Bayesian Machine Learning Project - Bayesian nonparametric models for ranked data

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Abstract

This document is mostly based on the work of Caron and Teh [4, 3]. We study here the Plackett-Luce choice model. Its goal is to estimate the desirability of items k, by looking on different top-m lists. The two versions that we present here allow us to estimate these weights on a static way or a dynamic way (i.e depending on time). We apply these methods on the African vote for the golden ball trophy 2019 and on the rankings of movies on box-office during the end of 2019 and the beginning of 2020.

1 Static Bayesian nonparametric ranking models

In this part we are going to present the Plackett-Luce model, its Bayesian version and the non parametric extension introduced in the article. We are going to apply it to the African vote for the golden ball trophy 2019.

1.1 The original Plackett-Luce model

The Original Plackett-Luce model was introduced in [5] to model ordered list of top-m items. Those type of data arise often, for example if you ask people to give their top-m best movies of all time. Throughout this part, let us suppose the data we observe are a finite number of partial rankings of length m indexed by $l \in [L]$. Are present in those rankings K different objects and for each object $k \in [K]$,let n_k be the number of time the object appears.

The finite model assumes that for each item $k \in [M] = \{1, 2, ..., M\}$, is assigned a positive rating parameter w_k, called desirability.

The model generates a top-m $\rho = (\rho_1, \cdots, \rho_m)$ of items of $[M]$ list the following way.

At each stage $i = 1, \cdots, m$, an item is chosen among the items that have not been chosen yet with probability proportional to its desirability. The probability that a list ρ is chosen is then :

$$P(\rho) = \prod_{i=1}^{m} \frac{w_{\rho_i}}{\left(\sum_{k=1}^{M} w_k\right) - \left(\sum_{j=1}^{i-1} w_{\rho_j}\right)} \tag{1}$$

1.2 Bayesian Plackett-Luce model with gamma prior

There exist another Thurstonian interpretation (that is theoretically important) to generate a ranking l. For each item $k \in [M]$ a latent variable $z_{lk} \sim \text{Exp}(w_k)$ is generated. We can think z_{lk} as the arrival time of item k in a race. Then the items are ordered by the time of arrival with ρ_{li} the index of the

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item that arrived in the ith position in the ranking l. It can be shown that this way of generating ranking is equivalent to the previous one and yield the same probability (1) for a given ranking.

This is illustrated by the toy example in the notebook where the probability of a ranking generated by both methods is estimated by a simple Monte-carlo and is compared to the true one. The data and the PSG players' score is given by France football for the first leg of PSG-Dortmund in the Round of 16 of Champions League 2020.

To compute the probabilities distribution, they define another latent variable $Z_{li} \triangleq z_{\rho_{li}-z_{\rho_{li-1}}}$

It can be shown that the joint distribution of $((\rho_l)^{l\in[L]}, (Z_{li})_{i\in[m]}^{l\in L} | (w_k)_{k\in[K]}$ follow

$$P\left((\rho_l)^{l\in[L]}, (Z_{li})_{i\in[M]}^{l\in[L]}) | (w_k)_{k\in[K]}\right) = \prod_{l=1}^{L}\prod_{i=1}^{m} w_{\rho_{li}} \exp\left(-Z_{li}\left(\sum_{k=1}^{M} w_k - \sum_{j=1}^{i-1} w_{\rho_{lj}}\right)\right) \tag{2}$$

Then by factorization, the posterior $Z_{li}|(\rho_l, w) \sim \text{Exp}\left(\sum_{k=1}^{M} w_k - \sum_{j=1}^{i-1} w_{\rho_{lj}}\right)$

Now if we suppose a prior on $w_k \sim \text{Gamma}\left(\frac{\alpha}{M}, \tau\right)$ the posterior distribution is :

$$w_k|(\rho_l), (Z_{li}), (w_{k'})_{k'\neq k} \sim \text{Gamma}\left(\frac{\alpha}{M} + n_k, \tau + \sum_{l=1}^{L}\sum_{i=1}^{m} \delta_{lik} Z_{li}\right) \tag{3}$$

Usually tau is set to one as we look at the ratio between the w. The rankings are invariant to the rescaling of the mass w

With δ_{lik} the occurrences indicator define by :

$$\delta_{lik} \triangleq \begin{cases} 0 & \text{if } \exists j < i \text{ with } Y_{lj} = X_k^* \\ 1 & \text{otherwise} \end{cases} \tag{4}$$

