Conic Duality

Yinyu Ye
Department of Management Science and Engineering
Stanford University
Stanford, CA 94305, U.S.A.

http://www.stanford.edu/~yyye

Vectors and Norms

- Real numbers: $\mathcal{R}, \quad \mathcal{R}_+, \quad \operatorname{int}\mathcal{R}_+$
- n-dimensional Euclidean space $\mathcal{R}^n, \quad \mathcal{R}^n_+, \quad \operatorname{int}\mathcal{R}^n_+$
- Component-wise: $\mathbf{x} \geq \mathbf{y}$ means $x_j \geq y_j$ for $j = 1, 2, ..., n$
- $\mathbf{0}$: vector of all zeros; and $\mathbf{e}$: vector of all ones
- Inner-product of two vectors:

$$\mathbf{x} \bullet \mathbf{y} := \mathbf{x}^T\mathbf{y} = \sum_{j=1}^{n} x_j y_j$$

- Euclidean norm: $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}}$,

  Infinity-norm: $\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, ..., |x_n|\}$,

  p-norm: $\|\mathbf{x}\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$

- The dual of the p norm, denoted by $\|.\|^*$, is the q norm, where $\frac{1}{p}+\frac{1}{q}=1$
- Column vector:

$$\mathbf{x}=(x_1;x_2;\ldots;x_n)$$

Row vector:

$$\mathbf{x}=(x_1,x_2,\ldots,x_n)$$

- Transpose operation: A^T
- A set of vectors $\mathbf{a}_1, ..., \mathbf{a}_m$ is said to be linearly dependent if there are scalars $\lambda_1, ..., \lambda_m$, not all zero, such that the linear combination

$$\sum_{i=1}^{m} \lambda_i \mathbf{a}_i = \mathbf{0}$$

- A linearly independent set of vectors that span R^n is a basis.

Hyper plane and Half-spaces

$$H = \{\mathbf{x} : \mathbf{a}\mathbf{x} = \sum_{j=1}^{n} a_j x_j = b\}$$

$$H^{+} = \{\mathbf{x} : \mathbf{a}\mathbf{x} = \sum_{j=1}^{n} a_j x_j \leq b\}$$

$$H^{-} = \{\mathbf{x} : \mathbf{a}\mathbf{x} = \sum_{j=1}^{n} a_j x_j \geq b\}$$

