'Financial Contagion'

Application of a Structural Model of Credit Risk to the Network of Interbank Loans

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Welcome to my talk about financial contagion.
In fact I am suspicious of this terminology,
which is why I have put those words in inverted commas.
I am going to propose a model that owes more to
pricing theory than to epidemiology.

The importance of pricing counterparty risk

> *Under Basel II, the risk of counterparty default and credit migration risk were addressed but mark-to-market losses due to credit valuation adjustments (CVA) were not. During the financial crisis, however, roughly two thirds of losses attributed to counterparty credit risk were due to CVA losses and only about one third were due to actual defaults.*

Basel Committee on Banking Supervision, 2011

The first of my two motivating quotes is from the Bank for International Settlements not far from here.

The key point here is the estimate that roughly two thirds of losses attributed to counterparty credit risk, during the financial crisis, were due to fear of default rather than to actual default.

The technology to account for the danger of default in a single counterparty is not new. The so-called structural model of credit risk dates back to the 1970s and is associated with names such as Merton, Black and Cox.

The problem of pricing risk in a network

> *The Bank's solvency contagion model examines how deteriorating capital positions lead to revaluation of interbank debt claims, which in turn can affect banks' capital positions further ... Bank staff's judgement is that ... the overall impact on the system via this channel remains immaterial ... See Bank of England, Staff Working Paper No. 662, 'The decline of solvency contagion risk', June 2017.*
>
> Bank of England stress testing results, 2017

My second quote is from the Bank of England. It recognises the need to account for this effect and it has a model. As regulator, it must concern itself with system-wide effects and has adopted the language of contagion. One thing affects another, affects another. It sounds like a fixed point argument and it is. This is the approach traditionally associated with network pricing, with papers by Eisenberg and Noe in 2001 and Rogers and Veraart in 2013. But they were modelling the impact of actual default. The Bank of England is applying the same logic, but to banks whose credit rating is merely impaired.

The conclusion of the Bank of England is that there is nothing to worry about. I don't have ~~their~~ its data, so I can't take a view on that. I am also not going to criticise ~~their~~ its assumptions. They may not be true but they are useful. I adopt them myself. But I don't like the model it derives from those assumptions.

Outline

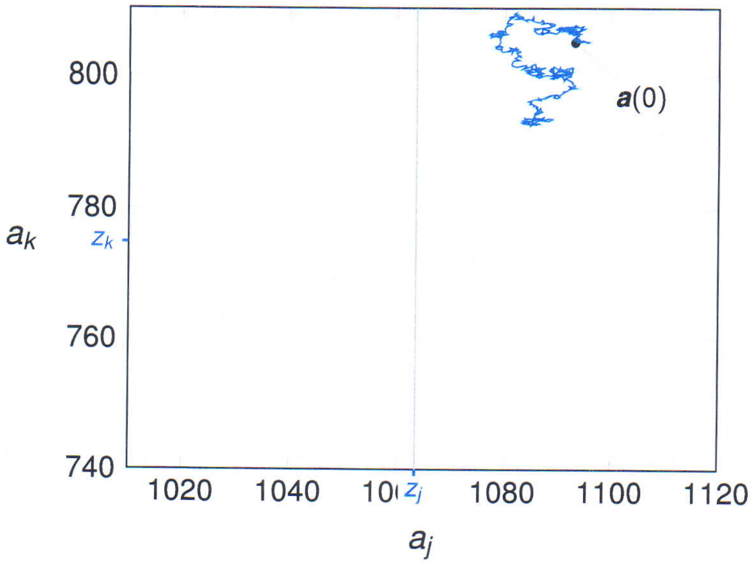
Illustrative plots
- Two banks and zero recovery rate
- The Bank of England model

Development
- Alternative maturity profile
- General solution structure for multiple banks
- Nonzero recovery rate

My aim today is to share my intuition for the way I think this should be handled. So to start off with at least, I confine myself to a very simple case, that of two banks and zero recovery rate. Two banks because two dimensions is as many as I can comfortably fit on a ~~page~~. slide. Zero recovery rate means that when a bank defaults, its creditors get nothing. That as well is to make the plots easier. Neither of these assumptions is required by the model itself as we'll see nearer the end.