And for items that do not appear in the list observed, we can define $w_* \triangleq \sum_{k|n_k=0} w_k$. w_* represent the sum of the score of all the items that are not yet observed, it is the cumulative mass of all the items not observed. It can also be interpreted as the probability (if normalize) of a new item arriving at the first place of a new ranking. Its distribution follows

$$w_*|(\rho_l), (Z_{li}), (w_k)_{k,n_k>0} \sim \text{Gamma}\left(\alpha, \tau + \sum_{l=1}^{L}\sum_{i=1}^{m} Z_{li}\right) \tag{5}$$

1.3 Non parametric Bayesian Plackett-Luce model

Now let's assume we have an infinite pool of items $(X_k)_{k\geq 1}$ with rankings $(w_k)_{k\geq 1}$. As in (1) we can compute the probability of a top m-list.

$$P(X_{\rho_1}, ..., X_{\rho_m}) = \prod_{i=1}^{m} \frac{w_{\rho_i}}{\left(\sum_{k=1}^{+\infty} w_k\right) - \left(\sum_{j=1}^{i-1} w_{\rho_j}\right)} \tag{6}$$

And to formalize this law and to prove the theorems, they define an atomic measure :

$$G \triangleq \sum_{k}^{+\infty} w_k \delta_{X_k}$$

Then by re-normalize G, we have a atomic probability distribution. The first item from a list X_{ρ_1} is drown according to G. Then the atom is removed. to draw the next one. There exist theory to treat those kind of problem on permutation of random measure. [6]

Then we can compute the joint probability of $((\rho_l)^{l\in[L]}, (Z_{li})_{i\in[m]}^{l\in L})$ knowing (w) i.e. G as in 2

$$P\left((\rho_l)^{l\in[L]}, (Z_{li})_{i\in[m]}^{l\in[L]})|G\right) = \prod_{l=1}^{L}\prod_{i=1}^{m} w_{\rho_{li}} \exp\left(-Z_{li}\left(\sum_{k=1}^{+\infty} w_k - \sum_{j=1}^{i-1} w_{\rho_{lj}}\right)\right) \tag{7}$$

The same goes for the posterior of $Z_l i|(\rho_l, w)$:

$$Z_{li}|(\rho_l, w) \sim \text{Exp}\left(\sum_{k=1}^{+\infty} w_k - \sum_{j=1}^{i-1} w_{\rho_{lj}}\right) \tag{8}$$

Then by taking heuristically $M \to +\infty$ in 3, this is shown in the appendix of the article. We have

$$w_k^*|(\rho_l), (Z_{li}), (w_{k'}^*)_{k'\neq k} \sim \text{Gamma}\left(n_k, \tau + \sum_{l=1}^{L}\sum_{i=1}^{m} \delta_{lik} Z_{li}\right) \tag{9}$$

The law of the sum of the weight of items that do not appear in our list stays the same.

1.4 Gibbs sampling

With the posterior distribution given by the equation (8),(9),(5). And a final one if on put a prior Gamma(a, b) on α with a,b 2 hyper parameters. We have :

$$\alpha|Z_{li} \sim \text{Gamma}\left(a + K, b + \log\left(1 + \frac{\sum_{l=1}^{L}\sum_{i=1} Z_{li}}{\tau}\right)\right) \tag{10}$$

We have the following Gibbs updates:

- update $Z_{li}|\text{rest} \sim \text{Exp}\left(w_*^* - \sum_k \delta_{lik} w_k\right)$ with (8)
- update $w_k^*|\text{rest} \sim \text{Gamma}\left(n_k, \tau + \sum_{l=1}^{L}\sum_{i=1}^{m} \delta_{lik} Z_{li}\right)$ with (9)
- update $\alpha|\text{rest} \sim \text{Gamma}\left(a + K, b + \log\left(1 + \frac{\sum_{l=1}^{L}\sum_{i=1} Z_{li}}{\tau}\right)\right)$ with (10)
- update $w_*^*|\text{rest} \sim \text{Gamma}\left(\alpha, \tau + \sum_{l=1}^{L}\sum_{i=1}^{m} Z_{li}\right)$ with (5)

We have the pseudo code for the Gibbs sampler algorithm 1

1.5 Numerical result

We applied the Gibbs sampler on data of the African vote for the golden ball. The data is accessible here : [1]. The rule for that election is that each country gives its top 5 players (among a pre-selected pool of 30 player which does not verify the assumption of the infinitely many items...). The first player is attributed 6 points, the second 4 points, the third 3 points the fourth 2 points and the last 1 point. The ranking is then computed based on the number of point collected by each player. In the 2019 edition the African continent elected Messi with 187 point in front of Mané with 170. Let see what the algorithm tells us.