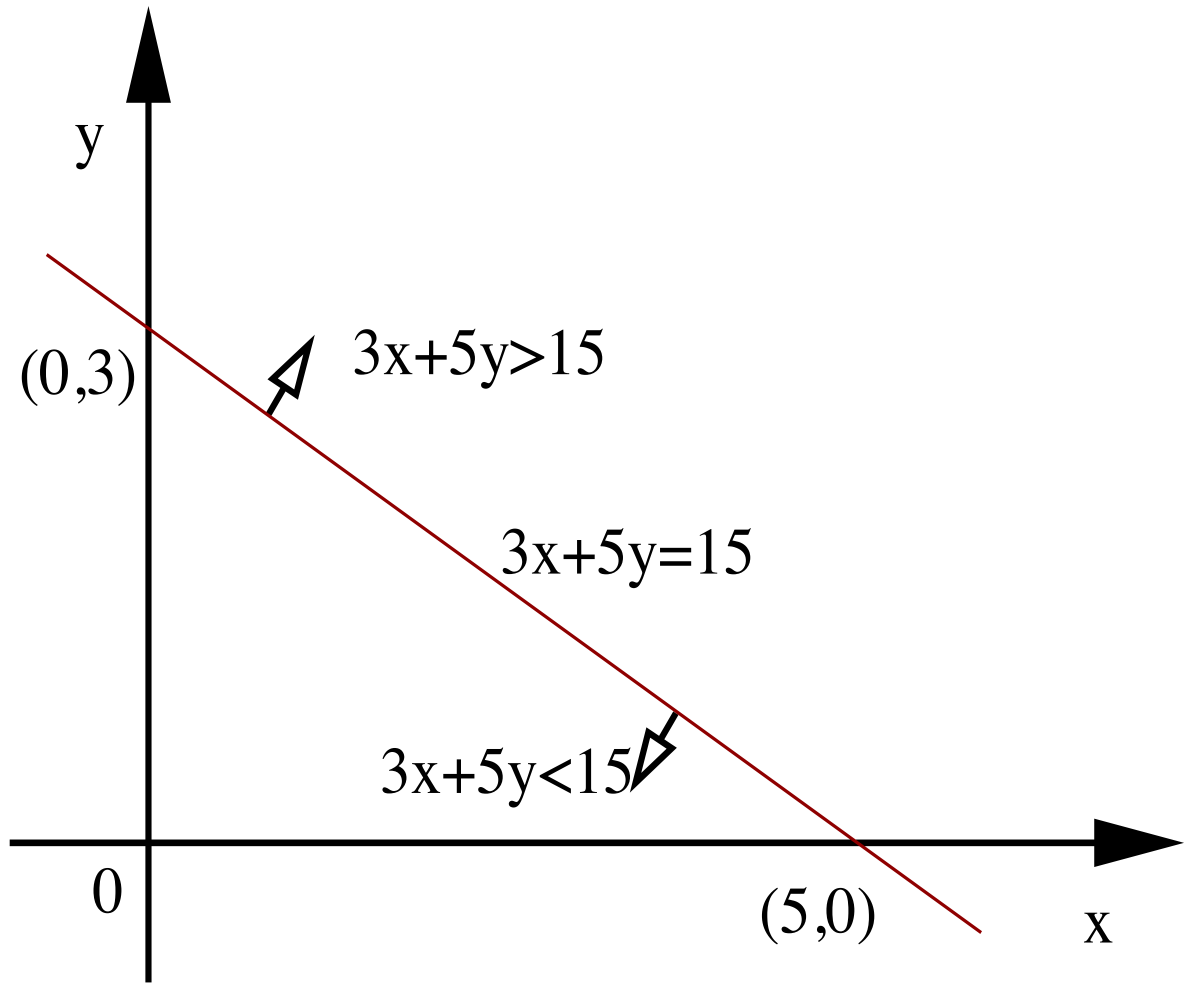

Figure 1: Plane and Half-Spaces

Matrices and Norms

- Matrix: $\mathcal{R}^{m\times n}$, ith row: $a_{i.}$, jth column: $a_{.j}$, ijth element: a_{ij}
- A_I denotes the submatrix of A whose rows belong to index set I, A_J denotes the submatrix whose columns belong to index set J, A_{IJ} denotes the submatrix whose rows belong to index set I and columns belong to index set J.
- Determinant: $\det(A)$, Trace: $\mathrm{tr}(A)$
- The operator norm of $\|A\|$,

$$\|A\|^2 := \max_{0\neq x\in\mathcal{R}^n} \frac{\|Ax\|^2}{\|x\|^2}$$

- All-zero matrix: $\mathbf{0}$, and identity matrix: I
- Symmetric matrix: $Q = Q^T$

- Positive Definite: $Q \succ 0$ iff $\mathbf{x}^T Q \mathbf{x} > 0, \quad \text{for all} \quad \mathbf{x} \neq \mathbf{0}$
- Positive Semidefinite: $Q \succeq 0$ iff $\mathbf{x}^T Q \mathbf{x} \geq 0, \quad \text{for all} \quad \mathbf{x}$
- Null space and Row space of matrix A:

Theorem 1 *The null space and row space of a matrix are perpenticular to each other, that is,*

$$\mathbf{x}^T \mathbf{s} = 0, \quad \forall\, A\mathbf{x} = \mathbf{0} \quad \textit{and} \quad \mathbf{s} = A^T \mathbf{y}.$$

Symmetric Matrix Space

- n-dimensional symmetric matrix space: $\mathcal{S}^n$
- Inner Product:

$$X \bullet Y = \mathrm{tr} X^T Y = \sum_{i,j} X_{i,j} Y_{i,j}$$

- Frobenius norm:

$$\|X\|_f = \sqrt{\mathrm{tr} X^T X}$$

- Positive semidefinite matrix set: $\mathcal{S}^n_+$, Positive definite matrix set: $\mathrm{int}\, \mathcal{S}^n_+$

- Decomposition of Symmetric Positive Semidefinite Matrices:

$$X = \sum_{i=1}^{r} \mathbf{x}_i \mathbf{x}_i^T$$

where r is the rank of X, and $\mathbf{x}_i^T \mathbf{x}_j = 0$ for $i \neq j$.