Evolution

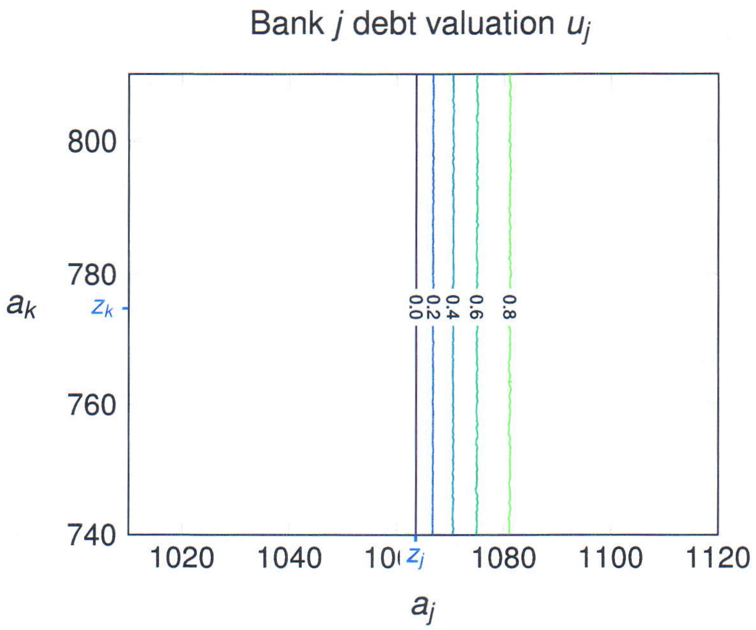
~~The rôle~~ of the real economy



The real economy I assume to evolve independently of the financial sector. It communicates with the financial sector through this vector a. This component a_j is the external assets of a bank j and this component a_k is the external assets of bank k. So j might be Barclays and k might be HSBC. These figures are in billions of pounds. This path is a sample path, or simulation, over a year. I assume a correlated Geometric Brownian Motion. It is a big assumption but it is the assumption of the Bank of England. Or weaker even, since ~~they have no correlation term~~ the Bank of England dispenses with the ~~it permits no~~ correlation term.

z_j stands for the total liabilities of bank j. So if at any point a_j were to fall below z_j, then bank j fails. Graphically, it would be if the path were to cross the line. This particular path didn't, but if I were to generate, say, fifteen thousand such paths then some fraction of them would. That fraction we call the probability of default at a(0).

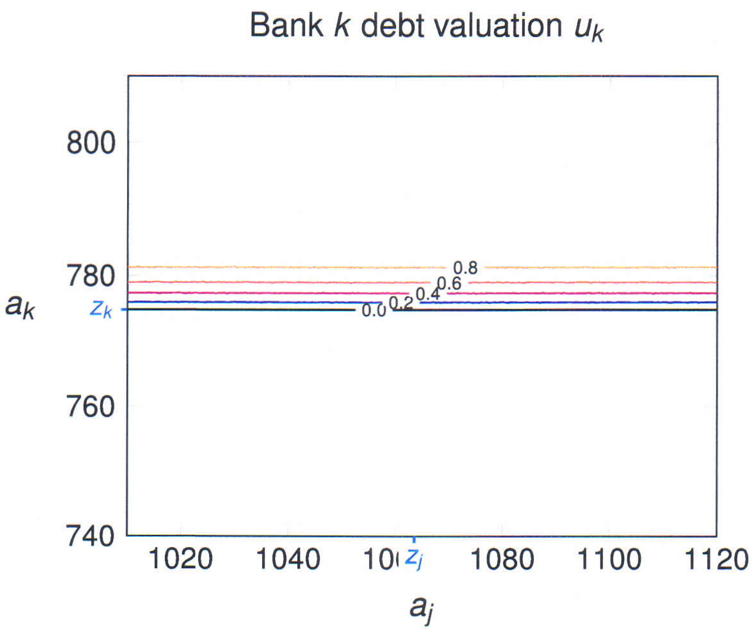
Pricing in the risk of counterparty failure



I have done exactly that, and calculated the probability of default from many different starting points. This contour plot shows u_j, one minus probability-of-default; the fair value per unit face value of bank j's debt with one year to maturity. Assume no coupons and an interest rate of zero.

