# Gradient-Free Optimal Postprocessing of MCMC Output

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#### Overview

#### Problem

Develop a computationally efficient algorithm for summarising the output of a Markov Chain Monte Carlo simulation.

#### Motivation

Uncertainty quantification in a multi-stage simulation of the functioning of the human heart.

### Existing solution

The optimisation algorithm of Riabiz et al. (2022) to select a subsample of MCMC output that minimises a measure of proximity to the target distribution (kernel Stein discrepancy), which requires the gradients of the log-posterior and is thus expensive.

#### Proposal

Modify the algorithm of Riabiz et al. (2022) to use the gradient-free kernel Stein discrepancy of Fisher and Oates (2024).

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### Markov Chain Monte Carlo

Markov chain Monte Carlo (MCMC) are a popular class of algorithms for sampling from complex probability distributions.

Given a target distribution P defined on a state space  $\mathcal{X}$ , an MCMC algorithm proceeds by constructing a chain of random variables  $(X_i)_{i=0}^{\infty}$  which satisfy the Markov property:

$$\mathbb{P}(X_{i+1} \in A | X_0, \dots, X_i) = \mathbb{P}(X_{i+1} \in A | X_i)$$
 for any measurable  $A \in \mathcal{X}$ .

Viewed as a function, the right-hand side above is called the Markov transition kernel and is denoted

$$R(A|x) := \mathbb{P}(X_{i+1} \in A|X_i = x).$$

The transition kernel R is selected so that it is easy to sample from and to ensure asymptotic convergence to the target distribution P:

$$P_i \xrightarrow{d} P$$
 as  $i \to \infty$ .

A sample of size n is a realisation  $(x_i)_{i=0}^n$  of the first n variables in the chain, which is constructed sequentially.

# Challenges of running MCMC

- 1 The choice of a starting point for a chain.
- Exploring the modes of a multimodal distribution.
- Calibrating the scale of the proposal distribution.
- Convergence detection.
- Oetecting and eliminating the burn-in.
- O Autocorrelation between samples in a chain.
- Compressing sample for further expensive processing.

# **Thinning**

#### Problem

Given MCMC output  $(x_i)_{i=1}^n$  of length n, identify  $m \ll n$  indices  $\pi(j) \in \{1, \ldots, n\}$  with  $j \in \{1, \ldots, m\}$ , such that the approximation provided by the subset of samples

$$\frac{1}{m}\sum_{j=1}^m \delta(x_{\pi(j)})$$

is closest to the target distribution.

We need a measure of proximity of the selected subsample to the target distribution.

# Measure of proximity

#### Integral probability metric

An integral probability metric between two distributions P and P' is defined as

$$\mathcal{D}_{\mathcal{F}}(P, P') := \sup_{f \in \mathcal{F}} \left| \int_{\mathcal{X}} f \, \mathrm{d}P - \int_{\mathcal{X}} f \, \mathrm{d}P' \right|,$$

where  $\mathcal{X}$  is a measurable space on which both P and P' are defined and  $\mathcal{F}$  is a set of test functions.

The metric is said to be measure-determining if

$$\mathcal{D}_{\mathcal{F}}(P, P') = 0$$
 iff  $P = P'$ ,

and it offers convergence control if

$$\mathcal{D}_{\mathcal{F}}(P, P'_m) \to 0$$
 implies  $P'_m \xrightarrow{d} P$ 

as  $m \to \infty$ , for any sequence of distributions  $P'_{m'}$ 

# Measure of proximity

### Integral probability metric

An integral probability metric between two distributions P and  $P^\prime$  is defined as

$$\mathcal{D}_{\mathcal{F}}(P, P') := \sup_{f \in \mathcal{F}} \left| \int_{\mathcal{X}} f \, \mathrm{d}P - \int_{\mathcal{X}} f \, \mathrm{d}P' \right|,$$

where  $\mathcal{X}$  is a measurable space on which both P and P' are defined and  $\mathcal{F}$  is a set of test functions.

However, it is difficult to compute in practice:

- the integral  $\int_{\mathcal{X}} f \, dP$  is often intractable,
- the supremum requires optimisation.

# Stein discrepancy

#### Integral probability metric

An integral probability metric between two distributions P and P' is defined as

$$\mathcal{D}_{\mathcal{F}}(P,P') := \sup_{f \in \mathcal{F}} \left| \int_{\mathcal{X}} f \, \mathrm{d}P - \int_{\mathcal{X}} f \, \mathrm{d}P' \right|,$$

where  $\mathcal{X}$  is a measurable space on which both P and P' are defined and  $\mathcal{F}$  is a set of test functions.

