n-Cells Identification Model

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1 Introduction

This document contains the theory underneath the TANI project [1], a Topological Analysis of nD Images toolkit for Python.

1.1 Binarity

The nD images considered in this theoretical study are binary. This is the same as thinking of every nD voxel as existing or not. This does not mean the toolkit works only for binary images: in every function performance there will always be a binarization step so that the theory you are reading fits the algorithms for grayscale images.

2 Hypothesis

Let V be a discrete n-dimensional space, where every point can be identified by its coordinates:

$$(u_0, u_1, \dots, u_n)$$
$$u_i \in \mathbb{N}$$

And J an extension of V such that "point density is doubled" and therefore coordinates can be multiple of $\frac{1}{2}$:

$$(\frac{1}{2}u'_0, \frac{1}{2}u'_1, \dots, \frac{1}{2}u'_n)$$
$$u'_i \in \mathbb{N}$$

Let $P \in J$ be ay point whose coordinates are such:

$$(x_0 + \lambda_0\frac{1}{2}, x_1 + \lambda_1\frac{1}{2}, \dots, x_n + \lambda_n\frac{1}{2})$$
$$x_i \in \mathbb{N}$$
$$\lambda_i \in 0, 1$$

Such coordinates will not identify P anymore; they will instead refer to the r-dimensional entity ("*cell*" from now on) E with:

$$r = n - \sum_{i=0}^{n} \lambda_i$$

This way, P happen to be "the mass center" of E. Note that:

- when $\lambda_i = 0\,\forall i$ then $dim(E) = n$, so E is not a point.
- when $\lambda_i = 1\,\forall i$ then $dim(E) = 0$ so E is a common 0-dimensional point.

3 Model Outcomes

An $r$dimensional cell (r-cell) E has some special relations with its r-or-less-dimensional surrounding cells.

3.1 Subcells

A cell S is called a *subcell of* E if its coordinates are obtainable from the ones of E by the only modification of λ_i values.

The number of w-dimensional subcells (w-subcells) of an r-cell is given by:

$$s(r,w) = 2^{r-w}\binom{r}{w};$$

Recall from 1.1 that cells can either exist or not. If E *exists* so do all its subcells. The subcells set $S(E)$ is the set of cells that are subcells to E.

Let $\mathbf{E}$ be a vector of k r-cells such $E_0, E_1, \ldots, E_k$, then:

$$S(\mathbf{E}) =< s : s \in S(E_i) > \;\forall i \in 0, 1, \ldots, k$$

The intersection dimension $i(\mathbb{E})$ is the maximum dimension of $S(\mathbf{E})$ elements.

3.2 Connectivity

A vector of r-cells $\mathbf{E}$ is called α-connected if $i(\mathbf{E}) \geq \alpha$

The potential connected set $\mathbf{P}(E, \alpha)$ is the set of r-cells that, if had existed would have been α-connected to E.

Note that the size of $\mathbf{P}(E)$ is equal to the maximum number of cells E can be α-connected and is given by:

$$|\mathbf{P}(E,\alpha)| = \sum_{i=\alpha}^{r-1} s(r,i)$$

3.3 Connected Component

A vector of k r-cells $\mathbf{P}$ is called an α-path if:

- there are only two cells $E_i, E_j \in \mathbf{P}$ α-connected to two other cells (one each) $E_p, E_q \in \mathbf{P}$ (the "beginning" and "ending" of the path).
- every other $E_l \in P$ is α-connected to $E_m \in P$ and $E_n \in P$ ($m, n \neq l$, $m \neq n$, $l, m, n, \in 0, 1, \ldots, k$).

A set of k r-cells C is called an α-connected component (α-CC) if for every cell $E_i \in C$ there is at least one α-path between E_i and E_j ($i, j \in 0, 1, \ldots, k$).

4 An intuitive 3D example

Let $n = 3$ (recall n is the space dimension), then:

- (3, 3, 3) is the cube (3-cell) centered in the equivalent Cartesian plane (3, 3, 3) point.
- (3, 3, 7/2) is the face (2-cell) centered in the equivalent Cartesian plane (3, 3, 7/2) point.
- (3, 7/2, 7/2) is the edge (1-cell) centered in the equivalent Cartesian plane (3, 7/2, 7/2) point.
- (7/2, 7/2, 7/2) is the point (0-cell) centered in the equivalent Cartesian plane (7/2, 7/2, 7/2) point.

Note that $\frac{7}{2} = 3 + \frac{1}{2}$.

Suppose now a 3-cell (cube) in whatever coordinates. Let's compute its number of w-subcells:

- for $w = 2$,

$$s(3,2) = 2^{3-2}\binom{3}{2} = 6$$

- for $w = 1$,

$$s(3,1) = 2^{3-1}\binom{3}{1} = 12$$

- for $w = 0$,

$$s(3,0) = 2^{3-0}\binom{3}{0} = 8$$

So a cube has 6 2-subcells (faces), 12 1-subcells (edges) and 8 0-subcells (vertices). For each subcell there is a neighbour 3-cell sharing it, so that the connectivity of a cube is:

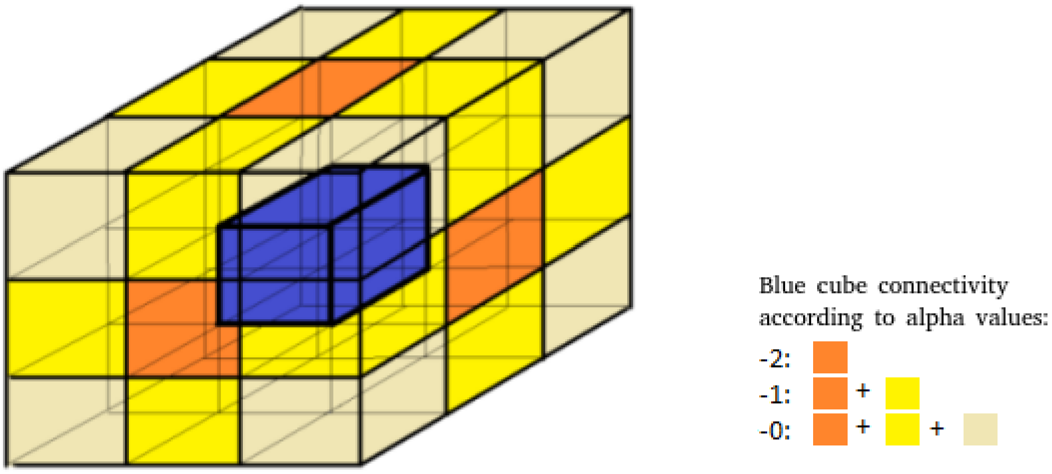

Figure 1: A 3D connectivity example

i.e. there are 6 2-connected cubes, (neighbour cubes sharing at least a face), 18 1-connected cubes, (neighbour cubes sharing at least an edge) and 26 0-connected cubes, (neighbour cubes sharing at least a vertex)

References

[1] GitHub. *Topological Analysis of nD Images*. URL: `https://github.com/agiulianomirabella/tani`.