Note that the valuation u_j falls smoothly to zero at the failure boundary, where probability of default is one. Naturally these contour lines are vertical and straight, since we have introduced no dependence on a_k.

Pricing in the risk of counterparty failure



Here I do the same for bank k. Same axes, same scale. z_k stands for the total liabilities of bank k. Bank k is assumed to fail, and its debt worthless, if a_k falls below z_k.

I have described a Monte Carlo pricing algorithm that is technically justified by the hedging argument and no-arbitrage principles introduced by Merton, Black and Scholes in the 1970s. However they tended to use the language of partial differential equations, and confined themselves to one dimension for the underlying. But the argument extends perfectly well to two or more underlyings.

Balance sheets in the unconnected case

Bank j

Assets	Liabilities
a_j	z_j
	E_j

Bank k

Assets	Liabilities
a_k	z_k
	E_k

The situation viewed in balance sheets. E_j is the difference between a_j and z_j. If it is positive we call it equity. If it's not then bank j has failed and is out of the game. This is an important point. In an early version of this model I made a mistake. I accidentally assumed that a bank would come back to life if its assets recovered. Not only was it a bad assumption, but it made the problem harder.

Balance sheets with interbank exposures

Bank j

Assets	Liabilities
a_j	z_j
	L_{jk}
$u_k L_{kj}$	E_j

Bank k

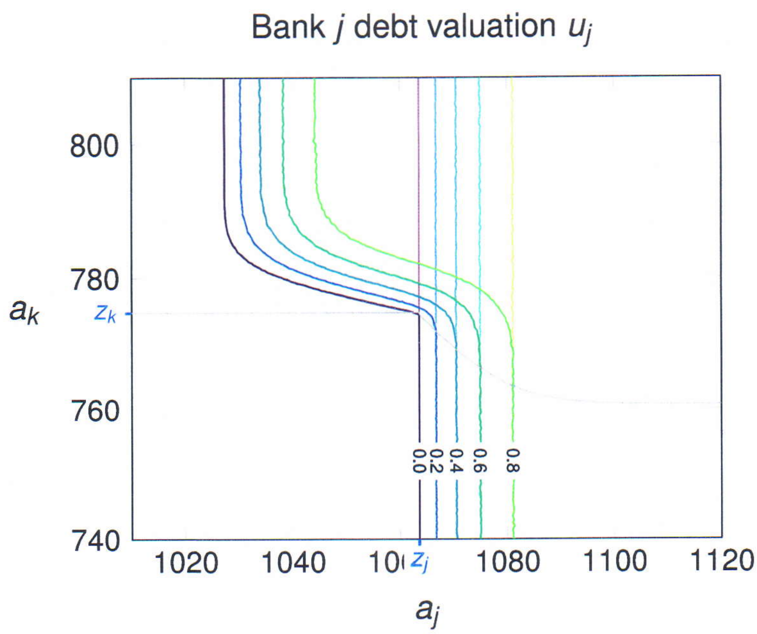
Assets	Liabilities
a_k	z_k
	L_{kj}
$u_j L_{jk}$	E_k

It is time to introduce interbank exposures. These entries in green could represent many kinds of exposure, but it is most straightforward simply to imagine debt. Bank j owes the amount L_{jk} to bank k, payable after a year. The liability will not change for the period, whatever happens to asset prices, whatever happens to k. So I can just absorb it into fixed total liabilities z_j. On the other hand where it reappears on the asset side of bank k's balance sheet, it is mediated by u_j, a function of space and time, and is counted as an additional asset.

Likewise for L_{kj} the amount owed by bank k to bank j.

Because they now have these additional assets, the external assets of each bank can fall further before it fails.