- Let X be a positive semidefinite matrix of rank r, A be any given symmetric matrix. Then, there is a decomposition of X

$$X = \sum_{j=1}^{r} \mathbf{x}_j \mathbf{x}_j^T,$$

such that for all j,

$$\mathbf{x}_j^T A \mathbf{x}_j = A \bullet (\mathbf{x}_j \mathbf{x}_j^T) = A \bullet X/r.$$

Affine and Convex Combination

$S \subset R^n$ is affine if

$$[\,\mathbf{x}, \mathbf{y} \in S \text{ and } \alpha \in R\,] \Longrightarrow \alpha x + (1-\alpha) y \in S.$$

When $\mathbf{x}$ and $\mathbf{y}$ are two distinct points in R^n and α runs over R ,

$$\{\mathbf{z} : \mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}\}$$

is the line set determined by $\mathbf{x}$ and $\mathbf{y}$.

When $0 \le \alpha \le 1$, it is called the convex combination of $\mathbf{x}$ and $\mathbf{y}$ and it is the line segment between $\mathbf{x}$ and $\mathbf{y}$.

Convex Sets

- Set notations: $x \in \Omega, \quad y \notin \Omega\, S \cup T, \quad S \cap T$
- Ω is said to be a convex set if for every $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$ and every real number $\alpha \in [0,1]$, the point $\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2 \in \Omega$.
- Intersection of convex sets is convex; the convex hull of a set Ω is the intersection of all convex sets containing Ω
- A point in a set is called an extreme point of the set if it cannot be represented as the convex combination of two distinct points of the set.
- A set is a polyhedral set if it has finitely many extreme points.

Set Combinations

Let C_1 and C_2 be convex sets in a same space. Then,

- $C_1 \cap C_2$ is convex.
- $C_1 + C_2$ is convex, where

$$C_1 + C_2 = \{\mathbf{b}_1 + \mathbf{b}_2 : \ \mathbf{b}_1 \in C_1 \text{ and } \mathbf{b}_2 \in C_2\}.$$

- $C_1 \oplus C_2$ is convex, where

$$C_1 \oplus C_2 = \{(\mathbf{b}_1; \mathbf{b}_2) : \ \mathbf{b}_1 \in C_1 \text{ and } \mathbf{b}_2 \in C_2\}.$$

Cones

- A set K is a cone if $\mathbf{x} \in K$ implies $\alpha\mathbf{x} \in K$ for all $\alpha > 0$
- A convex cone is cone and it's also a convex-set.
- Dual cone:

$$K^* := \{\mathbf{y} : \ \mathbf{y} \bullet \mathbf{x} \geq 0 \quad \text{for all} \quad \mathbf{x} \in K\}$$

  $-K^*$ is also called the polar of K.

- The dual of a cone is always a closed convex cone.

Cone Examples

- Example 2.1: The n-dimensional non-negative orthant, $\mathcal{R}^n_+ = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{x} \geq \mathbf{0}\}$, is a convex cone; and it's self dual.
- Example 2.2: The set of all positive semi-definite symmetric matrices in $\mathcal{S}^n$, $\mathcal{S}^n_+$, is a convex cone, called the positive semi-definite matrix cone; and it's self dual.
- Example 2.3: The set $\{\mathbf{x} \in \mathcal{R}^n : x_1 \geq \|\mathbf{x}_{-1}\|\}$, $\mathcal{N}^n_2$, is a convex cone in $\mathcal{R}^n$ called the second-order (norm) cone; and it's self dual.
- Example 2.4: The set $\{\mathbf{x} \in \mathcal{R}^n : x_1 \geq \|\mathbf{x}_{-1}\|_p\}$, $\mathcal{N}^n_p$, is a convex cone in $\mathcal{R}^n$ for $p \geq 1$ called the p-order (norm) cone; and its dual is the q-order cone where $\frac{1}{p} + \frac{1}{q} = 1$.