We run this algorithm 1 on the data with the following parameters :

- $\alpha = 10$
- $\beta = 1$
- $N_{\text{warmup}} = 10000$
- $N_{\text{iteration}} = 10000$

Algorithm 1 Gibbs Sampler (Static)

Require: L top-m $Y_l = (Y_{l1}, \cdots, Y_{lm})$ Priors α, β, N_{warmup},$N_{\text{iteration}}$.
  Initialization
  $K \leftarrow$ number of discovered item
  $L \leftarrow$ number of rankings
  $m \leftarrow$ length of a ranking
  $\tau \leftarrow 1$
  $n_k \leftarrow$ number of apparition of item k
  $\delta_{lik} \leftarrow$as defined in 4
  $\alpha \leftarrow \text{Gamma}(a + K, b)$.
  $w_k^* \leftarrow \text{Gamma}(n_k, \tau)$.
  $w_*^* \leftarrow \text{Gamma}(\alpha, \tau)$.

  for $j = 1, \cdots, N_{\text{warmup}} + N_{\text{iterations}}$ **do**
    Update Z_{li} :
    for $l, i \in [L] * [m]$ **do**
      $Z_{li} \leftarrow \text{Exp}\left(w_*^* - \sum_k \delta_{lik} w_k\right)$
    end for
    Update w_k^*
    for $k = 1, \cdots, K$ **do**
      $w_k^* \leftarrow \text{Gamma}\left(n_k, \tau + \sum_{l=1}^{L} \sum_{i=1}^{m} \delta_{lik} Z_{li}\right)$
    end for
    Update α
    $\alpha \leftarrow \text{Gamma}\left(a + K, b + \log\left(1 + \frac{\sum_{l=1}^{L} \sum_{i=1} Z_{li}}{\tau}\right)\right)$
    Update w_*^* $w_*^*| \leftarrow \text{Gamma}\left(\alpha, \tau + \sum_{l=1}^{L} \sum_{i=1}^{m} Z_{li}\right)$
  end for

