Irreducibility and convergence of nonlinear state-space models of Markov chains

Application: CMA-ES

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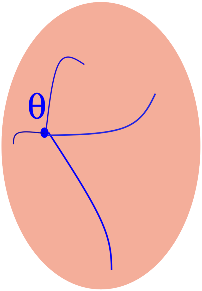

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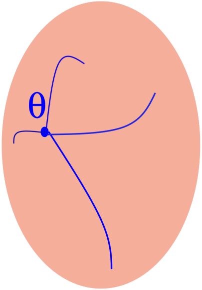

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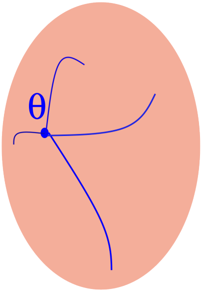

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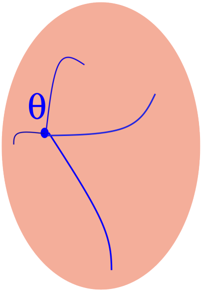

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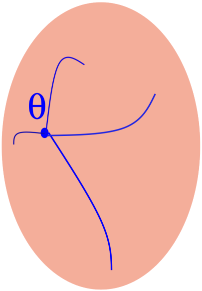

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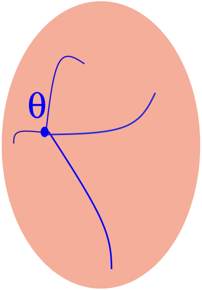

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Then:

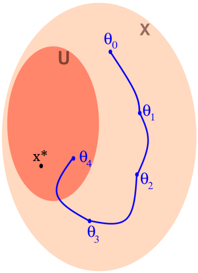
CM(F) is forward accessible

$\Leftrightarrow \forall\theta,\ \exists v_1, \dots, v_k \in \mathcal{O},\ \mathrm{rank}\ C_k = n.$

Globally attracting state

x^* is globally attracting when
$\forall \theta_0, \exists v_1, \ldots, v_k \in \mathcal{O} = \mathrm{supp}\ p$,

$$F_k(\theta_0, v_1, \ldots, v_k) \in \mathrm{Neighborhood}(x^*)$$



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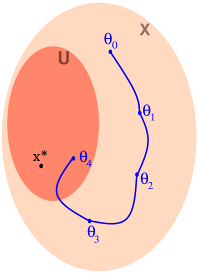

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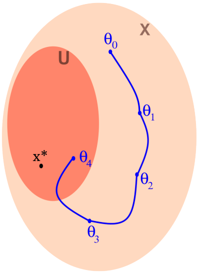

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then $\forall \theta$, $\exists u_1, \ldots, u_j \in \mathcal{O}$,

$$\mathrm{rank}\ C_k(\theta, u_1, \ldots, u_j) = n$$

Aperiodicity of $(\mathbf{CM}(\mathbf{F}))$

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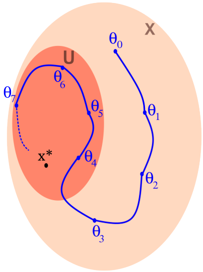
Aperiodicity of (CM(F))

(**CM**(**F**)) is aperiodic $\Leftrightarrow \exists$ a steadily attracting state

θ^* is steadily attracting when for any $\theta_0 \in \mathsf{X}$,
$\exists v_1, v_2, \cdots \in \mathcal{O} = \mathrm{supp}\ p$,

$$\lim_{k\to\infty} F_k(\theta_0, v_1, \ldots, v_k) = \theta^*$$



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ES approximate x^* by $\mathcal{N}(m_k, \sigma_k^2 I_d)$ by updating $\theta_k = (m_k, \sigma_k) \in \mathbb{R}^d \times \mathbb{R}_{++}$.

Algorithm: ES with stepsize adaptation

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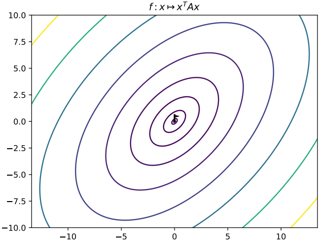

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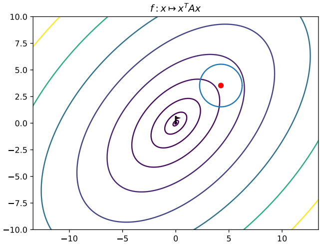

