Technical Report

Generalised Negative Binomial Distribution

Maciej J Swat

July 17, 2015

0.1 Introduction

The negative binomial distribution, NB, was first proposed almost 100 years ago in 1919 by Greenwood & Woods, [1], but its generalisation, called the *generalised negative binomial distribution*, GNB, was proved more then 50 years later by Jain & Consul (1971), [2].
The PMF of the GNB is defined for $0 < p < 1$ and $|\alpha\beta| < 1$ and reads in Greenwood & Woods paper

$$b_\beta(x, n, \alpha) = \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\,\alpha^x (1 - \alpha)^{n+\beta x-x}, n > 0, x = 0, 1, 2, 3, ...$$

such that $b_\beta(x, n, \alpha) = 0$ for $x \leq m$ if $n + \beta m < 0$. Interestingly, following distributions are special cases of the GNB distribution

- binomial, B(n,p)
- negative binomial, NB(r,p)[1]
- inverse binomial, IB(k,p)

which will be shown in the following sections.

0.2 GNB(α,β) $\rightarrow$ B(n,p)

According to Jain & Consul, [2], GNB reduces to B for $\beta = 0$ and indeed this can be shown (replacing α with p) as follows

$$b_\beta(x, n, \alpha) \rightarrow P_B(x; n, p) : \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\,\alpha^x (1 - \alpha)^{n+\beta x-x} \rightarrow \frac{n\,\Gamma(n)}{x!\,\Gamma(n - x + 1)}\,p^x (1 - p)^{n-x}$$

with the first term $\frac{n\,\Gamma(n)}{n!\,\Gamma(n-x+1)} = \frac{n(n-1)!}{x!(n-x)!} = \frac{n!}{x!(n-x)!} = \binom{n}{x}$ we get the expected result

$$P_B(x; n, p) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

0.3 GNB(α,β) $\rightarrow$ NB(r,p)

According also to Jain & Consul, [2], GNB reduces to NB for $\beta = 1$ and indeed this can be shown (replacing α with p and n with r) as follows

$$b_\beta(x, n, \alpha) \rightarrow P_{NB}(x; r, p) : \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\,\alpha^x (1 - \alpha)^{n+\beta x-x} \rightarrow \frac{r\,\Gamma(r + x)}{x!\,\Gamma(r + 1)}\,p^x (1 - p)^r$$

with the first term $\frac{r\,\Gamma(r+x)}{x!\,\Gamma(r+1)} = \frac{r\,(r+x-1)!}{x!\,r!} = \frac{(r+x-1)!}{x!\,(r-1)!} = \binom{r+x-1}{x}$ we get the correct PMF

$$P_{NB}(x; r, p) = \binom{r + x - 1}{x} (1 - p)^r p^x.$$

0.4 GNB(α,β) $\rightarrow$ IB(k,p)

Yanagimoto, [4], proposed the derivation of the new distribution IB from GNB for $\beta = 2, \alpha = 1 - p$ and $n = k$, which can be achieved as the following derivation shows

$$b_\beta(x, n, \alpha) \rightarrow P_{IB}(x; k, p) : \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\,\alpha^x (1 - \alpha)^{n+\beta x-x} \rightarrow \frac{k\,\Gamma(k + 2x)}{x!\,\Gamma(k + x + 1)}\,(1 - p)^x p^{k+x}$$

and the result follows in agreement with the formulation in [4]

$$P_{IB}(x; k, p) = \frac{k\,\Gamma(2x + k)}{\Gamma(x + 1)\,\Gamma(x + k + 1)}\,(1 - p)^x p^{k+x},$$

and from $|\alpha\beta| < 1$ we can derive the required condition for p, $1/2 < p < 1$.

[1] This corresponds to the NB1 parameterisation of the negative binomial distribution in ProbOnto, [3].

Bibliography

[1] Major Greenwood and G Udny Yule. An inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents. *Journal of the Royal statistical society*, pages 255–279, 1920.

[2] GC Jain and PC Consul. A generalized negative binomial distribution. *SIAM Journal on Applied Mathematics*, 21(4):501–513, 1971.

[3] Maciej J Swat, Pierre Grenon, Florent Yvon, Sarala Wimalaratne, and Niels Rode Kristensen. Extenstions in PharmML 0.7. Technical report, EMBL-EBI, July 2015.

[4] Takemi Yanagimoto. The inverse binomial distribution as a statistical model. *Communications in Statistics-Theory and Methods*, 18(10):3625–3633, 1989.