MAT-INF4110/MAT-INF9110 Mathematical optimization

Geir Dahl*

August 20, 2013

Convexity Part IV

Chapter 4 Representation of convex sets

- different representations of convex sets, boundary
- polyhedra and polytopes: facial structure

1 Faces of convex sets

- Face, Definition: Let C be a convex set. A convex subset F of C is a *face* of C if the following condition holds: if $x_1, x_2 \in C$ is such that $(1-\lambda)x_1 + \lambda x_2 \in F$ for some $0 < \lambda < 1$, then $x_1, x_2 \in F$.
- Remark: enough to check for $\lambda = 1/2$ here
- $\emptyset$, C: trivial faces
- Each nontrivial face is a subset of the relative boundary of C
- Exposed face, Definition: Let C be a convex set and H a supporting hyperplane of C. Then $C \cap H$ is called an *exposed face* of C.

*Department of Mathematics, University of Oslo, Norway. `geird@math.uio.no`

- Each exposed face is the set of optimal solutions in $\max\{c^T x : x \in C\}$ for some $c \in \mathbb{R}^n$.

- **Proposition** [Relation between faces and exposed faces] *Let C be a nonempty convex set in $\mathbb{R}^n$. Then each exposed face of C is also a face of C.*

- **Proof.** Let F be an exposed face of C. Then, for some suitable vector $c \in \mathbb{R}^n$, F is the set of optimal solutions in the problem of maximizing $c^T x$ over C. Define $v = \max\{c^T x : x \in C\}$. Thus, $F = \{x \in C : c^T x = v\}$. We noted above that F is convex, and we now verify the remaining face property. Let $x_1, x_2 \in C$ and assume that $(1-\lambda)x_1 + \lambda x_2 \in F$ for some $0 < \lambda < 1$. Then $c^T((1-\lambda)x_1 + \lambda x_2) = v$ and, moreover,

$$\begin{array}{ll} \text{(i)} & c^T x_1 \leq v \\ \text{(ii)} & c^T x_2 \leq v \end{array}$$

  by the definition of the optimal value v. Assume now that at least one of the inequalities (i) and (ii) above is strict. We multiply inequality (i) by $1-\lambda$, multiply inequality (ii) by λ and add the resulting inequalities. Since λlies strictly between 0 and 1 we then obtain

$$v > (1-\lambda)c^T x_1 + \lambda c^T x_2 = c^T((1-\lambda)x_1 + \lambda x_2) = v.$$

  From this contradiction, we conclude that $c^T x_1 = v$and $c^T x_2 = v$, so both $x_1$and $x_2$lie in F and we are done. $\square$

- **Proposition** [Face of face] *Let $C \subseteq \mathbb{R}^n$be a nonempty convex set. Let F_1 be a face of C and F_2 a face of F_1. Then F_2 is also a face of C.*

- Next: Extreme points and extreme halflines

- Extreme point, Definition: A face of F of C with $\dim(F) = 0$ consists of a single point x, and it is called an *extreme point*. Thus, there are no points $x^1, x^2 \in C$ satisfying $x^1, x^2 \neq x$ and $x = (1/2)x^1 + (1/2)x^2$.

- Notation: $\operatorname{ext}(C)$: set of extreme points for C

- For a polytope $P = \operatorname{conv}(\{x_1, \ldots, x_t\})$, then $\operatorname{ext}(C) \subseteq \{x_1, \ldots, x_t\}$.

- Face of dim. 1: either line segment, a line or a halfline. The latter called *extreme halfline*.

- The union of all extreme halflines of C is a set which we denote by $\text{exthl}\,(C)$.

- For a convex cone containing no line an extreme halfline is called an *extreme ray*.

2 The recession cone

- Question: How does an unbounded convex set "behave towards infinity"?

- For general sets this could be complicated, but convexity simplifies.

- A useful set: Let C be nonempty closed convex set in $\mathbb{R}^n$ and define

$$\text{rec}\,(C, x) = \{z \in \mathbb{R}^n : x + \lambda z \in C \ \text{ for all } \lambda \geq 0\}.$$

- $\text{rec}\,(C, x)$ is a nonempty closed convex cone.

- Now a surprising result!

