(i) The probability of picking six tickets of the same colour is given by

$$\text{Pr(six of the same colour)} = 3 \text{ x } \frac{6!}{6!0!0!} \ (\tfrac{1}{3})^6 \ (\tfrac{1}{3})^0 \ (\tfrac{1}{3})^0 \ = \ \frac{1}{243}.$$

The factor of 3 is present because there are three different colours.

(ii) The probability of picking five tickets of one colour and one ticket of another colour is

$$\text{Pr(five of one colour; one of another)} = 3 \text{ x } 2 \text{ x } \frac{6!}{5!1!0!} \ (\tfrac{1}{3})^5 \ (\tfrac{1}{3})^1 \ (\tfrac{1}{3})^0 = \ \frac{4}{81}.$$

The factors or 3 and 2 are included because there are three ways to choose the colour of the five matching tickets, and then two ways to choose the colour of the remaining ticket.

(iii) Finally, the probability of picking two tickets of each colour is

$$\text{Pr(two of each colour)} = \frac{6!}{2!2!2!} \ (\tfrac{1}{3})^2 \ (\tfrac{1}{3})^2 \ (\tfrac{1}{3})^2 \ = \ \frac{10}{81}.$$

Thus the expected return to any patron was, in pence,

$$100 \ (\tfrac{1}{243} + \tfrac{4}{81}) + (40 \times \tfrac{10}{81}) = 10.29.$$

A good time was had by all but the stallholder! ◄

30.15.2 The multivariate Gaussian distribution

A particularly interesting multivariate distribution is provided by the generalisation of the Gaussian distribution to multiple random variables $X_i, \ i = 1, 2, \ldots, n$. If the expectation value of X_i is $E(X_i) = \mu_i$ then the general form of the PDF is given by

$$f(x_1, x_2, \ldots, x_n) = N \text{ exp } [-\tfrac{1}{2} \textstyle\sum_i \sum_j a_{ij}(x_i - \mu_i)(x_j - \mu_j)],$$

where $a_{ij} = a_{ji}$ and N is a normalisation constant that we give below. If we write the column vectors x = $(x_1 \ \ x_2 \ \ \cdot \ \cdot \ \cdot \ \ x_n)^T$ and $\mu = (\mu_1 \ \ \mu_2 \ \ \cdots \ \ \mu_n)^T$, and denote the matrix with elements a_{ij} by A then

$$f(\text{x}) = f(x_1, x_2, \ldots, x_n) = N \text{ exp } [-\tfrac{1}{2}(\text{x}-\mu)^T A(\text{x}-\mu)],$$

where A is symmetric. Using the same method as that used to derive (30.145) it is straightforward to show that the MGF of $f(\text{x})$ is given by

$$M(t_1, t_2, \ldots, t_n) = \text{exp } (\mu^T \text{t} + \tfrac{1}{2} \text{t}^T \text{A}^{-1} \text{t}),$$

where the column matrix t = $(t_1 \ t_2 \cdots t_n)^T$. From the MGF, we find that

$$E[X_i X_j] = \frac{\partial^2 M(0,0,\ldots,0)}{\partial t_i \partial t_j} = \mu_i \mu_j + (\text{A}^{-1})_{ij},$$