

$$
\begin{aligned}
[BP_{ad}]: \quad & \min \quad \sum_{k=1}^{K} z_k && \\
& \text{s.t.} \quad \sum_{k=1}^{K} y_{kl} = 1 && l = 1, ..., L \\
& \sum_{l=1}^{L} D_l y_{kl} \leq V z_k && k = 1, ..., K \\
& y_{kl} \leq z_k && k = 1, ..., K, l = 1, ..., L \\
& y_{kl}, z_k \in \{0, 1\} && k = 1, ..., K, l = 1, ..., L.
\end{aligned}
$$

Column Generation /Set Covering Formulation

$$\alpha_h = \begin{cases} 1 & \text{if packing } h \text{ is used;} \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} [BP_{sc}]: \quad & \min \quad \sum_{h=1}^{H} \alpha_h \\ & \text{s.t.} \quad \sum_{h=1}^{H} a_l^h \alpha_h \geq 1 & l = 1, ..., L \\ & \alpha_h \in \{0,1\} & h = 1, ..., H \end{aligned}$$

$$\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}$$

Cover-inequality Formulation
(based on Cover Inequalities)

$$\sum_{l=1}^{L} D_l y_{kl} \leq V$$

are replaced with

$$\sum_{l \in C} y_{kl} \leq (|C| - 1) \qquad \text{for } C \subseteq \{1, 2, .., L\}, \text{where} \sum_{l \in C} D_l > V$$

$$
\begin{aligned}
[BP_{ci}]: \quad \min \quad & \sum_{k=1}^{K} z_k \\
\text{s.t.} \quad & \sum_{k=1}^{K} y_{kl} = 1 && l = 1, ..., L \\
& \sum_{l \in C} y_{kl} \leq (|C| - 1) && C \subseteq \{1, 2, .., L\}, \text{such that} \sum_{l \in C} D_l > V \\
& y_{kl} \leq z_k && k = 1, ..., K, l = 1, ..., L. \\
& y_{kl}, z_k \in \{0, 1\} && k = 1, ..., K, l = 1, ..., L.
\end{aligned}
$$

Generate Constraints Iteratively.

Start with:

$$
\begin{aligned}
\min \quad & \sum_{k=1}^{5} z_k = z_1 + z_2 + z_3 + z_4 + z_5 \\
\text{s.t.} \quad & y_{11} + y_{21} + y_{31} + y_{41} + y_{51} = 1; \\
& y_{12} + y_{22} + y_{23} + y_{24} + y_{25} = 1; \\
& y_{13} + y_{23} + y_{33} + y_{43} + y_{53} = 1; \\
& y_{14} + y_{24} + y_{34} + y_{44} + y_{54} = 1; \\
& y_{15} + y_{25} + y_{35} + y_{45} + y_{55} = 1; \\
& y_{11} \le z_1; y_{12} \le z_1; y_{13} \le z_1; y_{14} \le z_1; y_{15} \le z_1; \\
& y_{21} \le z_2; y_{22} \le z_2; y_{23} \le z_2; y_{24} \le z_2; y_{25} \le z_2; \\
& y_{31} \le z_3; y_{32} \le z_3; y_{33} \le z_3; y_{34} \le z_3; y_{35} \le z_3; \\
& y_{41} \le z_4; y_{42} \le z_4; y_{43} \le z_4; y_{44} \le z_4; y_{45} \le z_4; \\
& y_{51} \le z_5; y_{52} \le z_5; y_{53} \le z_5; y_{54} \le z_5; y_{55} \le z_5; \\
& y_{11}, y_{12}, \ldots, y_{55}, z_1, z_2, z_3, z_4, z_5 \in \{0, 1\}
\end{aligned}
$$

Solution:

$$\boxed{y_{11} = y_{12} = y_{13} = y_{14} = y_{15} = 1}$$

Add the cover inequalities:

$$\boxed{y_{k1} + y_{k2} + y_{k3} + y_{k4} + y_{k5} \le 4z_k, \quad k = 1, \ldots, 5.}$$

Continue iterating …

More on Set-covering/Set-packing and Column Generation Formulations

Example 1:

A city has six districts. The city's fire department must determine where to build fire stations. They want to build the minimum number of fire stations needed to ensure that at least one fire station is within 15 minutes (driving time) of each city. The times (in minutes) required to drive between the districts are shown below

	Dist. 1	Dist. 2	Dist. 3	Dist. 4	Dist. 5	Dist. 6
Dist. 1	0	10	20	30	30	20
Dist. 2		0	25	35	20	10
Dist. 3			0	15	30	20
Dist. 4				0	15	25
Dist. 5					0	14
Dist. 6						0

Formulate an IP that will tell the fire department the minimum number of fire stations to built and where they should be located? Provide two formulations: an assignment type and a column generation.

Example 2:

Air Canada wants to optimize its crew work schedule on a subnetwork in Ontario consisting of the airports of London, Toronto and Windsor. The fight schedule in a typical working day is given in Table 1.

Flight nb	Departure airport	Destination airport	Departure time	Arriva time
1	Toronto	London	09:00 am	10:00am
2	Windsor	London	10:00 am	10:30am
3	London	Toronto	11:00 am	11:30am
4	London	Toronto	11:30 am	12:30pm
5	Toronto	London	12:00 pm	12:30pm
6	London	Windsor	01:00 pm	01:30pm
7	Toronto	Windsor	01:00 pm	02:00pm
8	Windsor	Toronto	03:00 pm	04:00pm

There is one crew located at each airport at the beginning of a day. A duty is defined as a sequence of flights. The crews must return to their base airport at the end of their duty.
The cost of a duty is the total time elapsed from departure time of the first flight to the arrival time of the last flight in that duty. The objective of Air Canada is to minimize the total cost to cover each flight exactly once. Provide two types of formulations: an assignment type and a column generation type.