- **Proposition** [On recession cones] *The closed convex cone* $\text{rec}\,(C, x)$ *does not depend on* x*.*

- **Proof.** Let x_1 and x_2 be distinct points in C and let $z \in \text{rec}\,(C, x_1)$. Consider the two halflines $R_i = \{x_i + \lambda z : \lambda \geq 0\}$ for $i = 1, 2$. Then $R_1 \subseteq C$ and we want to prove that also $R_2 \subseteq C$. To do this consider a point $y_2 \in R_2$, where $y_2 \neq x_2$, and let $w \in [x_1, y_2)$ (this set is $[x_1, y_2] \setminus \{y_2\}$). The halfline from x_2 that goes through $w$must intersect the halfline R_1 (as R_1 and R_2 lie in the same two-dimensional plane). But then, as both this intersection point and $x_2$lie in C, convexity implies that $w \in C$. It follows that $[x_1, y_2) \subseteq C$ and since C is closed this implies that $y_2 \in C$. We have therefore proved that $R_2 \subseteq C$. Since this holds for every $z \in \text{rec}\,(C, x_1)$ we conclude that $\text{rec}\,(C, x_1) \subseteq \text{rec}\,(C, x_2)$. The converse inclusion follows by similar arguments (i.e., by symmetry), so $\text{rec}\,(C, x_1) = \text{rec}\,(C, x_2)$. $\square$

- Recession cone, Definition: $\text{rec}\,(C) := \text{rec}\,(C, x)$ is called the *recession cone* (or *asymptotic cone*) of C.

- One can show: $\text{rec}(C) = \{O\}$ if and only if C is bounded.

- If C is a closed convex cone, then $\text{rec}(C) = C$.

- Lineality space, Definition: $\text{lin}(C) := \text{rec}(C) \cap (-\text{rec}(C))$ is called the *lineality space* of C. It is a subspace.

- So $\text{lin}(C)$ consists of all direction vectors of lines that are contained in C (the zero vector is included)

- Polyhedra: $\text{rec}(\{x : Ax \leq b\}) = \{x : Ax \leq O\}$.

- Example: Hyperplane $H = \{x \in \mathbb{R}^n : a^T x = \alpha\}$. Then $\text{rec}(H) = \text{lin}(H) = \{a\}^\perp$

- Line-free (pointed), Definition: C is *line-free* (or *pointed*) if it contains no lines, i.e., $\text{lin}(C) = \{O\}$.

- Any closed convex set may be decomposed as the (direct) sum of its lineality space L and a line-free convex set (contained in $L^\perp$).

- For this reason it usually suffices to study line-free sets.

3 Inner representation and Minkowski's theorem

- A main goal is to prove results saying that any closed convex set C may be written as the convex hull $\text{conv}(S)$ of a certain "small" subset S of the boundary of C.

- Examples: circle disc, triangle

- **Lemma 4.3.1** *Let C be a line-free closed convex set and assume that $x \in \text{Rint}(C)$. Then there are two distinct points $x_1, x_2 \in \text{Rbd}(C)$ such that $x \in [x_1, x_2]$.*

- **Proof.** The affine hull of C, $\text{aff}(C)$, is parallel to a unique linear subspace L, so $\text{aff}(C) = L + x_0$ for some $x_0 \in C$. We observe that $\text{rec}(C) \subseteq L$ (for if $z \in \text{rec}(C)$, then $x_0 + z \in C \subseteq x_0 + L$, so $z \in L$). Moreover, $-\text{rec}(C) \subseteq L$ as $L$is a linear subspace. Thus, $\text{rec}(C) \cup$

$(-\text{rec}\,(C)) \subseteq L$. We now prove that this inclusion is strict. For if $\text{rec}\,(C) \cup (-\text{rec}\,(C)) = L$, then it is easy to see that both $\text{rec}\,(C)$ and $-\text{rec}\,(C)$must contain some line, and this contradicts that C is line-free. Therefore, there is a vector $z \in L \setminus (\text{rec}\,(C) \cup (-\text{rec}\,(C)))$. Consider the point $x(t) = x + tz$. Since $x \in \text{Rint}\,(C)$ there is some $\lambda_0 > 0$such that $x(t) \in C$ when $|t| \leq \lambda_0$. On ther other hand, if t is large enough $x(t) \notin C$as $z \notin \text{rec}\,(C)$. Since C is closed there is a maximal t, say $t = t_1$ such that $x(t)$ lies in C. Thus, $x(t_1) \in \text{Rbd}\,(C)$. Similarly (because $z \notin -\text{rec}\,(C)$) we can find $t_2$such that $x(t_2) \in \text{Rbd}\,(C)$. Moreover, we clearly have that $x \in [x_1, x_2]$ and the proof is complete. □

