UNIVERSITY OF WATERLOO

msci 435:
Advanced Optimization Techniques

Winter 2024

Samir Elhedhli
elhedhli@uwaterloo.ca
Department of Management Sciences,
University of Waterloo,
Canada

Lagrangian Relaxation

$$
\begin{array}{rrl}
\max & 16x_1 + 10x_2 & +4x_4 \\
\text{s.t.} & 8x_1 + 2x_2 & +x_3 + 4x_4 \leq 10 \\
& x_1 + x_2 \leq 1 & \\
& & x_3 + x_4 \leq 1 \\
& x_i \in \{0,1\}, & i = 1,2,3,4.
\end{array}
$$

Structure:

Both are relaxations:

- provide upper bounds for a maximization problem
- provide lower bounds for a minimization problem

Lagrangian Relaxation

M. Fisher, An Applications Oriented Guide to Lagrangian Relaxation, Interfaces vol 15 No 2 pp 10-21 (1985).

$$
\begin{array}{lll}
\max & 16x_1 + 10x_2 & +4x_4 \\
\text{s.t.} & 8x_1 + 2x_2 & +x_3 + 4x_4 \leq 10 \\
 & x_1 + x_2 \leq 1 & \\
 & & x_3 + x_4 \leq 1 \\
 & x_i \in \{0,1\}, & i = 1,2,3,4.
\end{array}
\qquad
\begin{array}{l}
\longrightarrow \quad u_1 \geq 0 \\
\longrightarrow \quad u_2 \geq 0
\end{array}
$$

u_1 and u_2 are called Lagrangian multipliers

Note: $u_1 \geq 0$ so use $(1 - x_1 - x_2)$ from $x_1 + x_2 \leq 1$

Add $u_1\,(1 - x_1 - x_2) + u_2\,(1 - x_3 - x_4)$

$$
\begin{array}{lll}
\max & 16x_1 + 10x_2 & +4x_4 + u_1(1 - x_1 - x_2) + u_2(1 - x_3 - x_4) \\
\text{s.t.} & 8x_1 + 2x_2 & +x_3 + 4x_4 \leq 10 \\
 & x_i \in \{0,1\}, & i = 1,2,3,4.
\end{array}
$$

Lagrangian Relaxtion

This is called the subproblem:

$$\begin{aligned} z_{SP} = \max \quad & (16-u_1)x_1 + (10-u_1)x_2 + (0-u_2)x_3 + (4-u_2)x_4 \\ \text{s.t.} \quad & 8x_1 + 2x_2 + x_3 + 4x_4 \leq 10 \\ & x_i \in \{0,1\}, \quad i = 1,2,3,4. \end{aligned}$$

An upper bound, called a Lagrangian bound is given by:

$$u_1 + u_2 + z_{SP}$$

The best Lagrangian bound is given by:

$$\min_{u_1 \geq 0, u_2 \geq 0} \{u_1 + u_2 + z_{SP}\}.$$

This problem is called the Lagrangian Dual problem

Lagrangian Relaxtion

Given $u_1, u_2 \geq 0$

Solve the subproblem

$$z_{SP} = \max\ (16-u_1)x_1 + (10-u_1)x_2 + (0-u_2)x_3 + (4-u_2)x_4$$
$$\text{s.t.}\quad 8x_1 + 2x_2 + x_3 + 4x_4 \leq 10$$
$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

Get Lagrangian upper bound

$$u_1 + u_2 + z_{SP}$$

Examples: $u_1 = 0;\ u_2 = 0$

Subproblem

$$z_{SP} = \max\ (16)x_1 + (10)x_2 + (0)x_3 + (4)x_4$$
$$\text{s.t.}\quad 8x_1 + 2x_2 + x_3 + 4x_4 \leq 10$$
$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

