Q1: Find the absolute maximum value of $f(x,y) = x+2y$ subject to the constraint $x^2+y^2=4$.

Answer: $g(x,y) = x^2+y^2-4=0$

$$\nabla f = \lambda \nabla g$$
$$\nabla f = \langle 1,2 \rangle$$
$$\nabla g = \langle 2x, 2y \rangle$$

$$\langle 1,2 \rangle = \lambda \langle 2x, 2y \rangle$$

1. $1 = \lambda 2x \Rightarrow \lambda = \frac{1}{2x}$
2. $2 = \lambda 2y \qquad \lambda = \frac{1}{y}$
3. $x^2+y^2-4=0$

$$\left.\begin{matrix}\lambda = \frac{1}{2x} \\ \lambda = \frac{1}{y}\end{matrix}\right\} \quad \frac{1}{2x} = \frac{1}{y} \qquad \boxed{2x=y}$$

$x \neq 0$
$y \neq 0$

$$x^2 + (2x)^2 - 4 = 0$$
$$x^2 + 4x^2 - 4 = 0$$
$$5x^2 - 4 = 0$$
$$5x^2 = 4$$
$$x^2 = \frac{4}{5}$$
$$x = \pm \frac{2}{\sqrt{5}}$$

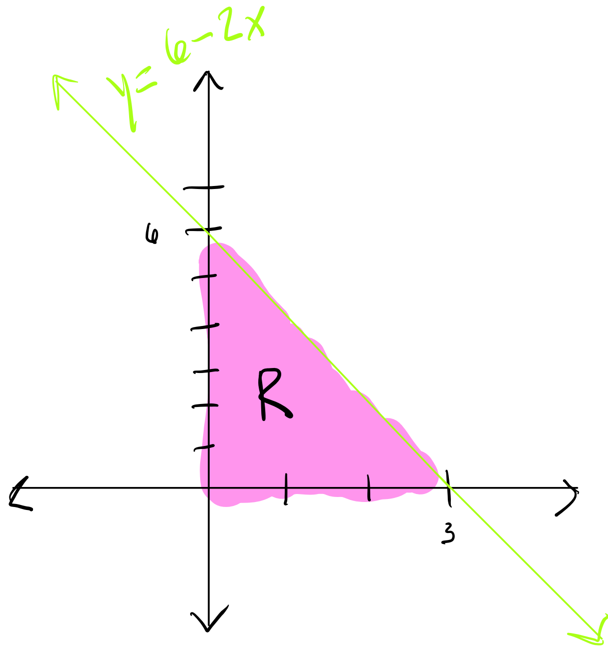
Table:

x	y	λ	$f(x,y) = x+2y$
$\frac{2}{\sqrt{5}}$	$\frac{4}{\sqrt{5}}$	$\frac{\sqrt{5}}{4}$	$\frac{10}{\sqrt{5}}$
$-\frac{2}{\sqrt{5}}$	$\frac{-4}{\sqrt{5}}$	$\frac{\sqrt{5}}{-4}$	$\frac{-10}{\sqrt{5}}$

Absolute max: $\frac{10}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$

Q2: Reverse the order of integration in the following integral.

$$\int_0^3 \int_0^{6-2x} F(x,y)\,dy\,dx$$

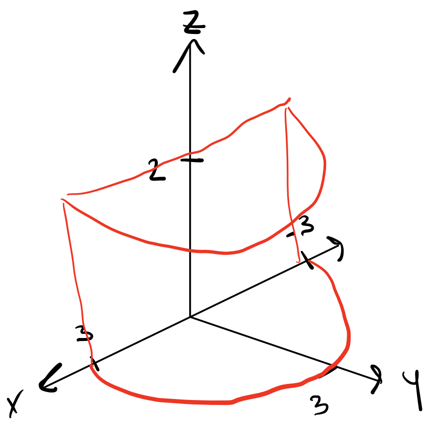


Answer:

$$\int_{x=0}^{x=3} \int_{y=0}^{y=6-2x} F(x,y)\,dy\,dx = \int_{y=0}^{y=6} \int_{x=0}^{x=-\frac{1}{2}y+3} f(x,y)\,dx\,dy$$