Cone and Dual Facts

Let K_1 and K_2 be both closed convex cones. Then

i) $(K_1^*)^* = K_1$.

ii) $K_1 \subset K_2 \Longrightarrow K_2^* \subset K_1^*$.

iii) $(K_1 \oplus K_2)^* = K_1^* \oplus K_2^*$.

iv) $(K_1 + K_2)^* = K_1^* \cap K_2^*$.

v) $(K_1 \cap K_2)^* = K_1^* + K_2^*$.

Convex Polyhedral Cones I

- A cone K is (convex) polyhedral if its intersection with a hyperplane is a polyhedral set.
- A convex cone K is polyhedral if and only if K can be represented by

$$K = \{\mathbf{x} : A\mathbf{x} \le 0\} \quad \text{or} \quad \{\mathbf{x} : \mathbf{x} = A\mathbf{y},\ \mathbf{y} \ge \mathbf{0}\}$$

for some matrix A. In the latter case, K is generated by the columns of A.

- The nonnegative orthant is a polyhedral cone but the second-order cone is not polyhedral.

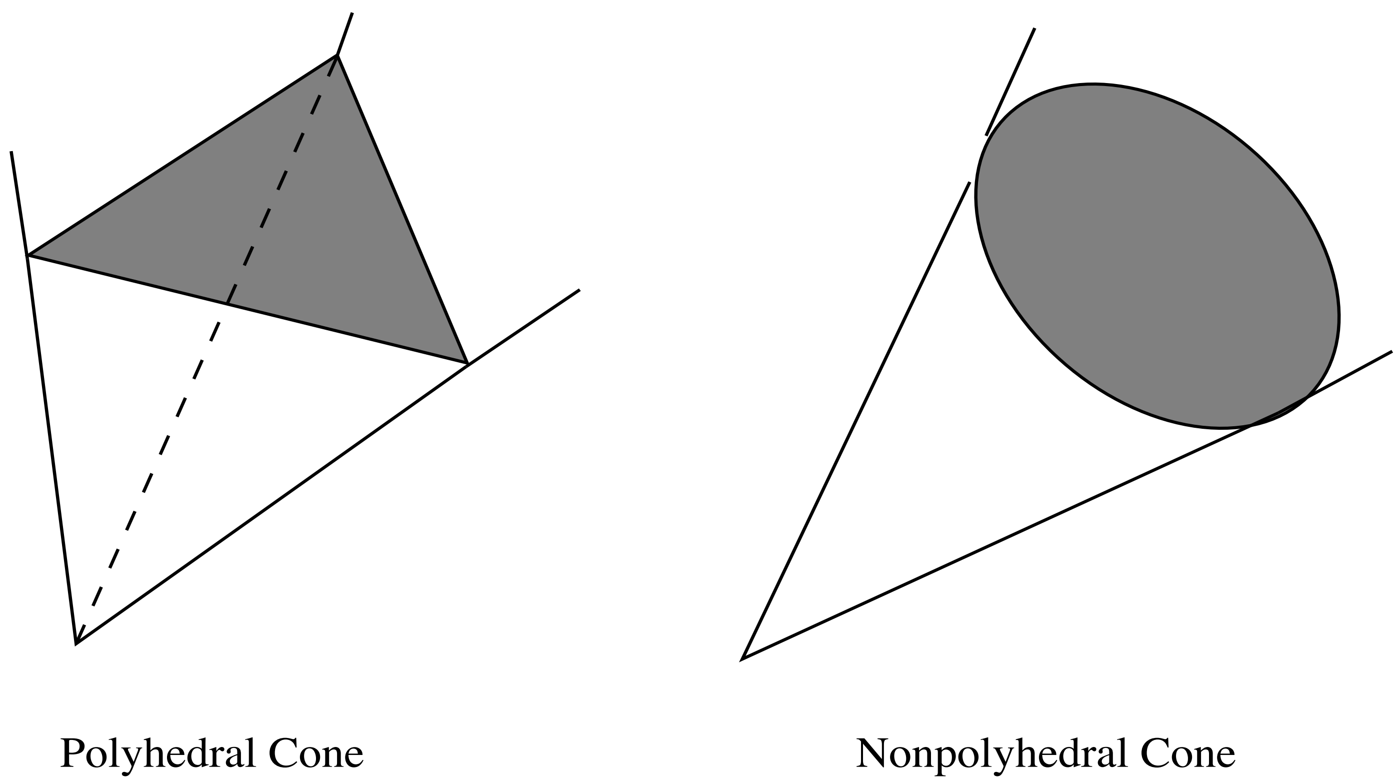

Figure 2: Polyhedral and non-polyhedral cones.

Convex Polyhedral Cones II

It has been proved that for cones the concepts of "polyhedral" and "finitely generated" are equivalent according to the following theorem.

Theorem 2 *(Minkowski and Weyl) A convex cone C is polyhedral if and only if it is finitely generated, that is, the cone is generated by a finite number of vectors* $\mathbf{b}_1,...,\mathbf{b}_m$*:*

$$C = \mathit{cone}(\mathbf{b}_1, ..., \mathbf{b}_m) := \left\{ \sum_{i=1}^{m} \mathbf{b}_i y_i : \ y_i \geq 0, \ i = 1, ..., m \right\}.$$

Carathéodory's theorem

The following theorem states that a polyhedral cone can be generated by a set of basic directional vectors.

Theorem 3 *Let convex polyhedral cone* $C = \textit{cone}(\mathbf{b}_1, \ldots, \mathbf{b}_m)$ *and* $\mathbf{x} \in C$. *Then,* $\mathbf{x} \in \textit{cone}(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_d})$ *for some linearly independent vectors* $\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_d}$ *chosen from* $\mathbf{b}_1, \ldots, \mathbf{b}_m$.

Some times we even have:

$$\left\{ \mathbf{x} \in \mathcal{R}^2_+ : \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} \le \mathbf{0} \right\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} y_1 + \begin{pmatrix} 2 \\ 1 \end{pmatrix} y_2 : \; y_1, y_2 \ge 0 \right\}.$$