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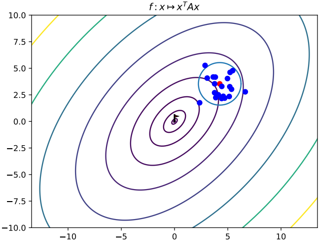

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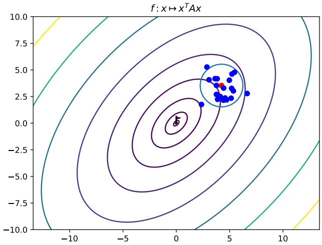

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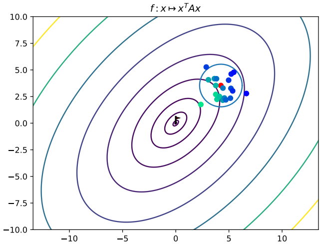

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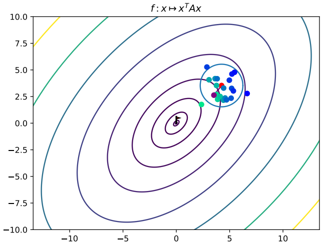

Algorithm: ES with stepsize adaptation

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Goal: $\min_{x \in \mathbb{R}^d} f(x)$

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For $k = 0, 1, 2, \dots$:

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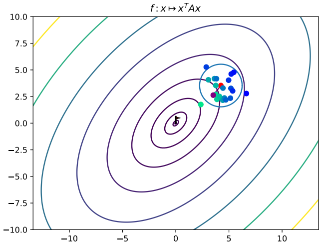

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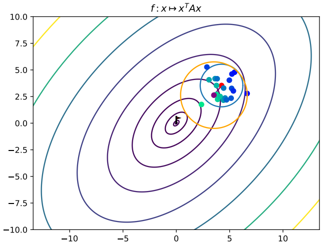

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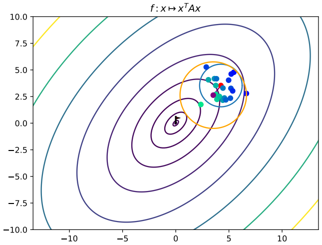

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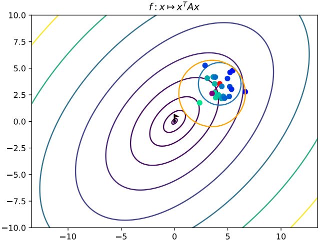

Convergence via analysis of CM(F)

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with $v_{k+1} = (u_{k+1}^{1:\lambda}, \ldots, u_{k+1}^{\lambda:\lambda})$.

Control model [Chotard & Auger 2019]

Consider
$$\theta_{k+1} = F(\theta_k, \alpha(\theta_k, u_{k+1})) \qquad \textbf{(CM(F))}$$

- $F \colon \mathsf{X} \times \mathsf{V} \to \mathsf{X}$ is $\mathcal{C}^1$
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Conclusion: ES converges *linearly* to x^*

Control model, [G., Auger & Durmus, soon submitted]

$$\theta_{k+1} = F(\theta_k, \alpha(\theta_k, u_{k+1})) \qquad \textbf{(CM(F))}$$

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Algorithm: ES with covariance matrix adapation

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Goal: $\min_{x \in \mathbb{R}^d} f(x)$

Given: $m_0 \in \mathbb{R}^d$, $\sigma_0 > 0$

For $k = 0, 1, 2, \ldots :$

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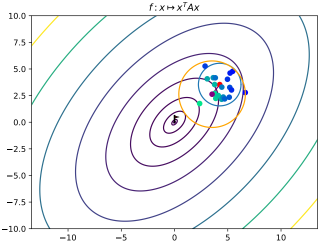

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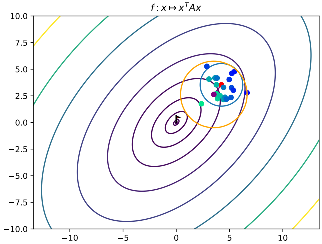

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Algorithm 1 CMA-ES

Goal: $\min_{x \in \mathbb{R}^d} f(x)$

Given: $m_0 \in \mathbb{R}^d$, $\sigma_0 > 0$, $C_0 \in \mathcal{S}^d_{++}$

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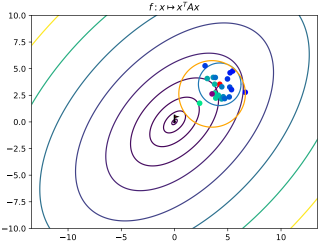