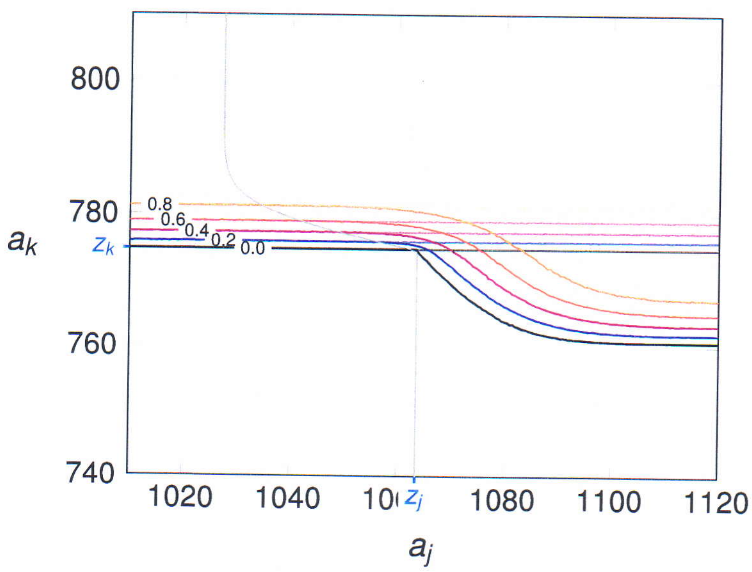
Pricing in the effect of interbank exposures



In the plot we can see that the failure boundary for bank j has shifted to the left, at least in the region where bank k survives. The contour lines all move in sympathy. If bank k does fail, if a_k falls below the thin grey line, even if it subsequently recovers, then we are back to our original problem. These vertical lines are exactly where they were in the earlier slides. So here are two functions plotted on the same axes, one for both banks surviving, one for just bank j surviving.
This suggests the need explicitly to track the set of surviving banks. It is a third argument to our debt valuation function, after asset values and time.

Pricing in the effect of interbank exposures

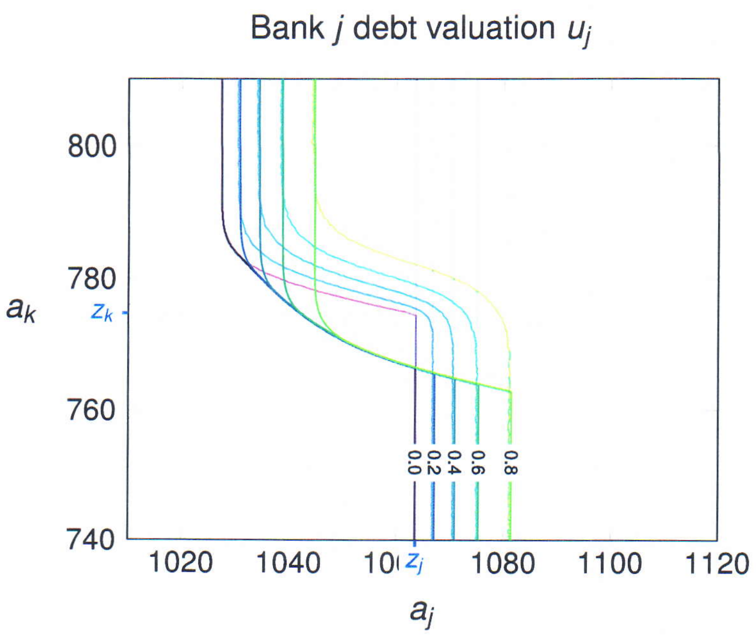
Bank k debt valuation u_k



The analogous plot for bank k. So long as asset prices stay to the right of the thin grey line, that is, so long as bank j remains solvent, the failure boundary for bank k is the lower one. But if the asset path crosses that line, ~~the~~ bank k's failure boundary reverts to the upper one, as in the earlier slide. Note that the two solutions are continuous with one another where they meet at the grey line. This is a natural consequence of our algorithm.

It is a property of random walks that, as one approaches a boundary, the probability of crossing it rises to one. So there can be no sudden jumps in expectation.

