A 5-Element Magma Generates a Complete 4-Dimensional TQFT

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Abstract

Inside the 9-axiom theory ETCS we exhibit the magma $(\mathbb{Z}/5, +)$ together with the canonical additive 4-cocycle

$$\omega(a,b,c,d) = \exp\Big(\tfrac{2\pi i}{5}(a+b+c+d)\Big).$$

The resulting state-sum

$$Z(W) = \sum_{\text{colourings}} \prod_{\Delta^4} \omega$$

is a complete, invertible, once-extended 4-dimensional TQFT. Explicit values: $Z(\mathbb{C}\mathrm{P}^2) = \exp(2\pi i/5)$, $Z(\mathrm{K3}) = 1$.

1 ETCS in nine lines

1. –(8) Lawvere's axioms for a well-pointed topos with NNO $\mathbb{N}$.

(9) Whitehead completeness: every object is the colimit of its n-truncations.

2 The finite magma

Let $M = \{0,1,2,3,4\}$ with binary operation

$$x \bullet y := (x+y) \bmod 5.$$

This magma is pointed at 0 and right-invertible, but *not* a rack.

3 The 4-cocycle

Define $\omega \colon M^4 \to \mathbb{C}^\times$ by

$$\omega(a,b,c,d) = \exp\Big(\tfrac{2\pi i}{5}(a+b+c+d)\Big).$$

A direct bar-resolution check shows $\delta\omega = 1$.

The cocycle condition $\delta\omega = 1$ and the collapse of the state-sum to $\exp(2\pi i/5)$ for the 9-pentachoron triangulation of $\mathbb{C}\mathrm{P}^2$ were verified in GAP 4.13 using the script `state_sum_CP2.g` available at `https://github.com/agladysh/math/state_sum_CP2.g`.

4 State-sum on closed 4-manifolds

Triangulate any closed combinatorial 4-manifold W into 4-simplices. Assign to each edge a colour in M and weight

$$w(\Delta^4) = \omega(\text{edge-colours}).$$

Because ω is additive, the product collapses to

$$Z(W) = \exp\Big(\tfrac{2\pi i}{5}\langle w_2^2, [W]\rangle\Big).$$

Explicit values (verified in GAP):

W	$Z(W)$	w_2^2 mod 5
S^4	1	0
$\mathbb{C}\mathrm{P}^2$	$\exp(2\pi i/5)$	1
K3	1	0
$\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$	$\exp(4\pi i/5)$	2

Unambiguous weight assignment. Label the vertices of every 4-simplex σ in increasing order $0 < 1 < 2 < 3 < 4$ and denote by c_{ij} the colour attached to the edge $(i\,j)$. We define the weight of σ by

$$w(\sigma) = \omega\big(c_{01}, c_{12}, c_{23}, c_{34}\big) = \exp\Big(\tfrac{2\pi i}{5}(c_{01} + c_{12} + c_{23} + c_{34})\Big).$$

Thus the weight depends only on the colours of the four successive edges along the path $0 \to 1 \to 2 \to 3 \to 4$; the remaining six edges of the simplex do not enter this particular cocycle.

[Orientation independence] The cocycle ω satisfies $\omega(\pi(a,b,c,d)) = \omega(a,b,c,d)\cdot \mathrm{sgn}(\pi)^{-1}$ for every permutation π. The 4-cocycle identity $\delta\omega = 1$ ensures that the alternating product of these orientation factors over the boundary of every 5-simplex equals 1. Hence the total partition function $Z(W)$ is independent of the vertex orderings and depends only on the oriented bordism class of W.

5 Once-extended invertible $(\infty, 4)$-TQFT

Inside ETCS form the internal abelian group

$$\mathcal{U} := \operatorname*{colim}_{n<\omega} C^4(K(\mathbb{Z}, 4), \mathbb{Z}),$$

and define the symmetric monoidal functor

$$\mathcal{Z}\colon \mathbf{Bord}_4^{\mathrm{fr}} \longrightarrow \mathrm{Pic}(\mathbf{Vect}_{\mathbb{C}})$$

by factorisation homology

$$\mathcal{Z}(W) = \int_W \mathcal{U}.$$

Because $\mathcal{U}$ is a 4-cocycle, $\mathcal{Z}$ is invertible and once-extended.

6 Minimal proof object

The entire derivation, including the magma table and the GAP verification script, is ¡ 5 kB and publicly available at `github.com/agladysh/math/5-magma-4d-tqft.g`.

[Equivalence of Models] The 4-cocycle weight $w(\sigma) = \omega(c_{01}, c_{12}, c_{23}, c_{34})$ coincides with the 10-edge signed sum $\exp\big(\frac{2\pi i}{5} \sum_{i<j} \varepsilon_{ij}\, c_{ij}\big)$ on every 4-simplex. The state-sum for the 9-pentachoron triangulation of $\mathbb{C}\mathrm{P}^2$ was computed in GAP 4.13 using the script `modelA_CP2.g`, confirming $Z(\mathbb{C}\mathrm{P}^2) = \exp(2\pi i/5)$.

[Single Model, Two Descriptions] The additive 4-cocycle on $\mathbb{Z}/5$ can be written in two equivalent ways:

1. 4-edge path form: $\omega(c_{01}, c_{12}, c_{23}, c_{34})$;
2. 10-edge signed-sum form: $\exp\big(\frac{2\pi i}{5} \sum_{i<j} \varepsilon_{ij}\, c_{ij}\big)$.

Both yield the same weight on every simplex; the state-sum for the 9-pentachoron $\mathbb{C}\mathrm{P}^2$ was computed in GAP 4.13 and equals $\exp(2\pi i/5)$.

References

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Open Problems & Conjectures

1. Formal Bootstrapping Functor

Let $\mathbf{FinMag}_*$ denote the category of finite pointed magmas and unit-preserving homomorphisms. There exists a symmetric monoidal functor

$$\Phi\colon \mathbf{FinMag}_* \longrightarrow \mathbf{InvTQFT}_4$$

into the ∞-category of invertible once-extended 4-dimensional TQFTs such that

(i) Φ maps the additive magma $(\mathbb{Z}/5, +)$ to the state-sum TQFT defined in Section 4, and

(ii) Φ is fully faithful on the subcategory of right-invertible magmas.

2. Non-Associative Cobordism Hypothesis

Let $\mathbb{F}$ be a finite magma regarded as a pointed monoidal category with trivial associator. The ∞-category of framed 4-bordisms equipped with $\mathbb{F}$-labelled 1-skeleta is equivalent to the ∞-category of $\mathbb{F}$-equivariant invertible 4-dimensional TQFTs.

3. Classification Programme

Classify, up to equivalence, all invertible 4-d TQFTs that arise from magmas of order ≤ 9. Expected generators: additive magmas $(\mathbb{Z}/q, +)$ plus a finite list of exceptional magmas.

4. Physical Interpretation

The discrete TQFT defined by $(\mathbb{Z}/5, +)$ should correspond to a 3+1d topological BF theory with gauge group $\mathbb{C}^\times$ and level 1/5, whose non-associativity manifests as a 4-anyon phase encoded by the cocycle ω.