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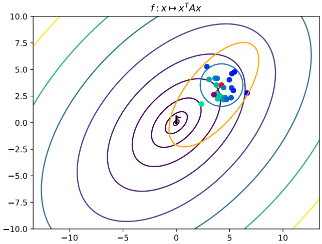
For $k = 0, 1, 2, \ldots$:

1. $u^1_{k+1}, \ldots, u^\lambda_{k+1} \sim \mathcal{N}(0, I_d)$,
   $x^i_{k+1} = x_k + \sigma_k \sqrt{C_k} u^i_{k+1}$
2. sort $f(x^i_{k+1})$:
   $f\left(x^{1:\lambda}_{k+1}\right) \leqslant \cdots \leqslant f\left(x^{\lambda:\lambda}_{k+1}\right)$
3. $m_{k+1} = \sum w_i x^{i:\lambda}_{k+1} = x_k + \sigma_k \sqrt{C_k} \sum w_i u^{i:\lambda}_{k+1}$
4. $\sigma_{k+1} = \sigma_k \times \exp\left(\frac{\left\|\sum w_i u^{i:\lambda}_{k+1}\right\|}{\mathbb{E}\left\|\sum w_i u^i_{k+1}\right\|} - 1\right)$
5. $C_{k+1} = (1-c) C_k + c \sum w_i \left[\sqrt{C_k} u^{i:\lambda}_{t+1}\right] \left[\sqrt{C_k} u^{i:\lambda}_{t+1}\right]^T$



CMA-ES follows CM(F)

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with $v_{k+1} = (u_{k+1}^{1:\lambda}, \ldots, u_{k+1}^{\lambda:\lambda})$.

Control model, [G., Auger & Durmus, soon submitted]

$$\theta_{k+1} = F(\theta_k, \alpha(\theta_k, u_{k+1})) \qquad \textbf{(CM(F))}$$

- $F : \mathsf{X} \times \mathsf{V} \to \mathsf{X}$ is **locally Lipschitz**
- $\{v_{k+1}\}$ such that $v_{k+1} = \alpha(\theta_k, u_{k+1}) \sim p_{\theta_k}$ l.s.c.
- X and V are **manifolds**

Theorem
Suppose that θ^* is steadily attracting, i.e., for $\theta_0 \in \mathsf{X}$,

$$\exists (v_1, v_2, \dots) \in \mathcal{O}^{\infty}_{\theta_0}, \quad \lim F_k(\theta_0, v_1, \dots, v_k) = \theta^*.$$

If $\exists (v_1, \dots, v_k) \in \mathcal{O}^k_{\theta^*}$, such that any

$$C_k \in \mathrm{Clarke}_{v_{1..k}} F_k(\theta^*, v_1, \dots, v_k)$$

is of rank n, then $\{\theta_k\}_k$ is an irreducible aperiodic T-chain.

Consequence

Corollary
Under additional assumptions on the objective function f,

$$(z_k, \Sigma_k)_{k\in\mathbb{N}}$$

is a irreducible, aperiodic T-chain.

Drift

Then (z_k, Σ_k) is positive recurrent and follows a LLN if there exists a drift $V \colon \mathsf{X} \to [0, +\infty]$ such that

$$\mathbb{E}\left[V(z_1, \Sigma_1)\right] \leqslant (1 - \varepsilon) \times V(z_0, \Sigma_0) + b \times \mathbf{1}_{(z_0, \Sigma_0) \in K}$$

Theorem (Drift for the normalized chain)
When minimizing a **spherical** *function* $f : x \mapsto g\left(x^T x\right)$ *(*$g : \mathbb{R} \to \mathbb{R}$ *increasing), then the irreducible, aperiodic Markov chain* $(z_k, \Sigma_k)_{k \in \mathbb{N}}$ *satisfies a Foster-Lyapunov condition with the potential defined by*

$$V(z, \Sigma) = \alpha \times \frac{\|\Sigma z\|^2}{\lambda_{\max}(\Sigma)^2} + \beta \times \lambda_{\max}(\Sigma)$$

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This can be generalized to when minimizing **ellipsoid** functions $f(x) = g(x^T H x)$ using the **affine-invariance** of CMA-ES.

Affine-Invariance

$$
\begin{tikzcd}
(m_0, C_0) \arrow[r, "\min f(x)"] \arrow[d, "\Psi"'] & (m_1, C_1) \\
(m_0', C_0') \arrow[r, "\min f(Bx+b)"] & (m_1', C_1') \arrow[u, "\Psi^{-1}"']
\end{tikzcd}
$$

Thank you!

Scaling-invariant functions