- Notation: $\text{ext}\,(C)$ is the set of all extreme points
- Notation: $\text{exthl}\,(C)$ is the union of all extreme halflines of C
- The next result is very central in convexity and its applications
- **Theorem 4.3.2** Inner description of closed convex sets] *Let $C \subseteq \mathbb{R}^n$ be a nonempty and line-free closed convex set. Then $C$is the convex hull of its extreme points and extreme halflines, i.e.,*

$$C = \text{conv}\,(\text{ext}\,(C) \cup \text{exthl}\,(C)).$$

- **Proof.** We prove this by induction on $d = \dim(C)$. If $\dim(C) = 0$, C must be a one-point set, and then the result is trivial. Assume that the result holds for all line-free closed convex sets in $\mathbb{R}^n$ having dimension strictly smaller than d. Let C be a line-free closed convex set with $d = \dim(C) > 0$. Let $x \in C$. We treat two possible cases separately.

  First, assume that $x \in \text{Rbd}\,(C)$. From Proposition 3.2.3 C has a nontrivial supporting hyperplane H at x. Then the exposed face $C' := C \cap H$ of C is a strict subset of C and $\dim(C') < d$ (as C contains a point which does not lie in $\text{aff}\,(C') \subseteq H$). Since C' is convex and has dimension less than d, and $x \in C'$, we conclude from our induction hypothesis that $x$may be written as a convex combination of points in $\text{ext}\,(C') \cup \text{exthl}\,(C')$. But $\text{ext}\,(C') \subseteq \text{ext}\,(C)$and $\text{exthl}\,(C') \subseteq \text{exthl}\,(C)$ so we are done.

  The remaining case is when $x \in \text{Rint}\,(C)$ (for $\text{Rbd}\,(C) = \text{cl}\,(C) \setminus \text{Rint}\,(C) = C \setminus \text{Rint}\,(C)$ as $C$is closed). We now use Lemma 4.3.1 and conclude that x may be written as a convex combination of two points x_1 and x_2 lying on the relative boundary of C. But, by the first part of our proof, both x_1 and x_2 may be written as convex combinations of points in $\text{ext}\,(C') \cup \text{exthl}\,(C')$. A small calculation then proves that also $x$is a convex combination of points in $\text{ext}\,(C') \cup \text{exthl}\,(C')$ and we are done. □

- **Corollary 4.3.3** [Inner description – another version] *Let $C \subseteq \mathbb{R}^n$ be a nonempty and line-free closed convex set. Choose a direction vector z for each extreme halfline of C and let Z be the set of these direction vectors. Then we have that*

$$C = \text{conv}\,(\text{ext}\,(C)) + \text{rec}\,(C) = \text{conv}\,(\text{ext}\,(C)) + \text{cone}\,(Z).$$

- **Proof.** Note first that $Z \subseteq \text{rec}\,(C)$ (see an exercise) and by convexity $\text{cone}\,(Z) \subseteq \text{rec}\,(C)$ (why?). Let $x \in C$, so by Theorem 4.3.2 we may write $x$as a convex combination of points $v_1, \ldots, v_t$ in $\text{ext}\,(C)$and points $w_1, \ldots, w_r$ in $\text{exthl}\,(C)$. But every point w_j in $\text{exthl}\,(C)$ may be written $w_j = x_j + z_j$ for some $x_j \in C$ and $z_j \in Z$. From this we obtain that $x$is a convex combination of $v_1, \ldots, v_t$ plus a nonnegative combination of points $z_1, \ldots, z_r$in Z. This proves that $C \subseteq \text{conv}\,(\text{ext}\,(C)) + \text{cone}\,(Z) \subseteq \text{conv}\,(\text{ext}\,(C)) + \text{rec}\,(C)$ (the last inclusion is due to what we noted initially). Moreover, the inclusion $\text{conv}\,(\text{ext}\,(C)) + \text{rec}\,(C) \subseteq C$, follows directly from the fact that $C$is convex and the definition of the recession cone. This proves the desired equalities. □

- If a convex set is bounded, then clearly $\text{rec}\,(C) = \{0\}$.