Solution: $x_1 = 1;\ x_2 = 1; x_3 = 0;\ x_4 = 0;\ z_{SP} = 26$

Lagrangian upper bound:

$$u_1 + u_2 + z_{SP} = 0+0+26 = 26$$

Lagrangian Relaxtion

Examples: $u_1=20;\ u_2=20$

Subproblem:

$$z_{SP} = \max\ (16\text{-}20)x_1 + (10\text{-}20)x_2 + (\text{-}20)x_3 + (4\text{-}20)x_4 = \max\ (\text{-}4)x_1 + (\text{-}10)x_2 + (\text{-}20)x_3 + (\text{-}16)x_4$$

$$\text{s.t.} \quad 8x_1 + 2x_2 + x_3 + 4x_4 \le 10$$

$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

Solution: $x_1=0;\ x_2=0; x_3=0;\ x_4=0;\ z_{SP}=0$

Lagrangian upper bound:

$$u_1 + u_2 + z_{SP} = 20+20+0 = 40$$

Examples: $u_1=13;\ u_2=0$

Subproblem:

$$z_{SP} = \max\ (16\text{-}13)x_1 + (10\text{-}130)x_2 + (0)x_3 + (4)x_4 = \max\ 3x_1 - 3x_2 + 4x_4$$

$$\text{s.t.} \quad 8x_1 + 2x_2 + x_3 + 4x_4 \le 10$$

$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

Solution: $x_1=0;\ x_2=0; x_3=0;\ x_4=1;\ z_{SP}=4$

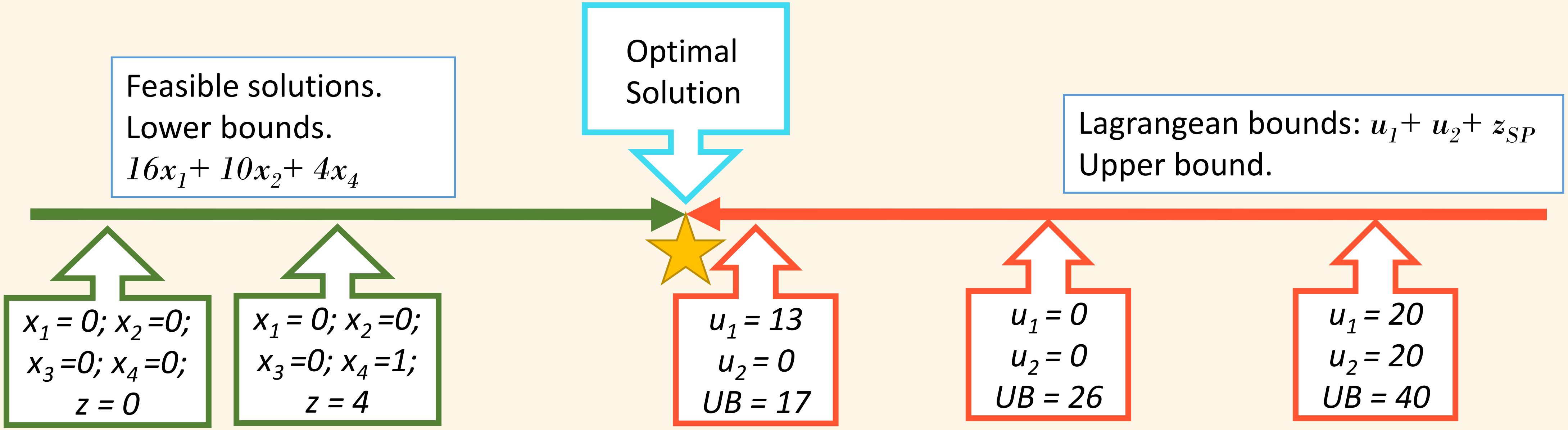
Lagrangian upper bound:

$$u_1 + u_2 + z_{SP} = 13+0+4 = 17$$

Lagrangian Relaxtion

u_1	u_2	SP objective	x_1	x_2	x_3	x_4	z_{SP}	UB
0	0	$16x_1 + 10x_2 + 4x_4$	1	1	0	0	26	26
20	20	$-4x_1 - 10x_2 - 20x_3 - 16x_4$	0	0	0	0	0	40
13	0	$3x_1 - 3x_2 + 4x_4$	0	0	0	1	4	17

- $u_1 = 13$; $u_2 = 0$ gave the best upper bound over the listed combinations.
- If subproblem solution is feasible to the original problem then its corresponding objective provides a lower bound.