#### Idea

Avoid the need to evaluate  $\int_{\mathcal{X}} f \, \mathrm{d}P$  by choosing a set of functions  $\mathcal{F}$  such that  $\int_{\mathcal{X}} f \, \mathrm{d}P = 0$  for all  $f \in \mathcal{F}$ .

# Stein discrepancy (continued)

Gorham and Mackey (2015) observed that the infinitesimal generator of a Markov process  $(Z_t)_{t\geq 0}$  given by

$$(\mathcal{L}u)(x) := \lim_{t \to 0} \frac{\mathbb{E}[u(Z_t)|Z_0 = x] - u(x)}{t} \quad \text{for } u : \mathbb{R}^d \to \mathbb{R}$$

satisfies

$$\mathbb{E}[(\mathcal{L}u)(Z)]=0$$

under mild conditions on  $\mathcal{L}$  and u.

In the specific case of an overdamped Langevin diffusion

$$\mathrm{d}Z_t = \frac{1}{2}\nabla\log p(Z_t)\,\mathrm{d}t + \mathrm{d}W_t,$$

where p is the density of P and  $W_t$  is the standard Brownian motion, the infinitesimal generator becomes

$$(\mathcal{L}_P u)(x) = \frac{1}{2} \langle \nabla u(x), \nabla \log p(x) \rangle + \frac{1}{2} \langle \nabla, \nabla u(x) \rangle.$$

# Stein discrepancy (continued)

The infinitesimal generator of an overdamped Langevin diffusion:

$$(\mathcal{L}_P u)(x) = \frac{1}{2} \langle \nabla u(x), \nabla \log p(x) \rangle + \frac{1}{2} \langle \nabla, \nabla u(x) \rangle.$$

Denoting  $g=\frac{1}{2}\nabla u$ , Gorham and Mackey (2015) obtain the Stein operator

$$\mathcal{A}_{P}g \coloneqq \langle g, \nabla \log p \rangle + \langle \nabla, g \rangle = \langle p^{-1}\nabla, pg \rangle,$$

and rewrite the expression for the integral probability metric as

$$\mathcal{D}_{P,\mathcal{G}}(P') = \sup_{g \in \mathcal{G}} \left| \int_{\mathcal{X}} \mathcal{A}_{P} g \, \mathrm{d}P' \right|$$

for a suitably chosen set G.

# Stein discrepancy (continued)

Using the Langevin Stein operator, the integral probability metric specialises to

### Stein discrepancy

$$\mathcal{D}_{P,\mathcal{G}}(P') = \sup_{g \in \mathcal{G}} \left| \int_{\mathcal{X}} \mathcal{A}_{P} g \, \mathrm{d}P' \right|$$

The difficulty evaluating the supremum still remains.

#### Idea

Employ the kernel trick to eliminate the supremum in the expression for the integral probability metric.

# Reproducing kernel Hilbert space

A Hilbert space is a vector space V equipped with the inner product operation  $\langle \cdot, \cdot \rangle$  and its induced norm  $\| \cdot \|$  satisfying  $\| v \|^2 = \langle v, v \rangle$  for all  $v \in V$ , if it is complete:

$$\sum_{i=1}^{\infty} \|v_i\| < \infty \quad \text{implies} \quad \sum_{i=1}^{\infty} v_i \in V$$

for any sequence  $v_i \in V$ .

A Hilbert space  $\mathcal{H}$  of real-valued functions defined on a set  $\mathcal{X}$  is called a reproducing kernel Hilbert space (RKHS) if there exists a function  $k: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$  such that:

- for every  $x \in \mathcal{X}$ , the function  $k(x, \cdot)$  belongs to  $\mathcal{H}$ ,
- k satisfies the reproducing property  $\langle f(\cdot), k(\cdot, x) \rangle = f(x)$  for any  $f \in \mathcal{H}$  and  $x \in \mathcal{X}$ .

We denote  $\mathcal{H}(k)$  the RKHS with kernel k.