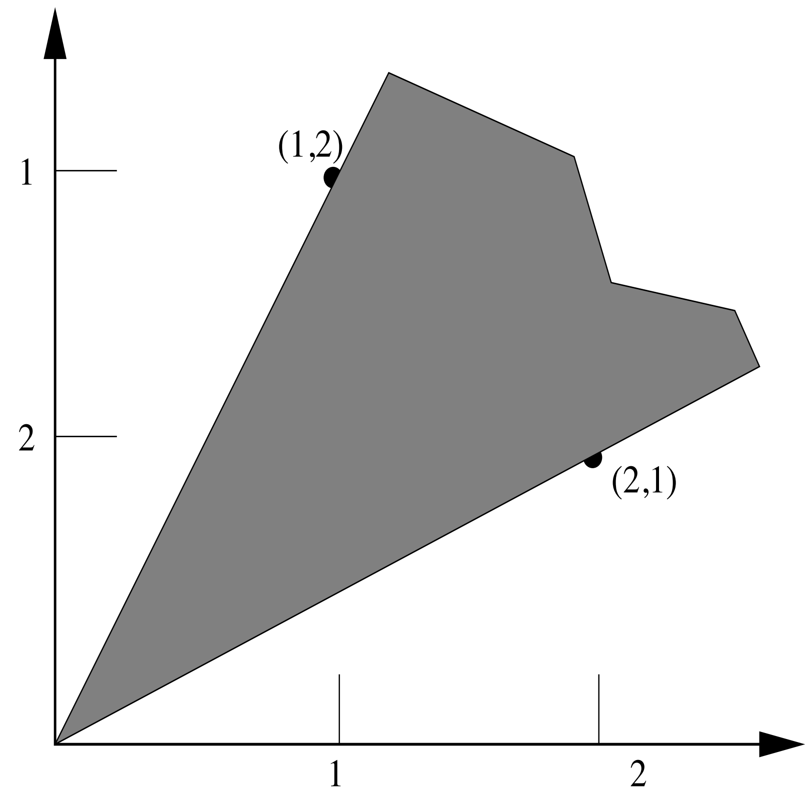

Figure 3: Representations of a polyhedral cone.

Separating hyperplane theorem

The most important theorem about the convex set is the following separating hyperplane theorem (Figure 4).

Theorem 4 *(Separating hyperplane theorem) Let $C \subset \mathcal{E}$, where $\mathcal{E}$ is either $\mathcal{R}^n$ or $\mathcal{S}^n$, be a closed convex set and let $\mathbf{b}$ be a point exterior to C. Then there is a vector $\mathbf{a} \in \mathcal{E}$ such that*

$$\mathbf{a} \bullet \mathbf{b} > \sup_{\mathbf{x} \in C} \mathbf{a} \bullet \mathbf{x}$$

where $\mathbf{a}$ is the norm direction of the hyperplane.

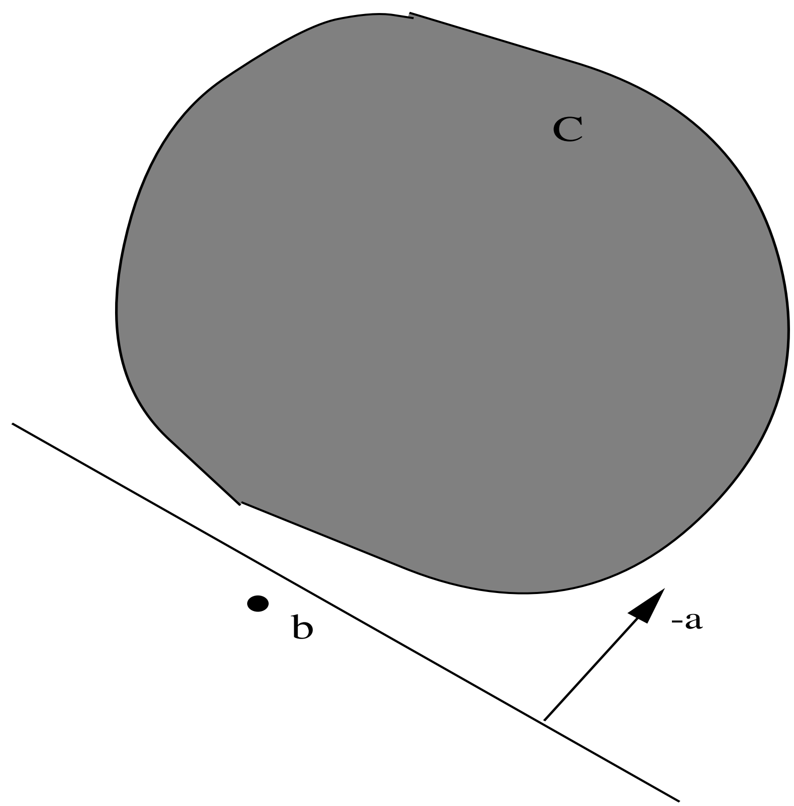

Figure 4: Illustration of the separating hyperplane theorem; an exterior point $\mathbf{b}$ is separated by a hyperplane from a convex set C.

Examples

Let C be a unit circle centered at point $(1;1)$. That is, $C = \{\mathbf{x} \in \mathcal{R}^2 :\ (x_1 - 1)^2 + (x_2 - 1)^2 \le 1\}$. If $\mathbf{b} = (2;0)$, $\mathbf{a} = (-1;1)$ is a separating hyperplane vector.

