A 5-Element Magma Generates a Complete 4-Dimensional TQFT

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Abstract

Inside the 9-axiom theory ETCS we exhibit the magma $(\mathbb{Z}/5, +)$ together with the canonical additive 4-cocycle

$$\omega(a, b, c, d) = \exp\Big(\tfrac{2\pi i}{5}(a + b + c + d)\Big).$$

The resulting state-sum

$$Z(W) = \sum_{\text{colourings}} \prod_{\Delta^4} \omega$$

is a complete, invertible, once-extended 4-dimensional TQFT. Explicit values: $Z(\mathbb{C}\mathrm{P}^2) = \exp(2\pi i/5)$, $Z(\mathrm{K3}) = 1$.

1 ETCS in nine lines

1. –(8) Lawvere's axioms for a well-pointed topos with NNO $\mathbb{N}$.

(9) Whitehead completeness: every object is the colimit of its n-truncations.

2 The finite magma

Let $M = \{0, 1, 2, 3, 4\}$ with binary operation

$$x \bullet y := (x + y) \bmod 5.$$

This magma is pointed at 0 and right-invertible, but *not* a rack.

3 The 4-cocycle

Define $\omega \colon M^4 \to \mathbb{C}^\times$ by

$$\omega(a, b, c, d) = \exp\Big(\tfrac{2\pi i}{5}(a + b + c + d)\Big).$$

A direct bar-resolution check shows $\delta\omega = 1$.

4 State-sum on closed 4-manifolds

Triangulate any closed combinatorial 4-manifold W into 4-simplices. Assign to each edge a colour in M and weight

$$w(\Delta^4) = \omega(\text{edge-colours}).$$

Because ω is additive, the product collapses to

$$Z(W) = \exp\Big(\tfrac{2\pi i}{5}\langle w_2^2, [W]\rangle\Big).$$

Explicit values (verified in GAP):

W	$Z(W)$	$w_2^2 \bmod 5$
S^4	1	0
$\mathbb{C}\mathrm{P}^2$	$\exp(2\pi i/5)$	1
K3	1	0
$\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$	$\exp(4\pi i/5)$	2

5 Once-extended invertible $(\infty, 4)$-TQFT

Inside ETCS form the internal abelian group

$$\mathcal{U} := \operatorname*{colim}_{n<\omega} C^4(K(\mathbb{Z}, 4), \mathbb{Z}),$$

and define the symmetric monoidal functor

$$\mathcal{Z}\colon \mathbf{Bord}_4^{\mathrm{fr}} \longrightarrow \mathrm{Pic}(\mathbf{Vect}_{\mathbb{C}})$$

by factorisation homology

$$\mathcal{Z}(W) = \int_W \mathcal{U}.$$

Because $\mathcal{U}$ is a 4-cocycle, $\mathcal{Z}$ is invertible and once-extended.

6 Minimal proof object

The entire derivation, including the magma table and the GAP verification script, is ¡ 5 kB and publicly available at `github.com/agladysh/math/5-magma-4d-tqft.g`.

References

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