Q3: Convert the following to cylindrical coordinates:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2} \frac{1}{1+x^2+y^2}\, dz\, dy\, dx$$



Ans:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dV = r\, dz\, dr\, d\theta$

$x^2+y^2 = r^2$

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=3}\int_{z=0}^{z=2} \frac{1}{1+r^2}\, r\, dz\, dr\, d\theta$$

Q4: Evaluate $\int_C F \cdot T\, ds$ where

$F = \langle -y, x \rangle$ on the parabola $y = x^2$

from $(0,0)$ to $(1,1)$.

Answer: $r(t) = \langle \overset{x}{t}, \overset{y}{t^2} \rangle$ $\quad r'(t) = \langle 1, 2t \rangle$

$0 \leq t \leq 1$

$$\int_C F \cdot T\, ds = \int_C F \cdot r'(t)\, dt = \int_{t=0}^{t=1} \langle -t^2, t \rangle \cdot \langle 1, 2t \rangle\, dt$$

$$= \int_{t=0}^{t=1} -t^2 + 2t^2\, dt = \int_{t=0}^{t=1} t^2\, dt = \left. \frac{t^3}{3} \right|_{t=0}^{t=1}$$

$$= \boxed{\frac{1}{3}}$$

<u>Q5:</u> Is $\langle \overset{f}{yz}, \overset{g}{xz}, \overset{h}{xy} \rangle$ conservative?

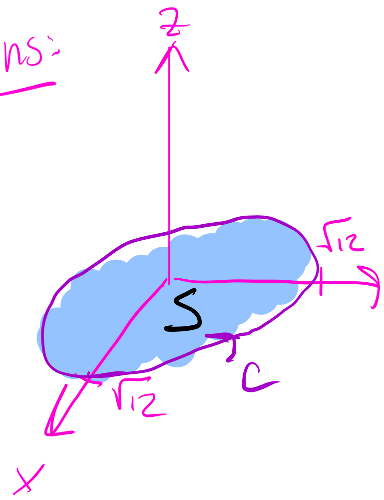
<u>Ans:</u>

$$\left.\begin{aligned} f_y &= z = g_x \\ f_z &= y = h_x \\ g_z &= x = h_y \end{aligned}\right\} \underline{\text{yes!}}$$

Q6: Set up the line integral $\oint F \cdot dr$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume C has counterclockwise orientation.

$F = \langle 2y, -z, x \rangle$; C is the circle $x^2+y^2=12$ in the plane $z=0$.

Ans:



$S = r(u,v) = \langle u\cos v, u\sin v, 0 \rangle$ $\quad 0 \le u \le \sqrt{12}$, $0 \le v \le 2\pi$

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n \, dS = \iint_R \nabla \times F \cdot t_u \times t_v \, dA$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & -z & x \end{vmatrix} = (0-(-1))i - (1-0)j + (0-2)k$$
$$= \langle 1, -1, -2 \rangle$$

$t_u = \langle \cos v, \sin v, 0 \rangle$

$t_v = \langle -u\sin v, u\cos v, 0 \rangle$

$$t_u \times t_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 0, 0, u \rangle \checkmark$$

$$\boxed{\int_{u=0}^{u=\sqrt{12}} \int_{v=0}^{v=2\pi} -2u \, du \, dv}$$

Q7: Use the divergence theorem to compute net outward flux of $F = \langle x, -2y, 3z \rangle$ where S is the sphere $x^2+y^2+z^2=6$.

Ans: $\iint_S F \cdot n \, dS = \iiint_D \nabla \cdot F \, dV$

$\nabla \cdot F = 1 + -2 + 3 = 2$

$= \iiint_D 2 \, dV = 2 \cdot$ volume of sphere of radius $\sqrt{6}$

$= 2 \cdot \frac{4}{3} \pi (\sqrt{6})^3$

$= \frac{2 \cdot 4 \cdot 6 \cdot \sqrt{6} \cdot \pi}{3}$

$= \boxed{16\sqrt{6}\pi}$