# Kernel Stein discrepancy

Taking the unit-ball in a Cartesian product of d copies  $\mathcal{H}(k)$ 

$$\mathcal{G} := \left\{ g: \mathbb{R}^d \rightarrow \mathbb{R}^d \left| \sum_{i=1}^d \|g_i\|_{\mathcal{H}(k)}^2 \leq 1 \right. \right\},$$

Proposition 2 in Gorham and Mackey (2017) shows that the Stein discrepancy becomes

$$\mathcal{D}_{P}^{2}(P') := \mathcal{D}_{P,\mathcal{G}}(P') = \iint_{\mathcal{X}} k_{P}(x,y) \,\mathrm{d}p'(x) \,\mathrm{d}p'(y),$$

where p' is the density of P', and  $k_P(x, y)$  is given by

$$k_{P}(x,y) := (\nabla_{x} \cdot \nabla_{y})k(x,y)$$

$$+ \langle \nabla_{x}k(x,y), \nabla_{y} \log p(y) \rangle + \langle \nabla_{y}k(x,y), \nabla_{x} \log p(x) \rangle$$

$$+ k(x,y)\langle \nabla_{x} \log p(x), \nabla_{y} \log p(y) \rangle.$$

# Kernel Stein discrepancy (continued)

### Kernel Stein discrepancy (KSD)

$$\mathcal{D}_P^2(P') := \iint_{\mathcal{X}} k_P(x,y) \, \mathrm{d} \rho'(x) \, \mathrm{d} \rho'(y),$$

If P' is the discrete distribution, this becomes

$$\mathcal{D}_P^2\left(\frac{1}{n}\sum_{i=1}^n\delta(x_i)\right)=\frac{1}{n^2}\sum_{i,j=1}^nk_P(x_i,x_j),$$

where

$$k_{P}(x,y) := (\nabla_{x} \cdot \nabla_{y})k(x,y)$$

$$+ \langle \nabla_{x}k(x,y), \nabla_{y} \log p(y) \rangle + \langle \nabla_{y}k(x,y), \nabla_{x} \log p(x) \rangle$$

$$+ k(x,y)\langle \nabla_{x} \log p(x), \nabla_{y} \log p(y) \rangle.$$

The typical choice for k(x, y) is the inverse multiquadric kernel:

$$k(x,y) = (c^2 + ||\Gamma^{-1/2}(x-y)||)^{\beta}.$$

# Inverse multiquadric kernel

The common choice of the kernel k is the inverse multiquadric kernel (IMQ)

$$k(x,y) = (c^2 + ||\Gamma^{-1/2}(x-y)||)^{\beta}.$$

When  $\beta \in (-1,0)$  and  $\Gamma = I$ , Gorham and Mackey (2017) demonstrate that  $\mathcal{D}_P(P')$  provides convergence control (Theorem 8). Theorem 4 in Chen et al. (2019) justifies the introduction of  $\Gamma$  in IMQ.

# Stein thinning

Riabiz et al. (2022) propose a greedy algorithm to select points from the sample that minimise the KSD at each iteration:

Algorithm 1: Stein thinning.

#### Data:

sample  $(x_i)_{i=1}^n$  from MCMC,

gradients  $(\nabla \log p(x_i))_{i=1}^n$ 

desired cardinality  $m \in \mathbb{N}$ 

**Result:** Indices  $\pi$  of a sequence  $(x_{\pi(j)})_{j=1}^m$  where  $\pi(j) \in \{1, \dots, n\}$ .

for 
$$j = 1, \ldots, m$$
 do

$$\pi(j) \in \operatorname*{arg\,min}_{i=1,\ldots,n} rac{k_P(x_i,x_i)}{2} + \sum_{j'=1}^{j-1} k_P(x_{\pi(j')},x_i)$$

end

# Stein thinning (continued)

The complication in using Stein thinning comes from the need to calculate gradients of the log-posterior to evaluate the kernel:

$$k_{P}(x,y) := (\nabla_{x} \cdot \nabla_{y})k(x,y)$$

$$+ \langle \nabla_{x}k(x,y), \nabla_{y} \log p(y) \rangle + \langle \nabla_{y}k(x,y), \nabla_{x} \log p(x) \rangle$$

$$+ k(x,y)\langle \nabla_{x} \log p(x), \nabla_{y} \log p(y) \rangle.$$

This might be expensive, although it can be easily parallelised.