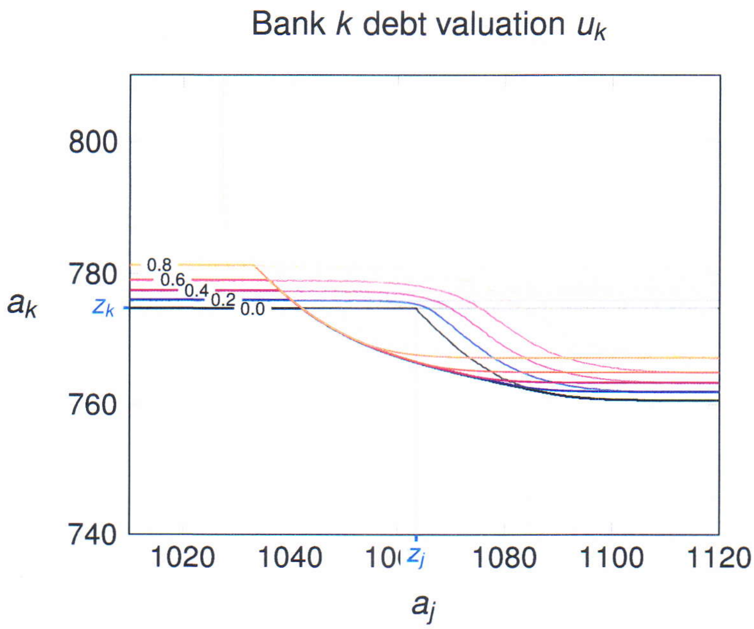
Comparison with the Bank of England model



Although the Bank of England's data are not public, its algorithm is. It has been coded up in Python and uploaded to GitHub. This plot shows the result when it is run against the estimated figures we have been using. My own solution is reproduced in fainter lines for comparison.

There is a large region where the Bank of England model says everything is fine, but where my model indicates failure for both banks. At the edge of that region, the Bank of England's model's results display a very sudden discontinuity, where the debt of both banks goes from close to 100% face value to worthless in no distance at all.

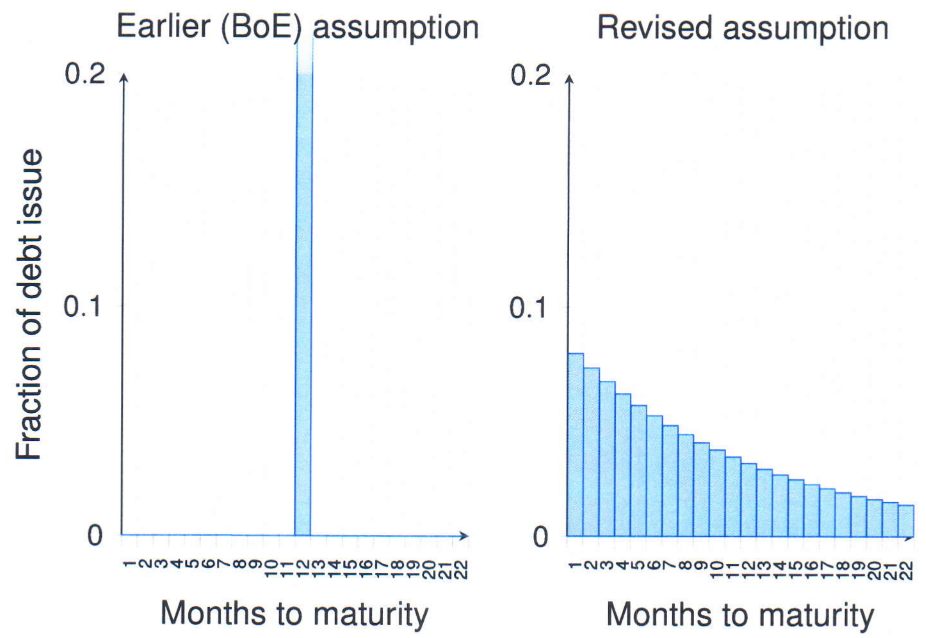
Comparison with the Bank of England model



The discontinuity appears in the same place when pricing bank k's debt. These discontinuities appear because ~~they~~ the Bank of England is ~~are~~ not, like us, solving the two-dimensional problem directly. Instead it applies a fixed point procedure to force together two copies of the solution to the one-dimensional problem, which has an analytic expression. For some sets of input, like this one, the procedure will have multiple solutions. The program selects the most optimistic one. So what we see here is something like hysteresis. There is no danger of multiple solutions in the approach I am describing today.

