TECHNICAL REPORT

GNB Distribution Family

Maciej J SWAT

July 20, 2015

Appendix A

Generalised Negative Binomial Distribution

A.1 Introduction

The *negative binomial* distribution, NB, was first proposed almost 100 years ago in 1920 by Greenwood & Woods, [1], but its generalisation for both *binomial* and NB distributions called the *generalised negative binomial distribution*, GNB, was discovered more then 50 years later by Jain & Consul (1971), [2]. The PMF of the GNB is defined for $0 < \alpha < 1$ and $|\alpha\beta| < 1$ and reads in Jain & Consul paper

$$b_\beta(x, n, \alpha) = \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\;\alpha^x (1-\alpha)^{n+\beta x - x}, n > 0, x = 0, 1, 2, 3, ...$$

such that $b_\beta(x, n, \alpha) = 0$ for $x \leq m$ if $n + \beta m < 0$.
Interestingly, following distributions are special cases of the GNB distribution

- binomial, B(n,p)
- negative binomial, NB(r,p)[1]
- inverse binomial, IB(k,p)

which will be shown in the following sections.

A.2 GNB(α,β) $\rightarrow$ B(n,p)

According to Jain & Consul, [2], GNB reduces to B for $\beta = 0$ and indeed this can be shown (replacing α with p) as follows

$$b_\beta(x, n, \alpha) \rightarrow P_B(x; n, p) : \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\;\alpha^x (1-\alpha)^{n+\beta x - x} \rightarrow \frac{n\,\Gamma(n)}{x!\,\Gamma(n - x + 1)}\;p^x (1-p)^{n-x}$$

with the first term in the last expression $\frac{n\,\Gamma(n)}{x!\,\Gamma(n-x+1)} = \frac{n(n-1)!}{x!(n-x)!} = \frac{n!}{x!(n-x)!} = \binom{n}{x}$ we get the expected result

$$P_B(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}.$$

A.3 GNB(α,β) $\rightarrow$ NB(r,p)

According also to Jain & Consul, [2], GNB reduces to NB for $\beta = 1$ and indeed this can be shown (replacing α with p and n with r) as follows

$$b_\beta(x, n, \alpha) \rightarrow P_{NB}(x; r, p) : \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\;\alpha^x (1-\alpha)^{n+\beta x - x} \rightarrow \frac{r\,\Gamma(r + x)}{x!\,\Gamma(r + 1)}\;p^x (1-p)^{r}$$

with the first term $\frac{r\,\Gamma(r+x)}{x!\,\Gamma(r+1)} = \frac{r\,(r+x-1)!}{x!\,r!} = \frac{(r+x-1)!}{x!\,(r-1)!} = \binom{r+x-1}{x}$ we get the correct PMF

$$P_{NB}(x; r, p) = \binom{r + x - 1}{x} p^x (1-p)^r.$$

[1]This corresponds to the NB1 parameterisation of the negative binomial distribution in ProbOnto, [3].

A.4 GNB(α,β) $\rightarrow$ IB(k,p)

Yanagimoto, [4], proposed the *inverse binomial* distribution as another special case of GNB for $\beta = 2, \alpha = 1 - p$ and $n = k$, which can be derived as the following shows

$$b_\beta(x, n, \alpha) \rightarrow P_{IB}(x; k, p) : \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\,\alpha^x (1-\alpha)^{n+\beta x - x} \rightarrow \frac{k\,\Gamma(k + 2x)}{x!\,\Gamma(k + x + 1)}\,(1-p)^x p^{k+x}$$

and the result follows in agreement with the formulation in [4], i.e.

$$P_{IB}(x; k, p) = \frac{k\,\Gamma(2x + k)}{\Gamma(x + 1)\,\Gamma(x + k + 1)}\,p^{k+x}(1-p)^x,$$

and from $|\alpha\beta| < 1$ and $0 < \alpha < 1$ one can derive the required condition for p, $1/2 < p < 1$.

A.5 Bios

Here short bios of the people behind these distributions:

- Greenwood and Yule (1920), 'An inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents':
  - Major Greenwood FRS (9 August 1880 - 5 October 1949) was an English epidemiologist and statistician born in Shoreditch in London's East End. He was elected President of the Royal Statistical Society in 1934 and awarded its Guy Medal in Gold in 1945.
  - Udny Yule FRS (18 February 1871 - 26 June 1951) was a Scottish statistician, born in Morham, near Haddington. He was active in the Royal Statistical Society, was also awarded its Guy Medal in Gold in 1911, and served as its president in 1924-26.
- Jain and Consul (1971), 'A generalized negative binomial distribution':
  - about Jain nothing is known on the web.
  - Prem C. Consul is a Canadian professor emeritus at the Department of Mathematics and Statistics, University of Calgary, and author of books on Generalised Poisson and Lagrangian distributions `http://math.ucalgary.ca/math_unitis/profiles/prem-c-consul`
- Yanagimoto (1989), 'The inverse binomial distribution as a statistical model':
  - Takemi Yanagimoto - Japanese professor at the Institute of Statistical Mathematics in Tokyo. `http://www.ism.ac.jp/~yanagmt/eng.html`

Bibliography

[1] Major Greenwood and G Udny Yule. An inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents. *Journal of the Royal statistical society*, pages 255–279, 1920.

[2] GC Jain and PC Consul. A generalized negative binomial distribution. *SIAM Journal on Applied Mathematics*, 21(4):501–513, 1971.

[3] Maciej J Swat, Pierre Grenon, Florent Yvon, Sarala Wimalaratne, and Niels Rode Kristensen. Extenstions in PharmML 0.7. Technical report, EMBL-EBI, July 2015.

[4] Takemi Yanagimoto. The inverse binomial distribution as a statistical model. *Communications in Statistics-Theory and Methods*, 18(10):3625–3633, 1989.