Finding the best Lagrange bound/multipliers

The subproblem:

$$\begin{array}{rll} z_{SP} = \max & (16-u_1)x_1 + (10-u_1)x_2 & +(0-u_2)x_3 + (4-u_2)x_4 \\ \text{s.t.} & 8x_1 + 2x_2 & +x_3 + 4x_4 \leq 10 \\ & x_i \in \{0,1\}, & i = 1,2,3,4. \end{array}$$

The Lagrangian Dual problem:

$$\min_{u_1 \geq 0, u_2 \geq 0} \{u_1 + u_2 + z_{SP}\}$$

$$\min_{u_1, u_2 \geq 0} \left\{ \begin{array}{rl} u_1 + u_2 + \max & (16-u_1)x_1 + (10-u_1)x_2 + (0-u_2)x_3 + (4-u_2)x_4 \\ \text{s.t.} & 8x_1 + 2x_2 + x_3 + 4x_4 \leq 10 \\ & x_1, x_2, x_3, x_4 \in \{0,1\} \end{array} \right\}$$

Finding the best Lagrange multipliers

The subproblem tries to maximize $(16-u_1)x_1+(10-u_1)x_2+(0-u_2)x_3+(4-u_2)x_4$ *over set*
$X=\{x_1, x_2, x_3, x_4 \text{ binary} : 8x_1+2x_2+x_3+4x_4 \leq 10\}$

Find the maximum over all elements of set X*. There are 11 possibilities:*

There are 11 possibilities:

$x_1 =$	0	0	0	0	0	0	0	0	1	1	1
$x_2 =$	0	0	0	0	1	1	1	1	0	0	1
$x_3 =$	0	0	1	1	0	0	1	1	0	1	0
$x_4 =$	0	1	0	1	0	1	0	1	0	0	0
	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$	$h=9$	$h=10$	$h=11$

In general, find the maximum over elements of set X*, denoted as:*

$$\begin{bmatrix} x_1^{h=1} \\ x_2^{h=1} \\ x_3^{h=1} \\ x_4^{h=1} \end{bmatrix}, \begin{bmatrix} x_1^{h=2} \\ x_2^{h=2} \\ x_3^{h=2} \\ x_4^{h=2} \end{bmatrix}, \ldots, \begin{bmatrix} x_1^{h} \\ x_2^{h} \\ x_3^{h} \\ x_4^{h} \end{bmatrix}, \ldots, \begin{bmatrix} x_1^{H} \\ x_2^{H} \\ x_3^{H} \\ x_4^{H} \end{bmatrix}$$

$$= \max_{h=1,\ldots,H} \{(16-u_1)x_1^h + (10-u_1)x_2^h + (-u_2)x_3^h + (4-u_2)x_4^h\}.$$

Deriving the Master Problem:

The Lagrangian Dual problem:

$$\min_{u_1\geq 0, u_2 \geq 0} \{u_1 + u_2 + z_{SP}\}$$

$$\min_{u_1,u_2} \left\{ u_1 + u_2 + \max_{h=1,\ldots,H} \{(16-u_1)x_1^h + (10-u_1)x_2^h + (-u_2)x_3^h + (4-u_2)x_4^h\} \right\}$$

$$\theta = \max_{h=1,\ldots,H} \{(16-u_1)x_1^h + (10-u_1)x_2^h + (-u_2)x_3^h + (4-u_2)x_4^h\}$$

$$\begin{aligned} \theta &\geq (16-u_1)x_1^h + (10-u_1)x_2^h + (-u_2)x_3^h + (4-u_2)x_4^h \text{ for all } h = 1, \ldots, H. \\ &\Rightarrow \theta + (x_1^h + x_2^h)u_1 + (x_3^h + x_4^h)u_2 \geq 16x_1^h + 10x_2^h + 4x_4^h \text{ for all } h = 1, \ldots, H \end{aligned}$$