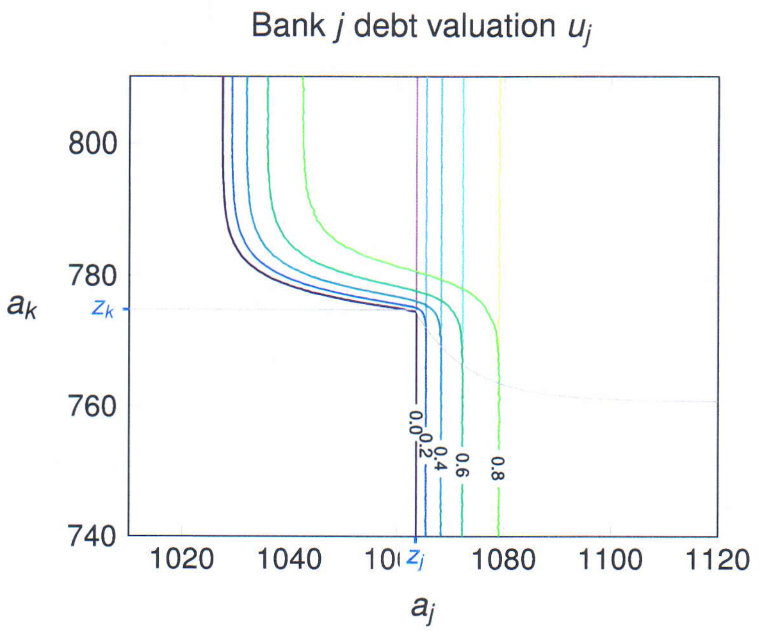
Our solutions do match at either extreme because I have kept as close as possible to the assumptions of the Bank of England.

Alternative maturity profile



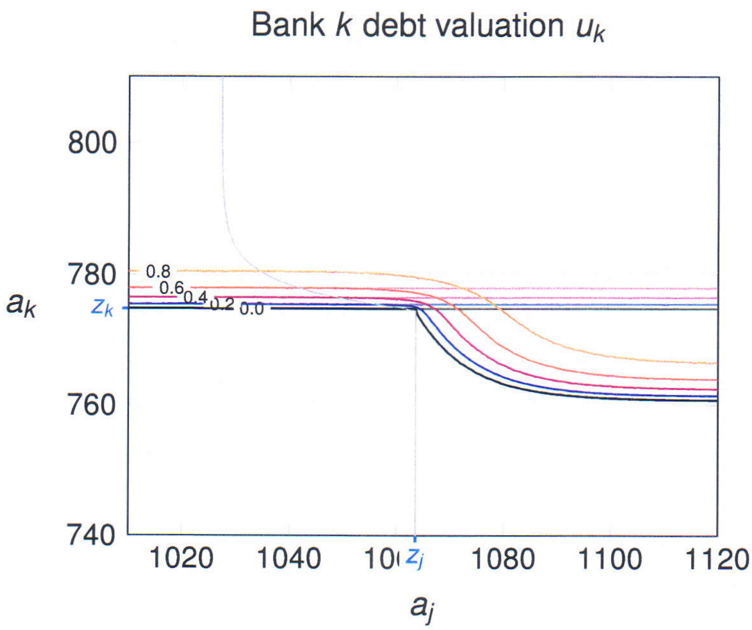
In particular I have kept to the assumption that all interbank debt has exactly twelve months to maturity and that it somehow stays out at twelve months to maturity. I am replacing this now with my own arbitrary assumption, that maturity falls off smoothly. Both assumptions have the same average maturity, twelve months. My assumption has the virtue that it makes calculations easier when we pass to differential equations.

Pricing with revised maturity profile



The revised plot looks qualitatively similar to my earlier one.