- We get the following very important consequence of Corollary 4.3.3

- **Corollary 4.3.4** [Minkowski's theorem] *If $C \subseteq \mathbb{R}^n$is a compact convex set, then C is the convex hull of its extreme points, i.e.,* $C = \text{conv}\,(\text{ext}\,(C))$.

4 Polytopes and polyhedra

- Goal: specialize results like Minkowski to polyhedra

- Throughout: we consider nonempty line-free (pointed) polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

- Then $\text{rank}\,(A) = n$ and $m \geq n$. Let a_i be the ith row in A (treated as a column vector).

- Vertex, Definition: A point $x_0 \in P$ (polyhedron as above) is called a *vertex* of P if x_0 is the (unique) solution of n linearly independent equations from the system $Ax = b$.

- So: x_0 unique solution of a subsystem $A_0x = b_0$ for an invertible $n \times n$ submatrix A_0 of A.

- **Lemma 4.4.1** [Extreme point = vertex] *Let $x_0 \in P$. Then x_0 is a vertex of Pif and only if x_0 is an extreme point of P.*

- **Proof.** Let x_0 be a vertex of P, so there is a $n \times n$ submatrix A_0 of A and a corresponding subvector b_0 of b such that $A_0x_0 = b_0$. Assume that $x_0 = (1/2)x_1 + (1/2)x_2$ where $x_1, x_2 \in P$. Let a_i be a row of A_0. We have that $a_i^T x_1 \leq b_i$ and $a_i^T x_2 \leq b_i$ (as both points lie in P). But if one of these inequalities were strict we would get $a_i^T x_0 = (1/2)a_i^T x_1 + (1/2)a_i^T x_2 < b_i$ which is impossible (because $A_0x_0 = b_0$). This proves that $A_0x_1 = b_0$ and $A_0x_2 = b_0$. But $A_0$is nonsingular so we get $x_1 = x_2 = x_0$. This shows that x_0 is an extreme point of P.

  Conversely, assume that $x_0 \in P$ is not a vertex of P, and consider all the indices i such that $a_i^T x_0 = b_i$. Let A_0 be the submatrix of A containing the corresponding rows a_i, and let b_0 be the corresponding subvector of b. Thus, $A_0x_0 = b_0$and, since x_0 is not a vertex, the rank of $A_0$is less than n, so the kernel (nullspace) of A_0 contains a nonzero vector z, i.e., $A_0z = O$. We may now find a "small" $\epsilon > 0$ such that the two points $x_1 = x_0 + \epsilon \cdot z$ and $x_2 = x_0 - \epsilon \cdot z$ both lie in P. (For each row a_i which is not in A_0 we have $a_i^T x_0 < b_i$ and a small ϵ assures that $a_i^T(x_0 \pm \epsilon z) < b_i$). But $x_0 = (1/2)x_1 + (1/2)x_2$so $x_0$is not an extreme point. □

- An *extreme halfline* is an unbounded face of dimension 1.

- It has the form $F = x_0 + \text{cone}\,(\{z\})$ such that there are no two non-parallel vectors $z_1, z_2 \in \text{rec}\,(P)$ such that $z = z_1 + z_2$.

- **Lemma 4.4.2** [Extreme halfline] *Let $R = x_0 + \text{cone}\,(\{z\})$ be a halfline in P. Then R is an extreme halfline if and only if $A_0z = O$ for some $(n-1) \times n$ submatrix $A_0$of a with* $\text{rank}\,(A_0) = n - 1$.

- **Corollary 4.4.3** [Finiteness] *Each pointed polyhedron has a finite number of extreme points and extreme halflines.*

- **Proof.** According to Lemma 4.4.1 each vertex of a pointed polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is obtained by setting n linear inequalities inequalities among the m inequalities in the defining system $Ax \leq b$ to equality. But there are only a finite number of such choices of subsystems (in fact, at most $\begin{pmatrix} m \\ n \end{pmatrix} = m!/(n!(m-n)!)$), so the number of vertices is finite. For similar reasons the number of extreme halflines is finite (at most $m!/((n-1)!(m-n+1)!)$). $\square$

- Next: The representation theorem for polyhedra:

- This is the main theorem for polyhedra, and it is due to Motzkin, Farkas, Minkowski, Weyl

- It holds for all (also non-pointed) polyhedra.