If $\mathbf{b} = (0;-1)$, $\mathbf{a} = (0;1)$ is a separating hyperplane vector. It is worth noting that these separating hyperplanes are not unique.

Farkas' Lemma for Polyhedral Cone

Theorem 5 *Let* $A \in \mathcal{R}^{m\times n}$ *and* $\mathbf{b} \in \mathcal{R}^m$*. Then, the system* $\{\mathbf{x}:\ A\mathbf{x}=\mathbf{b},\ \mathbf{x}\in\mathcal{R}^n_+\}$ *has a feasible solution* $\mathbf{x}$ *if and only if that* $\{\mathbf{y}:\ -A^T\mathbf{y}\in\mathcal{R}^n_+,\ \mathbf{b}^T\mathbf{y}>0,\ (\mathbf{b}^T\mathbf{y}=1)\}$ *has no feasible solution.*

A vector $\mathbf{y}$, with $A^T\mathbf{y}\le\mathbf{0}$ and $\mathbf{b}^T\mathbf{y}>0$, is called a infeasibility certificate for the system $\{\mathbf{x}:\ A\mathbf{x}=\mathbf{b},\ \mathbf{x}\ge\mathbf{0}\}$.

Example: Let $A=(1,1)$ and $b=-1$. Then, $y=-1$ is an infeasibility certificate for $\{\mathbf{x}:\ A\mathbf{x}=b,\ \mathbf{x}\ge\mathbf{0}\}$.

Alternative Systems

Farkas' lemma is also called the alternative theorem, that is, exactly one of the two systems:

$$\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$$

and

$$\{\mathbf{y} : \ A^T\mathbf{y} \leq \mathbf{0}, \ \mathbf{b}^T\mathbf{y} > 0, \ (\mathbf{b}^T\mathbf{y} = 1)\},$$

is feasible.

Geometrically, Farkas' lemma means that if a vector $\mathbf{b} \in \mathcal{R}^m$ does not belong to the cone generated by $\mathbf{a}_{.1}, ..., \mathbf{a}_{.n}$, then there is a hyperplane separating $\mathbf{b}$ from cone$(\mathbf{a}_{.1}, ..., \mathbf{a}_{.n})$, that is,

$$\mathbf{b} \notin C := \{A\mathbf{x} : \ \mathbf{x} \geq \mathbf{0}\},$$

which is a closed convex set(?).

Proof

Let $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ have a feasible solution, say $\bar{\mathbf{x}}$. Then, $\{\mathbf{y} : \ A^T\mathbf{y} \leq \mathbf{0}, \ \mathbf{b}^T\mathbf{y} > 0\}$ is infeasible, since otherwise,

$$0 < \mathbf{b}^T\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T(A^T\mathbf{y}) \leq 0$$

since $\mathbf{x} \geq \mathbf{0}$ and $A^T\mathbf{y} \leq \mathbf{0}$.

Now let $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ have no feasible solution, that is, $\mathbf{b} \notin C := \{A\mathbf{x} : \ \mathbf{x} \geq \mathbf{0}\}$. Then, by the separating hyperplane theorem, there is $\mathbf{y}$ such that

$$\mathbf{y} \bullet \mathbf{b} > \sup_{\mathbf{c} \in C} \mathbf{y} \bullet \mathbf{c}$$

or

$$\mathbf{y} \bullet \mathbf{b} > \sup_{\mathbf{x} \geq \mathbf{0}} \mathbf{y} \bullet (A\mathbf{x}) = \sup_{\mathbf{x} \geq \mathbf{0}} A^T\mathbf{y} \bullet \mathbf{x}. \quad (1)$$

Since $\mathbf{0} \in C$ we have $\mathbf{y} \bullet \mathbf{b} > 0$.