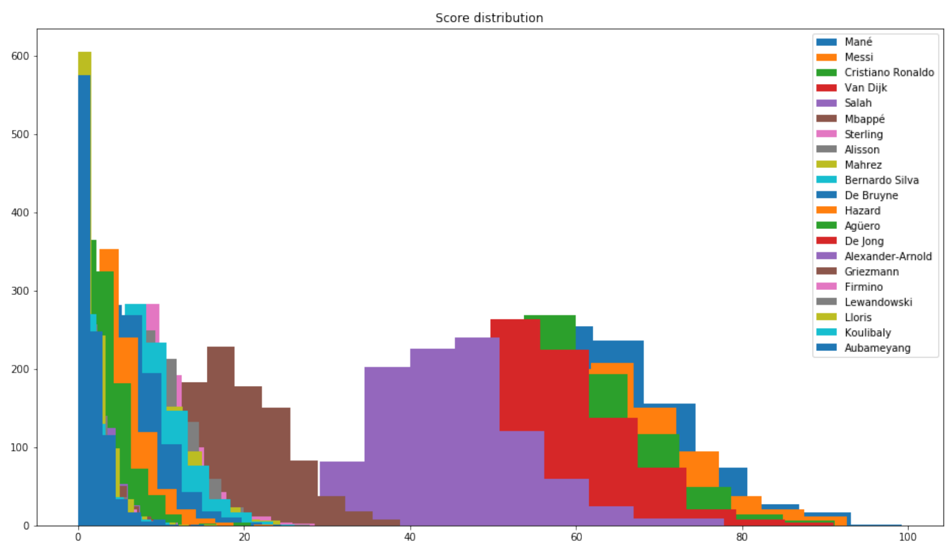

Figure 1: Distribution of weights for each player

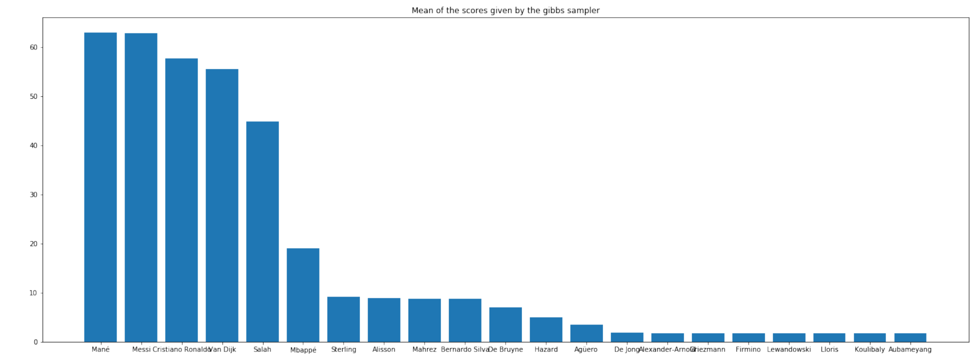

Figure 2: Mean of the distribution for each player ranked

The result are presented in fig. 1 and 2

We can add that we have a $w_*^* = 85$ that 21 out of 30 player appeared in the rankings.

We can remark that the model gives Mané first before Messi but are still very close. What may seem strange is that w_*^* is very high compared to the the score of Mane for example. If we draw a top 5 ranking for example by asking the top 5 players from a African journalist, I am not sure that he will put a new a player on top of the his list with this high probability. This is very unintuitive. Maybe this is due to the fact that the pool of player is not infinite (Which I think won't change a lot of thing as the top-5 of many country will stay the same). Or maybe this model is not very suited to the golden ball vote as each ranking is given a different country and each country may have different preferences. Another hypothesis is that we do not have enough ranking data.

2 Dynamic Bayesian nonparametric ranking models

2.1 Pitt-Walker dependence model

Suppose $G_t \sim \Gamma(\alpha, \tau, H)$ is a Gamma process. Then, we can write G_t under the following form :

$$G_t = \sum_{k=1}^{\infty} w_{tk} \delta_{X_{tk}}$$

Let us define the random measure C_t conditionnally to G_t :

$$C_t | G_t = \sum_{k=1}^{\infty} c_{tk} \delta_{X_{tk}} \quad \text{where} \quad c_{tk} | G_t \sim \text{Poisson}\left(\phi_t w_{tk}\right)$$

Then, the law of G_t given C_t is given by :

$$G_t = G_t^* + \sum_{k=1}^{\infty} w_{tk}^* \delta_{X_{tk}}$$

where G_t^* and $(w_{tk}^*)_k$ are mutually independents and :

$$G_t^* | C_t \sim \Gamma(\alpha, \tau + \phi_t, H) \qquad w_{tk}^* | C_t \sim \text{Gamma}(c_{tk}, \tau + \phi_t)$$

Define then $G_{t+1} = G_{t+1}^* + \sum_{k=1}^{\infty} w_{t+1,k} \delta_{X_{tk}}$, where $G_{t+1}^* \sim \Gamma(c_{tk}, \tau + \phi_t)$. Then, for an atome X with mass $w > 0$, the probability that it is dead at time t is given by :

$$\mathbb{P}\left[G_t(\{X\}) = 0 | w\right] = \exp(-y_{t|1} w)$$

where $y_{t|t-1} = \phi_{t-1}$ and $y_{t|s-1} = \frac{y_{t|s} \phi_{s-1}}{\phi_{s-1} + \tau + y_{t|s}}$.

2.2 Posterior characterization and Gibbs sampling

At each time $t = 1, \cdots, T$, suppose that we observe one top-m list $Y_t = (Y_{t1}, \cdots, Y_{tm})$.

Let $X^* = (X_k^*)_{k=1,\cdots,K}$ be the set of unique items observed in $Y_1, \cdots, Y_T$.

Let $n_{tk} \in \{0, 1\}$ be the number of times that X_k^* appears at time t in Y.