- **Theorem 4.4.4** [Main theorem for polyhedra] *Each polyhedron $P \subseteq \mathbb{R}^n$ may be written as*

$$P = \text{conv}\,(V) + \text{cone}\,(Z)$$

for finite sets $V, Z \subset \mathbb{R}^n$. In particular, if P is pointed, we may here let V be the set of vertices and let Z consist of a direction vector of each extreme halfline of P.

Conversely, if V and Z are finite sets in $\mathbb{R}^n$, then the set $P = \text{conv}\,(V) + \text{cone}\,(Z)$ is a polyhedron. i.e., there is a matrix $A \in \mathbb{R}^{m,n}$ and a vector $b \in \mathbb{R}^m$ for some m such that

$$\text{conv}\,(V) + \text{cone}\,(Z) = \{x \in \mathbb{R}^n : Ax \leq b\}.$$

- **Proof.** Consider first a pointed polyhedron $P \subseteq \mathbb{R}^n$. Due to Corollary 4.4.3 P has a finite number of extreme halflines. Moreover the set V of vertices is finite (Lemma 4.4.1). Let Z be the finite set consisting of a direction vector of each of these extreme halflines. It follows from Corollary 4.3.3 that

$$P = \text{conv}\,(V) + \text{cone}\,(Z).$$

This proves the first part of the theorem when $P$is pointed. If $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is not pointed, we recall that $P = P' \oplus \text{lin}\,(P)$ where $\text{lin}\,(P)$ is the lineality space of P and $P'$is pointed. In fact, we may let P' be the pointed polyhedron

$$P' = \{x \in \mathbb{R}^n : Ax \leq b,\ Bx = O\}$$

where the rows $b_1, \ldots, b_k \in \mathbb{R}^n$ of the $k \times n$-matrix B is a basis of the linear subspace $\text{lin}\,(P)$. By the first part of the theorem (as P' is pointed) there are finite sets V and Z'such that $P = \text{conv}\,(V) + \text{cone}\,(Z')$. We now note that $\text{lin}\,(P) = \text{cone}\,(b_0, b_1, \ldots, b_k)$ where $b_0 = -\sum_{j=1}^{k} b_j$ (see an exercise). But then it is easy to check that $P = \text{conv}\,(V) + \text{cone}\,(Z)$ where $Z = Z' \cup \{b_0, b_1, \ldots, b_k\}$ so the first part of the theorem is shown.

We shall prove the second part by using what we just showed in a certain (clever!) way.

First, we prove the result for convex cones, so assume that $C$is a finitely generated cone, say $C = \text{cone}\,(\{z_1, \ldots, z_t\})$. We introduce the set

$$C^\circ = \{a \in \mathbb{R}^n : z_j^T a \leq 0 \ \text{ for } j = 1, \ldots, t\}.$$

The main observation is that $C^\circ$is a polyhedral cone (and therefore a polyhedron) in $\mathbb{R}^n$: it is defined by the linear and homogeneous inequalities $z_j^T a \leq 0$ for $j = 1, \ldots, t$ (here a is the variable vector!). Thus, by the first part of the theorem, there is a finite set of vectors $a_1, \ldots, a_s \in C^\circ$ such that $C^\circ = \text{cone}\,(\{a_1, \ldots, a_s\})$ (because any polyhedral cone has only one vertex, namely O, see an exercise). We shall prove that

$$(*) \quad C = \{x \in \mathbb{R}^n : a_i^T x \leq 0 \ \text{ for } i = 1, \ldots, s\}.$$

If $x_0 \in C$, then $x_0 = \sum_{j=1}^{t} \mu_j z_j$for some $\mu_j \geq 0$, $j \leq t$. For each $i \leq s$ and $j \leq t$ we have from the definition of C° that $a^T z_j \leq 0$ and therefore $a^T x_0 = \sum_{j=1}^{t} \mu_j a^T z_j \leq 0$. This shows the inclusion "$\subseteq$" in $(*)$. Assume next that $x_0 \notin C$. As $C = \text{cone}\,(\{z_1, \ldots, z_t\})$ it follows from Farkas' lemma (Theorem 3.2.5) that there is a vector $y \in \mathbb{R}^n$ such that $y^T x_0 > 0$and $y^T z_j < 0$for each j. Therefore $y \in C^\circ$ so $y = \sum_i \lambda_i a_i$ for nonnegative numbers λ_i $(i \leq s)$. But x_0 violates the inequality