Furthermore, $A^T\mathbf{y} \leq \mathbf{0}$. Since otherwise, say $(A^T\mathbf{y})_1 > 0$, one can have a vector $\bar{\mathbf{x}} \geq \mathbf{0}$ such that $\bar{x}_1 = \alpha > 0, \bar{x}_2 = \ldots = \bar{x}_n = 0$, from which

$$\sup_{\mathbf{x}\geq\mathbf{0}} A^T\mathbf{y} \bullet \mathbf{x} \geq A^T\mathbf{y} \bullet \bar{\mathbf{x}} = (A^T\mathbf{y})_1 \cdot \alpha$$

and it tends to ∞ as $\alpha \rightarrow \infty$. This is a contradiction because $\sup_{\mathbf{x}\geq\mathbf{0}} A^T\mathbf{y} \bullet \mathbf{x}$ is bounded from above by (1).

Farkas' Lemma variant

Theorem 6 *Let $A \in \mathcal{R}^{m \times n}$ and $\mathbf{c} \in \mathcal{R}^n$. Then, the system $\{\mathbf{y} : \ A^T\mathbf{y} \le \mathbf{c}\}$ has a solution $\mathbf{y}$ if and only if that $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} \ge \mathbf{0}$, $\mathbf{c}^T\mathbf{x} < 0$ has no feasible solution $\mathbf{x}$.*

Again, a vector $\mathbf{x} \ge \mathbf{0}$, with $A\mathbf{x} = \mathbf{0}$ and $\mathbf{c}^T\mathbf{x} < 0$, is called a infeasibility certificate for the system $\{\mathbf{y} : \ A^T\mathbf{y} \le \mathbf{c}\}$.

Example: Let $A = (1; -1)$ and $\mathbf{c} = (1; -2)$. Then, $\mathbf{x} = (1; 1)$ is an infeasibility certificate for $\{y : \ A^Ty \le \mathbf{c}\}$.

Alternative Systems for General Cone?

$$\{\mathbf{x}:\ A\mathbf{x}=\mathbf{b},\ \mathbf{x}\in C\}$$

and

$$\{\mathbf{y}:\ -A^T\mathbf{y}\in C^*,\ \mathbf{b}^T\mathbf{y}>0\}?$$

Counterexample:

$$A_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ A_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

and

$$\mathbf{b}=\begin{pmatrix}0\\ 2\end{pmatrix},\ C=\mathcal{S}^2_+.$$

Farkas Lemma for General Convex Cone

Theorem 7 *Consider system* $\{\mathbf{x} :\ A\mathbf{x} = \mathbf{b},\ \mathbf{x} \in K\}$ *for a (closed) convex cone* K. *Suppose that there exists vector* $\bar{\mathbf{y}}$ *such that* $-A^T\bar{\mathbf{y}} \in \operatorname{int} K^*$. *Then,*

- *Set* $C := \{A\mathbf{x} :\ \mathbf{x} \in K\}$ *is a closed convex set.*
- *The system* $\{\mathbf{x} :\ A\mathbf{x} = \mathbf{b},\ \mathbf{x} \in K\}$ *has a feasible solution* $\mathbf{x}$ *if and only if that* $\{\mathbf{y} :\ -A^T\mathbf{y} \in K^*,\ \mathbf{b}^T\mathbf{y} > 0,\ (\mathbf{b}^T\mathbf{y} = 1)\}$ *has no feasible solution.*

Corollary 1 *Consider system* $\{(\mathbf{y}, \mathbf{s}) :\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{s} \in K\}$ *for a (closed) convex cone* K. *Suppose that there exists vector* $\bar{\mathbf{x}}$ *such that* $A\bar{\mathbf{x}} = \mathbf{0},\ \bar{\mathbf{x}} \in \operatorname{int} K^*$. *Then,*

- *Set* $C := \{A^T\mathbf{y} + \mathbf{s} :\ \mathbf{s} \in K\}$ *is a closed convex set.*
- *The system* $\{\mathbf{x} :\ A\mathbf{x} = \mathbf{0},\ \mathbf{c} \bullet \mathbf{x} = -1,\ \mathbf{x} \in K^*\}$ *has a feasible solution* $\mathbf{x}$ *if and only if that* $\{(\mathbf{y}, \mathbf{s}) :\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{s} \in K\}$ *has no feasible solution.*