Define finally ρ_i as $Y_t = (X_{\rho_1}^*, \cdots, X_{\rho_m}^*)$. Let us denote :

$$\begin{aligned}
&w_{tk} \doteq G_t(\{X_k^*\}) \\
&w_t^* \doteq G_t(\mathbb{X} \setminus X^*) \\
&c_{tk} \doteq C_t(\{X_k^*\}) \\
&c_t^* \doteq C_t(\mathbb{X} \setminus X^*) \\
&\forall t, \forall i, Z_{ti} \sim \text{Exp}\left(w_t^* + \sum_{k=1}^{K} w_{tk}^* - \sum_{j=1}^{i-1} w_{t\rho_j}\right)
\end{aligned}$$

A Gibbs sampler step is as follows :

- Update Z_{ti}.
- Update $(w_{tk}) | (Z_{ti}, (c_{tk}), (c_t^*)$.
- Update $(c_{tk}), (c_t^*)$.
- Update $\alpha, (w_t^*), (c_t^*)$.
- Update (ϕ_t).

2.3 Numerical experiment

2.3.1 Gibbs sampler

We executed the following algorithm :

Algorithm 2 Gibbs Sampler (Dynamic Plackett-Luce model)

Require: Set of items $X^* = (X_k^*)_{k=1,\cdots,K}$, L top-m lists at each time $t = 1, \cdots, T$: $Y_{tl} = (Y_{tl1}, \cdots, Y_{tlm})$.

Initialization

$\alpha \leftarrow \text{Gamma}(a, b)$.

$\phi \leftarrow \phi_{\text{init}}$.

$G_t(\mathbb{X}) \leftarrow 1$.

$w_{tk} \leftarrow \frac{1}{K}$.

$w_{t*} \leftarrow 0$.

$c_{tk} \leftarrow \text{Poisson}(\phi w_{tk})$.

$c_{t*} \leftarrow 0$.

$Z_{til} \leftarrow \text{Exponential}(1 - (i-1)/K)$.

$n_{tk} \leftarrow \sum_{l=0}^{L} \sum_{i=1}^{m} \mathbb{1}_{X_k^* = Y_{tli}}$.

for $j = 1, \cdots, N_{\text{iterations}}$ **do**

  Update $G_t(\mathbb{X})$:

  $G_0(\mathbb{X}) \leftarrow \text{Gamma}(\alpha, \tau)$.

  for $t = 1, \cdots, T-1$ **do**

    $M_t \leftarrow \text{Poisson}(\phi G_t(\mathbb{X}))$

    $G_{t+1} \leftarrow \text{Gamma}(\alpha + M_t, \tau + \phi)$.

  end for

  Change w_{tk} and w_{t*}, in order to have $G(\mathbb{X}) = \sum_{k=1}^{K} w_{tk} + w_{t*}$.

  Update c_{tk}

  Define τ_k^- as $\sup\{t = 1, \cdots, T | \forall u \leqslant t, n_{uk} = 0\}$.

  Define τ_k^+ as $\inf\{t = 1, \cdots, T | \forall u \geqslant t, n_{uk} = 0\}$.

  Define $x_T = \sum_{kl} Z_{Tkl}$ and $x_t = \sum_{kl} Z_{tkl} + \frac{\phi x_{t+1}}{1+\phi+x_{t+1}}$.

  for $t = \tau_k^-$ **do**

    $c_{tk} \leftarrow \text{Poisson}((1+\phi)/(1+\phi+x_t)\phi w_{tk})$.

    $w_{t+1,k} \leftarrow \text{Gamma}(c_{tk}, \tau + \phi + x_t)$.

  end for

  for $t = \tau_k^- + 1, \cdots, \tau_k^+ - 1$ **do**

    if $w_{t+1,k} > 0$ **then**

      Propose $\tilde{c}_{tk} = \text{zeroTruncatedPoisson}(\phi w_{tk})$.

      $c_{tk} \leftarrow \tilde{c}_{tk}$ with probability $\frac{\text{Gamma}(w_{t+1,k}; \tilde{c}_{tk}, \tau+\phi)}{\text{Gamma}(w_{t+1,k}; c_{tk}, \tau+\phi)}$.

    else

      $c_{tk} \leftarrow 0$ with probability $\frac{1}{1+\phi w_{tk}(\tau+\phi)}$.

      $c_{tk} \leftarrow 1$ otherwise.

    end if

  end for

  for $t = \tau_k^+$ **do**

    $c_{tk} \leftarrow \text{Poisson}((1+\phi)/(1+\phi+x_t)\phi w_{tk})$.