The Master problem:

$$\begin{array}{ll} [MP]: \min & u_1 + u_2 + \theta \\ \text{s.t.} & \theta + (x_1^h + x_2^h)u_1 + (x_3^h + x_4^h)u_2 \geq 16x_1^h + 10x_2^h + 4x_4^h; \quad h = 1, \ldots, H \\ & u_1 \geq 0, u_2 \geq 0 \end{array}$$

Lagrangian Relaxation: The algorithm

$[MP]$:

$$\begin{array}{ll} \min & u_1 + u_2 + \theta \\ \text{s.t.} & \theta + (x_1^h + x_2^h)u_1 + (x_3^h + x_4^h)u_2 \geq 16x_1^h + 10x_2^h + 4x_4^h; \\ & h = 1, \ldots, H \end{array}$$

$\downarrow\downarrow u_1, u_2$ $\qquad\qquad$ $\uparrow\uparrow x_1^h, x_2^h, x_3^h, x_4^h$

$[SP]$:

$$\begin{array}{ll} \max & (16 - u_1)x_1 + (10 - u_1)x_2 + (0 - u_2)x_3 + (4 - u_2)x_4 \\ \text{s.t.} & 8x_1 + 2x_2 + x_3 + 4x_4 \leq 10 \\ & x_i \in \{0, 1\}, i = 1, 2, 3, 4. \end{array}$$

	1. Start with any values of $u \geq 0$
While	$LB \neq UB$
	2. Solve SP, get a solution x^h and an upper bound UB_h
	3.Update the upper bound $UB = \min(UB, UB_h)$
	4. Use the solution x^h to add one more cut to MP
	5. Solve the new MP, get a solution u and a lower bound LB
End	while

Matlab implmentation

```
f=[1;1;1];
A=[];lb=[0;0;0];ub=[inf;inf;inf];b=[];
z_master=-inf;z_lag=inf;

options=optimoptions('linprog','display','off');options2 = optimoptions('intlinprog','display','off');

%initializations
u1=0;u2=0; iter=0;
while (abs(z_lag-z_master)>0.0001)
        iter=iter+1;
        %solve SP
        obj=[16-u1;10-u1;-u2;4-u2]';
        weight=[8,2,1,4];
        x=zeros(4,1);
        x=intlinprog(-obj,[1:4],weight,10,[],[],zeros(4,1),ones(4,1),[],options2);
        z_lag=min(z_lag,u1+u2+obj*x);
        new_const=[x(1)+x(2),x(3)+x(4),1];
        A=[A;new_const];
        b=[b;[16,10,0,4]*x];
        [uutheta,z_master]=linprog(f,-A,-b,[],[],lb,ub,options);
        u1=uutheta(1);
        u2=uutheta(2);
        fprintf('iter=%g: u1= %2.2f, u2 = %2.2f, x1 = %2g,x2 = %2g,x3 = %2g,x4 = %2g, z_master =%2.2f , z_lag=%2.2f\n',iter, u1,u2,x(1),x(2),x(3),x(4),z_master,z_lag);
end;
```

Output:

```
iter=1: u1= 13.00, u2 = 0.00,  x1 =  1,x2 =  1,x3 = -0,x4 =  0, z_lower =13.00 , z_upper=26.00
iter=2: u1= 11.00, u2 = 0.00,  x1 =  0,x2 = -0,x3 = -0,x4 =  1, z_lower =15.00 , z_upper=17.00
iter=3: u1= 12.00, u2 = 0.00,  x1 =  1,x2 = -0,x3 = -0,x4 = -0, z_lower =16.00 , z_upper=16.0
```

Lagrangian Relaxation

Summary:

- Introduced Lagrangian Relaxation
  - Subporblem
  - Master problem
  - Lagrangian dual

Next →

More on Lagrangian Relaxation +Lagrangian heuristics