# Gradient-free kernel Stein discrepancy

Fisher and Oates (2024) introduce a gradient-free version of KSD. An auxiliary distribution Q need to be chosen by the user, then the gradient-free KSD is given by

$$k_{P,Q}(x,y) = \frac{q(x)}{p(x)} \frac{q(y)}{p(y)} k_Q(x,y),$$

where

$$k_{Q}(x,y) := (\nabla_{x} \cdot \nabla_{y})k(x,y)$$

$$+ \langle \nabla_{x}k(x,y), \nabla_{y} \log q(y) \rangle + \langle \nabla_{y}k(x,y), \nabla_{x} \log q(x) \rangle$$

$$+ k(x,y)\langle \nabla_{x} \log q(x), \nabla_{y} \log q(y) \rangle.$$

When k(x, y) is the inverse multiquadric kernel, the gradient-free KSD offers convergence control (Theorem 2 in Fisher and Oates (2024)).

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# Gradient-free Stein thinning

We modify the algorithm of Riabiz et al. (2022) to minimise the gradient-free KSD of Fisher and Oates (2024):

Algorithm 2: Gradient-free Stein thinning.

#### Data:

sample  $(x_i)_{i=1}^n$  from MCMC, target log-densities  $(\log p(x_i))_{i=1}^n$  auxiliary log-densities  $(\log q(x_i))_{i=1}^n$  auxiliary gradients  $(\nabla \log q(x_i))_{i=1}^n$  desired cardinality  $m \in \mathbb{N}$ 

**Result:** Indices  $\pi$  of a sequence  $(x_{\pi(j)})_{j=1}^m$  where  $\pi(j) \in \{1, \ldots, n\}$ .

for 
$$j=1,\ldots,m$$
 do

$$\pi(j) \in \operatorname*{arg\,min}_{i=1,...,n} rac{k_{P,Q}(x_i,x_i)}{2} + \sum_{i'=1}^{j-1} k_{P,Q}(x_{\pi(j')},x_i)$$

end

### Algorithm 3: Optimised gradient-free Stein thinning.

#### Data:

```
sample (x_i)_{i=1}^n from MCMC, target log-densities (\log p(x_i))_{i=1}^n auxiliary log-densities (\log q(x_i))_{i=1}^n auxiliary gradients (\nabla \log q(x_i))_{i=1}^n desired cardinality m \in \mathbb{N}.
```

**Result:** Indices  $\pi$  of a sequence  $(x_{\pi(j)})_{j=1}^m$  where  $\pi(j) \in \{1, \ldots, n\}$ . Initialise an array A[i] of size nSet  $A[i] = k_{P,Q}(x_i, x_i)$  for  $i = 1, \ldots, n$ Set  $\pi(1) = \arg\min_i A[i]$  **for**  $j = 2, \ldots, m$  **do** Update  $A[i] = A[i] + 2k_{P,Q}(x_{\pi(j-1)}, x_i)$  for  $i = 1, \ldots, n$ Set  $\pi(j) = \arg\min_i A[i]$ 

end

# **Evaluation protocol**

The following protocol was used in the evaluating the new method:

- obtain a sample from the target distribution (depending on the test case, the sampling is done either i.i.d. or via MCMC),
- apply naïve thinning, Stein thinning and the proposed algorithm to get a thinned sample of a given cardinality,
- evaluate the result of thinning using an impartial metric.

### **Energy distance**

In order to assess how well the selected sample approximates the target distribution, we use the energy distance.

### Energy distance (Rizzo and Székely (2016))

The squared energy distance is defined for two distributions P and Q as

$$D_{e}^{2}(P,Q) := 2\mathbb{E}||X - Y|| - \mathbb{E}||X - X'|| - \mathbb{E}||Y - Y'||,$$

where  $X, X' \sim P$  and  $Y, Y' \sim Q$ .

For samples  $x_1, \ldots, x_n$  and  $y_1, \ldots, y_m$  from X and Y, respectively, the corresponding statistic is given by

$$\mathcal{E}_{n,m}(P,Q) := \frac{2}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \|x_i - y_j\| - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \|x_i - x_j\| - \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \|y_i - y_j\|.$$

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#### Bivariate Gaussian mixture

We use the bivariate Gaussian mixture with means

$$\mu_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \qquad \mu_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

covariance matrices

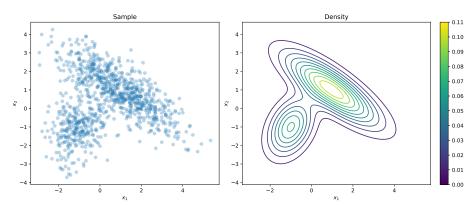
$$\Sigma_1 = \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 1 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 2 & -0.8\sqrt{3} \\ -0.8\sqrt{3} & 1.5 \end{pmatrix}$$

and weights

$$w = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}.$$

### Bivariate Gaussian mixture: sample

We obtain 1000 points by directly drawing from the target distribution:

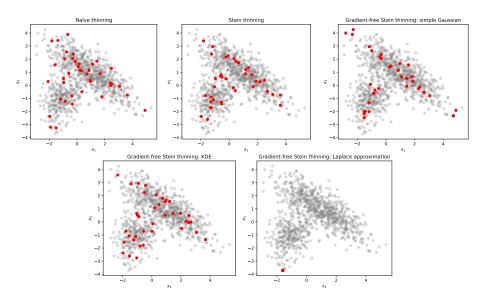


# Bivariate Gaussian mixture: thinning approaches

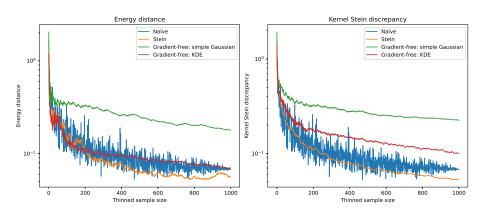
We evaluate the following approaches:

- naïve thinning,
- standard Stein thinning,
- gradient-free Stein thinning with different choices of *Q*:
  - multivariate Gaussian using the sample mean and covariance,
  - Laplace approximation,
  - KDE approximation.

# Bivariate Gaussian mixture: thinning results



# Bivariate Gaussian mixture: comparison of approaches



### Lotka-Volterra inverse problem

The Lotka-Volterra model describes the evolution of an idealised ecosystem with two species: predator and prey.

Let  $u_1$  be the size of the prey population and  $u_2$  the size of the predator population. The model then postulates the following dynamic:

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = \theta_1 u_1 - \theta_2 u_1 u_2,$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = -\theta_3 u_2 + \theta_4 u_1 u_2,$$

with  $\theta_1, \ldots, \theta_4 > 0$ .

The inverse problem: given a noisy realisation

$$y(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \varepsilon(t),$$

infer  $\theta = (\theta_1, \dots, \theta_4)^T$  that best describes the observed behaviour.

# Lotka-Volterra inverse problem: synthetic data

#### Lotka-Volterra model

$$\begin{split} \frac{\mathrm{d}u_1}{\mathrm{d}t} &= \theta_1 u_1 - \theta_2 u_1 u_2, \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} &= -\theta_3 u_2 + \theta_4 u_1 u_2, \end{split}$$

We solve the model with parameters

$$\theta^* = (0.67, 1.33, 1, 1)^T$$

and initial values

$$u(0) = (1,1)^T$$

for  $t \in [0, 25]$  discretised into N = 2400 points.

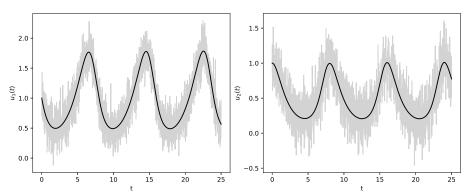
We then add bivariate i.i.d. Gaussian noise:

$$\varepsilon(t) \sim \mathcal{N}(0, \operatorname{diag}(0.2^2, 0.2^2)).$$



### Lotka-Volterra inverse problem: synthetic data

The resulting noisy data is used as the input for the inverse problem:



### Lotka-Volterra inverse problem: Bayesian inference

Assuming independent observations, we take the likelihood to be

$$\mathcal{L}(\theta) = \prod_{i=1}^{N} \phi_i(u(t_i; \theta)),$$

where

$$\phi_i(u(t_i;\theta)) \propto \exp\left(-\frac{1}{2}(y(t_i)-u(t_i;\theta))^T C^{-1}(y(t_i)-u(t_i;\theta))\right)$$

with  $C = diag(0.2^2, 0.2^2)$ .

Since  $\theta_k > 0$ , we put independent log-normal priors on each  $\theta_k$ :

$$\pi(\theta) \propto \exp\left(-\frac{1}{2}(\log \theta)^T(\log \theta)\right).$$

By the Bayes theorem, the posterior is then

$$p(\theta) \propto \mathcal{L}(\theta)\pi(\theta)$$
.