    $w_{t+1,k} \leftarrow \text{Gamma}(c_{tk}, \tau + \phi + x_t)$.

  end for

  Update α

  Define $y_T = 0$ and $y_{t-1} = y_t - \log \frac{1+\phi}{1+\phi+x_t}$.

  $\alpha \leftarrow \text{Gamma}(a + K, b + y_1 + \log(1 + x_1))$.

  Update c_{t*} **and** w_{t*}

  $w_{1*} \leftarrow \text{Gamma}(\alpha, \tau + x_1)$.

  for $t = 1, \cdots, T-1$ **do**

    $c_{t*} \leftarrow \text{Poisson}((1+\phi)/(1+\phi+x_{t+1})\phi w_{t*})$.

    $w_{t+1*} \leftarrow \text{Gamma}(\alpha + c_{t*}, \tau + \phi + x_{t+1})$.

  end for

for $j = 1, \cdots, N_{\text{iterations}}$ **do**
  Update w_{tk}, w_{t*}.
  for $t = 1, \cdots, T$ **do**
    for $k = 1, \cdots, K$ **do**
      if $c_{tk} + c_{t-1,k} + n_{tk} = 0$ **then**
        $w_{tk} \leftarrow 0$.
      else
        Define $\delta_{tkil} = 0$ if $\exists \hat{i} \leqslant i, Y_{tl\hat{i}} = X_k^*$, and 1 otherwise.
        $w_{tk} \leftarrow \text{Gamma}(n_{tk} + c_{t-1,k} + c_{tk}, \tau + 2\phi + \sum_{i,l} \delta_{tkil} Z_{til})$.
      end if
    end for
    $w_{t*} \leftarrow \text{Gamma}(\alpha + c_{t*} + c_{t-1*}, \tau + 2\phi + \sum_{i,l} Z_{til})$.
  end for

  Update Z_{til}
  $Z_{til} \leftarrow \text{Exponential}(1/(w_{t*} + \sum_k \delta_{tikl} w_{tk}))$.

  Update ϕ
  Propose $\tilde{\phi} = \phi \exp(\sigma\varepsilon)$, for $\varepsilon \sim \mathcal{N}(0, 1)$.
  $\phi \leftarrow \tilde{\phi}$ with probability :

$$\min\left(1, \frac{p(\tilde{\phi})}{p(\phi)} \frac{\phi}{\tilde{\phi}} \prod_{t=1}^{T-1} \exp\left(w_{t+1,*} \frac{\phi - \tilde{\phi}}{(\tau + \tilde{\phi} + x_t)(\tau + \phi + x_t)}\right) \left(\frac{\tau + \phi + x_t}{\tau + \tilde{\phi} + x_t}\right)^{\alpha + c_{t*}} \right.$$
$$\left. \prod_{k: w_{tk} > 0} \exp\left(w_{t+1,k} \frac{\phi - \tilde{\phi}}{(\tau + \tilde{\phi} + x_t)(\tau + \phi + x_t)}\right) \left(\frac{\tau + \phi + x_t}{\tau + \tilde{\phi} + x_t}\right)^{\alpha + c_{tk}} \right)$$

  if $j \geqslant N_{\text{iterations}} - N_{\text{sample}}$ **then**
    $w_{t,k}^{(j - N_{\text{sample}})} \leftarrow w_{t,k}$.
    $w_{t*}^{(j - N_{\text{sample}})} \leftarrow w_{t*}$.
  end if
end for
return $w_{t,k}^{(j)}, w_{t*}^{(j)}$ for $j = 0, \cdots, N_{\text{sample}} - 1$

2.3.2 Parameters

We have used :

- As a data set, the top-5 box-office movies, each week between October 2019 and February 2020, in US, France, UK and Germany[2].
- $a = 0.01, b = 0.01$.
- $\phi_{\text{init}} = 10^{-5}$.
- $\tau = 1$.
- $N_{\text{iterations}} = 10^5$, $N_{\text{sample}} = 1/2 \times N_{\text{iterations}}$.

2.3.3 Results

We obtained the following weights w_{tk}, after renormalization :

Figure 3: Evolution of the mean of samples of $w_{t,k}$ and w_{t*}

It looks like that the weights w_{t*} seem to be over-estimated again – even if we expected large w_{t*} : if we pick any random country, it is likely that national movies are in the top-5 more successful movies only in this country. Maybe if we used more lists of rankings the results would be more accurate.

References

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