$y^T x \leq 0$ (as $y^T x_0 > 0$). This implies that $x_0 \notin \{x \in \mathbb{R}^n : a_i^T x \leq 0 \ \text{ for } i = 1, \ldots, s\}$. This proves $(*)$ and we have shown that every finitely generated convex cone is polyhedral.

More generally, let $P = \mathrm{conv}\,(V) + \mathrm{cone}\,(Z)$ where V and $Z$are finite sets in $\mathbb{R}^n$. Let $V = \{v_1, \ldots, v_k\}$ and $Z = \{z_1, \ldots, z_m\}$. Let $C = \mathrm{cone}\,(\{(v_1, 1), \ldots, (v_k, 1), (z_1, 0), \ldots, (z_m, 0)\})$, so this is a finitely generated convex cone in $\mathbb{R}^{n+1}$. By what we just showed this cone is polyhedral, so there is a matrix $A \in \mathbb{R}^{m,n+1}$ such that $C = \{(x, x_{n+1}) : \sum_{j=1}^n a^j x_j + a^{n+1} x_{n+1} \leq O\}$ (here a^j is the jth column of A). Note that $x \in P$ if and only if $(x, 1) \in C$ (see an exercise). Therefore, $x \in P$ if and only if $Ax + a^{n+1} \cdot 1 \leq O$, i.e., $Ax \leq b$ where $b = -a^{n+1}$. This proves that P is a polyhedron and our proof is complete. $\square$

- **Corollary 4.4.5** [Polytopes] *A set is a polytope if and only if it is a bounded polyhedron.*

- Example: Consider the set

$$P = \{x \in \mathbb{R}^n : 1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$

which is a polyhedron. Actually, it is bounded (as P is contained in the unit cube $K = \{x : 0 \leq x_i \leq 1 \ (i \leq n)\}$). So, by Corollary 4.4.5 P is also a polytope, i.e., the convex hull of a finite set of points, say V. We may here use V as the set of extreme points, or, as we just saw, the set of vertices of P. Let us find the vertices! Since P is defined by $n + 1$ inequalities, any vertex is obtained by removing one inequality and solve for the remaining n equations. If we delete the first inequality $1 \geq x_1$, and solve the remaining equations, we get $x_1 = x_2 = \cdots = x_n = 0$, so $x = O$ is a vertex (as we see that it also lies in P). Further, if we delete the inequality $x_k \geq x_{k+1}$ and solve the remaining equations, we get $x_1 = \cdots = x_k = 1$ and $x_{k+1} = \cdots = x_n = 0$. This gives the vertex $(1, \ldots, 1, 0, \ldots, 0)$ with k leading ones. We have then found all the vertices; there were $n + 1$ such points. Actually, one can show that these points are affinely independent, so P is a simplex. Try the same method for finding the extreme points when P is modified by adding the inequality $2x_1 + x_2 \geq 1$ (we assume here that $n \geq 3$).

- We next discuss an important application

- ... which is Combinatorial optimization and $(0, 1)$-polytopes:

- CO: class $\mathcal{F}$ of (feasible) subsets of a finite set E; let $n = |E|$. And a weight/cost function $c : E \to \mathbb{R}$. Find a set $F \in \mathcal{F}$ with $\sum_{e \in F} w_e$ largest possible (or smallest possible).

- Represent each subset F by its incidence vector (indicator vector/function) χ^F, χ^F_j is 1 if $j \in E$, and equals 0 otherwise.

- Define $P_{\mathcal{F}}$ as the convex hull of all these incidence vectors. A polytope. The vertices are these incidence vectors (all of them; why?).

- By the Main Theorem, $P_{\mathcal{F}}$ is a bounded polyhedron, so its points are the solutions to some linear system $Ax \leq b$.

- Thus: CO is, in principle, reduced to an LP problem since optimal solutions are in the vertices and $c^T \chi^F = \sum_{j \in F} c_j$.