# Lotka-Volterra inverse problem: MCMC

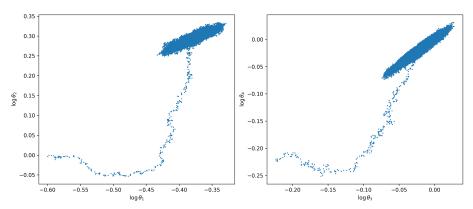
The inference is performed using the Metropolis-Hastings algorithm with the starting points taken from Riabiz et al. (2022):

Chain	$\theta_1$	$\theta_2$	$\theta_3$	$\theta_4$
1	0.55	1	8.0	0.8
2	1.5	1	8.0	8.0
3	1.3	1.33	0.5	8.0
4	0.55	3	3.	8.0
5	0.55	1	1.5	1.5

Since the parameters  $\theta_k$  of the Lotka-Volterra model are positive, we run MCMC in the log-space by applying the reparameterisation  $\zeta_k = \log \theta_k$ . We run 500,000 iterations of the algorithm for each chain.

### Lotka-Volterra inverse problem: MCMC sample

The sample from the first chain is shown here for illustration:



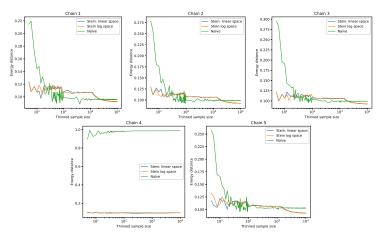
## Lotka-Volterra inverse problem: thinning approaches

#### We evaluate the following approaches:

- naïve thinning,
- standard Stein thinning,
- gradient-free Stein thinning with different choices of Q:
  - multivariate Gaussian using the sample mean and covariance,
  - Laplace approximation fails as in bivariate Gaussian case,
  - KDE approximation computationally infeasible.

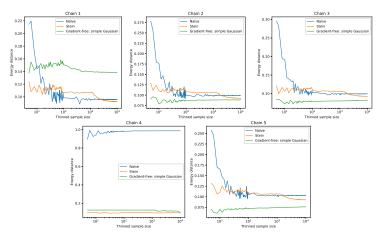
# Lotka-Volterra inverse problem: Stein thinning

We have a choice of applying thinning in linear or logarithmic space, however the energy distance comparison indicates no discernible difference, so we proceed to use the logarithmic space.



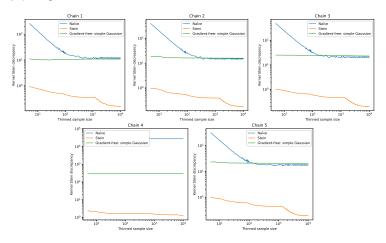
# Lotka-Volterra inverse problem: comparison of approaches

Gradient-free thinning using a multivariate Gaussian with the sample mean and covariance matrix performs comparably with the gradient-based approach, notably for chain 4, when compared using the energy distance:



# Lotka-Volterra inverse problem: comparison of approaches

Gradient-free thinning offers a significant improvement over naïve thinning in terms of KSD:



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#### Contribution

#### The project makes three contributions:

- implementation of the gradient-free Stein thinning algorithm in the Python library stein-thinning,
- evaluation of the performance of the proposed algorithm,
- improvement of the computational efficiency of the existing Stein thinning algorithm from  $O(nm^2)$  to O(nm), where n is the input sample size and m is the desired thinned sample size.

#### Conclusions

- The gradient-free approach is feasible and performs similarly to the Stein thinning algorithm of Riabiz et al. (2022) for small thinned sample sizes,
- The performance of the algorithm depends crucially on the choice of the auxiliary distribution. For example, even in the highly favourable setting of i.i.d. samples from a Gaussian mixture, choosing the auxiliary distribution based on the Laplace approximation fails to produce a thinned sample.
- The simple multivariate Gaussian distribution using the sample mean and covariance offered a good starting point in our experiments, however bespoke treatment might be required for more complex problems.
- In deciding whether to use the new algorithm as opposed to the gradient-based approach, the effort involved in selecting a good auxiliary distribution must be weighed against the computational cost of obtaining gradients.

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### Further Research

- Evaluate the choices of KDE kernels other than Gaussian for constructing the auxiliary distribution.
- Parallelise the computation of KDE.
- Perform thinning in a lower-dimensional space.
- Investigate the behaviour of Stein thinning for large thinned sample sizes.
- Compare the performance of the approaches in terms of estimating the true parameters of the Lotka-Volterra model.
- Run an experiment with randomised starting points.

# Further Research (continued)

- Repeat the experiments with more advanced MCMC algorithms.
- Check how running a gradient-free MCMC sampling algorithm (such the random-walk Metropolis-Hastings) followed by Stein thinning of the sample compares to running a gradient-based sampling algorithm (e.g. HMC).
- Provide theoretical justification for gradient-free Stein thinning.
- Explore other gradient-free alternatives.

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