Pricing with revised maturity profile



The most observant among you may have noticed that I have not addressed the question of how to find the failure boundary in the first place. It would seem that we need to know the debt valuation function to find the boundary, and that we need to know the boundary to solve for the debt valuation function. But we are saved from having to resolve this with some sort of fixed point analysis by that property of continuity. At the failure boundary itself, wherever it may be, we know that the solution for both banks surviving is continuous with the solution for just one bank surviving. Bank k in this plot. So we can use the solution for just one bank surviving as a proxy for the solution for both banks surviving when it comes to calculating the exact location of the boundary.

Solution dependency structure in general

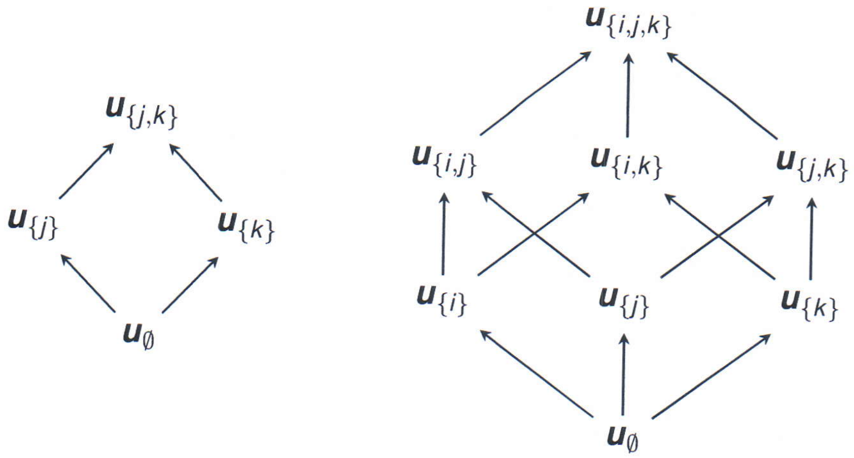


Figure: Progression of the algorithm for a system of two banks and for a system of three.

I am describing a computation with the structure represented on the left here. In order to compute the solution for both banks surviving, I need boundary conditions supplied by the two solutions for just one bank surviving. One might also say that each of those two solutions depends on conditions supplied by the solution for no banks surviving, although that is trivial. Therefore the algorithm must proceed upwards from the bottom of this graph to the top, from the solution associated with the set of no banks to the solution associated with the set of both banks, just as I structured this talk. These ideas extend in a straightforward way to a larger number of banks. The number of solutions required rises exponentially and the dimensions of each also rises with the number of banks. Nonetheless I have found it possible to perform this exercise for a system of seven banks, the number assumed in the UK sector. That code solves differential equations rather than running Monte Carlo, because Monte Carlo is not good for calculating boundaries.

