Derivation of the alternate form of the Quadratic Formula

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February 2023

1 Derivation

We begin with the well-known form of the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

We then multiply the numerator and denominator by $-b \mp \sqrt{b^2 - 4ac}$ and simplify to get the following:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \left(\frac{-b \mp \sqrt{b^2 - 4ac}}{-b \mp \sqrt{b^2 - 4ac}} \right) \\
&= \frac{b^2 \mp b\sqrt{b^2 - 4ac} \pm b\sqrt{b^2 - 4ac} - \left(b^2 - 4ac\right)}{2a\left(-b \mp \sqrt{b^2 - 4ac}\right)} \\
&= \frac{b^2 - \left(b^2 - 4ac\right)}{2a\left(-b \mp \sqrt{b^2 - 4ac}\right)} = \frac{(2a)\,(2c)}{2a\left(-b \mp \sqrt{b^2 - 4ac}\right)} \\
x &= \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}
\end{aligned} \tag{2}$$

So we are left with an alternate form of the quadratic equation, which moves the radical to the denominator.

2 Discussion

These two forms of the quadratic equation are mathematically equivalent, but implementation of these two forms in Python yields slightly different results. Owing to the underlying methods of storing floats in Python, the accumulation of round-off errors when multiplying, dividing, adding, and subtracting numbers can output numerically distinct results for mathematically equivalent processes. As it turns out, the subtraction method for both forms gives the more accurate results (that is, closest to the true result). For the example given in the problem prompt, $0 = 0.001x^2 + 1000x + 0.001$, the true roots are given by $x = -1 \cdot 10^6$ and $x = -1 \cdot 10^{-6}$, which can be verified by calculating these by hand or using a higher level program such as Desmos or Mathematica. The most accurate results in Python are, as explained above, obtained by using the 'minus' root of both forms of the quadratic equation.