- Example: TSP (Traveling Salesman Problem): graph, tour through all points, length, minimize length.

- The *TSP polytope* P_{TSP} is the convex hull of incidence vectors of tours. Well studied, but very complicated boundary (faces)!

- This approach to CO problems is an area called *polyhedral combinatorics*.

- More about this approach in the second part of the course!

- Now, we consider faces more closely

- Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

- Valid inequality, Definition: an inequality $c^T x \leq \alpha$ is *valid* for P if $P \subseteq \{x \in \mathbb{R}^n : c^T x \leq \alpha\}$, i.e., $c^T x \leq \alpha$ for all $x \in P$.

- The next result says how exposed faces arise from subsystems of the inequalities in $Ax \leq b$.

- **Theorem 4.4.6** *Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. A nonempty set F is an exposed face of P if and only if*

  $$(*) \quad F = \{x \in P : A'x = b'\}$$

  for some subsystem $A'x \leq b'$ of $Ax \leq b$.

- **Proof.** Let a_i denote the ith row of A (viewed as a column vector). Let F be a nonempty exposed face of P, so $F = \{x \in P : c^T x = \alpha\}$ where $c^T x = \alpha$ defines a supporting halfspace for P. Then the optimal value v^* of the linear programming problem (P) max $\{c^T x : Ax \leq b\}$ satisfies $v^* = \alpha < \infty$ and we have $F = \{x \in P : c^T x = v^*\}$. From the LP duality theorem we have that

  $$v^* = \max\{c^T x : Ax \leq b\} = \min\{y^T b : y^T A = c^T, \ \ y \geq O\}$$

  and the dual LP must be feasible (as the primal optimal value is finite). Let y^* be an optimal dual solution, so $(y^*)^T A = c^T$, $y^* \geq O$ and $(y^*)^T b = v^*$, and define $I' = \{i \leq m : y_i > 0\}$. We claim that $(*)$ holds with the subsystem $A'x \leq b'$ consisting of the inequalities $a_i^T x \leq b_i$ for $i \in I'$. To see this, note that for each $x \in P$ we have

  $$c^T x = (y^*)^T Ax = \sum_{i \in I} y_i^* (Ax)_i = \sum_{i \in I'} y_i^* (Ax)_i \leq \sum_{i \in I'} y_i^* b_i = v^*.$$

  Thus we have $c^T x = v^*$ if and only if $a_i^T x = b_i$ for each $i \in I'$, and $(*)$ holds.

  Conversely, assume that the set F satisfies $(*)$ for some subsystem $A'x \leq b'$ of $Ax \leq b$, say that $A'x \leq b'$ consists of the inequalities $a_i^T x \leq b_i$ for $i \in I'$ (where $I' \subseteq \{1, \ldots, m\}$). Let $c = \sum_{i \in I'} a_i$ and $\alpha = \sum_{i \in I'} b_i$. Then $c^T x \leq \alpha$ is a valid inequality for P (it is a sum of other valid inequalities, see an exercise. Furthermore F is the face induced by $c^T x \leq \alpha$, i.e., $F = \{x \in P : c^T x = \alpha\}$. In fact, a point $x \in P$ satisfies $c^T x = \alpha$ if and only if $a_i^T x = b_i$ for each $i \in I'$. $\square$

- **Theorem 4.4.7** *Let P be a polyhedron in $\mathbb{R}^n$. Then every face of P is also an exposed face of P. Thus, for polyhedra, these two notions coincide.*

- Facet, definition: a *facet* of a polyhedron $P \subseteq \mathbb{R}^n$ may be defined as a face F of P with $\dim(F) = \dim(P) - 1$.

- Let P be a full-dimeensional polyhedron P in $\mathbb{R}^3$: *Euler's relation* says that

  $$v - e + f = 2.$$

  where v, e and f are the number of vertices, edges and facets, respectively.

- May be generalized to higher dimensions.
- Facets are interesting because they are related to minimial representations of polyhedra: you only need one inequality for every facet (plus, possibly, equations for defining the affine hull).

SUMMARY OF NEW CONCEPTS AND RESULTS:

- face, exposed face
- extreme point, extreme halfline, ray
- recession cone
- inner representation, Minkowski's theorem
- main theorem for polyhedra