Some supporting theory

Theorem

For s and s' sets of (surviving) banks,

$$s' \subseteq s \implies \boldsymbol{u}_{s'} \leq \boldsymbol{u}_s.$$

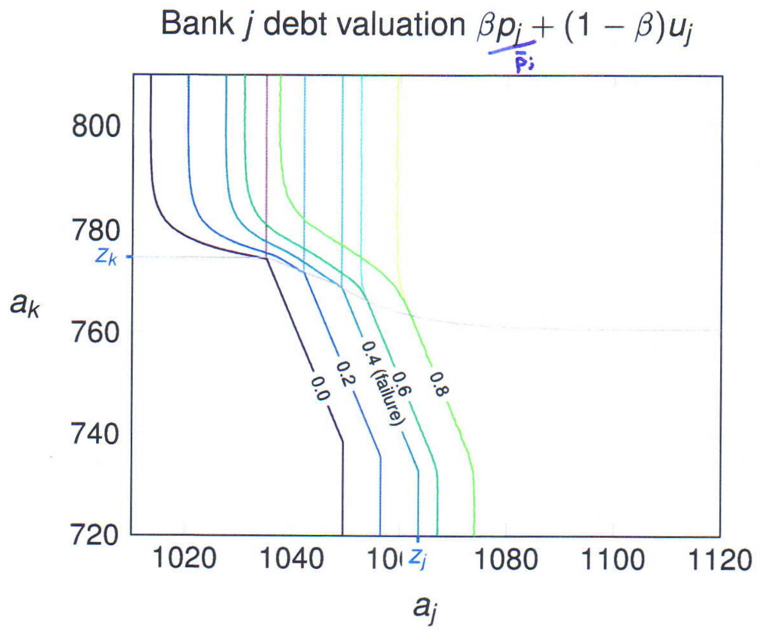
Corollary

With $\boldsymbol{E}^*$ the equity valuation function indicating solvency,

$$\exists s' \subseteq s \quad \forall i \in s \quad 0 < E_i^*(\boldsymbol{u}_{s'}, \boldsymbol{a}, t) \implies \forall i \in s \quad 0 < E_i^*(\boldsymbol{u}_s, \boldsymbol{a}, t).$$

There is a theory behind all this, and here is a taste of it. The theorem tells us that if one set of banks is included in another then the debt valuation function associated with the first takes lower values than the debt valuation function associated with the second. These bold-face us are vector-valued so the inequality applies in every component. It is not a surprising result, since it is hard to see how an increased number of surviving banks could damage the health of any party. This corollary is what's really useful. If every bank in a set s is solvent according to the solution associated with some subset of s, then every bank in s is solvent according to the solution associated with s itself. Together with the principle of continuity, this is what allows us to use solutions associated with subsets of s to determine the boundary conditions we need to set up the problem for set s itself.

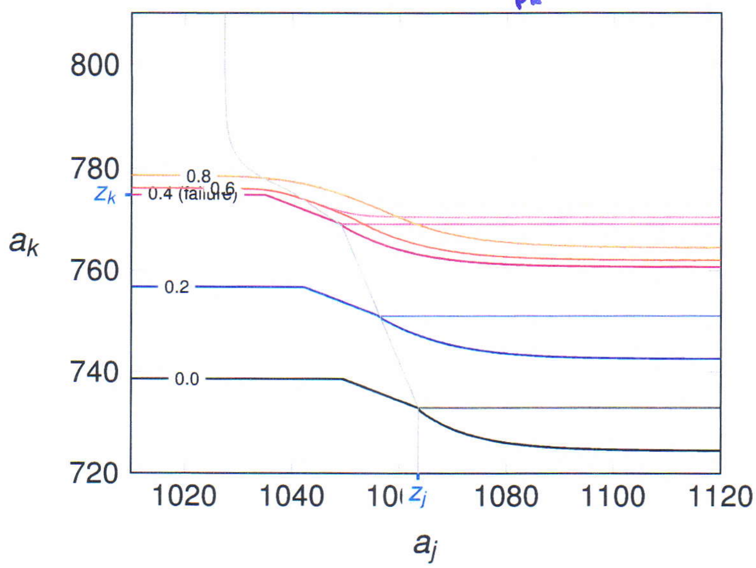
Pricing with recovery rate $\beta = 0.4$



These are my last couple of slides. Nonzero recovery rates have been an important consideration in the existing literature on financial contagion. Eisenberg and Noe had in effect a recovery rate of one, while Rogers and Veraart generalised that to values of β between zero and one. Here I have adopted a Rogers and Veraart-type model to value the debt of banks in default, while using the procedure I have described for banks ~~not yet in default~~ still surviving. So the first ~~component~~ term in this expression comes from Rogers and Veraart and is responsible for these straight lines and sharp corners we see to the left of the failure boundary. The second ~~component~~ term comes from our solution to the diffusion equation and gives rise to these smoother contours above and to the right of the failure boundaries.

Pricing with recovery rate $\beta = 0.4$

Bank k debt valuation $\beta p_k + (1-\beta) u_k$



In summary: I propose a model for the valuation of interbank debt in a network, and hence for the assessment of financial contagion. The problem and the assumptions are not new, but as far as I am aware I have arrived at a new methodology. Results suggest that if one is to take my model seriously then the Bank of England's model is overoptimistic in a large region of its domain. I end with the disclaimer that, having so many assumptions, any such model can only be considered indicative; as one small piece in the puzzle.